# Optimal embeddings of Bessel-potential-type spaces into generalized Hölder spaces

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   - Better target spaces
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   - Banach function space $X(\mathbb{R}^n)$
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3. Bessel-potential-type spaces and optimal embeddings
   - Bessel-potential-type spaces
   - Generalized Hölder spaces
   - Optimal Embeddings: super-limiting case
   - Optimal Embeddings: limiting case
   - Bessel-potential spaces $H^\sigma X(\mathbb{R}^n)$ and optimal embeddings
   - Optimal Embeddings: limiting case-Again!
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4. References

# Classical results

## Sobolev's classical embedding theorem

$\Omega \subset \mathbb{R}^n$ is a domain with a sufficient smooth boundary

- Limiting case, *i.e.*, $p = \frac{n}{k} \geq 1$

  $$W_p^k(\Omega) \hookrightarrow L_q(\Omega), \quad \text{for all} \quad q \in [p, +\infty),$$

  ($q = +\infty$, if $p = 1$ so that $k = n$).
- Super-limiting case, *i.e.*, $p > n/k$, $W_p^k(\Omega) \hookrightarrow C_B(\Omega)$.

  When $k = 1 + n/p \in \mathbb{N}$,

  $$W_p^k(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \quad \text{for all} \quad \alpha \in (0, 1).$$

  If $p = 1$, we can have $\alpha = 1$ - Lipschitz space.

# Better target spaces

## Super-Limiting case: "almost" Lipschitz functions

Brézis-Wainger [1980],

$$H_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n})$$

with $\lambda(t) := t|\log t|^{\frac{1}{p'}}$, $t \in (0, \frac{1}{2})$, which implies, for some positive constant $c$,

$$|f(x) - f(y)| \leq c\|f|H_p^{1+n/p}\| \, |x-y| \, |\log|x-y||^{\frac{1}{p'}}$$

for all $f \in H_p^{1+n/p}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$ such that $0 < |x-y| < \frac{1}{2}$.

# Banach function norm $\rho$

$\rho : \mathcal{M}^+(\mathbb{R}^n) \longrightarrow [0, \infty]$

- $\rho(f) = 0 \iff f = 0$ a.e. in $\mathbb{R}^n$
- $\rho(\alpha f) = \alpha \rho(f)$ if $\alpha \geq 0$
- $\rho(f + g) \leq \rho(f) + \rho(g)$
- $0 \leq g \leq f$ a.e. $\Longrightarrow \rho(g) \leq \rho(f)$ (lattice property)
- $0 \leq f_n \nearrow f$ a.e. $\Longrightarrow \rho(f_n) \nearrow \rho(f)$ (Fatou property)
- $|E|_n < \infty \Longrightarrow \rho(\chi_E) < \infty$
- $|E|_n < \infty \Longrightarrow \int_E f(x)\,dx \leq C_E\,\rho(f)\ \forall f \in \mathcal{M}^+(\mathbb{R}^n)$, $0 < C_E < \infty$

# Banach function space $X(\mathbb{R}^n)$

$\rho$ Banach function norm on $\mathcal{M}(\mathbb{R}^n)$

BFS $X(\mathbb{R}^n) = X_\rho(\mathbb{R}^n) := \{f \in \mathcal{M}(\mathbb{R}^n) : \rho(|f|) < \infty\}$

$\|f\|_X := \rho(|f|)$

associate BFS $X'(\mathbb{R}^n) := X_{\rho'}(\mathbb{R}^n)$

associate norm $\rho'$ $\rho'(g) := sup\{\int_{\mathbb{R}^n} f(x)g(x)\,dx \colon \rho(f) \leq 1\}$

## Example

$L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $\rho_p(f) := \begin{cases} \left(\int_{\mathbb{R}^n} f^p(x)\,dx\right)^{1/p} & \text{if } 1 \leq p < \infty \\ ess\,sup_{\mathbb{R}^n} f(x) & \text{if } p = \infty \end{cases}$

$(L^p(\Omega))' = L^{p'}(\mathbb{R}^n)$, $1/p + 1/p' = 1$

# Non-increasing rearrangement $f^*$

## Definition (the distribution function)

$$\mu_f(\lambda) := |\{x \in \mathbb{R}^n : \ |f(x)| > \lambda\}|_n, \quad \lambda \geq 0$$

## Theorem

$$\int_\Omega |f(x)|^q \, dx = q \int_0^\infty \lambda^{q-1} \mu_f(\lambda) \, d\lambda \quad \forall \, q \in (0, \infty).$$

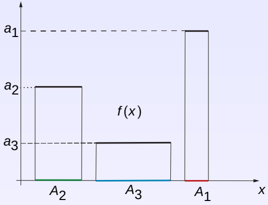
$g$ equimeasurable with $f$ if $\mu_g(\lambda) = \mu_f(\lambda), \quad \lambda \geq 0$

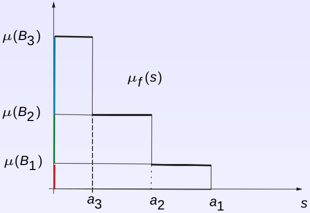
## Definition (the non-increasing rearrangement)

$$f^*(t) := \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\} = |\{\lambda \geq 0; \ |\mu_f(\lambda)| > t\}|_1, \quad t \geq 0$$

$f$ and $f^*$ are equimeasurable: $\mu_f(\lambda) = \mu_{f^*}(\lambda), \qquad \lambda \geq 0.$

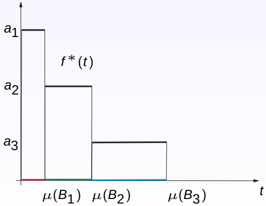
# Non-increasing rearrangement $f^*$: Example



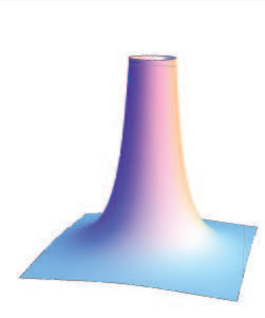

$$f(x) = \sum_{j=1}^{3} a_j \chi_{A_j}(x)$$

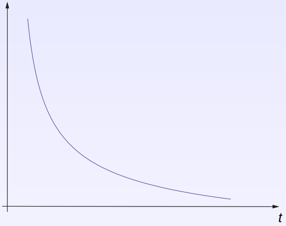
$$\mu_f(s) = \sum_{k=1}^{3} \underbrace{\sum_{j=1}^{k} \mu(A_j)}_{=:\mu(B_k)} \chi_{[a_{k+1},a_k)}(s) \qquad (a_4 := 0)$$




$$f^*(t) = \sum_{j=1}^{3} a_j \chi_{[\mu(B_{j-1}),\mu(B_j))}(t) \qquad (\mu(B_0) := 0)$$

# Non-increasing rearrangement $f^*$: Example






$$I_1(x) = |x|^{-1}, \quad x \in \mathbb{R}^2 \setminus \{0\}$$

$$(I_1)^*(t) = \left(\frac{t}{\pi}\right)^{-1/2}, \quad t > 0.$$

$$I_\sigma(x) = |x|^{\sigma - n}, \quad x \in \mathbb{R}^n \setminus \{0\},$$
$$0 < \sigma < n$$

$$(I_\sigma)^*(t) = \left(\frac{t}{\omega_n}\right)^{\sigma/n - 1}, \quad t > 0.$$

# r.i. Banach function space

BFS $X$ is rearrangement invariant (r.i.) $\Longleftrightarrow$

$f \in X$, $g$ equimeasurable with $f$, then $g \in X$ and $\|g\|_X = \|f\|_X$

Luxemburg representation theorem $\Longrightarrow$

If $X(\mathbb{R}^n)$ is a r.i. BFS, then $\exists$ a r.i. BFS $\overline{X}$ over $((0,\infty),\mu_1)$ s.t.

$$\|f\|_X = \|f^*\|_{\overline{X}} \quad \forall\, f \in X(\mathbb{R}^n).$$

$\overline{X}$ representation space of $X(\mathbb{R}^n)$

**Example**

$$\int_{\mathbf{R}^n} |f(x)|^p\,dx = \int_0^\infty f^*(t)^p\,dt, \quad 0 < p < \infty, \quad f \in \mathcal{M}(\mathbb{R}^n)$$

$$ess\,sup_{\mathbf{R}^n} f = f^*(0), \quad f \in \mathcal{M}(\mathbb{R}^n)$$

$L^p(\mathbb{R}^n)$, $1 \le p \le \infty$ are r.i. Banach function spaces

# Lorentz-Karamata spaces

## Definition

$b$ slowly varying function on $(0, +\infty)$ ($b \in SV(0, +\infty)$):

- Lebesgue measurable function $b : (0, +\infty) \longrightarrow (0, +\infty)$;
- for each $\epsilon > 0$,

$$t^{\epsilon} b(t) \approx f_{\epsilon}(t) \nearrow \text{ and } t^{-\epsilon} b(t) \approx f_{-\epsilon}(t) \searrow .$$

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## Examples

Let $\alpha, \beta \in \mathbb{R}$:

- $b(t) = (1 + |\log t|)^{\alpha}(1 + \log(1 + |\log t|))^{\beta}$;
- $b(t) = \exp(|\log t|^{\alpha})$, $0 < \alpha < 1$.

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## Lorentz-Karamata space $L_{p,q;b}(\mathbb{R}^n)$

$$\|f\|_{p,q;b} := \|t^{\frac{1}{p}-\frac{1}{q}} b(t)\, f^*(t)\|_{q;(0,+\infty)}$$

## Example

$b = \ell^{\boldsymbol{\alpha}}$ - product of powers of iterated “logs”

Logarithmic Bessel-potential-type spaces (Edmunds, Gurka and Opic, 1997).

# Bessel-potential-type spaces

Let $\sigma > 0$, $p \in (1, +\infty)$, $q \in [1, +\infty]$ and $b \in SV(0, +\infty)$. Let $g_\sigma$ be *Bessel kernel*.

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) = \{u : u = g_\sigma * f,\ f \in L_{p,q;b}(\mathbb{R}^n)\}$$

endowed with the (quasi-)norm $\|u\|_{\sigma;p,q;b} := \|f\|_{p,q;b}$.

Notation: $H^0 L_{p,q;b}(\mathbb{R}^n) = L_{p,q;b}(\mathbb{R}^n)$.

## Bessel Kernel

$g_\sigma$, com $\sigma > 0$:

$$\widehat{g_\sigma}(\xi) = (2\pi)^{-n/2}(1 + |\xi|^2)^{-\sigma/2}, \xi \in \mathbb{R}^n.$$

where the Fourier transform $\hat{f}$ of a function $f$ is given by

$\hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx$

# Bessel-potential-type spaces

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**Theorem** [Neves 2004 (Edmunds, Gurka and Opic, 1997 - Logarithmic Bessel-potential-type spaces)]

If $k \in \mathbb{N}$, $p \in (1, +\infty)$ and $q \in (1, +\infty)$

$$H^k L_{p,q;b}(\mathbb{R}^n) = W^k L_{p,q;b}(\mathbb{R}^n)$$

and the corresponding (quasi-)norms are equivalent, where

$$W^k L_{p,q;b}(\mathbb{R}^n) := \{u :\ D^\alpha u \in L_{p,q;b}(\mathbb{R}^n), |\alpha| \leq k\}$$

and

$$\|u\|_{W^k L_{p,q;b}(\mathbb{R}^n)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p,q;b}.$$

## Notation

$h \in \mathbb{R}^n, \quad f \in C_B(\mathbb{R}^n)$

$\Delta_h f(x) := f(x+h) - f(x), \quad x \in \mathbb{R}^n$

$\Delta_h^{k+1} f(x) := \Delta_h(\Delta_h^k f)(x), \quad x \in \mathbb{R}^n, \ k \in \mathbb{N}$

$\omega_k(f,t) := \sup_{|h|\le t} \|\Delta_h^k f\|_\infty, \quad t \ge 0.$
$\omega(f,t) := \omega_1(f,t).$

## Definition (the class $\mathcal{L}_r^k$ )

*the class* $\mathcal{L}_r^k$, $k \in \mathbb{N}$, $r \in (0,+\infty]$, *consists of all continuous functions* $\lambda : (0,1) \to (0,+\infty)$ *such that*

$$\left\| t^{-1/r} \frac{1}{\lambda(t)} \right\|_{r;(0,1)} = +\infty \tag{1}$$

$$\left\| t^{-1/r} \frac{t^k}{\lambda(t)} \right\|_{r;(0,1)} < +\infty \tag{2}$$

- $k$, $r$ and $\lambda$ are connected by (1) and (2)

# the generalized Hölder space

## Definition

$k \in \mathbb{N}$, $r \in (0, +\infty]$, $\lambda \in \mathcal{L}_r^k$

$$\Lambda_{\infty,r}^{k;\lambda(\cdot)}(\overline{\mathbb{R}^n}) := \{f \in C_B(\mathbb{R}^n);\ \|f|\Lambda_{\infty,r}^{k,\lambda(\cdot)}(\overline{\mathbb{R}^n})\| < +\infty\}$$

$$\|f|\Lambda_{\infty,r}^{k,\lambda(\cdot)}(\overline{\mathbb{R}^n})\| := \|f\|_\infty + \left\| t^{-1/r}\frac{\omega_k(f,t)}{\lambda(t)} \right\|_{r;(0,1)}$$

- If $k = 1$, we sometimes use $\Lambda_{\infty,r}^{\lambda(\cdot)}(\overline{\mathbb{R}^n}) := \Lambda_{\infty,r}^{1;\lambda(\cdot)}(\overline{\mathbb{R}^n})$ and $C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n}) := \Lambda_{\infty,\infty}^{\lambda(\cdot)}(\overline{\mathbb{R}^n})$.
- conditions (1) and (2) are natural
- the scale $\Lambda_{\infty,r}^{k,\lambda(\cdot)}$ contains both Hölder and Zygmund spaces
- $\lambda(t) \equiv t^\alpha,\ \alpha \in (0,1] \Rightarrow \Lambda_{\infty,\infty}^{1,\lambda(\cdot)}(\overline{\mathbb{R}^n}) \equiv C^{0,\alpha}(\overline{\mathbb{R}^n})$
- $\lambda(t) \equiv t^\alpha,\ \alpha > 0,\ k > \alpha \Rightarrow \Lambda_{\infty,\infty}^{k,\lambda(\cdot)}(\overline{\mathbb{R}^n}) \equiv Z^\alpha(\overline{\mathbb{R}^n})$

# Optimal Embeddings: super-limiting case

**Theorem** [Gogatishvili, N. & Opic (2005)]

Let $\sigma \in [1, n+1)$, $\max\{1, n/\sigma\} < p < n/(\sigma - 1)$, $q \in (1, +\infty)$, $r \in [q, +\infty]$ and let $b \in SV(0, +\infty)$. $\Omega \subset \mathbb{R}^n$ a nonempty domain. $\lambda(t) = t^{\sigma - n/p}[b(t^n)]^{-1}, \quad t \in (0, 1]$.

(i) Then $H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$.

(ii) If $\lim\limits_{t \to 0_+} \dfrac{\frac{t}{\lambda(t)}}{\|\tau^{-1/r} \frac{\tau}{\mu(\tau)}\|_{r;(0,t)}} = 0$, $H^\sigma L_{p,q;b}(\mathbb{R}^n) \not\hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$

(iii) Let $\overline{q} \in (0, q)$. $H^\sigma L_{p,q;b}(\mathbb{R}^n) \not\hookrightarrow \Lambda^{\lambda(\cdot)}_{\infty,\overline{q}}(\overline{\Omega})$.

(i) - N. (2004); (i),(ii)- Logarithmic Bessel potential space-Edmunds, Gurka and Opic (1997,2000) - $r = +\infty$;

# Optimal Embeddings: super-limiting case

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Let $\sigma \in (1, n+1)$, $p = n/(\sigma - 1)$, $q \in (1, +\infty)$, $r \in [q, +\infty]$ and $b \in SV(0, +\infty)$ such that $\| t^{-1/q'} [b(t)]^{-1} \|_{q';(0,1)} = +\infty$.

$$\lambda_r(t) = t\ [b(t^n)]^{q'/r} \left( \int_{t^n}^{2} \tau^{-1} [b(\tau)]^{-q'} d\tau \right)^{1/q' + 1/r}, \quad t \in (0,1].$$

(i) Then $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\lambda_r(.)}(\overline{\mathbb{R}^n})$.

(ii) If $\lim\limits_{t \to 0_+} \dfrac{\|\tau^{-1/r} \frac{\tau}{\lambda_r(\tau)}\|_{r;(0,t)}}{\|\tau^{-1/r} \frac{\tau}{\mu(\tau)}\|_{r;(0,t)}} = 0$, $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \not\hookrightarrow \Lambda_{\infty,r}^{\mu(.)}(\overline{\Omega})$

(iii) Let $\overline{q} \in (0, q)$. $H^\sigma L_{n/(\sigma-1),q;b}(\mathbb{R}^n) \not\hookrightarrow \Lambda_{\infty,\overline{q}}^{\lambda_{\overline{q}}(.)}(\overline{\Omega})$.

(i) - N. (2004); (i),(ii) - Logarithmic Bessel potential space-Edmunds, Gurka and Opic (1997,2000) - $r = +\infty$;

# Optimal Embeddings: super-limiting case

## [(i)] general ideas

- $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^\sigma L_{p,q;b}(\mathbb{R}^n)$.
- Let $u \in \mathcal{S}(\mathbb{R}^n) \subset H^\sigma L_{p,q;b}(\mathbb{R}^n)$.
- (Lifting argument) $\frac{\partial u}{\partial x_i} \in H^{\sigma-1} L_{p,q;b}(\mathbb{R}^n)$, for $i = 1, \dots, n$.
- Use embedding results for limiting case and inequality of DeVore and Sharpley inequality

$$\omega(u,t) \precsim \int_0^t |\nabla u|^*(\sigma^n)\, d\sigma, \quad t > 0.$$

- Use of appropriate Hardy-type inequalities.
- Hence $\|u|\Lambda^{\lambda(\cdot)}_{\infty,r}\| \precsim \|u\|_{\sigma;p,q;b}$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

## Proofs: [(ii), (iii)] general ideas

- Use of appropriate extremal functions!

# Optimal Embeddings: super-limiting case-Examples

## Example

Let $n \in \mathbb{N}$ such that $n \geq 2$ and let $k \in \mathbb{N}$ such that $2 \leq k \leq n$. Let $p = \frac{n}{k-1}$, $q \in (1, +\infty)$ and $r \in [q, +\infty]$.
Let $\alpha \in \mathbb{R}$ and let $b \in SV(0, +\infty)$ be defined by $b(t) = (1 + |\log t|)^{\alpha}, \quad t > 0$.

- If $\alpha < \frac{1}{q'}$, then
$$W^k L^{p,q}(\log L)^{\alpha}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,q}^{\lambda_q(\cdot)}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda_{\infty,r}^{\lambda_r(\cdot)}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_\infty(\cdot)}(\overline{\mathbb{R}^n}),$$
with
$$\lambda_r(t) = t(1 + |\log t|)^{\frac{1}{r} + \frac{1}{q'} - \alpha} \quad \text{and} \quad \lambda_\infty(t) = t(1 + |\log t|)^{\frac{1}{q'} - \alpha}, \quad t \in (0, 1].$$
- If $\alpha = \frac{1}{q'}$, then
$$W^k L^{p,q}(\log L)^{\alpha}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,q}^{\lambda_q(\cdot)}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda_{\infty,r}^{\lambda_r(\cdot)}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_\infty(\cdot)}(\overline{\mathbb{R}^n}),$$
with
$$\lambda_r(t) = t(1 + |\log t|)^{\frac{1}{r}}(1 + \log(1 + |\log t|))^{\frac{1}{r} + \frac{1}{q'} - \beta}, \quad t \in (0, 1];$$
$$\lambda_\infty(t) = t(1 + \log(1 + |\log t|))^{\frac{1}{q'} - \beta}, \quad t \in (0, 1].$$
- If $\alpha > \frac{1}{q'}$ then $W^k L^{p,q}(\log L)^{\alpha}(\mathbb{R}^n) \hookrightarrow Lip(\mathbb{R}^n)$.

# Optimal Embeddings: limiting case

**Theorem (Gogatishvili, N. and Opic (2007))**

*Let* $0 < \sigma < n$, $p = n/\sigma$, $q \in (1, +\infty)$ *and* $b \in SV(0, +\infty)$ *such that* $\|t^{-1/q'}[b(t)]^{-1}\|_{q';(0,1)} < +\infty$. *Then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n}).$$

*where*

$$\lambda(t) := \left( \int_0^{t^n} [b(\tau)]^{-q'} \frac{dt}{t} \right)^{1/q'}, \quad t > 0.$$

Logarithmic Bessel potential space - Edmunds, Gurka and Opic (2005);

$$H^\sigma L_{p,q;\frac{1}{q'},\cdots,\frac{1}{q'},\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n}),$$

with $\alpha_m > \frac{1}{q'}$ and $\lambda(t) = \ell_m^{\frac{1}{q'}-\alpha_m}(t)$.

# Bessel-potential spaces $H^\sigma X(\mathbb{R}^n)$

Let $\sigma > 0$

$X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n) \ldots$ r. i. BFS over $(\mathbb{R}^n, \mu_n)$ $\Longrightarrow$

$X \hookrightarrow L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ $\Longrightarrow$

$u = f * g_\sigma$ is well defined for all $f \in X$

$H^\sigma X(\mathbb{R}^n) := \{u : u = f * g_\sigma,\ f \in X(\mathbb{R}^n)\}$

$\|u\|_{H^\sigma X} := \|f\|_X$

Notation: $H^0 X(\mathbb{R}^n) = X(\mathbb{R}^n)$.

# Optimal Embeddings: smoothness $\sigma \in (0, n)$

**Theorem** [Gogatishvili, N. & Opic (2009) $\sigma < 1$, (2010) $\sigma \in (0, n)$]

Let $\sigma \in (0, n)$ and let $X = X(\mathbb{R}^n)$ be a r. i. BFS. Assume that $\Omega$ is a domain in $\mathbb{R}^n$. Then $H^\sigma X(\mathbb{R}^n) \hookrightarrow C(\overline{\Omega})$ if and only if $\|g_\sigma\|_{X'} < +\infty$.

**Lemma** [Gogatishvili, N. & Opic (2009) $\sigma < 1$, (2010) $\sigma \in (0, n)$ ]

Let $\sigma \in (0, n)$, $p \in (1, +\infty)$, $q \in [1, +\infty]$ and $b \in SV(0, +\infty)$. If $X = L_{p,q;b}(\mathbb{R}^n)$, then

$$g_\sigma \in X'$$

if and only if either

$$p > \frac{n}{\sigma} \tag{3}$$

or

$$p = \frac{n}{\sigma} \quad \text{and} \quad \|t^{-\frac{1}{q'}}(b(t))^{-1}\|_{q';(0,1)} < +\infty. \tag{4}$$

# Optimal Embeddings: smoothness $\sigma \in (0, n)$

**Theorem** [Gogatishvili, N. & Opic (2010)]

Let $\sigma \in (0, n)$ and let $X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$ be a r. i. BFS such that $\|g_\sigma\|_{X'} < +\infty$. Put $k := [\sigma] + 1$, assume that $r \in (0, +\infty]$ and $\mu \in \mathcal{L}_r^k$. Then

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$$

if and only if

$$\left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \int_0^{t^n} \tau^{\frac{\sigma}{n}-1} f^*(\tau)\, d\tau \right\|_{r;(0,1)} \precsim \|f\|_X \quad \text{for all } f \in X.$$

# Optimal Embeddings: smoothness $\sigma \in (0, n)$

## Proof: uses key estimates

Let $\sigma \in (0, n)$ and let $X = X(\mathbb{R}^n)$ be a r. i. BFS such that $\|g_\sigma\|_{X'} < +\infty$.
Then $f * g_\sigma \in C(\overline{\mathbb{R}^n})$ for all $f \in X$ and

$$\omega_k(f * g_\sigma, t) \lesssim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \;\forall t \in (0,1), \forall f \in X,$$

where $k \geq [\sigma] + 1$.
Moreover, this estimate is sharp: given $k \in \mathbb{N}$, there are $\delta \in (0, 1)$ and $\alpha > 0$ such that

$$\omega_k(\overline{f} * g_\sigma, t) \gtrsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \;\forall t \in (0,1), \forall f \in X,$$

where $\overline{f}(x) := f^*(\beta_n |x|^n)\chi_{C_\alpha(0,\delta)}(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$,
$C_\alpha(0, \delta) := C_\alpha \cap B(0, \delta)$ with $C_\alpha := \{y \in \mathbb{R}^n : y_1 > 0,\ y_1^2 > \alpha \sum_{i=2}^n y_i^2\}$.

# Optimal Embeddings: limiting case-Example

## Theorem [Gogatishvili, N. & Opic (2009) $\sigma < 1$, [Gogatishvili, N. & Opic (2010) $\sigma \in (0, n)$ ]

Let $\sigma \in (0, n)$, $p = \frac{n}{\sigma}$, $q \in (1, +\infty]$, $r \in (0, +\infty]$, $k = [\sigma] + 1$, $\mu \in \mathcal{L}_r^k$ and let $b \in SV(0, +\infty)$: $\|t^{-\frac{1}{q'}}(b(t))^{-1}\|_{q';(0,1)} < +\infty$. Let

$$\lambda_{qr}(x) := b^{q'/r}(x^n)\left(\int_0^{x^n} b^{-q'}(t)\,\frac{dt}{t}\right)^{\frac{1}{q'}+\frac{1}{r}}, \quad x \in (0, 1].$$

If $1 < q \le r \le +\infty$, then

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\mathbb{R}^n}) \Longleftrightarrow \overline{\lim_{x\to 0_+}} \frac{\|t^{-\frac{1}{r}}(\mu(t))^{-1}\|_{r;(x,1)}}{\|t^{-\frac{1}{r}}(\lambda_{qr}(t))^{-1}\|_{r;(x,1)}} < +\infty.$$

## Remark

When $r \in [1, +\infty]$, then

$$\Lambda_{\infty,r}^{k,\lambda_{qr}(\cdot)}(\overline{\mathbb{R}^n}) = \Lambda_{\infty,r}^{1,\lambda_{qr}(\cdot)}(\overline{\mathbb{R}^n}).$$

# Embeddings: Examples in the limiting case

## Example

Let $\sigma \in (0, n)$, $p = \frac{n}{\sigma}$, $q \in (1, +\infty]$ and $r \in [q, +\infty]$.
If $\alpha > \frac{1}{q'}$, $\beta \in \mathbb{R}$, then

$$H^\sigma L^{p,q}(\log L)^\alpha(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow \Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n}),$$

$$\lambda_{qr}(t) = (1 + |\log t|)^{\frac{1}{r}+\frac{1}{q'}-\alpha}(1 + \log(1 + |\log t|))^{-\beta}, \quad t \in (0, 1];$$
$$\lambda_{q\infty}(t) = (1 + |\log t|)^{\frac{1}{q'}-\alpha}(1 + \log(1 + |\log t|))^{-\beta}, \quad t \in (0, 1].$$

If $\alpha = \frac{1}{q'}$, $\beta > \frac{1}{q'}$, then

$$H^\sigma L^{p,q}(\log L)^\alpha(\log\log L)^\beta(\mathbb{R}^n) \hookrightarrow \Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow C^{0,\lambda_{q\infty}(\cdot)}(\overline{\mathbb{R}^n}),$$

$$\lambda_{qr}(t) = (1 + |\log t|)^{\frac{1}{r}}(1 + \log(1 + |\log t|))^{\frac{1}{r}+\frac{1}{q'}-\beta}, \quad t \in (0, 1];$$
$$\lambda_{q\infty}(t) = (1 + \log(1 + |\log t|))^{\frac{1}{q'}-\beta}, \quad t \in (0, 1].$$

# Optimal Embeddings: super-limiting case-Again!

**Theorem** [Gogatishvili, N. & Opic (2009) $\sigma < 1$, [Gogatishvili, N. & Opic (2010) $\sigma \in (0,n)$ ]

Let $\sigma \in (0,n)$, $p \in (\frac{n}{\sigma}, +\infty)$, $q \in [1,+\infty]$, $b \in SV(0,+\infty)$, $r \in (0,+\infty]$, $k = [\sigma]+1$, and $\mu \in \mathcal{L}_r^k$.
Let

$$\lambda(x) := x^{\sigma - \frac{n}{p}}(b(x^n))^{-1}, \quad x \in (0,1].$$

If $1 \le q \le r \le +\infty$, then

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \iff \overline{\lim_{x\to 0_+}} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1} \right\|_{r;(x,1)} \lambda(x) < +\infty.$$

**Remark**

When $r \in [1,+\infty]$, then

$$\Lambda^{k,\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) = \Lambda^{[\sigma-\frac{n}{p}]+1,\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}).$$

If $\sigma = \frac{n}{p} + 1$, then

$$\Lambda^{k,\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) = \Lambda^{2,\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}).$$

# Optimal Embeddings: super-limiting case-Again!

## Corollary [Gogatishvili, N. & Opic (2010)]

If $\sigma \in (1, n)$, $p = \frac{n}{\sigma - 1}$, $q \in (1, +\infty]$, $r \in [q, +\infty]$ and $b \in SV(0, +\infty)$ be such that $\| t^{-1/q'} [b(t)]^{-1} \|_{q';(0,1)} = +\infty$.
Let

$$\lambda(t) := t\,(b(t^n))^{-1}, \quad t \in (0, 1].$$

and let

$$\lambda_{qr}(t) := t\ [b(t^n)]^{q'/r} \left( \int_{t^n}^{2} \tau^{-1} [b(\tau)]^{-q'} d\tau \right)^{1/q' + 1/r}, \quad t \in (0, 1].$$

Then

$$H^{\sigma} L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{2,\lambda(\cdot)}_{\infty,q}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}).$$

## Remark

Previous Corollary improves Theorem 3.2 of (GNO, 2005), provided $\sigma = 1 + \frac{n}{p} < n$ and shows that the Brézis-Wainger embedding of the Sobolev space $H_p^{1+\frac{n}{p}}(\mathbb{R}^n)$, $\sigma = 1 + \frac{n}{p} < n$, into the space of "almost" Lipschitz functions is a consequence of a better embedding whose target is the Zygmund space.

## Example

if $n > 1$, $p = q$, $p > \frac{n}{n-1}$, $b(t) = \ell_1^{\alpha}(t)$, $t \in (0, +\infty)$, $\alpha < \frac{1}{p'}$,

$$H^{1+\frac{n}{p}} L^p (\log L)^{\alpha}(\mathbb{R}^n) \hookrightarrow \Lambda^{2,\lambda(\cdot)}_{\infty,p}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{1,\lambda_r(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \hookrightarrow \Lambda^{1,\lambda_\infty(\cdot)}_{\infty,\infty}(\overline{\mathbb{R}^n}),$$

with

$$\lambda(x) := x\, \ell_1^{-\alpha}(x) \quad \text{for all} \quad x \in (0,1],$$

and

$$\lambda_{pr}(x) = x \ (\ell_1(x))^{1/p' + 1/r - \alpha}, \quad x \in (0,1],\ r \in [p, +\infty].$$

If $\alpha = 0$, this example shows that the Brézis-Wainger embedding of the Sobolev space $H_p^{1+\frac{n}{p}}(\mathbb{R}^n)$, $\sigma = 1 + \frac{n}{p} < n$, into the space of "almost" Lipschitz functions is a consequence of a better embedding whose target is the Zygmund space $\Lambda^{2,Id(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n})$ ($Id(\cdot)$ stands for the identity map).

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