Regularity of the vanishing ideal over a parallel composition of paths

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Joint work with A. Macchia, M. Vaz Pinto and R. Villarreal

Combinatorial Commutative Algebra,
Mathematical Congress of the Americas
Montreal, July 27, 2017

Definition of I_G

Vanishing ideals over graphs

- G is a simple graph without isolated vertices.
- K a finite field of cardinality q.
- $V_G = \{1, 2, \ldots, n\}$.
- $S = K[t_{ij} \mid \{i,j\} \in E_G]$.
- Let $I_G \subset S$ be the ideal generated by:

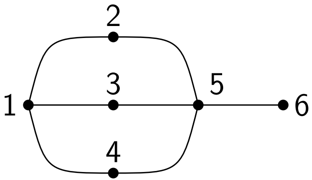
$$\{f \text{ homogeneous} \mid f(\ldots, x_i x_j, \ldots) = 0,\ \forall x_i \in K^*\}$$

where, $f(\ldots, x_i x_j, \ldots)$ is obtained by: $t_{ij} \mapsto x_i x_j$.

[Rentería, Simis, Villarreal, 2011]

Example [Using Macaulay2]

Generators of I_G



I_G has the following minimal generating set:

- $t_{12}^{q-1} - t_{56}^{q-1}, \ldots, t_{45}^{q-1} - t_{56}^{q-1}$

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Generators of I_G

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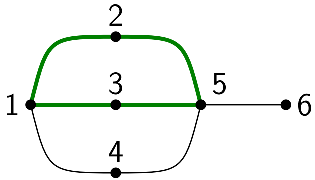

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- $t_{12}^{q-1} - t_{56}^{q-1}, \ldots, t_{45}^{q-1} - t_{56}^{q-1}$
- $t_{13}t_{25} - t_{12}t_{35}$,

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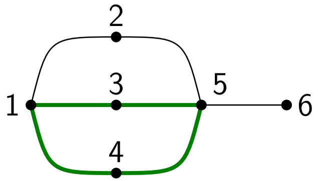

- $t_{12}^{q-1} - t_{56}^{q-1}, \ldots, t_{45}^{q-1} - t_{56}^{q-1}$
- $t_{13}t_{25} - t_{12}t_{35},\ \ t_{14}t_{35} - t_{13}t_{45},$

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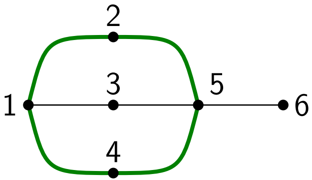

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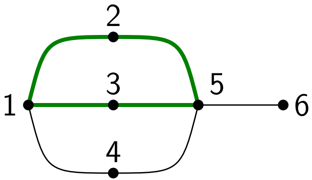

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- $t_{12}^{a}t_{25}^{b} - t_{13}^{a}t_{35}^{b}$, with $a + b \equiv 0 \pmod{q-1}$

Example [Using Macaulay2]

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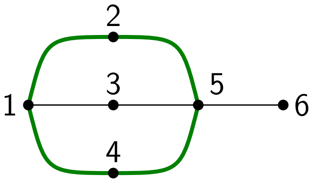

- $t_{12}^{q-1} - t_{56}^{q-1}, \ldots, t_{45}^{q-1} - t_{56}^{q-1}$
- $t_{13}t_{25} - t_{12}t_{35},\ t_{14}t_{35} - t_{13}t_{45},\ t_{14}t_{25} - t_{12}t_{45}$
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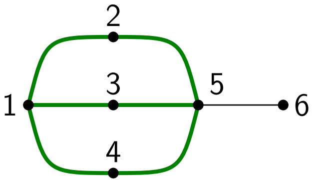

- $t_{12}^{q-1} - t_{56}^{q-1}, \dots, t_{45}^{q-1} - t_{56}^{q-1}$
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- $t_{12}^{a}t_{25}^{b} - t_{13}^{a}t_{35}^{b},\ t_{12}^{a}t_{25}^{b} - t_{14}^{a}t_{45}^{b}$ with $a + b \equiv 0 \pmod{q-1}$
- $t_{12}^{a}t_{13}^{b}t_{14}^{c} - t_{25}^{a}t_{35}^{b}t_{45}^{c}$, with $a + b + c \equiv 0 \pmod{q-1}$.

(Can assume $0 \le a, b, c \le q - 2$.)

Properties of I_G

in [Rentería, Simis, Villarreal, 2011]

- I_G is a radical, binomial, graded ideal of $K[t_{ij} \mid_{\{i,j\} \in E_G}]$;
- Letting $s = |E_G| = \dim K[t_{ij} \mid_{\{i,j\} \in E_G}]$,

$$\left\{ t_{ij}^{q-1} - t_{kl}^{q-1} \mid_{\{i,j\} \in E_G} \right\} \subset I_G \implies \mathrm{ht}(I_G) = s - 1;$$

- Since any variable, t_{ij}, is (S/I_G)-regular, S/I_G is C–M.
- If G is connected or bipartite,

$$I_G = (P_G + (t_{ij}^{q-1} - t_{kl}^{q-1} \mid \{i,j\} \in E_G)) : \left(\prod_{\{i,j\} \in E_G} t_{ij} \right)^{\infty}$$

where P_G is the toric ideal of G.

Castelnuovo–Mumford Regularity

Given a minimial free graded resolution:

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{b_{(s-1)j}} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{b_{2j}} \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{b_{1j}} \to S \to S/I_G$$

$$\operatorname{reg}(G) := \operatorname{reg}(S/I_G) := \max_{i,j} \{j - i \mid b_{ij} \neq 0\}.$$

If the Hilbert series of S/I_G as reduced rational fraction is:

$$F(T) = \frac{1 + sT + h_2 T^2 + \cdots + h_r T^r}{1 - T} \quad \text{then,}$$

$$S/I_G \text{ C–M} \implies \operatorname{reg}(G) = \deg F(T) + \dim S/I_G = r.$$

Known values of reg(G)

- reg $\mathcal{K}_{a,b} = (\max\{a,b\} - 1)(q-2)$;
  [González, Rentería, 2008]
- G = tree or $\mathcal{C}_{2k+1}$, reg $G = (s-1)(q-2)$;
  [Sarmiento, Vaz Pinto, Villarreal, 2011]
- reg $\mathcal{K}_n = \lceil (n-1)(q-2)/2 \rceil$;
  [González, Rentería, Sarmiento, 2013]
- reg $\mathcal{C}_{2k} = (k-1)(q-2)$;
  [N., Vaz Pinto, Villarreal, 2015]
- G bipartite and $H_1, \ldots, H_m$ its blocks,

  reg $G = \sum$ reg $H_i + (m-1)(q-2)$;
  [N., Vaz Pinto, Villarreal, 2014]

Main Result

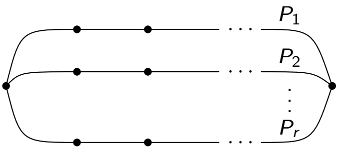
Theorem (Macchia, N., Vaz Pinto, Villarreal)

(i) *If* $\forall_i, k_i \notin 2\mathbb{N}$, $\operatorname{reg} G = (\lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2)$;

(ii) *If* $\forall_i, k_i \in 2\mathbb{N}$, $\operatorname{reg} G = (k_1/2 + \cdots + k_r/2 - 1)(q-2)$;

(iii) $\operatorname{reg} G = \operatorname{reg} H_1 + \operatorname{reg} H_2 + (q-2)$.

- Let G be the graph:



$H_1 = \text{Pc(odd paths)}$,
$H_2 = \text{Pc(even paths)}$.

Proof and other results

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- The inequalities $\geq$ in (i) and (ii) come from:

$$G \subset \mathcal{K}_{a,b} \implies \operatorname{reg} G \geq (\max\{a, b\} - 1)(q-2)$$

[Vaz Pinto, Villarreal, 2013]

and the computation of a and b for a *bipartite* parallel composition of paths.

Proof and other results

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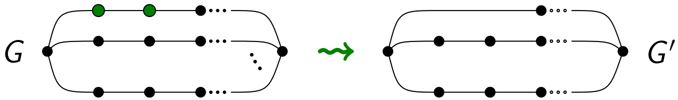
- The inequality $\leq$ in (i) amounts to showing that, if



then $\operatorname{reg}(G) \leq \operatorname{reg}(G') + (q-2)$.

Proof and other results

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(iii) $\operatorname{reg} G = \operatorname{reg} H_1 + \operatorname{reg} H_2 + (q-2)$.

- The inequalities $\leq$ in (ii) and (iii) are obtained using:

Proposition. If $H_1, H_2 \subset G$ are subgraphs such that

$$E_G = E_{H_1} \cup E_{H_2} \quad \text{and} \quad E_{H_1} \cap E_{H_2} \neq \emptyset$$

then $\operatorname{reg} G \leq \operatorname{reg} H_1 + \operatorname{reg} H_2$.

Proof and other results

Theorem (Macchia, N., Vaz Pinto, Villarreal)

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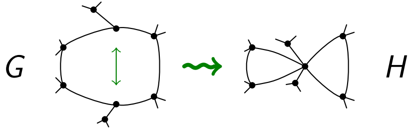
- The inequality $\geq$ in (iii) is obtained using:

Proposition. If H is obtained from G by identifying two nonadjacent vertices, then $\operatorname{reg} G \geq \operatorname{reg} H$.



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