Taxonomies of Geometric Problems[1]

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Abstract

In the current Information Society the organisation of the information is key to ensure the information safekeeping and retrieval. It is of utmost importance that each and every user can find the information he/she is looking for, presented in such a way that best fit his/her needs. Geometry is no exception, the servers of geometric information should be easily and successfully searchable. By classifying the information contained in the servers of geometric information accordingly to several taxonomies, it will be possible to begin applying filters to the users' queries, adjusting them to the perceived user's needs. Having that in mind, the introduction of an adaptive filtering mechanisms into servers of geometric information is considered.

Different taxonomies for different goals are presented. For educational purposes, a classification like *Common Core Standards* should be considered, but other considerations like the complexity of the construction, the provability, by a geometry automatic theorem prover, of a given conjecture and the readability of the resulting proof, should be taken into account. For research in automated deduction purposes, other issues must be considered, e.g. efficiency and applicability of the available automated provers.

To validate the usefulness of these taxonomies it will be used, as a case study, their application to a server of geometric information. In particular, Thousands of Geometric problems for geometric Theorem Provers will be considered. *TGTP* is a Web-based repository of geometric problems being developed to support the testing and evaluation of geometric automated theorem proving systems. Using this system it will be analyse how the taxonomies could help to tailor the search for information adapted to each and every geometer.

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1. Introduction

In the area of geometry there are now a large number of computational tools that can be used to perform many different tasks, dynamic geometry systems (DGS), computer algebra systems (CAS), geometry automatic theorem provers (GATP), among others (Quaresma, 2017). All these tools are clients of geometric information—information that can be found on repositories of geometric knowledge such as: *Intergeo*;[2] *TGTP*;[3] *GeoGebra Materials*,[4] among others.

It can be claimed that the usefulness of such servers of geometric information, is directly related with the possibility of an easy retrieval of the information a given user is looking for. Therefore the information should be organised in such a way that it will be possible the design of filters adjusted to the user's preferences.

The organisation of information through the taxonomy concept allows to allocate, retrieve and communicate information within a system in a logical way, that is, in classes, subclasses, sub-subclasses, and so on. Each of these levels aggregate information about the existing documents in the repository. An advantage of this form of access is the user's guarantee of best selection of searched term, since the classes contain mutually exclusive topics.

Different taxonomies would answer to different users' needs. The problems in the servers must be classified in such a way that, in response to a client query, only the problems in the user's level and/or interest and/or language are returned.

If the organisation of a large field of knowledge like mathematics are to be considered, taxonomies like *Mathematics Subject Classification* (MSC)[5] can be found. In this paper the much narrower scope of constructive geometry is addressed, i.e. geometric constructions made by dynamic geometry systems and geometric problems, eventually with an associated construction, manipulated by geometric automated theorem provers.

The term "geometric problem" is here used in a general way, it is used in relation to the case study, i.e. taxonomies for *TGTP*. There was and there is an important debate on the distinction between problems and theorems, some scholars consider that they should be distinguished, others consider that problems can be reduced to theorems, others consider that theorems can be reduced to problems. This debate is very old and continues today: see for example Morrow (1992); Heath (1908); Martin (1998); Hartshorne (2000); Sidoli (2018). However, this paper is a case study about taxonomies for *TGTP*, therefore theorems, their proofs, construction problems and their solutions are considered from a single and uniform taxonomic point of view.

For researchers in geometric automated theorem proving it will be interesting to look for: conjectures not yet proved by GATPs; theorems with readable proofs; theorems proved efficiently; etc. (Chou et al., 1996a,b; Janičić et al., 2012; Jiang and Zhang, 2012; Stojanović et al., 2011; Wang and Su, 2015).

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[2]Interoperable Interactive Geometry for Europe: http://i2geo.net/

[3]Thousand of Geometric problems for geometric Theorem Provers: http://hilbert.mat.uc.pt/TGTP/

[4]GeoGebra Materials: https://www.geogebra.org/materials/

[5]http://msc2010.org/, the MSC2020 revision is under way

If a taxonomy based on those criteria would please the researchers in automatic geometric reasoning, it might not be completely suitable for the educational community. When designing filters for educational purposes, education levels (International Standard Classification of Education (UNESCO, 2012)), levels of geometry reasoning (Usiskin, 1982) and also personal preferences must be considered. Adding to other approaches, a new approach to geometrography (Lemoine, 1902; Mackay, 1893; Pinheiro, 1974), is considered, taking into account a very interesting point of view on geometrical constructions classification. Applying geometrography's principles to the dynamic geometry systems, it is possible to (re)define the concepts of *coefficient of simplicity* and a new *coefficient of freedom* to measure the complexity and dynamics of a DGS construction.

With reference to the paper's case study, Thousands of Geometric problems for geometric Theorem Provers (*TGTP*) is a Web-based repository of geometric problems with integrated GATPs. It is being developed to support the testing and evaluation of geometric automated theorem proving systems (Quaresma, 2011). The list of problems in *TGTP* can be explored with some powerful textual and geometric search mechanisms (Haralambous and Quaresma, 2018), but, if adapting the search to each user's needs is pursued, it is necessary to introduce a classification for each *TGTP* problem, stating their characteristics, in face of one or more intended users' expectations.

Originally *TGTP* was aimed to the geometric automatic theorem provers community, as said above to support the testing and evaluation of geometric automated theorem proving systems, so its expected audience is mostly researchers whose background is mathematics and/or computer science and whose research focus is automatic reasoning, formalisation of mathematics, artificial intelligence, among others. The interest in proofs and proving in mathematics education and the application of GATPs in mathematics education (Janičić and Quaresma, 2007; Hanna and de Villiers, 2012; Quaresma and Santos, 2016) opens a new community of potential users of *TGTP* (Quaresma et al., 2018a).

The classification of each *TGTP* problem at a given educational level, should be possible and not very difficult. The classification accordingly to a level of geometry reasoning would be more difficult. Nevertheless such classifications are useful in any educational environment, e.g. when linking with educational platforms like the *Web Geometry Laboratory* (*WGL*) (Quaresma et al., 2018b; Santos et al., 2018).

The current search mechanisms in *TGTP* allow its users to search for a given specific problem, or set of problems, e.g. look for Ceva's theorem, look for all problems with the word "circumcircle" in its description, look for problems containing some given geometric configuration (Quaresma, 2011; Haralambous and Quaresma, 2018). The introduction of taxonomies in *TGTP* can add a filtering step that, together with the text and geometric search mechanisms, will allow to tailor *TGTP*'s usefulness to each user's needs. For example, a secondary school teacher preparing a class about circumcenter centre would choose filters: CCS classes, CO.A.1 and C.A.3; construction complexity, simple; proofs in education, verification: good, filtering the *TGTP* database, or any other geometry knowledge repository, in such a way that a good set of examples could be browsed and choose as teaching materials.

An interesting development of this research is the application of the taxonomies presented in this paper, to proof assistant area. We do not consider this application here, however we give some suggestions about it (see Section 2.2).

Overview of the paper.. The paper is organised as follows: first, in §2, taxonomies for GATP research and taxonomies for education will be discussed. In §3, the application of taxonomies to

TGTP will be analysed providing some examples, also a new approach to *Geometrography* will be presented. In §4, conclusions are drawn and future work will be discussed.

2. Taxonomies

Taxonomies are a crucial components for any application of information retrieval, natural language processing and knowledge management. The design of an hierarchical faceted meta-data system, allow users to browse the information accordingly to multiple categories simultaneously (Hearst, 2009). Designing such a faceted system to geometry will allow geometers to browse through the repositories of geometric knowledge in a more fulfilling way. For example an automatic deduction researcher will look for provability, for the methods used, for the readability of the proof produced and for efficiency measures. An educator will look for a given class of geometric problems, for the construction complexity and for the validation of geometric conjectures, among others.

Mathematics Subject Classification. Using the *Mathematics Subject Classification* (MSC) the following classes can be considered:

Geometry generic classes

- **51M05** Euclidean geometries (general) and generalisations
- **51N10** Affine analytic geometry
- **51N15** Projective analytic geometry
- **51N20** Euclidean analytic geometry
- **51N25** Analytic geometry with other transformation groups
- **68U05** Computer graphics; computational geometry
- **70G55** Algebraic geometry methods

GATP geometric automated theorem proving

- **03B35** Mechanisation of proofs and logical operations
- **68T15** Theorem proving (deduction, resolution, etc.)
- **70G55** Algebraic geometry methods
- **94B27** Geometric methods (including applications of algebraic geometry)
- **97G70** Analytic geometry. Vector algebra
- **97E50** Reasoning and proving in the mathematics classroom

Education generic and deductive reasoning in education

- **97-XX** Mathematics education
- **97E50** Reasoning and proving in the mathematics classroom
- **97Uxx** Educational material and media, educational technology
- **68Q32** Computational learning theory
- **68T05** Learning and adaptive systems

68T35 Languages and software systems (knowledge-based systems, expert systems, etc.)

97U50 Computer assisted instruction; e-learning

Although this classification is too coarse to allow a fine grade classification adaptable to the users needs, it is nevertheless useful as a way to add a level of meta information to the geometric elements in the repositories of geometric information.

2.1. Taxonomies for GATP Research

For more than half a century, mechanical theorem proving in geometry is an active research topic (Chou et al., 1994; Chou and Gao, 2001; Jiang and Zhang, 2012). This research has led to three types of GATPs: synthetic, semi-synthetic and algebraic. Synthetic GATPs have proven to be impractical for non-trivial geometry theorems (Gelernter, 1995; Coelho and Pereira, 1986). As for the semi-synthetic ones, such as the area method and the full-angle method developed by Chou, Gao and Zhang, they are capable of proving many theorems, providing readable proofs (Chou et al., 1996a,b; Janičić et al., 2012). Algebraic methods, such as the Wu's method and the Gröbner method, have been used to prove many theorems, some of them non-trivial, within seconds, but generate very hard to read proofs (Chou et al., 1994).

It is then clear that producing fast, short and readable proofs remains a challenge to researchers in the field of automated theorem proving (Boutry et al., 2014; Chou et al., 1994, 2000; Jiang and Zhang, 2012; Pąk, 2015; Stojanović et al., 2011; Wang and Su, 2015). As such, a taxonomy for GATP research must deal with the following classes: *GATP provability*, *readability* and *efficiency*.

GATP Provability. Given a conjecture and n GATPs, if the conjecture is successfully proved by p of those GATPs, the ratio p/n provides a simple method to measure this criteria.

Apart this numerical criteria one might consider a classification of the geometric conjectures accordingly to the different methods. If synthetic methods are to be considered, relying in heuristics, most of the implementations are only able to solve a certain kind of problems. Also the algebraic methods and the semi-synthetic methods have their limitations, defining classes of problems they can solve (Chou and Gao, 2001).

Readability. As far as the authors of this paper know, there are two proposals to measure the readability of a proof.

Chou et al. (1994, p.442) proposed a way to measure how difficult a formal proof is (using the area method). To measure this criteria the triple *(time, maxt, lems)* is used, where:

- *time* is the time needed to complete the proof;
- *maxt* is the number of terms of the maximal polynomial[6] occurring in the proof;
- *lems* is the number of elimination lemmas used to eliminate points from geometry quantities. In other words, *lems* is the number of deduction steps in the proof.

Using this criteria and analysing all the proofs done by the GATP they have implemented, they reach to the following thresholds[7] of proof readability.

[6] The polynomial of highest degree, as clear by the examples presented in the paper by Chou et al. 1994.

[7] Accordingly to their data: 66.9% of the proofs has maxt ≤ 5, 42.6% has lems ≤ 10 and 73.2% has lems ≤ 20.

- According to Chou et al. (1994, p.442) a formal proof, done using the area method, is considered readable if one of the following conditions holds:
  - the maximal term in the proof is less than or equal to 5;
  - the number of deduction steps of the proof is less than or equal to 10;
  - the maximal term in the proof is less than or equal to 10 and the deduction step is less than or equal to 20.
- The de Bruijn factor (de Bruijn, 1994; Wiedijk, 2000), the quotient of the *size* of corresponding informal proof and the *size* of the formal proof, could also be used as a measure of readability. Using this quotient a proof can be considered readable if the value is less than or equal to 2 (the formal proof is at most twice as larger then a given informal proof).

In both cases it is somehow assumed, *readability by experts*, i.e. a given geometer, expert in the language of the prover that produced the proof. If non-experts, from the geometer not familiarised with the prover, to the complete novice on the area of the automatic deduction, are to be considered the need of synthetic proofs with natural language descriptions is felt. There are some recent approaches to this, there are even visual approaches, where the formal proof can also followed by a correspondent visual manipulation (Stojanović et al., 2011; Ye et al., 2010a,b).

The question here is how to classify the problems against this new criteria. The following schema may be considered:

1. no readable proof;
2. non-synthetic proof (i.e. a proof without a correspondent geometric description, e.g. algebraic methods);
3. semi-synthetic proof with a corresponding prover's language rendering;
4. (semi-)synthetic proof with a corresponding natural language rendering;
5. (semi-)synthetic proof with a corresponding natural language and visual rendering;

The Chou's conditions allows to define a threshold for area method semi-synthetic proofs. Along this lines it is possible to define coefficients of readability for other semi-synthetic methods (levels 3, 4 and 5). The de Bruijn factor can be used in all levels, but it will be more meaningful on levels above 3.

Efficiency. The amount of time needed to complete a proof should be considered, by itself, a way to measure the difficulty of a formal proof.

The space complexity is less important, because the physical constraints are nowadays less important and because the time spent waiting for the proof, or for the validation of a geometric property, is the important factor for the user.

2.2. Taxonomies in Education

Defining taxonomies in an education setting depends on many factors, the type of geometry being considered, in secondary schools most of the times will be Euclidean geometry, studied in a synthetic way (without coordinates) and analytically (with coordinates), the axiomatic system used, the type of computational tools being used, the educational level and also the development of the geometric ideas of the students depending on their behaviour, their level of geometric knowledge and personal preferences.

An interesting taxonomy to be considered in education is the USA initiative, *Common Core Standards*[8] (CCS) (McCallum, 2015; Rivera, 2013; Wu, 2012). According to it, the classes of geometry[9] are: congruence (CO); similarity, right triangles, and trigonometry (SRT); circles (C); expressing geometric properties with equations (GPE); geometric measurement and dimension (GMD) and modelling with geometry (MG). Each class has its own subclasses, defined by degrees of difficulty, for example, in the class of congruence, it has the subclass, *lower degree of difficulty*, with the sub-subclasses: experiment with transformations in the plane; understand congruence in terms of rigid motions; prove geometric theorems, and the subclass, *higher degree of difficulty*, with the sub-subclass: make geometric constructions.

The major classes (abbreviations, in capital letters) are divided in subclasses with two levels, the first one designated by a capital letter and the second by a number. Below the different class/subclass/sub-subclass used to classify the examples presented in section 3.1.

C.A.3 Circles / Understand and apply theorems about circles / Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

CO.A.1 Congruence / Experiment with transformations in the plane / Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

CO.A.4 Congruence / Experiment with transformations in the plane / Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

CO.C.10 Congruence / Prove geometric theorems / Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

CO.D.12 Congruence / Make geometric constructions / Make formal geometric constructions with a variety of tools and methods (compass and straight edge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

SRT.B.4 Similarity, right triangles, and trigonometry / Prove theorems involving similarity / Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

SRT.B.5 Similarity, right triangles, and trigonometry / Prove theorems involving similarity / Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

[8] `http://www.corestandards.org/Math/`. The CCS was launched in 2009 and it is an effort of the *National Governors Association* and *Council of Chief State School Officers*.

[9] `http://www.corestandards.org/Math/Content/HSG/`

Geometrography. Another approach that can be useful in this context is given by the geometrography approach (Lemoine, 1902; Mackay, 1893; Pinheiro, 1974). Geometrography, "alias the art of geometric constructions," aims at providing a tool: to designate every geometric construction by a symbol that manifests its simplicity and exactitude; to teach the simplest way to execute an assigned construction; to discuss a known solution to a problem and eventually replacing it with a better solution; to compare different solutions for a problem, by deciding which is the most exact and the simplest solution from the point of view of geometrography (Loria, 1908). In this perspective, geometrography offers a very interesting point of view on geometrical problems and their taxonomies in education. In Lemoine's geometrography two coefficients are defined to measure the relative difficulty to perform some geometric constructions. The approach is applied to ruler and compass geometry, i.e. geometric constructions made with the help of a ruler and a compass only. Considering the modifications proposed by Mackay (1893) the following **R**uler and **C**ompass constructions[10] and the corresponding coefficients can be considered.

To place the edge of the ruler in coincidence with one point . . . . R_1

To place the edge of the ruler in coincidence with two points . . . . $2R_1$

To draw a straight line . . . . R_2

To put one point of the compasses on a determinate point . . . . C_1

To put one point of the compasses on two determinate points . . . . $2C_1$

To describe a circle . . . . C_2

Then a given construction is measured against the number of those elementary steps. For example, for the construction of a triangle, given its three vertices A, B and C, Mackay (1893) estimate $4R_1 + 3R_2$: to put the ruler in contact with A and B is $2R_1$; to draw AB is R_2; with the ruler in contact with B to put it also in contact with C is R_1; to draw BC is R_2; repeat that for C and A is R_1 and finally to draw CA is R_2. In all, $4R_1 + 3R_2$.

For a given construction expressed by the equation:

$$l_1R_1 + l_2R_2 + m_1C_1 + m_2C_2$$

where l_i and m_j are coefficients denoting the number of times any particular operation is performed. The number $(l_1 + l_2 + m_1 + m_2)$ is called the *coefficient of simplicity* of the construction, it denotes the total number of operations. The number $(l_1 + m_1)$ is called the *coefficient of exactitude* of the construction, it denotes the number of preparatory operations on which the exactitude of the construction depends (Mackay, 1893; Merikoski and Tossavainen, 2010).

Some variants of Lemoine's geometrography can be defined, e.g. by adding rules for other idealised tools/operations (e.g. carpenter's square, graduated rulers, etc.), or by adding a value for the change of the instrument/operation, or by considering different values for different operations (Grüttner, 1912; Loria, 1908).

Extrapolating (modernising) geometrography, considering the "tools" of dynamic geometry systems, the *coefficient of exactitude* loose its meaning, the construction will be executed by

[10] Lemoine (1902) considers the following basic operations: L1. Place the ruler through a given point; L2. draw a line; C1. place one leg of the compass on a given point; C2. place one leg of the compass on an indeterminate point of a given line; C3. draw a circle.

the DGS, so exact (minus floating point representation considerations), but the *coefficient of simplicity* of the constructions can still be useful, it can be used to classify the constructions by levels of simplicity and, in this way, adding another level of meta-information. Also a new dimension can be added, the *coefficient of freedom*, given by the degree of freedom a given geometric object has, e.g. "a point in a line" has one degree of freedom, a point in the plane has two degrees of freedom. This new coefficient will give a value for the dynamism of the geometric construction.

The need of a classification for the complexity of construction is felt in educational settings. For example in the *Euclidea*[11] and *Geometriagon*[12] platforms the constructions contained in the respective repositories are classified by levels of difficulty.

Euclidea count actions with tools (L), e.g. build a line, a perpendicular line, etc. and also count movements (E), such as a construction was made using a ruler and a compass. Defining *costs*, (L, E), for each tool. *Geometriagon* defines the *difficulty level*, $1 \leq n \leq 5$, awarded by the proposer and other users. Both system would benefit from the adoption of a geometrography approach.

In the following the geometrography approach to the classification of the geometric construction made using the *Geometry Constructions LaTeX Converter* (*GCLC*)[13] (Janičić, 2006) is presented (*GCLC* provides three of the four GATPs embedded in *TGTP*).

Geometrography in GCLC. Apart the rendering of geometric constructions *GCLC* also incorporate GATPs based on the area method, Wu's method and Gröbner Basis method, all those tools use the GCL formal language (Janičić, 2010).

Considering the operations: *define a point, anywhere in the plane*, D and *define a given object, using other objects*, C, the following values for the *GCLC* basic constructions are obtained:[14]

`point` – fix a point in the plane .... D

`line` – uses two points .... $2C$

`circle` – uses two points .... $2C$

`intersec` – uses two lines .... $2C$

`intersec` – uses four points .... $4C$

`intersec2` – uses a circle and a circle or line .... $2C$

`midpoint` – uses two points .... $2C$

`med` – uses two points .... $2C$

`bis` – uses three points .... $3C$

[11] `https://www.euclidea.xyz/`, also available as an application for smartphones.
[12] `http://polarprof-001-site1.htempurl.com/geometriagon/`
[13] `http://poincare.matf.bg.ac.rs/~janicic/gclc/`
[14] A similar approach based on the DGS *GeoGebra* was presented at the conference CADGME2018, *Vanda Santos, Nuno Baeta and Pedro Quaresma, Geometrography in GeoGebra*, `http://www.uc.pt/en/congressos/cadgme2018/ficheiros/CADGME-2018_paper_34`

`perp` – uses a point and a line .... $2C$

`foot` – uses a point and a line .... $2C$

`parallel` – uses a point and a line .... $2C$

`onsegment` – uses two points .... $2C$

`online` – uses two points .... $2C$

`oncircle` – uses two points .... $2C$

The degrees of freedom are measured against the point definitions. The `point` definition, define a point with two degrees of freedom, the `onsegment`, `online` and `oncircle` constructions, define points with one degree of freedom.

A script that analyse any GCLC construction was implemented, giving its coefficients of simplicity (cs) and freedom (cf).

This research also open an interesting application, a geometrography approach regarding proofs and proof assistants, considering for example a combination of dynamic geometry software with proof assistants. An interesting case study could be, for example, *GeoCoq* (Pham and Bertot, 2012), a combination of *GeoGebra* and *Coq*. This will be object of future research.

GATPs in Education. There are complex problems, which a secondary school student will find difficult to prove, that can be automatically proved. For example, the *Geometry Expert* program[15] (Chou et al., 1994), proved automatically about 400 problems extracted from a typical geometry secondary school book (Altshiller-Court, 2007). Chou et al. (1994, p.453) also proposed a classification of level of difficulty of geometrical theorems in terms of various types of geometric constructions involved: collinearity; parallelism; proportionality; perpendicularity; circle; angle, in ascending order of difficulty.

In Proceedings of the ICMI Study 19 Conference: *Proof and Proving in Mathematics Education* (Lin et al., 2009a,b) many articles exploring the use of proofs in a learning environment can be found. In (Hanna, 2000; Hanna and Sidoli, 2007; de Villiers, 1990) Michael de Villiers and Gila Hanna gave a list of the usefulness of proofs and proving in a learning environment. For the purpose of problems classification the following two points emerge:

- verification (concerned with the truth of a statement);
- explanation (providing insight into why it is true);

Considering the formal validation of properties of geometric constructions the time needed to get an answer should be used as a classification criteria.

Many of the current DGSs have already the capability of a formal validation of properties: *Cinderella*[16] (Richter-Gebert and Kortenkamp, 1999) contains a randomised theorem checker; *GCLC*, *Java Geometry Expert* (*JGEX*)[17] and *GeoGebra*[18] (version 5) (Hohenwarter, 2002) have

[15] `http://www.cs.wichita.edu/~ye/examples.html`
[16] `https://cinderella.de`
[17] `http://www.cs.wichita.edu/~ye/`
[18] `https://www.geogebra.org/`

a number of automated theorem provers incorporated in them, thus opening the possibility to give a formal answer to a given validation question (Botana et al., 2015; Janičić and Quaresma, 2007; Ye et al., 2011).

Extrapolating from the concept of "wait-time"[19]—periods of silence that followed teacher questions and students' completed responses (Rowe, 1972)—or more precisely the category of periods of silence "Post-Teacher Question Wait-Time" (Stahl, 1994), the following classes of "GATP validation time" could be defined, in terms of time, t, taken by the GATP to answer:

- good: $t \leq 1.5s$;
- fair: $1.5s < t \leq 3s$;
- poor: $t > 3s$.

The use of formal proofs in an educational setting is a more complex issue. As stated in (Hanna and de Villiers, 2012; Quaresma and Santos, 2016), formal statements have an important role in the area of education. GATPs implementing the Wu's method, the Gröbner bases method, the area method or full-angle method, are capable of producing proofs and are very efficient, but the proofs produced are far from the level of non-expert readable proofs (Botana et al., 2015).

For a classification propose the criteria of readability (of the proof produced), efficiency, interaction between the GATP and its users, student-level, must be considered. This is an open and active research area. The following classes could be considered:

- yes: a small (in length) and fast (in processing time), readable synthetic GATP proof exist;
- maybe: a small (in length) and fast (in processing time), non-synthetic (e.g. semi-synthetic) readable GATP proof exist;
- no: no GATP proof exist or the GATP proof is neither small nor fast nor readable.

3. Taxonomies in *TGTP*

Applying the above defined taxonomies to the *TGTP* server of geometric knowledge poses a few technical problems that should not be difficult to solve. Indeed *TGTP*, in its current version, already provides, for each conjecture, the ratio of successful proofs to proof's attempts, as well as other items of information needed to classify a given conjecture. *TGTP* also provides a proof of a conjecture if one is provided by the method's implementation, a notable exception is the implementation of the area method using the *Coq* proof assistant[20] (Janičić et al., 2012; Narboux, 2007), which only provides a `proved` or `not proved` output.

For the *GCLC* constructions contained in *TGTP* an average value of simplicity (cs) of 22.5 was obtained. Having that in mind, three classes of geometric constructions for an increasing level of complexity were defined: simple constructions, $1 \leq cs < 15$; average complexity constructions, $15 \leq cs < 30$; complex constructions, $cs \geq 30$.

[19]Note that the "wait-time" concept will be used to define the threshold values for the response times of GATPs, it is not any new kind of taxonomy.

[20]`http://dpt-info.u-strasbg.fr/~narboux/area_method.html`

TGTP contain 36 simple constructions; 119 average complexity constructions; 30 complex constructions.

If applied to *TGTP* the educational taxonomies should also take into account the objectives of teaching, a short response time and readable demonstrations.

A first step would be to make the necessary modifications to *TGTP* to be able to add the missing items of information: modifying the forms used to manually manipulate the information about the geometric problems; implementing scripts capable of manipulate the information already contained in *TGTP*, adding, automatically, those needed items of information. A final step would be the use of all that information into filters that, by manual selection or automatic processing of users' needs, could be used to improve the users' queries.

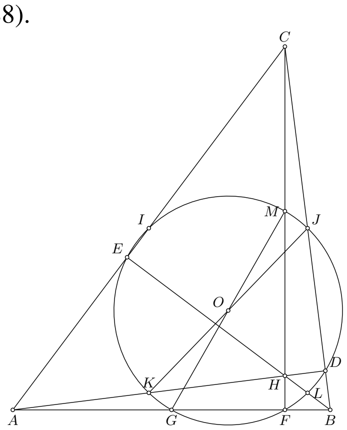
3.1. TGTP Examples

In the following, some selected examples are presented. For each example its classification, accordingly to the taxonomies defined above, is shown.

Below, the *GATP provability* ratio is given by the number of proofs successfully concluded over the number of attempts.

The *complexity* is given by the *GCLC* geometrography complexity coefficient.

GEO0316—Nine Points Circle Prove that in any triangle midpoints of each side, feet of each altitude and midpoints of the segments of each altitude from its vertex to the orthocenter lie on a circle (Chou, 1988).



MSC: 51M05, 70G55, 94B27.

GATP Provability: 1/3.

Semi-synthetic methods: 0/1: *GCLC* area method, "The conjecture is out of scope of the prover".

Algebraic methods: 1/2: *GCLC* Wu's method, "The conjecture successfully proved"; *GCLC* Gröbner basis method, "The conjecture not proved - timeout".

Readability (Chou et al., 1994): non-synthetic proof

Wu's Method, 16 pages long proof.

Readability (de Bruijn, 1994): no readable proof.

de Bruijn factor: 16/6.

See section Appendix A.1 for the informal proof. The formal proof is not included given its size, it can be consulted in *TGTP*.

Efficiency (CPU time): 0.17s.

GCLC Wu's Method: 0.17s.

CCS: C.A.3; CO.A.1; CO.C.10; CO.D.12.

Construction Complexity: complex (cs=41).

Coefficient of simplicity: $3 \times D + 3 \times 2C + 3 \times 2C + 3 \times 2C + 2 \times 2C + 2C + 3 \times 2C + 2 \times 2C + 2C + 2C = 41$.

Coefficient of freedom: $3 \times 2 = 6$.

Proofs in Education:

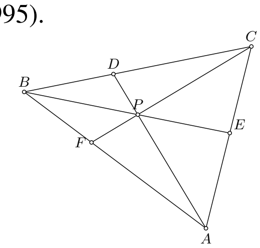
Verification: good

GCLC Wu's Method: 0.17s.

Explanation: no

only an algebraic, long (16 pages) GATP proof, exist.

GEO0001—Ceva's Theorem Let ΔABC be a triangle and P be any point in the plane. Let $D = AP \cap CB$, $E = BP \cap AC$, and $F = CP \cap AB$. Show that: $\frac{\overline{AF}}{\overline{FB}} \times \frac{\overline{BD}}{\overline{DC}} \times \frac{\overline{CE}}{\overline{EA}} = 1$. P should not be in the lines parallels to AC, AB and BC and passing through B, C and A respectively (Zhang et al., 1995).



MSC: 51M05, 70G55, 94B27, 97G70, 03B35.

GATP Provability: 4/4.

Semi-synthetic methods: 2/2: *GCLC* area method, "The conjecture successfully proved"; *Coq* area method, proved.

Algebraic methods: 2/2: *GCLC* Wu's method, "The conjecture successfully proved"; *GCLC* Gröbner basis method, "The conjecture successfully proved".

Readability (Chou et al., 1994): semi-synthetic proof with a corresponding prover's language description;

GCLC area method deduction steps = 9 < 10;

GCLC area method, 4 pages long proof; *GCLC* Wu's method, 7 pages long proof; *GCLC* Gröbner basis method, 8 pages long proof.

Readability (de Bruijn, 1994): semi-synthetic proof with a corresponding prover's language description;

de Bruijn factor: 1/1;

See section Appendix A.2 for the informal proof and section Appendix B.1 for the area method formal proof.

Efficiency (CPU time): 0.04s

GCLC area method: 0.04s; *Coq* area method: 3.32s; *GCLC* Wu's method: 0.33s; *GCLC* Gröbner basis method: 0.08s.

CCS: CO.A.4; CO.C.10; CO.D.12; SRT.B.5.

Construction Complexity: average (cs=22).

Coefficient of simplicity: $4 \times D + 6 \times 2C + 3 \times 2C = 22$.

Coefficient of freedom: $4 \times 2 = 8$.

Proofs in Education:

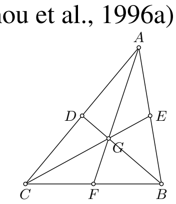
Verification: good

GCLC area method: 0.04s; *GCLC* Gröbner basis method: 0.08s.

Explanation: maybe

the area method, semi-synthetic, GATP readable proof is 1 page long (4 pages in total if title and proof information pages are to be considered) and it is produced in 0.04s.

GEO0013—Centroid Theorem The three medians of a triangle meet in a point, and each median is trisected by this point (Chou et al., 1996a).



MSC: 51M05, 70G55, 94B27, 97G70, 03B35.

GATP Provability: 4/4.

Semi-synthetic methods: 2/2: *GCLC* area method, "The conjecture successfully proved"; *Coq* area method, proved.

Algebraic methods: 2/2: *GCLC* Wu's method, "The conjecture successfully proved"; *GCLC* Gröbner basis method, "The conjecture successfully proved".

Readability (Chou et al., 1994): semi-synthetic proof with a corresponding prover's language description;

GCLC area method: maximal term $4 < 5$.

GCLC area method, 4 pages long proof; Wu's method, 7 pages long proof; Gröbner basis method, 12 pages long proof.

Readability (de Bruijn, 1994): semi-synthetic proof with a corresponding prover's language description.

de Bruijn factor: 2/1.

See section Appendix A.3 for the informal proof and section Appendix B.2 for the area method formal proof.

Efficiency (CPU time): 0.05s

GCLC area method: 0.05s; *Coq* area method: 3.32s; *GCLC* Wu's method: 0.08s; *GCLC* Gröbner basis method: 0.09s.

CCS: C.A.3; CO.A.1; CO.A.4; CO.C.10; SRT.B.4.

Construction Complexity: average (cs=17).

Coefficient of simplicity: $3 \times D + 3 \times 2C + 2 \times 2C + 2 \times 2C = 17$.

Coefficient of freedom: $3 \times 2 = 6$.

Proofs in Education:

Verification: good

GCLC area method: 0.05s; *GCLC* Wu's method: 0.08s.

Explanation: maybe

the area method, semi-synthetic, GATP readable proof is 2 pages long (4 pages in total if title and proof information pages are to be considered) and it is produced in 0.05s.

3.2. TGTP Queries

Querying the *TGTP* repository is possible in three ways: a simple textual query, a more comprehensive textual search, and a geometric search (Quaresma, 2011; Haralambous and Quaresma, 2018).

The simple textual query is done using *MySQL* regular expressions queries,[21] over the `name` attribute of the `Conjectures` table, it will provide the list of conjectures with names containing the query as a sub-string. Another, more powerful, textual query is available, using the *full-text search* of *MySQL*.[21] The attributes `name, description, shortDescription, keyword` of the `theorems` and `keywords` tables are used, allowing, for a given input sentence, to get the list of most similar sentences in any attribute of the different problem descriptions.

Based on some preliminary work on geometric search (Haralambous and Quaresma, 2014) a geometric search mechanism is being developed. The queries are constructed using *GeoGebra* dynamic geometry system and the constructed figure is semantically compared with the figures in the repository (Haralambous and Quaresma, 2018).

The developed taxonomies will allow to apply filters to any given query, e.g. in the *Intergeo* repository, the "complex text search" filter, already implements an adaptive filtering, alongside the query, the user can choose some filters (e.g. type of activity).

Finally, together with some machine learning mechanisms capable of finding the level of geometric knowledge and/or preferences of *TGTP* users, the proposed taxonomies will open the possibility of implementing adaptive queries, where the user's profile will be used to find the filters that best adjust to the users' needs.

4. Conclusions & Future Work

The definition of taxonomies for geometry will enable to incorp orate many useful information in the geometric objects (constructions, conjectures, proofs). All the collected information will constitute a meta-information block that would became part of the geometric objects in the repositories, in a similar way as the meta-information contained in a digital photography.

The next step is the construction of meta-information reports and filters capable of using that information to enhance the user's queries.

The filtering mechanism will allow improved and adapted user's queries. It will open the possibility to query for classes of geometric information, e.g. conjectures with semi-synthetic proofs; geometric construction with a coefficient of simplicity lower then 30, etc.

Considering repositories of geometric knowledge, the implementation of a, taxonomies aware filtering mechanism, must be preceded by populating the repositories databases with the meta information needed for the classification of the problems. Using *TGTP* as an example, adding the information can be seen as a three stages process:

[21] `https://dev.mysql.com/doc/refman/8.0/en/`

Manual Stage: the MSC classification; the CCS classification; the de Bruijn's factor of readability; availability of proofs for educational purposes. The procedures for inserting or updating geometric information must be changed in such a way that the meta-information can be manually added.

Automatic Stage: GATP provability; Chou's factor of readability; efficiency; *GCLC* geometrography. The information already contained in the database is enough, it must be, automatically, parsed and transformed in meta-information whenever modifications are being made to the database.

Machine Learning Stage: some of the considered taxonomies are not absolute, e.g the readability is strongly dependant of the reader: GATP experts; secondary students; etc. The introduction of users' profiles build interactively through analysis of users' interaction with the system, will allow to adapt the filters to each users' needs. In a first adaptive stage the users modelling can be implemented with simple quantitative or qualitative rule-based approaches, in a later stage, machine learning techniques over users' interactions with the system should be used to have an automatic and continuous fit of the system to the user's needs.

The presented taxonomies must be validated, improved if necessary, and other taxonomies must be taken into consideration as well.

The filtering mechanism introduced by the taxonomies, the possibility of combining different filters and the combination of this filtering stage with the search mechanism must be implemented in the repositories, or in any system that remotely access the repository, e.g. *WGL* (Quaresma et al., 2018a).

References

Nathan Altshiller-Court. *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle (Dover Books on Mathematics)*. Dover Publications, 2 edition, 2007. ISBN 0-486-45805-9. unabridged republication of the second edition of the work.

Francisco Botana, Markus Hohenwarter, Predrag Janičić, Zoltán Kovács, Ivan Petrović, Tomás Recio, and Simon Weitzhofer. Automated theorem proving in GeoGebra: Current achievements. *Journal of Automated Reasoning*, 55(1):39–59, 2015. ISSN 0168-7433. doi: 10.1007/s10817-015-9326-4.

Pierre Boutry, Julien Narboux, Pascal Schreck, and Gabriel Braun. Using small scale automation to improve both accessibility and readability of formal proofs in geometry. In Francisco Botana and Pedro Quaresma, editors, *Preliminary Proceedings of the 10th International Workshop on Automated Deduction in Geometry (ADG 2014)*, number 2014/001 in CISUC Technical Reports, pages 31–49, Coimbra, Portugal, 2014. CISUC.

Shang-Ching Chou. *Mechanical Geometry Theorem Proving*. D. Reidel Publishing Company, 1988.

Shang-Ching Chou and Xiao-Shan Gao. Automated reasoning in geometry. In John Alan Robinson and Andrei Voronkov, editors, *Handbook of Automated Reasoning*, pages 707–749. Elsevier Science Publishers B.V., 2001. ISBN 0-444-50813-9.

Shang-Ching Chou, Xiao-Shan Gao, and Jing-Zhong Zhang. *Machine Proofs in Geometry*. World Scientific, 1994.

Shang-Ching Chou, Xiao-Shan Gao, and Jing-Zhong Zhang. Automated generation of readable proofs with geometric invariants, I. Multiple and shortest proof generation. *Journal of Automated Reasoning*, 17(3):325–347, 1996a. doi: 10.1007/bf00283133.

Shang-Ching Chou, Xiao-Shan Gao, and Jing-Zhong Zhang. Automated generation of readable proofs with geometric invariants, II. Theorem proving with full-angles. *Journal of Automated Reasoning*, 17(3):349–370, 1996b. doi: 10.1007/BF00283134.

Shang-Ching Chou, Xiao-Shan Gao, and Jing-Zhong Zhang. A deductive database approach to automated geometry theorem proving and discovering. *Journal of Automated Reasoning*, 25(3):219—-246, 2000. doi: 10.1023/A:1006171315513.

Helder Coelho and Luis Moniz Pereira. Automated reasoning in geometry theorem proving with Prolog. *Journal of Automated Reasoning*, 2(4):329–390, December 1986. doi: 10.1007/BF00248249.

Nicolaas Govert de Bruijn. *Selected Papers on Automath*, volume 133 of *Studies in Logic and the Foundations of Mathematics*, chapter A survey of the project Automath, pages 141–161. Elsevier, Amsterdam, 1994. doi: 10.1016/S0049-237X(08)70203-9.

Michael de Villiers. The role and function of proof in mathematics. *Pythagoras*, 24:17–24, 1990.

Herbert Gelernter. Realization of a geometry-theorem proving machine. In *Computers & Thought*, pages 134–152, Cambridge, MA, USA, 1995. MIT Press. ISBN 0-262-56092-5. URL `http://dl.acm.org/citation.cfm?id=216408.216418`.

Adalbert Grüttner. *Die Grundlagen der Geometrographie*. Breslau: Nischkowsky, 1912. URL `https://www.e-periodica.ch/cntmng?pid=ens-001:1914:16::37`.

Gila Hanna. Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44(1-2):5–23, 2000. ISSN 0013-1954. doi: 10.1023/A:1012737223465.

Gila Hanna and Michael de Villiers, editors. *Proof and Proving in Mathematics Education*, volume 15 of *New ICMI Study Series*. Springer, 2012. ISBN 978-94-007-2128-9. doi: 10.1007/978-94-007-2129-6.

Gila Hanna and Nathan Sidoli. Visualisation and proof: a brief survey of philosophical perspectives. *ZDM*, 39(1-2): 73–78, 2007. doi: 10.1007/s11858-006-0005-0.

Yannis Haralambous and Pedro Quaresma. Querying geometric figures using a controlled language, ontological graphs and dependency lattices. In S. M. Watt, J.H. Davenport, A. P. Sexton, P. Sojka, and J. Urban, editors, *CICM 2014*, volume 8543 of *LNAI*, pages 298–311. Springer, 2014. doi: 10.1007/978-3-319-08434-3_22.

Yannis Haralambous and Pedro Quaresma. Geometric search in TGTP. In Hongbo Li, editor, *Proceedings of the 12th International Conference on Automated Deduction in Geometry*, pages 19–25, 2018. URL `http://adg2018.cc4cm.org/ADG2018Proceedings`.

Robin Hartshorne. *Geometry: Euclid and Beyond*. Springer, 2000. ISBN 978-0-387-22676-7. doi: 10.1007/978-0-387-22676-7.

Marti A. Hearst. *Search User Interfaces*. Cambridge University Press, 2009. ISBN 9780521113793.

Thomas L. Heath. *The Thirteen Books of Euclid's Elements, 3 vols.* Cambridge University Press, 1908.

M Hohenwarter. Geogebra - a software system for dynamic geometry and algebra in the plane. Master's thesis, University of Salzburg, Austria, 2002.

Predrag Janičić. Geometry constructions language. *Journal of Automated Reasoning*, 44:3–24, 2010. doi: 10.1007/s10817-009-9135-8.

Predrag Janičić. GCLC — A tool for constructive euclidean geometry and more than that. In Andrés Iglesias and Nobuki Takayama, editors, *Mathematical Software - ICMS 2006*, volume 4151 of *Lecture Notes in Computer Science*, pages 58–73. Springer, 2006. ISBN 978-3-540-38084-9. doi: 10.1007/11832225_6.

Predrag Janičić and Pedro Quaresma. Automatic verification of regular constructions in dynamic geometry systems. In Francisco Botana and Tomás Recio, editors, *Automated Deduction in Geometry*, volume 4869 of *Lecture Notes in Computer Science*, pages 39–51. Springer, 2007. ISBN 978-3-540-77355-9. doi: 10.1007/978-3-540-77356-6_3.

Predrag Janičić, Julien Narboux, and Pedro Quaresma. The Area Method: A Recapitulation. *Journal of Automated Reasoning*, 48(4):489–532, 2012. doi: 10.1007/s10817-010-9209-7.

Jianguo Jiang and Jingzhong Zhang. A review and prospect of readable machine proofs for geometry theorems. *Journal of Systems Science and Complexity*, 25(4):802–820, 2012. ISSN 1009-6124. doi: 10.1007/s11424-012-2048-3.

Émile Lemoine. *Géométrographie ou Art des constructions géométriques*. Number 18 in Scientia, Série Physico-Mathématique. C. Naud, Éditeur, Paris, Février 1902. URL `http://catalogue.bnf.fr/ark:/12148/cb36049032t`.

Fou-Lai Lin, Feng-Jui Hsieh, Gila Hanna, and Michael de Villiers, editors. *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, volume 1, 2009a. The Department of Mathematics, National Taiwan Normal University.

Fou-Lai Lin, Feng-Jui Hsieh, Gila Hanna, and Michael de Villiers, editors. *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*, volume 2, 2009b. The Department of Mathematics, National Taiwan Normal University.

Gino Loria. La geometrografia e le sue trasfigurazioni. *Periodico di Matematiche*, 3(6):114–122, 1908.

John Sturgeon Mackay. The geometrography of Euclid's problems. *Proceedings of the Edinburgh Mathematical Society*, 12:2–16, 1893. doi: 10.1017/S0013091500001565.

George Martin. *Geometric Constructions*. Springer, 1998. ISBN 978-1-4612-0629-3. doi: 10.1007/978-1-4612-0629-3.

William McCallum. *The Common Core State Standards in Mathematics*, pages 547–560. Springer, Cham, 2015. ISBN 978-3-319-17187-6. doi: 10.1007/978-3-319-17187-6_31.

Jorma K. Merikoski and Timo Tossavainen. Two approaches to geometrography. *Journal for Geometry and Graphics*, 13(1):15–28, 2010.

Glenn R. Morrow. *Proclus, A Commentary on the First Book of Euclid's Elements*. Classics/History of mathematics. Princeton University Press, 1992. ISBN 9780691020907. URL `https://books.google.pt/books?id=4-E9DwAAQBAJ`.

Julien Narboux. A graphical user interface for formal proofs in geometry. *Journal of Automated Reasoning*, 39:161–180, 2007. doi: 10.1007/s10817-007-9071-4.

Karol Pąk. Improving legibility of formal proofs based on the close reference principle is NP-Hard. *Journal of Automated Reasoning*, 55(3):295–306, Oct 2015. ISSN 1573-0670. doi: 10.1007/s10817-015-9337-1.

Tuan Minh Pham and Yves Bertot. A combination of a dynamic geometry software with a proof assistant for interactive formal proofs. *Electronic Notes in Theoretical Computer Science*, 285:43 – 55, 2012. ISSN 1571-0661. doi: 10.1016/j.entcs.2012.06.005. Proceedings of the 9th International Workshop On User Interfaces for Theorem Provers (UITP10).

Vergílio Athayde Pinheiro. *Geometrografia 1*. Gráfica Editora Bahiense, 1974.

Pedro Quaresma. Thousands of Geometric problems for geometric Theorem Provers (TGTP). In Pascal Schreck, Julien Narboux, and Jürgen Richter-Gebert, editors, *Automated Deduction in Geometry*, volume 6877 of *Lecture Notes in Computer Science*, pages 169–181. Springer, 2011. ISBN 978-3-642-25069-9. doi: 10.1007/978-3-642-25070-5_10.

Pedro Quaresma. Towards an intelligent and dynamic geometry book. *Mathematics in Computer Science*, 11(3–4): 427–437, Dec 2017. ISSN 1661-8289. doi: 10.1007/s11786-017-0302-8.

Pedro Quaresma and Vanda Santos. Visual geometry proofs in a learning context. In Walther Neuper and Pedro Quaresma, editors, *Proceedings of ThEdu'15*, volume 2016001 of *CISUC Technical Reports*, pages 1–6. CISUC, 2016. URL `https://www.cisuc.uc.pt/ckfinder/userfiles/files/TR2016-01.pdf`.

Pedro Quaresma, Vanda Santos, and Nuno Baeta. Exchange of geometric information between applications. In Pedro Quaresma and Walther Neuper, editors, *ThEdu'17 PostProceedings*, volume 267 of *EPTCS*, pages 108–119. Open Publishing Association, 2018a. doi: 10.4204/EPTCS.267.7.

Pedro Quaresma, Vanda Santos, and Milena Marić. WGL, a web laboratory for geometry. *Education and Information Technologies*, 23(1):237–252, Jan 2018b. ISSN 1573-7608. doi: 10.1007/s10639-017-9597-y.

Jürgen Richter-Gebert and Ulrich Kortenkamp. *The Interactive Geometry Software Cinderella*. Springer, 1999. ISBN 978-3540147190.

Ferdinand D. Rivera. *Teaching to the Math Common Core State Standards*. Sense Publishers, 2013. ISBN 978-94-6209-501-4.

Mary Budd Rowe. Wait-time and rewards as instructional variables: Their influence on language, logic, and fate control. Technical report, National Association for Research in Science Teaching, 1972. URL `https://files.eric.ed.gov/fulltext/ED061103.pdf`.

Vanda Santos, Pedro Quaresma, Helena Campos, and Milena Marić. Web Geometry Laboratory: Case Studies in Portugal and Serbia. *Interactive Learning Environments*, 26(1):3–21, 2018. doi: 10.1080/10494820.2016.1258715.

Nathan Sidoli. Uses of construction in problems and theorems in Euclid's Elements I–VI. *Archive for History of Exact Sciences*, 72(4):403–452, 7 2018. ISSN 0003-9519. doi: 10.1007/s00407-018-0212-4.

Robert J. Stahl. Using "think-time" and "wait-time" skillfully in the classroom. Technical report, ERIC Digest, 1994. URL `http://files.eric.ed.gov/fulltext/ED370885.pdf`.

Sana Stojanović, Vesna Pavlović, and Predrag Janičić. A coherent logic based geometry theorem prover capable of producing formal and readable proofs. In Pascal Schreck, Julien Narboux, and Jürgen Richter-Gebert, editors, *Automated Deduction in Geometry*, volume 6877 of *Lecture Notes in Computer Science*, pages 201–220. Springer, Berlin, Heidelberg, 2011. doi: 10.1007/978-3-642-25070-5_12.

UNESCO. *International Standard Classification of Education: ISCED 2011*. UIS, Montreal, Quebec, 2012.

Zalman Usiskin. van Hiele levels and achievement in secondary school geometry. Technical report, University of Chicago, 1982.

Ke Wang and Zhendong Su. Automated geometry theorem proving for human-readable proofs. In *Proceedings of the 24th International Conference on Artificial Intelligence*, IJCAI'15, pages 1193–1199. AAAI Press, 2015. ISBN 978-1-57735-738-4. URL `http://dl.acm.org/citation.cfm?id=2832249.2832414`.

Freek Wiedijk. The de Bruijn factor. Poster at International Conference on Theorem Proving in Higher Order Logics (TPHOL2000), 2000. Portland, Oregon, USA, 14-18 August 2000.

Hung-Hsi Wu. Teaching geometry according to the common core standards, 2012. URL `https://math.berkeley.edu/~wu/Progressions_Geometry.pdf`.

Zheng Ye, Shang-Ching Chou, and Xiao-Shan Gao. Visually dynamic presentation of proofs in plane geometry: Part 1. Basic features and the manual input method. *Journal of Automated Reasoning*, 45(3):213–241, October 2010a. ISSN 0168-7433. doi: 10.1007/s10817-009-9162-5.

Zheng Ye, Shang-Ching Chou, and Xiao-Shan Gao. Visually dynamic presentation of proofs in plane geometry: Part 2. Automated generation of visually dynamic presentations with the full-angle method and the deductive database method. *Journal of Automated Reasoning*, 45(3):243–266, October 2010b. ISSN 0168-7433. doi: 10.1007/s10817-009-9163-4.

Zheng Ye, Shang-Ching Chou, and Xiao-Shan Gao. An introduction to java geometry expert. In Thomas Sturm and

Christoph Zengler, editors, *Automated Deduction in Geometry*, volume 6301 of *Lecture Notes in Computer Science*, pages 189–195. Springer Berlin Heidelberg, 2011. ISBN 978-3-642-21045-7. doi: 10.1007/978-3-642-21046-4_10.

Jing-Zhong Zhang, Shang-Ching Chou, and Xiao-Shan Gao. Automated production of traditional proofs for theorems in Euclidean geometry i. The Hilbert intersection point theorems. *Annals of Mathematics and Artificial Intelligence*, 13 (1–2):109–137, Mar 1995. ISSN 1573-7470. doi: 10.1007/BF01531326.

Appendix A. Informal Proofs

Appendix A.1. Nine Points Circle Informal Proof

Taken from:
`www.ma.utexas.edu/users/shirley/a333l/Handouts/euclidean-geometry-ii/9_pt_circle_thm.pdf`, consulted 2017/12/23. Proof done with the help of lemmas to reduce the size of the proof.

<u>The Nine-Point Circle Theorem</u>:

For any triangle the following nine points lie on the same circle:

1) The three midpoints of the sides of the triangle,

2) The feet of the three altitudes of the triangle,

3) The midpoints of the three segments connecting vertices to the orthocenter.

This circle is called the Nine-point Circle of the triangle. Its center is the midpoint of the segment between the orthocenter and the circumcenter and its radius is ½ the radius of the circumcircle.

In the following discussion, the following three theorems are frequently applied :

Theorem 4.2.15, The Midpoint Connection Theorem: If a line segment has as its endpoints the midpoints of two sides of a triangle then the segment is contained in a line that is parallel to the third side and the segment is one-half the length of the third side.

Theorem (NIB) 4.6, The Hypotenuse Diameter Theorem: For any right triangle, the circle which has the hypotenuse as diameter contains the vertex with the right angle.

The Perpendicularity Statement: If a line is perpendicular to one of two parallel lines, then it is perpendicular to the other.

As shown before, in Neutral Geometry the Perpendicularity Statement is equivalent to the Euclidean Parallel Postulate, so the Perpendicularity Statement is true in Euclidean Geometry.

Two Lemmas are used below and their proofs are left as exercises:

Lemma 1: A parallelogram with at least one right angle is a rectangle.

Lemma 2: A rectangle is circumscribed by a circle and each diagonal of the rectangle is a diameter of the circle.

Recall that, in Euclidean Geometry, Parallelism is Transitive (by Theorem 4.2.9):

For given lines $\ell, \boldsymbol{m}$, and $\boldsymbol{n}$, if $\ell \parallel \boldsymbol{m}$ and $\boldsymbol{m} \parallel \boldsymbol{n}$ then $\ell \parallel \boldsymbol{n}$.

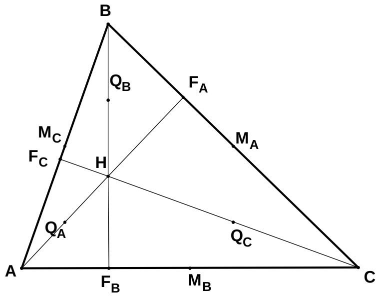
Also, by the Perpendicularity Statement, for given lines $\ell, \boldsymbol{m}$, and $\boldsymbol{t}$,

$$\text{if } \ell \parallel \boldsymbol{m} \text{ and } \boldsymbol{t} \perp \ell, \text{ then } \boldsymbol{t} \perp \boldsymbol{m}.$$

Proof of the Nine-Point Circle Theorem:

Proof: Let ΔABC be a given triangle.
Let H denote the orthocenter and let S denote the circumcenter.

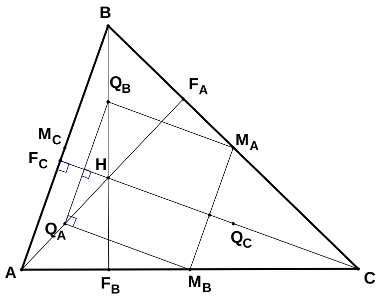
Let the following notations be defined:



1) M_A, M_B, and M_C are the midpoints of the sides of the triangle opposite vertices A, B, and C, respectively.

2) F_A, F_B, and F_C are the feet of the altitudes from vertices A, B and C, respectively.

3) Q_A, Q_B, and Q_C are the midpoints of the segments between the orthocenter H of the triangle and the vertices A, B and C, respectively.



$\overline{Q_AQ_B} \parallel \overline{AB}$ by the Midpoint Connection Theorem applied to ΔHAB.

$\overline{M_AM_B} \parallel \overline{AB}$ by the Midpoint Connection Theorem applied to ΔABC.

$\overline{Q_AQ_B} \parallel \overline{M_AM_B}$ by the transitivity of parallelism.

$\overline{Q_AM_B} \parallel \overline{CH}$ by the Midpoint Connection Theorem applied to ΔHAC.

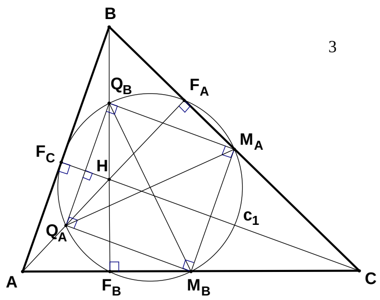
$\overline{Q_BM_A} \parallel \overline{CH}$ by the Midpoint Connection Theorem applied to ΔHBC.

$\overline{Q_AM_B} \parallel \overline{Q_BM_A}$ by the transitivity of parallelism. $\therefore$ $Q_A\,Q_B\,M_A\,M_B$ is a parallelogram .

$\overline{CF_C} \perp \overline{AB}$ by definition of "altitude". $\therefore \overline{CF_C} \perp \overline{Q_AQ_B}$ by the Perpendicularity Statement.

$\therefore \overline{Q_AM_B} \parallel \overline{CF_C}$ since C – H – F_C . $\therefore \overline{Q_AM_B} \perp \overline{Q_AQ_B}$ by the Perpendicularity Statement.

$\therefore \angle Q_BQ_AM_B$ is a right angle. $\therefore$ By Lemma 1, $Q_A\,Q_B\,M_A\,M_B$ is a rectangle.



Recall that $Q_A\, Q_B\, M_A\, M_B$ is a rectangle.

By Lemma 2, there exists a circle c_1 such that

circle c_1 circumscribes $Q_A\, Q_B\, M_A\, M_B$ and
c_1 = C(diameter = $\overline{Q_A M_A}$) and
c_1 = C(diameter = $\overline{Q_B M_B}$) .

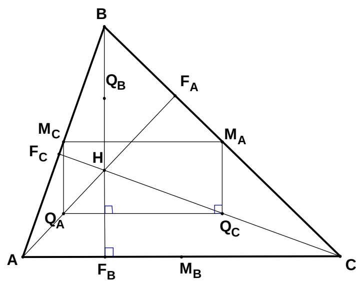
By definition of "altitude", $\angle Q_A F_A M_A$ is a right angle .
$\therefore$ $\Delta F_A Q_A M_A$ is a right triangle with hypotenuse $\overline{Q_A M_A}$.
$\therefore$ Circle c_1 contains F_A by the Hypotenuse Diameter Theorem.

By definition of "altitude", $\angle Q_B F_B M_B$ is a right angle.
$\therefore$ $\Delta F_B Q_B M_B$ is a right triangle with hypotenuse $\overline{Q_B M_B}$.
$\therefore$ Circle c_1 contains F_B by the Hypotenuse Diameter Theorem.

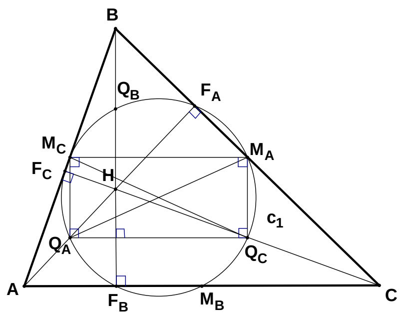
$\therefore$ Circle c_1 contains the six points: Q_A , Q_B , M_A , M_B , F_A , F_B

A similar argument shows that

$Q_A\, Q_C\, M_A\, M_C$ is a rectangle



Then, a similar argument shows that
the circle c_1 = C(diameter = $\overline{Q_A M_A}$)
circumscribes $Q_A\, Q_C\, M_A\, M_C$ and that
circle c_1 contains the point F_C .



Note also that all three rectangle diagonals serve as diameters for circle c_1 , that is,

$\overline{Q_A M_A}$, $\overline{Q_B M_B}$, and $\overline{Q_C M_C}$, all three are diameters of circle c_1 .

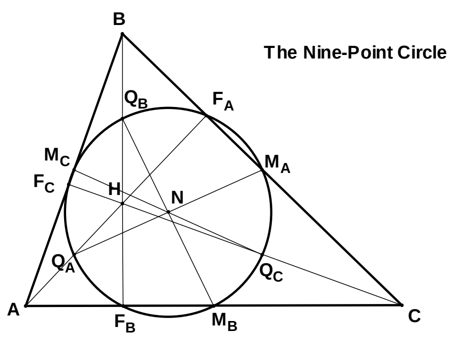
$\therefore$ **Circle c_1 contains the nine points: Q_A , Q_B , Q_C , M_A , M_B , M_C , F_A , F_B , F_C**

and it has $\overline{Q_A M_A}$, $\overline{Q_B M_B}$, and $\overline{Q_C M_C}$ **as diameters .**

The circle c_1 is called the **Nine-Point Circle.**

Let N be the midpoint of $\overline{Q_A M_A}$. Then, N is the center of the Nine-Point Circle.

We will call this circle the Nine-point Circle.



It remains only to locate the center on the Euler Line and to determine

the radius of the Nine-point Circle.

Recall that S denotes the Circumcenter of $\Delta\, ABC$

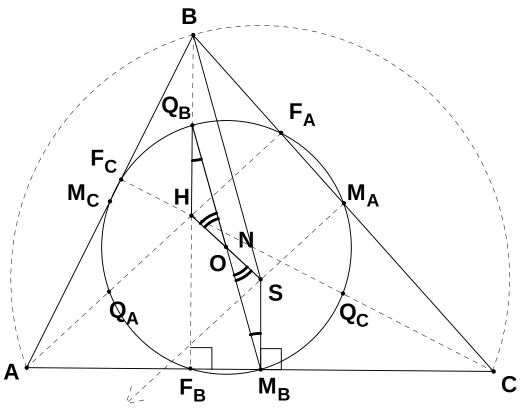
and that N denotes the center of the Nine-point Circle.

Let O be the point where Nine-Point Circle diameter $\overline{Q_B M_B}$ intersects $\overline{HS}$.

(A technical argument, which shows that $\overline{Q_B M_B}$ and $\overline{HS}$ must intersect, is omitted here.)

We show that point O and point N are the same point.

It will suffice to show that O is the midpoint $\overline{Q_B M_B}$ since $\overline{Q_B M_B}$ is a diameter of the Nine-Point Circle.



To do this, we show that $\Delta\, O Q_B H$ is congruent to $\Delta\, O M_B S$:

Now, $\overline{SM_B}$ lies along the perpendicular bisector of side $\overline{AC}$, so $\overline{SM_B} \perp \overline{AC}$.

Since $\overleftrightarrow{BH} \perp \overline{AC}$ also, $\overline{SM_B} \parallel \overline{BH}$.

Since $\overline{SM_B} \parallel \overline{BH}$, $\angle OQ_B H$ and $\angle OM_B S$ are alternate interior angles formed where transversal $\overleftrightarrow{Q_B M_B}$ intersects parallel lines.

$\therefore$ By the Converse of the Alternate Interior Angle Theorem , $\angle OQ_B H \;\cong\; \angle OM_B S$.

Also, $\angle HOQ_B \;\cong\; \angle SOM_B$ by the Vertical Angles Theorem.

Since Q_B is, by definition, the midpoint of $\overline{BH}$, $HQ_B = ½ (BH)$.

By the Altitude Segment Theorem, Theorem 4.7.1, $BH = 2\,(SM_B)$; so, $SM_B = ½ (BH)$.

Therefore, $HQ_B = SM_B$ and so, $\overline{HQ_B} \cong \overline{SM_B}$.

$\therefore \Delta\, O Q_B H \;\cong\; \Delta\, O M_B S$ by AAS.

Then, by CPCF, $\overline{Q_B O} \cong \overline{M_B O}$, and so, O is the midpoint of $\overline{Q_B M_B}$.

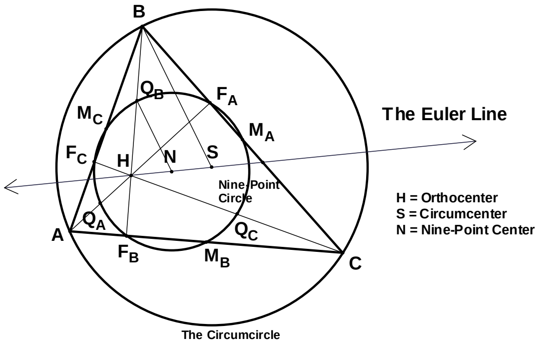
Since $\overline{Q_B M_B}$ is a diameter of the Nine-Point Circle, O = N, the center of the Nine-point Circle.

Also, by CPCF, $\overline{HO} \cong \overline{SO}$, so point O = N is also the midpoint of $\overline{HS}$.

This locates the center N of the Nine-Point Circle at the point midway between the Orthocenter H and the Circumcenter S . .

Thus, the center N of the Nine-point circle is the midpoint of the segment between the orthocenter H and the circumcenter S.

Therefore, the center of the Nine-Point Circle is on the Euler Line $\overleftrightarrow{HS}$



We show that the radius of the Nine-Point Circle is one-half the radius of the Circumcircle.

Segment $\overline{SB}$ is a radius of the circumcircle (since S is the circumcenter)
and segment $\overline{NQ_B}$ is a radius of the Nine-point circle.

Also, N is the midpoint of $\overline{HS}$, so $\overline{NQ_B}$ connects the midpoints of two of the sides of ΔSBH .

$\therefore$ By the Midpoint Connection Theorem applied to ΔSBH , $NQ_B = \frac{1}{2}(SB)$.

Therefore, since SB is the radius of the Circumcircle,

the radius of the Nine-point circle is ½ the radius of the Circumcircle.

QED

Appendix A.2. Ceva's Theorem Informal Proof

Taken from: `artofproblemsolving.com/wiki/index.php?title=Ceva's_Theorem`, consulted on 2017/12/18.

Proof

We will use the notation $[ABC]$ to denote the area of a triangle with vertices A, B, C.

First, suppose AD, BE, CF meet at a point X.We note that triangles ABD, ADC have the same altitude to line BC, but bases BD and DC. It follows that $\frac{BD}{DC} = \frac{[ABD]}{[ADC]}$. The same is true for triangles XBD, XDC, so

$$\frac{BD}{DC} = \frac{[ABD]}{[ADC]} = \frac{[XBD]}{[XDC]} = \frac{[ABD] - [XBD]}{[ADC] - [XDC]} = \frac{[ABX]}{[AXC]}.$$

Similarly, $\frac{CE}{EA} = \frac{[BCX]}{[BXA]}$ and $\frac{AF}{FB} = \frac{[CAX]}{[CXB]}$, so

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{[ABX]}{[AXC]} \cdot \frac{[BCX]}{[BXA]} \cdot \frac{[CAX]}{[CXB]} = 1$$

Now, suppose D, E, F satisfy Ceva's criterion, and suppose AD, BE intersect at X. Suppose the line CX intersects line AB at F'. We have proven that F' must satisfy Ceva's criterion. This means that

$$\frac{AF'}{F'B} = \frac{AF}{FB},$$

so

$$F' = F,$$

and line CF concurs with AD and BE. □

Appendix A.3. Centroid Theorem Informal Proof

Taken from: `new.math.uiuc.edu/public403/affine/centroid.html`, consulted on 2017/12/14.

Let G be the point where medians BB' and CC' of ΔABC intersect. We shall show that G trisects the two medians in the sense that $BG : GB' = 2 : 1$ and $CG : GC' = 2 : 1$. This means that any two medians meet at their point two-thirds of the way from the vertex to the midpoint of the opposite side. So all three do.

Prior to the place in Euclid's Elements are theorems about similar triangles and about angles made by transversals of two parallel lines. The ones we'll use here are

simAAA Two triangles are similar if and only if their angles are pairwise equal.

simSAS Two triangles are similar if and only if two pairs of corresponding sides have the same proportion and the included angles are equal.

altIA Two lines are parallel if and only if two alternate interior angles they make with a transversal are equal.

sameSA Two lines are parallel if and only if corresponding angles on the same side of a transversal are equal.

Step 1:. Apply *simSAS* to $\Delta AC'B'$ and ΔABC by noting that they have $\angle A$ in common, and the adjacent sides are in the ratio of $1:2$. So the two triangles are similar. That, by definition of similarity, implies that $\angle B'C'A = \angle CBA$ and $C'B' : BC = 1:2$.

Step 2:. Apply *sameSA* to the two lines $C'B'$ and BC to the first consequence: $\angle B'C'A = \angle CBA$. Therefore $C'B' \| BC$. That in turn, by *sameSA*, implies that $\angle GC'B' = \angle GCB$ and $\angle C'B'G = \angle CBG$.

Step 3:. Apply *simAAA* to the two triangles $\Delta GB'C'$ and ΔGBC. Two pairs of their angles have already been shown to be equal. The third pair, $\angle B'GC'$ and $\angle BGC$, are equal because they are "opposite angles". Thus the two triangles are similar.

Step 4:. From the second conclusion in Step 1 we know the ratio of to be $2:1$. So $1:2 = GB' : GB$ and $1:2 = GC' : GC$. So G does "trisect" two of the medians, as predicted. □

Appendix B. Formal Proofs

Appendix B.1. Ceva's Theorem Formal Proof

Proof produced by *GCLC* area method.

$$\left(\left(\frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right) \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1 \quad \text{by the statement} \tag{0}$$

$$\left(\left(\left(-1 \cdot \frac{\overrightarrow{AF}}{\overrightarrow{BF}}\right) \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right) \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1 \quad \text{by geometric simplifications} \tag{1}$$

$$\left(-1 \cdot \left(\frac{\overrightarrow{AF}}{\overrightarrow{BF}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1 \quad \text{by algebraic simplifications} \tag{2}$$

$$\left(-1 \cdot \left(\frac{S_{APC}}{S_{BPC}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1 \quad \text{by Lemma 8 (point } F \text{ eliminated)} \tag{3}$$

$$\left(-1 \cdot \left(\frac{S_{APC}}{S_{BPC}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \left(-1 \cdot \frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)\right)\right) = 1 \quad \text{by geometric simplifications} \tag{4}$$

$$\frac{\left(S_{APC} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)}{S_{BPC}} = 1 \quad \text{by algebraic simplifications} \tag{5}$$

$$\frac{\left(S_{APC} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{S_{CPB}}{S_{APB}}\right)\right)}{S_{BPC}} = 1 \quad \text{by Lemma 8 (point } E \text{ eliminated)} \tag{6}$$

$$\frac{\left(S_{APC} \cdot \left(\left(-1 \cdot \frac{\overrightarrow{BD}}{\overrightarrow{CD}}\right) \cdot \frac{S_{CPB}}{S_{APB}}\right)\right)}{(-1 \cdot S_{CPB})} = 1 \quad \text{by geometric simplifications} \tag{7}$$

$$\frac{\left(S_{APC} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{CD}}\right)}{S_{APB}} = 1 \quad \text{by algebraic simplifications} \quad (8)$$

$$\frac{\left(S_{APC} \cdot \frac{S_{BPA}}{S_{CPA}}\right)}{S_{APB}} = 1 \quad \text{by Lemma 8 (point } D \text{ eliminated)} \quad (9)$$

$$\frac{\left(S_{APC} \cdot \frac{S_{BPA}}{(-1 \cdot S_{APC})}\right)}{(-1 \cdot S_{BPA})} = 1 \quad \text{by geometric simplifications} \quad (10)$$

$$1 = 1 \quad \text{by algebraic simplifications} \quad (11)$$

NDG conditions are:
$S_{BPA} \neq S_{CPA}$ i.e., lines BC and PA are not parallel (construction based assumption)
$S_{APB} \neq S_{CPB}$ i.e., lines AC and PB are not parallel (construction based assumption)
$S_{APC} \neq S_{BPC}$ i.e., lines AB and PC are not parallel (construction based assumption)
$P_{FBF} \neq 0$ i.e., points F and B are not identical (conjecture based assumption)
$P_{DCD} \neq 0$ i.e., points D and C are not identical (conjecture based assumption)
$P_{EAE} \neq 0$ i.e., points E and A are not identical (conjecture based assumption)

Appendix B.2. Centroid Theorem Formal Proof

Proof produced by *GCLC* area method.

$$\left(-2 \cdot \frac{\overrightarrow{GD}}{\overrightarrow{GB}}\right) = 1 \quad \text{by the statement} \quad (0)$$

$$\left(-2 \cdot \left(-1 \cdot \left(-1 \cdot \frac{\overrightarrow{DG}}{\overrightarrow{BG}}\right)\right)\right) = 1 \quad \text{by geometric simplifications} \quad (1)$$

$$\left(-2 \cdot \frac{\overrightarrow{DG}}{\overrightarrow{BG}}\right) = 1 \quad \text{by algebraic simplifications} \quad (2)$$

$$\left(-2 \cdot \frac{S_{DEC}}{S_{BEC}}\right) = 1 \quad \text{by Lemma 8 (point } G \text{ eliminated)} \quad (3)$$

$$\frac{(-2 \cdot S_{DEC})}{S_{BEC}} = 1 \quad \text{by algebraic simplifications} \quad (4)$$

$$\frac{(-2 \cdot S_{CDE})}{S_{CBE}} = 1 \quad \text{by geometric simplifications} \quad (5)$$

$$\frac{\left(-2 \cdot \left(S_{CDA} + \left(\frac{1}{2} \cdot (S_{CDB} + (-1 \cdot S_{CDA}))\right)\right)\right)}{S_{CBE}} = 1 \quad \text{by Lemma 29 (point } E \text{ eliminated)} \quad (6)$$

$$\frac{((-1 \cdot S_{CDA}) + (-1 \cdot S_{CDB}))}{S_{CBE}} = 1 \quad \text{by algebraic simplifications} \quad (7)$$

$$\frac{((-1 \cdot S_{CDA}) + (-1 \cdot S_{CDB}))}{\left(S_{CBA} + \left(\frac{1}{2} \cdot (S_{CBB} + (-1 \cdot S_{CBA}))\right)\right)} = 1 \quad \text{by Lemma 29 (point } E \text{ eliminated)} \quad (8)$$

$$\frac{((-1 \cdot S_{ACD}) + (-1 \cdot S_{BCD}))}{\left(S_{CBA} + \left(\frac{1}{2} \cdot (0 + (-1 \cdot S_{CBA}))\right)\right)} = 1 \quad \text{by geometric simplifications} \quad (9)$$

$$\frac{((-1 \cdot S_{ACD}) + (-1 \cdot S_{BCD}))}{\left(\frac{1}{2} \cdot S_{CBA}\right)} = 1 \quad \text{by algebraic simplifications} \quad (10)$$

$$\frac{\left(\left(-1 \cdot \left(S_{ACA} + \left(\frac{1}{2} \cdot (S_{ACC} + (-1 \cdot S_{ACA}))\right)\right)\right) + (-1 \cdot S_{BCD})\right)}{\left(\frac{1}{2} \cdot S_{CBA}\right)} = 1 \quad \text{by Lemma 29 (point } D \text{ eliminated)} \quad (11)$$

$$\frac{\left(\left(-1 \cdot \left(0 + \left(\frac{1}{2} \cdot (0 + (-1 \cdot 0))\right)\right)\right) + (-1 \cdot S_{BCD})\right)}{\left(\frac{1}{2} \cdot S_{CBA}\right)} = 1 \quad \text{by geometric simplifications} \quad (12)$$

$$\frac{(-1 \cdot S_{BCD})}{\left(\frac{1}{2} \cdot S_{CBA}\right)} = 1 \quad \text{by algebraic simplifications} \quad (13)$$

$$\frac{\left(-1 \cdot \left(S_{BCA} + \left(\frac{1}{2} \cdot (S_{BCC} + (-1 \cdot S_{BCA}))\right)\right)\right)}{\left(\frac{1}{2} \cdot S_{CBA}\right)} = 1 \quad \text{by Lemma 29 (point } D \text{ eliminated)} \quad (14)$$

$$\frac{\left(-1 \cdot \left(S_{BCA} + \left(\frac{1}{2} \cdot (0 + (-1 \cdot S_{BCA}))\right)\right)\right)}{\left(\frac{1}{2} \cdot (-1 \cdot S_{BCA})\right)} = 1 \quad \text{by geometric simplifications} \quad (15)$$

$$1 = 1 \quad \text{by algebraic simplifications} \quad (16)$$

NDG conditions are: $S_{DEC} \neq S_{BEC}$ i.e., lines DB and EC are not parallel (construction based assumption); $P_{GBG} \neq 0$ i.e., points G and B are not identical (conjecture based assumption).