TR 2006/002 ISSN 0874-338X

GeoThms - Geometry Framework

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April 24, 2006

[1]This work was partially supported by CISUC/FCT.
[2]This work was partially supported by the CISUC/FCT, Centro International de Matemática (CIM), under the programme "Research in Pairs", while on visit to Coimbra University under the Coimbra Group Hospitality Scheme

Abstract

GeoThms is a system that integrates Automatic Theorem Provers (ATP), Dynamic Geometry Tools (DGT) and a database, providing a framework for exploring geometrical knowledge. A GeoThms user can browse through a list of available geometric problems, their statements, illustrations, and proofs. He/she can also interactively produce new geometrical constructions, theorems, and proofs and add new results to the existing ones. GeoThms framework provides an environment suitable for new ways of studying and teaching geometry at different levels.

keywords: Automated geometry theorem proving, Euclidean traditional proof, Area method, constructive geometry statements.

Contents

Chapter 1

Introduction

Aiming to build a framework for constructive geometry We have extended GCLC [DJ04, JT03], a widely used dynamic geometry package, with a module that allows formal deductive reasoning about constructions made in the (main) drawing module. The built-in theorem prover, GCLCprover, is based on the area method [CGZ93, CGZ96, Nar04, QJ06]. It produces proofs that are human-readable (in LaTeX form), and with a clear justification for each proof step. It is also possible, via a conversion tool, to reason about constructions made with Eukleides [Obr, QP06]. Closely linked to the mentioned tools is GeoDB — a database of geometry theorems. It stores geometric problems, their statements, illustrations, and proofs.

The theorem prover, the visualisation tools, and the database are "packed" in the GeoThms framework. This framework provides an Web interface where the users can easily browse through the list of geometric problems, their statements, illustrations and proofs, and also to interactively use the drawing and automatic proof tools. GeoThms is a set of PHP scripts of top of a MySQL database, and is accessible from: http://hilbert.mat.uc.pt/~geothms.

In this report we begin to present the Web interface and the MySQL database. Then we present the first ten geometric results contained in the database.

Chapter 2

The Framework

GeoThms, is a framework that link dynamic geometry tools (GCLC, Eukleides), geometry automatic theorem provers (GCLCprover), and a repository of geometry problems (geoDB) (see Figure 2.1).

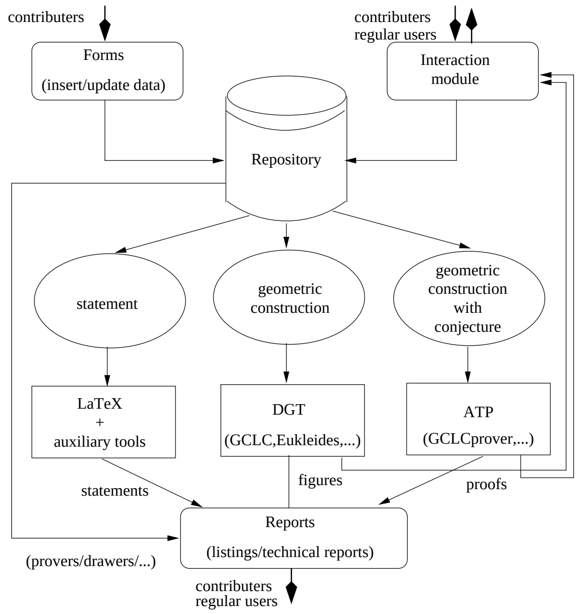


Figure 2.1: The GeoThms framework

GeoThms provides a Web workbench in the field of constructive problems in Euclidean geometry. Its tight integration with dynamic geometry tools and automatic theorem provers (GCLC [DJ04, JT03], Eukleides [Obr, QP06], and GCLCprover [QJ06], for the moment) and its repository of theorems, figures and proofs, give the user the possibility to easily browse through the list of geometric problems, their statements, illustrations and proofs, and also to interactively use the drawing and automatic proof tools.

The structure of the web interface has two main levels of interaction (see Figure 2.2).

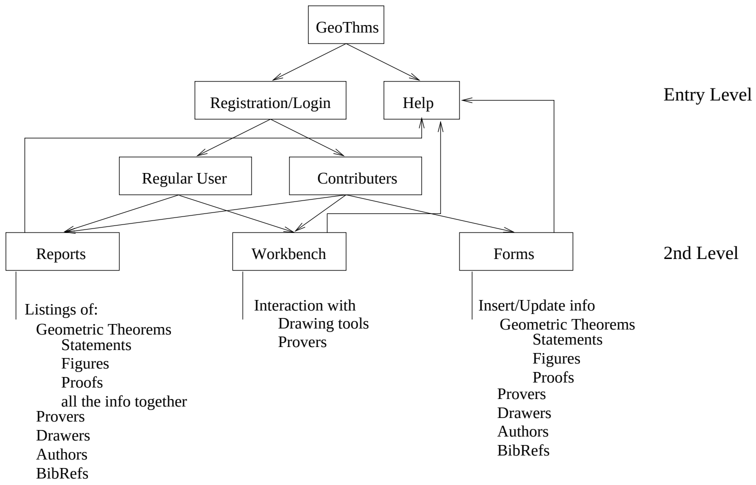


Figure 2.2: GeoThms — Web Interface

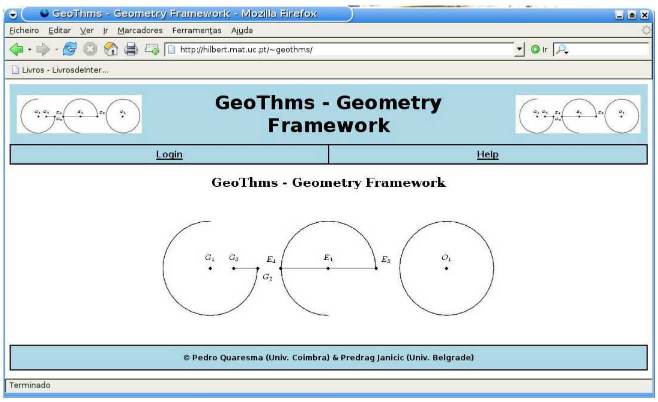


Figure 2.3: GeoThms – Home Page

The entry level (see Figure 2.3), accessible to all web-users, has some basic information about GeoThms, offers the possibility of registration to anyone interested in using GeoThms, and it gives access to the other levels. A (registered) regular user has access to a second level (see Figure 2.4) where he/she can browse the data from the database (see Figure 2.5), (in a formatted form, or in a textual form) (see Figure 2.6) and use the drawing/proof tools in an interactive way (see Figure 2.7).

A regular user can apply to the status of *contributer* (see Figure 2.8) in which case he/she will have the possibility to insert new data, and/or to update the data he/she had inserted previously (see Figure 2.9).

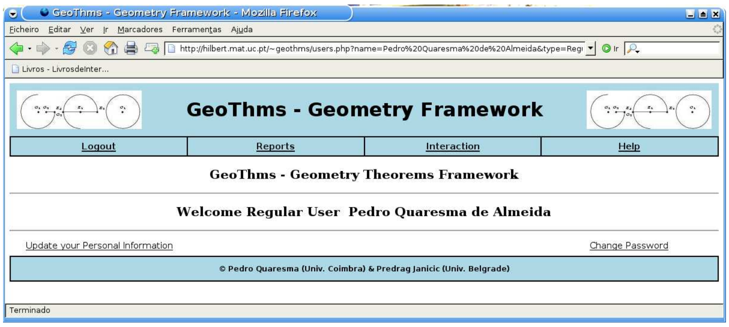


Figure 2.4: GeoThms – Regular Users Page

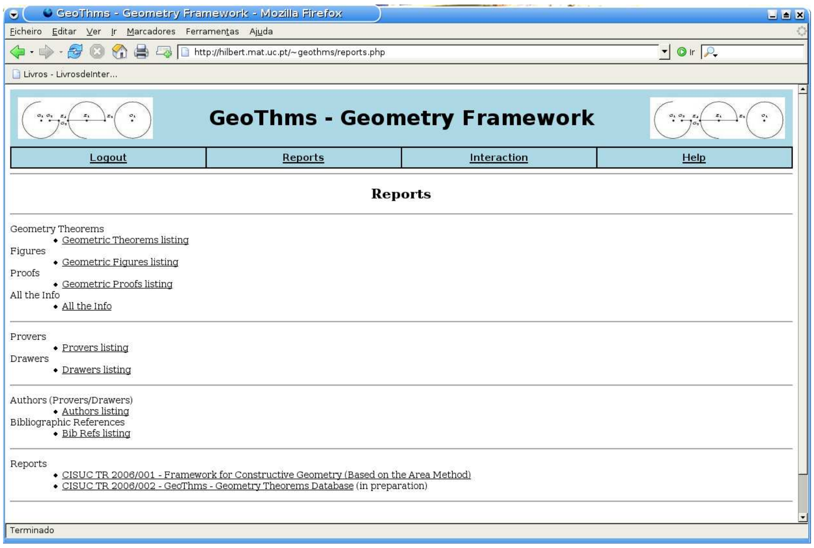


Figure 2.5: GeoThms – Reports Page

Constructions are described and stored in declarative languages of dynamic geometry tools such as GCLC and Eukleides. Figures are generated directly on the basis of geometric specifications, by GCLC and Eukleides and stored as Jpeg files. Conjectures are described and stored in a a form that extend geometric specifications. The specifications of conjectures are used (directly or via a converter) by GCLCprover. Proofs are generated by GCLCprover and stored as PDF files (produced by LaTeX from the ATP output and using a specific layout, specified by `gclc_proof` LaTeX style file).

The framework can be augmented by other dynamic geometry tools, and other geometry theorem provers.

GeoThms gives the user a complex framework suitable for new ways of communicating geometric knowledge, it provides an open system where one can learn from the existing knowledge base and seek for new results. GeoThms also provides a system for storing geometric knowledge (in a strict,

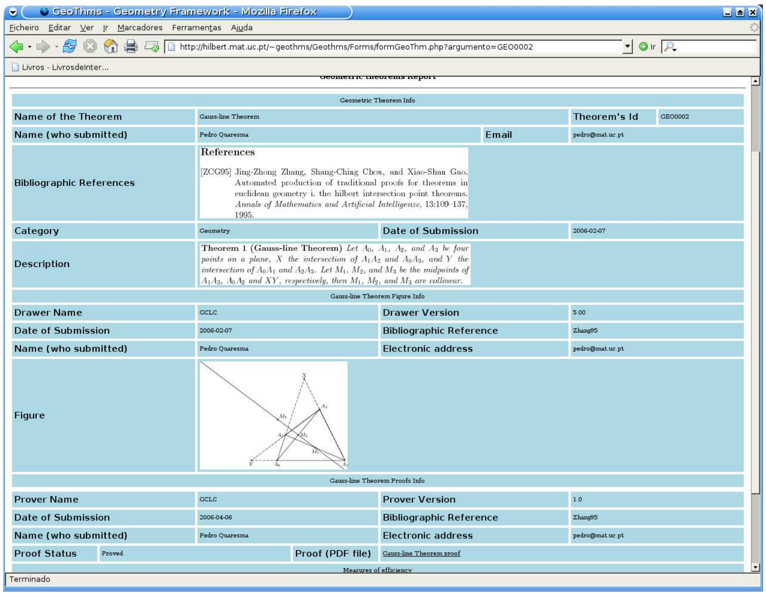


Figure 2.6: GeoThms – Theorem Report

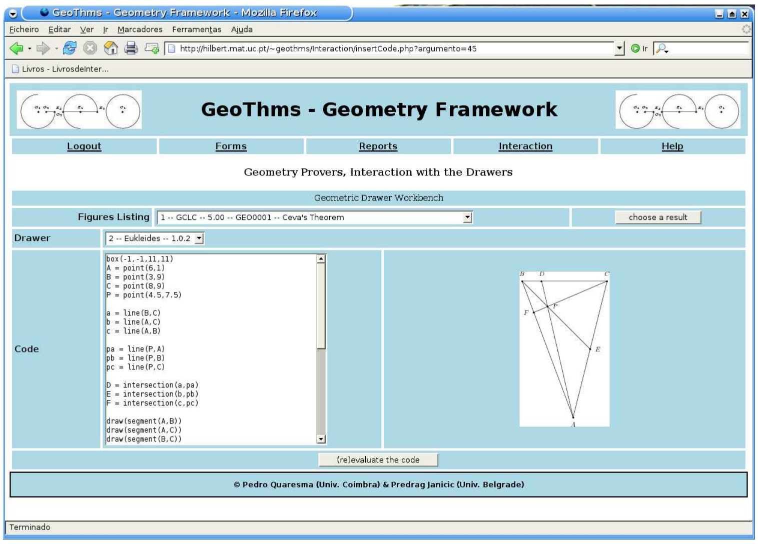


Figure 2.7: GeoThms – Interaction with the DGTs

declarative form) — not only theorem statements, but also their (automatically generated) proofs and corresponding figures, i.e., visualisations.

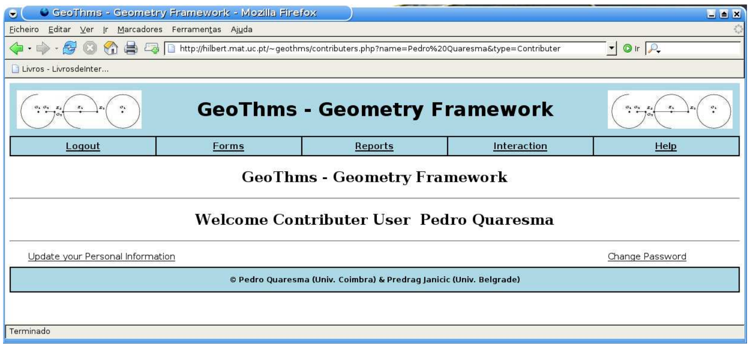


Figure 2.8: GeoThms – Contributers Page

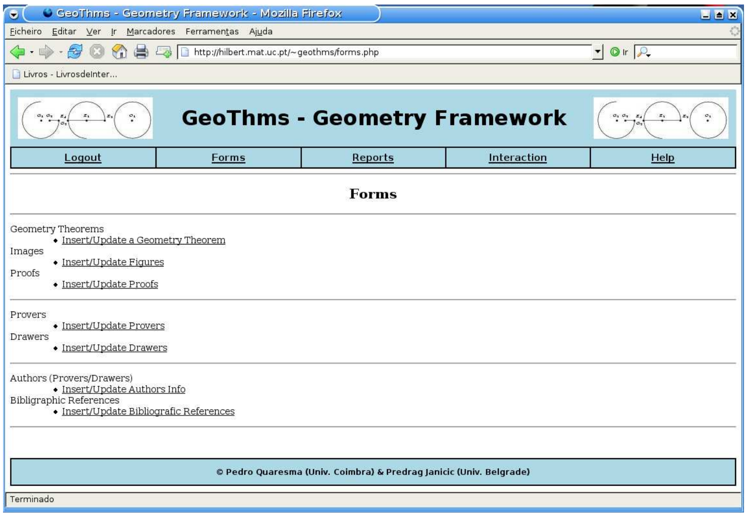


Figure 2.9: GeoThms – Forms Page

2.1 The geoDB database

The geoDB database gives support to the other tools, keeping the information, and allowing for its fast retrieving whenever necessary. The database is organised in the following form (see the entity-relationship diagram for details – Figure 2.10):

theorems — statements of theorems, in natural-language form, formatted in LaTeX;

figures — descriptions of geometrical constructions, in DGT's code (GCLC, Eukleides, or other drawing tools), they can be used for producing the corresponding figures;

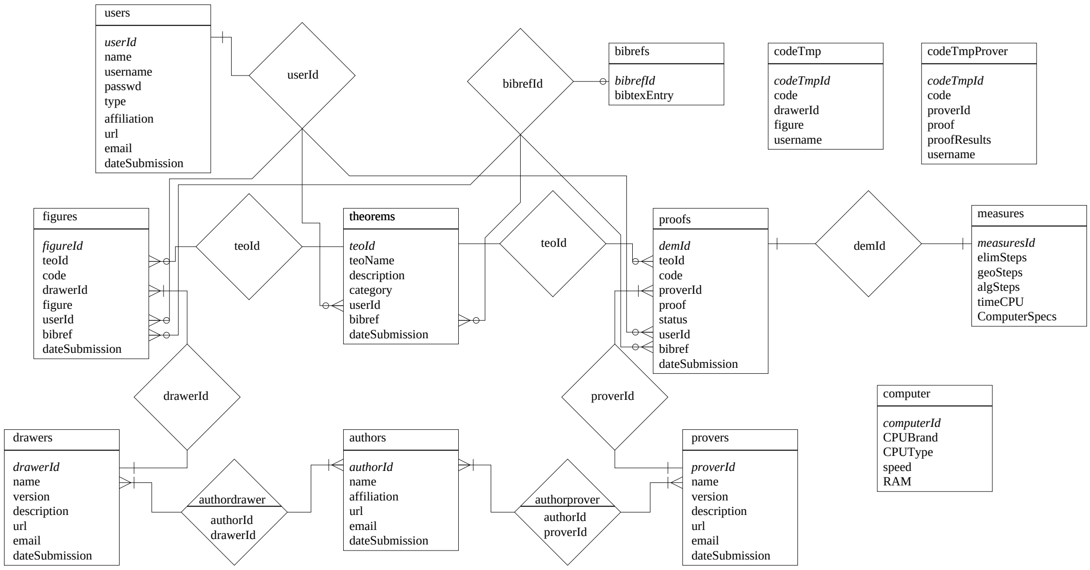


Figure 2.10: geoDB — Entity-relationship diagram

proofs — geometrical constructions with conjectures in ATP's code (GCLCprover, or other provers), they are used for producing the corresponding proofs;

A geometric theorem can have different figures and/or proofs, made by different tools, made by different users. This fact is expressed by the 1 to n relationships between the entities "theorems" and the other two entities (see Figure 2.10).

The database also has the following auxiliary entities:

bibrefs — bibliographic references, in BIBTEX format;

drawers & provers — information about the programs whose code is kept in the database, and with which the user can interact;

authors — information about the authors of the programs;

users — information about registered users.

computer — information about the computer used as the test bench.

The `codeTmp` and `codeTmpProver` tables are used to store temporary information, deleted after each session ends, for the interactive section of GeoThms.

The geoDB database is implemented in MySQL, with InnoDB transition safe type of tables, and with foreign key constraints.

Appendix A

The Database contents

In this appendix the first ten geometric results contained in the database are presented, all the info was automatically generated using a report from the GeoThms interface.

A.1 GEO0001 — Ceva's Theorem

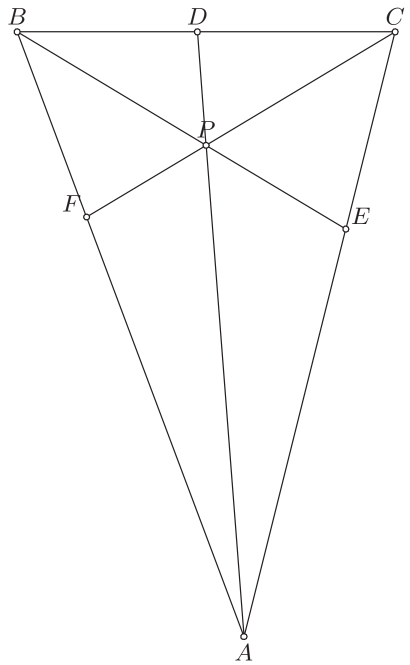
The Theorem Statement [CGZ93]

Theorem 1 (Ceva's Theorem) *Let ΔABC be a triangle and P be any point in the plane. Let $D = AP \cap CB$, $E = BP \cap AC$, and $F = CP \cap AB$. Show that:*

$$\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} = 1$$

P should not be in the lines parallels to AC, AB and BC and passing through B, C and A respectively.

The Image – GCLC 5.0



Prover's Code

```
point A 60 10
point B 30 90
point C 80 90
point P 55 75

line a B C
line b A C
line c A B

line pa P A
line pb P B
line pc P C

intersec D a pa
intersec E b pb
intersec F c pc

drawsegment A B
drawsegment A C
drawsegment B C

drawsegment A D
```

```
drawsegment B E
drawsegment C F

cmark_b A
cmark_t B
cmark_t C
cmark_t D
cmark_lt F
cmark_rt E
cmark_t P

prove { equal { mult { mult { sratio A F F B } { sratio B D D C } } { sratio C E E A } } 1
```

Proved — Proof, made with GCLC, v1.0

(A.1) $$\left(\left(\frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right) \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1$$, by the statement

(A.2) $$\left(\left(\left(-1 \cdot \frac{\overrightarrow{AF}}{\overrightarrow{BF}}\right) \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right) \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1$$, by geometric simplifications

(A.3) $$\left(-1 \cdot \left(\frac{\overrightarrow{AF}}{\overrightarrow{BF}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1$$, by algebraic simplifications

(A.4) $$\left(-1 \cdot \left(\frac{S_{APC}}{S_{BPC}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1$$, by Lemma 8 (point F eliminated)

(A.5) $$\left(-1 \cdot \left(\frac{S_{APC}}{S_{BPC}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \left(-1 \cdot \frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)\right)\right) = 1$$, by geometric simplifications

(A.6) $$\frac{\left(S_{APC} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)}{S_{BPC}} = 1$$, by algebraic simplifications

(A.7) $$\frac{\left(S_{APC} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{S_{CPB}}{S_{APB}}\right)\right)}{S_{BPC}} = 1$$, by Lemma 8 (point E eliminated)

(A.8) $$\frac{\left(S_{APC} \cdot \left(\left(-1 \cdot \frac{\overrightarrow{BD}}{\overrightarrow{CD}}\right) \cdot \frac{S_{CPB}}{S_{APB}}\right)\right)}{(-1 \cdot S_{CPB})} = 1$$, by geometric simplifications

(A.9) $$\frac{\left(S_{APC} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{CD}}\right)}{S_{APB}} = 1$$, by algebraic simplifications

(A.10) $$\frac{\left(S_{APC} \cdot \frac{S_{BPA}}{S_{CPA}}\right)}{S_{APB}} = 1$$, by Lemma 8 (point D eliminated)

(A.11) $$\frac{\left(S_{APC} \cdot \frac{S_{BPA}}{(-1 \cdot S_{APC})}\right)}{(-1 \cdot S_{BPA})} = 1$$, by geometric simplifications

(A.12) $$1 = 1$$, by algebraic simplifications

Q.E.D.

NDG conditions are:

$S_{BPA} \neq S_{CPA}$ i.e., lines BC and PA are not parallel (construction based assumption)

$S_{APB} \neq S_{CPB}$ i.e., lines AC and PB are not parallel (construction based assumption)

$S_{APC} \neq S_{BPC}$ i.e., lines AB and PC are not parallel (construction based assumption)

$P_{FBF} \neq 0$ i.e., points F and B are not identical (conjecture based assumption)

$P_{DCD} \neq 0$ i.e., points D and C are not identical (conjecture based assumption)

$P_{EAE} \neq 0$ i.e., points E and A are not identical (conjecture based assumption)

Number of elimination proof steps: 3

Number of geometric proof steps: 6

Number of algebraic proof steps: 23

Total number of proof steps: 32

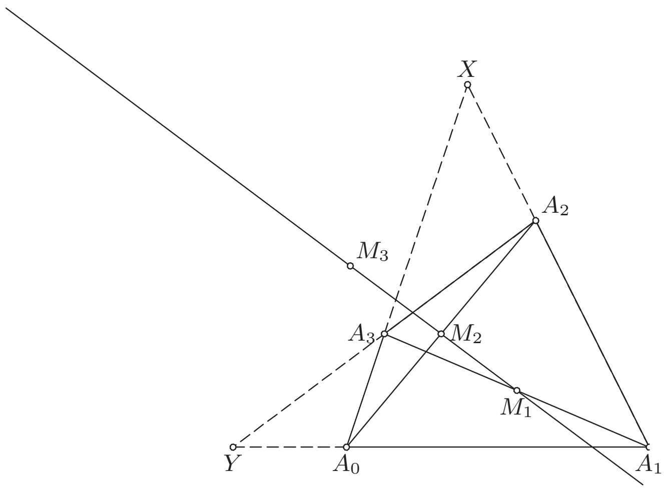
Time spent by the prover: 0.000 seconds

A.2 GEO0002 — Gauss-line Theorem

The Theorem Statement [ZCG95]

Theorem 2 (Gauss-line Theorem) *Let A_0, A_1, A_2, and A_3 be four points on a plane, X the intersection of A_1A_2 and A_0A_3, and Y the intersection of A_0A_1 and A_2A_3. Let M_1, M_2, and M_3 be the midpoints of A_1A_3, A_0A_2 and XY, respectively, then M_1, M_2, and M_3 are collinear.*

The Image – GCLC 5.0



Prover's Code

```
area 5 5 90 90

point A_0 50 10
point A_1 90 10
point A_2 75 40
point A_3 55 25

line a12 A_1 A_2
line a03 A_0 A_3

line a01 A_0 A_1
line a23 A_2 A_3
```

```
intersec X a12 a03
intersec Y a01 a23


midpoint M_1 A_1 A_3
midpoint M_2 A_0 A_2
midpoint M_3 X Y

cmark_b A_0
cmark_b A_1
cmark_rt A_2
cmark_l A_3

cmark_t X
cmark_b Y

cmark_b M_1
cmark_r M_2
cmark_rt M_3

drawsegment A_0 A_1
drawsegment A_0 A_2
drawsegment A_0 A_3
drawsegment A_1 A_2
drawsegment A_1 A_3
drawsegment A_2 A_3

drawdashsegment A_1 X
drawdashsegment A_3 X

drawdashsegment A_0 Y
drawdashsegment A_2 Y

drawline M_1 M_2

prooflevel 7
prove { equal { signed_area3 M_1 M_2 M_3 } { 0 } }
```

Proved — Proof, made with GCLC, v1.0

$$S_{M_1 M_2 M_3} = 0 \tag{A.13}$$
, by the statement

$$\left(S_{M_1 M_2 X} + \left(\frac{1}{2} \cdot (S_{M_1 M_2 Y} + (-1 \cdot S_{M_1 M_2 X}))\right)\right) = 0 \tag{A.14}$$
, by Lemma 29 (point M_3 eliminated)

$$\left(S_{X M_1 M_2} + \left(\frac{1}{2} \cdot (S_{M_1 M_2 Y} + (-1 \cdot S_{M_1 M_2 X}))\right)\right) = 0 \tag{A.15}$$
, by Lemma 1

$$\left(S_{X M_1 M_2} + \left(\frac{1}{2} \cdot (S_{Y M_1 M_2} + (-1 \cdot S_{M_1 M_2 X}))\right)\right) = 0 \tag{A.16}$$
, by Lemma 1

$$\left(S_{X M_1 M_2} + \left(\frac{1}{2} \cdot (S_{Y M_1 M_2} + (-1 \cdot S_{X M_1 M_2}))\right)\right) = 0 \tag{A.17}$$
, by Lemma 1

$$\left(S_{X M_1 M_2} + \left(\left(\frac{1}{2} \cdot S_{Y M_1 M_2}\right) + \left(\frac{1}{2} \cdot (-1 \cdot S_{X M_1 M_2})\right)\right)\right) = 0 \tag{A.18}$$
, by distribution of multiplication over addition

$$\left(S_{X M_1 M_2} + \left(\left(\frac{1}{2} \cdot S_{Y M_1 M_2}\right) + \left(-\frac{1}{2} \cdot S_{X M_1 M_2}\right)\right)\right) = 0 \tag{A.19}$$
, by multiplication of constants

$$\left(\left(\frac{1}{2} \cdot S_{X M_1 M_2}\right) + \left(\left(\frac{1}{2} \cdot S_{Y M_1 M_2}\right) + 0\right)\right) = 0 \tag{A.20}$$
, by similar summands

$$\left(\left(\frac{1}{2} \cdot S_{X M_1 M_2}\right) + \left(\frac{1}{2} \cdot S_{Y M_1 M_2}\right)\right) = 0 \tag{A.21}$$
, by addition with 0

$$((1 \cdot S_{XM_1M_2}) + (1 \cdot S_{YM_1M_2})) = 0 \quad \text{, by cancellation rule} \tag{A.22}$$

$$(S_{XM_1M_2} + (1 \cdot S_{YM_1M_2})) = 0 \quad \text{, by multiplication by 1} \tag{A.23}$$

$$(S_{XM_1M_2} + S_{YM_1M_2}) = 0 \quad \text{, by multiplication by 1} \tag{A.24}$$

$$\left(\left(S_{XM_1A_0} + \left(\frac{1}{2} \cdot (S_{XM_1A_2} + (-1 \cdot S_{XM_1A_0}))\right)\right) + S_{YM_1M_2}\right) = 0 \quad \text{, by Lemma 29 (point } M_2 \text{ eliminated)} \tag{A.25}$$

$$\left(\left(S_{XM_1A_0} + \left(\left(\frac{1}{2} \cdot S_{XM_1A_2}\right) + \left(\frac{1}{2} \cdot (-1 \cdot S_{XM_1A_0})\right)\right)\right) + S_{YM_1M_2}\right) = 0 \quad \text{, by distribution of multiplication over addition} \tag{A.26}$$

$$\left(\left(S_{XM_1A_0} + \left(\left(\frac{1}{2} \cdot S_{XM_1A_2}\right) + \left(-\frac{1}{2} \cdot S_{XM_1A_0}\right)\right)\right) + S_{YM_1M_2}\right) = 0 \quad \text{, by multiplication of constants} \tag{A.27}$$

$$\left(\left(\left(\frac{1}{2} \cdot S_{XM_1A_0}\right) + \left(\left(\frac{1}{2} \cdot S_{XM_1A_2}\right) + 0\right)\right) + S_{YM_1M_2}\right) = 0 \quad \text{, by similar summands} \tag{A.28}$$

$$\left(\left(\left(\frac{1}{2} \cdot S_{XM_1A_0}\right) + \left(\frac{1}{2} \cdot S_{XM_1A_2}\right)\right) + S_{YM_1M_2}\right) = 0 \quad \text{, by addition with 0} \tag{A.29}$$

$$\left(\left(\left(\frac{1}{2} \cdot S_{XM_1A_0}\right) + \left(\frac{1}{2} \cdot S_{XM_1A_2}\right)\right) + \left(S_{YM_1A_0} + \left(\frac{1}{2} \cdot (S_{YM_1A_2} + (-1 \cdot S_{YM_1A_0}))\right)\right)\right) = 0, \quad \text{by Lemma 29 (point } M_2 \text{ eliminated)} \tag{A.30}$$

(A.31)
$$\left(\left(\left(\frac{1}{2}\cdot S_{A_0XM_1}\right)+\left(\frac{1}{2}\cdot S_{XM_1A_2}\right)\right)+\left(S_{YM_1A_0}+\left(\frac{1}{2}\cdot(S_{YM_1A_2}+(-1\cdot S_{YM_1A_0}))\right)\right)\right)=0,\ \text{by Lemma 1}$$

(A.32)
$$\left(\left(\left(\frac{1}{2}\cdot S_{A_0XM_1}\right)+\left(\frac{1}{2}\cdot S_{A_2XM_1}\right)\right)+\left(S_{YM_1A_0}+\left(\frac{1}{2}\cdot(S_{YM_1A_2}+(-1\cdot S_{YM_1A_0}))\right)\right)\right)=0,\ \text{by Lemma 1}$$

(A.33)
$$\left(\left(\left(\frac{1}{2}\cdot S_{A_0XM_1}\right)+\left(\frac{1}{2}\cdot S_{A_2XM_1}\right)\right)+\left(S_{A_0YM_1}+\left(\frac{1}{2}\cdot(S_{YM_1A_2}+(-1\cdot S_{YM_1A_0}))\right)\right)\right)=0,\ \text{by Lemma 1}$$

(A.34)
$$\left(\left(\left(\frac{1}{2}\cdot S_{A_0XM_1}\right)+\left(\frac{1}{2}\cdot S_{A_2XM_1}\right)\right)+\left(S_{A_0YM_1}+\left(\frac{1}{2}\cdot(S_{A_2YM_1}+(-1\cdot S_{YM_1A_0}))\right)\right)\right)=0,\ \text{by Lemma 1}$$

(A.35)
$$\left(\left(\left(\frac{1}{2}\cdot S_{A_0XM_1}\right)+\left(\frac{1}{2}\cdot S_{A_2XM_1}\right)\right)+\left(S_{A_0YM_1}+\left(\frac{1}{2}\cdot(S_{A_2YM_1}+(-1\cdot S_{A_0YM_1}))\right)\right)\right)=0,\ \text{by Lemma 1}$$

(A.36)
$$\left(\left(\left(\frac{1}{2}\cdot S_{A_0XM_1}\right)+\left(\frac{1}{2}\cdot S_{A_2XM_1}\right)\right)+\left(S_{A_0YM_1}+\left(\left(\frac{1}{2}\cdot S_{A_2YM_1}\right)+\left(\frac{1}{2}\cdot(-1\cdot S_{A_0YM_1})\right)\right)\right)\right)=0 \quad , \quad \text{by distribution of multiplication over addition}$$

(A.37)
$$\left(\left(\left(\frac{1}{2}\cdot S_{A_0XM_1}\right)+\left(\frac{1}{2}\cdot S_{A_2XM_1}\right)\right)+\left(S_{A_0YM_1}+\left(\left(\frac{1}{2}\cdot S_{A_2YM_1}\right)+\left(-\frac{1}{2}\cdot S_{A_0YM_1}\right)\right)\right)\right)=0 \quad \text{, by multiplication of constants}$$

(A.38)
$$\left(\left(\left(\frac{1}{2}\cdot S_{A_0XM_1}\right)+\left(\frac{1}{2}\cdot S_{A_2XM_1}\right)\right)+\left(\left(\frac{1}{2}\cdot S_{A_0YM_1}\right)+\left(\left(\frac{1}{2}\cdot S_{A_2YM_1}\right)+0\right)\right)\right)=0, \quad \text{by similar summands}$$

(A.39)
$$\left(\left(\left(\frac{1}{2}\cdot S_{A_0XM_1}\right)+\left(\frac{1}{2}\cdot S_{A_2XM_1}\right)\right)+\left(\left(\frac{1}{2}\cdot S_{A_0YM_1}\right)+\left(\frac{1}{2}\cdot S_{A_2YM_1}\right)\right)\right)=0, \quad \text{by addition with 0}$$

(A.40)
$$(((1\cdot S_{A_0XM_1})+(1\cdot S_{A_2XM_1}))+((1\cdot S_{A_0YM_1})+(1\cdot S_{A_2YM_1})))=0 \quad \text{, by cancellation rule}$$

(A.41)
$$((S_{A_0XM_1}+(1\cdot S_{A_2XM_1}))+((1\cdot S_{A_0YM_1})+(1\cdot S_{A_2YM_1})))=0 \quad \text{, by multiplication by 1}$$

(A.42)
$$((S_{A_0XM_1}+S_{A_2XM_1})+((1\cdot S_{A_0YM_1})+(1\cdot S_{A_2YM_1})))=0 \quad \text{, by multiplication by 1}$$

(A.43)
$$((S_{A_0XM_1}+S_{A_2XM_1})+(S_{A_0YM_1}+(1\cdot S_{A_2YM_1})))=0 \quad \text{, by multiplication by 1}$$

(A.44)
$$((S_{A_0XM_1}+S_{A_2XM_1})+(S_{A_0YM_1}+S_{A_2YM_1}))=0 \quad \text{, by multiplication by 1}$$

(A.45)
$$\left(\left(\left(S_{A_0XA_1}+\left(\frac{1}{2}\cdot(S_{A_0XA_3}+(-1\cdot S_{A_0XA_1}))\right)\right)+S_{A_2XM_1}\right)+(S_{A_0YM_1}+S_{A_2YM_1})\right)=0, \quad \text{by Lemma 29 (point } M_1 \text{ eliminated)}$$

(A.46)
$$\left(\left(\left(S_{A_0XA_1} + \left(\left(\frac{1}{2} \cdot S_{A_0XA_3}\right) + \left(\frac{1}{2} \cdot (-1 \cdot S_{A_0XA_1})\right)\right)\right) + S_{A_2XM_1}\right) + (S_{A_0YM_1} + S_{A_2YM_1})\right) = 0 \quad \text{, by distribution of multiplication over addition}$$

(A.47)
$$\left(\left(\left(S_{A_0XA_1} + \left(\left(\frac{1}{2} \cdot S_{A_0XA_3}\right) + \left(-\frac{1}{2} \cdot S_{A_0XA_1}\right)\right)\right) + S_{A_2XM_1}\right) + (S_{A_0YM_1} + S_{A_2YM_1})\right) = 0 \quad \text{, by multiplication of constants}$$

(A.48)
$$\left(\left(\left(\left(\frac{1}{2} \cdot S_{A_0XA_1}\right) + \left(\left(\frac{1}{2} \cdot S_{A_0XA_3}\right) + 0\right)\right) + S_{A_2XM_1}\right) + (S_{A_0YM_1} + S_{A_2YM_1})\right) = 0, \quad \text{by similar summands}$$

(A.49)
$$\left(\left(\left(\left(\frac{1}{2} \cdot S_{A_0XA_1}\right) + \left(\frac{1}{2} \cdot S_{A_0XA_3}\right)\right) + S_{A_2XM_1}\right) + (S_{A_0YM_1} + S_{A_2YM_1})\right) = 0, \quad \text{by addition with 0}$$

(A.50)
$$\left(\left(\left(\left(\frac{1}{2} \cdot S_{A_0XA_1}\right) + \left(\frac{1}{2} \cdot S_{A_0XA_3}\right)\right) + \left(S_{A_2XA_1} + \left(\frac{1}{2} \cdot (S_{A_2XA_3} + (-1 \cdot S_{A_2XA_1}))\right)\right)\right) + (S_{A_0YM_1} + S_{A_2YM_1})\right) = 0 \quad \text{, by Lemma 29 (point } M_1 \text{ eliminated)}$$

(A.51)
$$\left(\left(\left(\left(\frac{1}{2} \cdot S_{A_0XA_1}\right) + \left(\frac{1}{2} \cdot S_{A_0XA_3}\right)\right) + \left(0 + \left(\frac{1}{2} \cdot (S_{A_2XA_3} + (-1 \cdot S_{A_2XA_1}))\right)\right)\right) + (S_{A_0YM_1} + S_{A_2YM_1})\right) = 0 \quad \text{, by Lemma 2 (collinearity)}$$

$$
\begin{aligned}
&\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(0+\left(\frac{1}{2}\cdot\left(S_{A_2XA_3}+(-1\cdot 0)\right)\right)\right)\right)\right.\\
&\left.+\left(S_{A_0YM_1}+S_{A_2YM_1}\right)\right)=0 \qquad\text{, by Lemma 2 (collinearity)}
\end{aligned}
\tag{A.52}
$$

$$
\begin{aligned}
&\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(0+\left(\frac{1}{2}\cdot\left(S_{A_2XA_3}+0\right)\right)\right)\right)\right.\\
&\left.+\left(S_{A_0YM_1}+S_{A_2YM_1}\right)\right)=0 \qquad\text{, by multiplication by 0}
\end{aligned}
\tag{A.53}
$$

$$
\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot\left(S_{A_2XA_3}+0\right)\right)\right)+\left(S_{A_0YM_1}+S_{A_2YM_1}\right)\right)=0,\ \text{by addition with 0}
\tag{A.54}
$$

$$
\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(S_{A_0YM_1}+S_{A_2YM_1}\right)\right)=0,\ \text{by addition with 0}
\tag{A.55}
$$

$$
\begin{aligned}
&\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)\right.\\
&\left.+\left(\left(S_{A_0YA_1}+\left(\frac{1}{2}\cdot\left(S_{A_0YA_3}+(-1\cdot S_{A_0YA_1})\right)\right)\right)+S_{A_2YM_1}\right)\right)=0,\quad\text{by Lemma 29 (point } M_1 \text{ eliminated)}
\end{aligned}
\tag{A.56}
$$

$$
\begin{aligned}
&\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(S_{A_0YA_1}+\left(\left(\frac{1}{2}\cdot S_{A_0YA_3}\right)+\left(\frac{1}{2}\cdot\left(-1\cdot S_{A_0YA_1}\right)\right)\right)\right)\right.\right.\\
&\left.\left.+S_{A_2YM_1}\right)\right)=0 \qquad\text{, by distribution of multiplication over addition}
\end{aligned}
\tag{A.57}
$$

$$\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)\right.$$
$$\left.+\left(\left(S_{A_0YA_1}+\left(\left(\frac{1}{2}\cdot S_{A_0YA_3}\right)+\left(-\frac{1}{2}\cdot S_{A_0YA_1}\right)\right)\right)+S_{A_2YM_1}\right)\right)=0,\quad \text{by multiplication of constants} \tag{A.58}$$

$$\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)\right.$$
$$\left.+\left(\left(\left(\frac{1}{2}\cdot S_{A_0YA_1}\right)+\left(\left(\frac{1}{2}\cdot S_{A_0YA_3}\right)+0\right)\right)+S_{A_2YM_1}\right)\right)=0\quad,\quad \text{by similar summands} \tag{A.59}$$

$$\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)\right.$$
$$\left.+\left(\left(\left(\frac{1}{2}\cdot S_{A_0YA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0YA_3}\right)\right)+S_{A_2YM_1}\right)\right)=0,\quad \text{by addition with 0} \tag{A.60}$$

$$\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(\left(\frac{1}{2}\cdot S_{A_0YA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0YA_3}\right)\right)\right.\right.$$
$$\left.\left.+\left(S_{A_2YA_1}+\left(\frac{1}{2}\cdot(S_{A_2YA_3}+(-1\cdot S_{A_2YA_1}))\right)\right)\right)\right)=0\quad,\quad \text{by Lemma 29 (point } M_1 \text{ eliminated)} \tag{A.61}$$

$$\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(\left(\frac{1}{2}\cdot S_{A_1A_0Y}\right)+\left(\frac{1}{2}\cdot S_{A_0YA_3}\right)\right)\right.\right.$$
$$\left.\left.+\left(S_{A_2YA_1}+\left(\frac{1}{2}\cdot(S_{A_2YA_3}+(-1\cdot S_{A_2YA_1}))\right)\right)\right)\right)=0\quad,\quad \text{by Lemma 1} \tag{A.62}$$

$$\begin{aligned}(A.63)\quad & \left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(\left(\frac{1}{2}\cdot S_{A_1A_0Y}\right)+\left(\frac{1}{2}\cdot S_{A_3A_0Y}\right)\right)\right.\right.\\ & \left.\left.+\left(S_{A_2YA_1}+\left(\frac{1}{2}\cdot (S_{A_2YA_3}+(-1\cdot S_{A_2YA_1}))\right)\right)\right)\right)=0 \qquad \text{, by Lemma 1}\end{aligned}$$

$$\begin{aligned}(A.64)\quad & \left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(\left(\frac{1}{2}\cdot S_{A_1A_0Y}\right)+\left(\frac{1}{2}\cdot S_{A_3A_0Y}\right)\right)\right.\right.\\ & \left.\left.+\left(S_{A_1A_2Y}+\left(\frac{1}{2}\cdot (S_{A_2YA_3}+(-1\cdot S_{A_2YA_1}))\right)\right)\right)\right)=0 \qquad \text{, by Lemma 1}\end{aligned}$$

$$\begin{aligned}(A.65)\quad & \left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(\left(\frac{1}{2}\cdot S_{A_1A_0Y}\right)+\left(\frac{1}{2}\cdot S_{A_3A_0Y}\right)\right)\right.\right.\\ & \left.\left.+\left(S_{A_1A_2Y}+\left(\frac{1}{2}\cdot (S_{A_3A_2Y}+(-1\cdot S_{A_2YA_1}))\right)\right)\right)\right)=0 \qquad \text{, by Lemma 1}\end{aligned}$$

$$\begin{aligned}(A.66)\quad & \left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(\left(\frac{1}{2}\cdot S_{A_1A_0Y}\right)+\left(\frac{1}{2}\cdot S_{A_3A_0Y}\right)\right)\right.\right.\\ & \left.\left.+\left(S_{A_1A_2Y}+\left(\frac{1}{2}\cdot (S_{A_3A_2Y}+(-1\cdot S_{A_1A_2Y}))\right)\right)\right)\right)=0 \qquad \text{, by Lemma 1}\end{aligned}$$

$$\begin{aligned}(A.67)\quad & \left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(\left(\frac{1}{2}\cdot S_{A_1A_0Y}\right)+\left(\frac{1}{2}\cdot S_{A_3A_0Y}\right)\right)\right.\right.\\ & \left.\left.+\left(S_{A_1A_2Y}+\left(\left(\frac{1}{2}\cdot S_{A_3A_2Y}\right)+\left(\frac{1}{2}\cdot (-1\cdot S_{A_1A_2Y})\right)\right)\right)\right)\right)=0 \qquad \text{, by distribution of multiplication over addition}\end{aligned}$$

$$\begin{aligned}(A.68)\quad &\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(\left(\frac{1}{2}\cdot S_{A_1A_0Y}\right)+\left(\frac{1}{2}\cdot S_{A_3A_0Y}\right)\right)\right.\right.\\ &\left.\left.+\left(S_{A_1A_2Y}+\left(\left(\frac{1}{2}\cdot S_{A_3A_2Y}\right)+\left(-\frac{1}{2}\cdot S_{A_1A_2Y}\right)\right)\right)\right)\right)=0 \qquad , \quad \text{by multiplication of constants}\end{aligned}$$

$$\begin{aligned}(A.69)\quad &\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)+\left(\left(\left(\frac{1}{2}\cdot S_{A_1A_0Y}\right)+\left(\frac{1}{2}\cdot S_{A_3A_0Y}\right)\right)\right.\right.\\ &\left.\left.+\left(\left(\frac{1}{2}\cdot S_{A_1A_2Y}\right)+\left(\left(\frac{1}{2}\cdot S_{A_3A_2Y}\right)+0\right)\right)\right)\right)=0 \qquad , \quad \text{by similar summands}\end{aligned}$$

$$\begin{aligned}(A.70)\quad &\left(\left(\left(\left(\frac{1}{2}\cdot S_{A_0XA_1}\right)+\left(\frac{1}{2}\cdot S_{A_0XA_3}\right)\right)+\left(\frac{1}{2}\cdot S_{A_2XA_3}\right)\right)\right.\\ &\left.+\left(\left(\left(\frac{1}{2}\cdot S_{A_1A_0Y}\right)+\left(\frac{1}{2}\cdot S_{A_3A_0Y}\right)\right)+\left(\left(\frac{1}{2}\cdot S_{A_1A_2Y}\right)+\left(\frac{1}{2}\cdot S_{A_3A_2Y}\right)\right)\right)\right)=0, \quad \text{by addition with 0}\end{aligned}$$

$$\begin{aligned}(A.71)\quad &((((1\cdot S_{A_0XA_1})+(1\cdot S_{A_0XA_3}))+(1\cdot S_{A_2XA_3}))\\ &+(((1\cdot S_{A_1A_0Y})+(1\cdot S_{A_3A_0Y}))+((1\cdot S_{A_1A_2Y})+(1\cdot S_{A_3A_2Y}))))=0, \quad \text{by cancellation rule}\end{aligned}$$

$$\begin{aligned}(A.72)\quad &(((S_{A_0XA_1}+(1\cdot S_{A_0XA_3}))+(1\cdot S_{A_2XA_3}))\\ &+(((1\cdot S_{A_1A_0Y})+(1\cdot S_{A_3A_0Y}))+((1\cdot S_{A_1A_2Y})+(1\cdot S_{A_3A_2Y}))))=0, \quad \text{by multiplication by 1}\end{aligned}$$

$$\begin{aligned}(A.73)\quad &(((S_{A_0XA_1}+S_{A_0XA_3})+(1\cdot S_{A_2XA_3}))\\ &+(((1\cdot S_{A_1A_0Y})+(1\cdot S_{A_3A_0Y}))+((1\cdot S_{A_1A_2Y})+(1\cdot S_{A_3A_2Y}))))=0, \quad \text{by multiplication by 1}\end{aligned}$$

$$\begin{aligned}(A.74)\quad &(((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})\\ &+(((1\cdot S_{A_1A_0Y})+(1\cdot S_{A_3A_0Y}))+((1\cdot S_{A_1A_2Y})+(1\cdot S_{A_3A_2Y}))))=0, \quad \text{by multiplication by 1}\end{aligned}$$

(A.75)
$$(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + ((S_{A_1A_0Y} + (1 \cdot S_{A_3A_0Y})) + ((1 \cdot S_{A_1A_2Y}) + (1 \cdot S_{A_3A_2Y})))) = 0, \quad \text{by multiplication by 1}$$

(A.76)
$$(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + ((S_{A_1A_0Y} + S_{A_3A_0Y}) + ((1 \cdot S_{A_1A_2Y}) + (1 \cdot S_{A_3A_2Y})))) = 0, \quad \text{by multiplication by 1}$$

(A.77)
$$(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + ((S_{A_1A_0Y} + S_{A_3A_0Y}) + (S_{A_1A_2Y} + (1 \cdot S_{A_3A_2Y})))) = 0, \quad \text{by multiplication by 1}$$

(A.78)
$$(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + ((S_{A_1A_0Y} + S_{A_3A_0Y}) + (S_{A_1A_2Y} + S_{A_3A_2Y}))) = 0, \quad \text{by multiplication by 1}$$

(A.79)
$$\left(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + \left(\left(\frac{((S_{A_0A_2A_3} \cdot S_{A_1A_0A_1}) + (-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_0})))}{S_{A_0A_2A_1A_3}} + S_{A_3A_0Y}\right) + (S_{A_1A_2Y} + S_{A_3A_2Y})\right)\right) = 0 \quad , \text{ by Lemma 30 (point } Y \text{ eliminated)}$$

(A.80)
$$\left(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + \left(\left(\frac{((S_{A_0A_2A_3} \cdot 0) + (-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_0})))}{S_{A_0A_2A_1A_3}} + S_{A_3A_0Y}\right) + (S_{A_1A_2Y} + S_{A_3A_2Y})\right)\right) = 0 \quad , \text{ by Lemma 2 (equal)}$$

(A.81)
$$\left(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + \left(\left(\frac{((S_{A_0A_2A_3} \cdot 0) + (-1 \cdot (S_{A_1A_2A_3} \cdot 0)))}{S_{A_0A_2A_1A_3}} + S_{A_3A_0Y}\right) + (S_{A_1A_2Y} + S_{A_3A_2Y})\right)\right) = 0 \quad , \text{ by Lemma 2 (equal)}$$

$$\begin{aligned}&\Big(((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})\\&+\Big(\Big(\frac{(0+(-1\cdot(S_{A_1A_2A_3}\cdot 0)))}{S_{A_0A_2A_1A_3}}+S_{A_3A_0Y}\Big)+(S_{A_1A_2Y}+S_{A_3A_2Y})\Big)\Big)=0,\ \text{by multiplication by 0}\end{aligned}\tag{A.82}$$

$$\begin{aligned}&\Big(((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})\\&+\Big(\Big(\frac{(0+(-1\cdot 0))}{S_{A_0A_2A_1A_3}}+S_{A_3A_0Y}\Big)+(S_{A_1A_2Y}+S_{A_3A_2Y})\Big)\Big)=0\quad,\quad\text{by multiplication by 0}\end{aligned}\tag{A.83}$$

$$\Big(((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})+\Big(\Big(\frac{(0+0)}{S_{A_0A_2A_1A_3}}+S_{A_3A_0Y}\Big)+(S_{A_1A_2Y}+S_{A_3A_2Y})\Big)\Big)=0,\ \text{by multiplication by 0}\tag{A.84}$$

$$\Big(((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})+\Big(\Big(\frac{0}{S_{A_0A_2A_1A_3}}+S_{A_3A_0Y}\Big)+(S_{A_1A_2Y}+S_{A_3A_2Y})\Big)\Big)=0,\ \text{by addition with 0}\tag{A.85}$$

$$(((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})+((0+S_{A_3A_0Y})+(S_{A_1A_2Y}+S_{A_3A_2Y})))=0\quad,\quad\text{by 0 numerator in fraction}\tag{A.86}$$

$$(((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})+(S_{A_3A_0Y}+(S_{A_1A_2Y}+S_{A_3A_2Y})))=0\quad,\quad\text{by addition with 0}\tag{A.87}$$

$$\begin{aligned}&\Big(((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})+\Big(\frac{((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})+(-1\cdot(S_{A_1A_2A_3}\cdot S_{A_3A_0A_0})))}{S_{A_0A_2A_1A_3}}\\&+(S_{A_1A_2Y}+S_{A_3A_2Y})\Big)\Big)=0\quad,\quad\text{by Lemma 30 (point } Y \text{ eliminated)}\end{aligned}\tag{A.88}$$

$$\left(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + \left(\frac{((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + (-1 \cdot (S_{A_1A_2A_3} \cdot 0)))}{S_{A_0A_2A_1A_3}} + (S_{A_1A_2Y} + S_{A_3A_2Y})\right)\right) = 0 \quad \text{, by Lemma 2 (equal)} \tag{A.89}$$

$$\left(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + \left(\frac{((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + (-1 \cdot 0))}{S_{A_0A_2A_1A_3}} + (S_{A_1A_2Y} + S_{A_3A_2Y})\right)\right) = 0\,, \quad \text{by multiplication by 0} \tag{A.90}$$

$$\left(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + \left(\frac{((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + 0)}{S_{A_0A_2A_1A_3}} + (S_{A_1A_2Y} + S_{A_3A_2Y})\right)\right) = 0 \quad \text{, by multiplication by 0} \tag{A.91}$$

$$\left(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + \left(\frac{(S_{A_0A_2A_3} \cdot S_{A_3A_0A_1})}{S_{A_0A_2A_1A_3}} + (S_{A_1A_2Y} + S_{A_3A_2Y})\right)\right) = 0, \quad \text{by addition with 0} \tag{A.92}$$

$$\left(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + \frac{((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + ((S_{A_1A_2Y} + S_{A_3A_2Y}) \cdot S_{A_0A_2A_1A_3}))}{S_{A_0A_2A_1A_3}}\right) = 0, \quad \text{by sum of fractions} \tag{A.93}$$

$$\left(((S_{A_0XA_1} + S_{A_0XA_3}) + S_{A_2XA_3}) + \frac{((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + ((S_{A_0A_2A_1A_3} \cdot S_{A_1A_2Y}) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2Y})))}{S_{A_0A_2A_1A_3}}\right) = 0 \quad \text{, by distribution of multiplication over addition} \tag{A.94}$$

(A.95)
$$\frac{((((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})\cdot S_{A_0A_2A_1A_3})+((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})+((S_{A_0A_2A_1A_3}\cdot S_{A_1A_2Y})+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y}))))}{S_{A_0A_2A_1A_3}} = 0 \quad \text{, by sum of fractions}$$

(A.96)
$$((((S_{A_0XA_1}+S_{A_0XA_3})+S_{A_2XA_3})\cdot S_{A_0A_2A_1A_3})+((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1}) + ((S_{A_0A_2A_1A_3}\cdot S_{A_1A_2Y})+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})))) = 0\,, \quad \text{by fraction equal 0}$$

(A.97)
$$(((S_{A_0A_2A_1A_3}\cdot(S_{A_0XA_1}+S_{A_0XA_3}))+(S_{A_0A_2A_1A_3}\cdot S_{A_2XA_3})) + ((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})+((S_{A_0A_2A_1A_3}\cdot S_{A_1A_2Y})+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})))) = 0, \quad \text{by distribution of multiplication over addition}$$

(A.98)
$$((((S_{A_0A_2A_1A_3}\cdot S_{A_0XA_1})+(S_{A_0A_2A_1A_3}\cdot S_{A_0XA_3}))+(S_{A_0A_2A_1A_3}\cdot S_{A_2XA_3})) + ((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})+((S_{A_0A_2A_1A_3}\cdot S_{A_1A_2Y})+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})))) = 0, \quad \text{by distribution of multiplication over addition}$$

(A.99)
$$\left((((S_{A_0A_2A_1A_3}\cdot S_{A_0XA_1})+(S_{A_0A_2A_1A_3}\cdot S_{A_0XA_3}))+(S_{A_0A_2A_1A_3}\cdot S_{A_2XA_3})) + \left((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})+\left(\left(S_{A_0A_2A_1A_3}\cdot\frac{((S_{A_0A_2A_3}\cdot S_{A_1A_2A_1})+(-1\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_0})))}{S_{A_0A_2A_1A_3}}\right)+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})\right)\right)\right) = 0 \quad \text{, by Lemma 30 (point } Y \text{ eliminated)}$$

(A.100)
$$\left((((S_{A_0A_2A_1A_3}\cdot S_{A_0XA_1})+(S_{A_0A_2A_1A_3}\cdot S_{A_0XA_3}))+(S_{A_0A_2A_1A_3}\cdot S_{A_2XA_3})) + \left((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})+\left(\left(S_{A_0A_2A_1A_3}\cdot\frac{((S_{A_0A_2A_3}\cdot 0)+(-1\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_0})))}{S_{A_0A_2A_1A_3}}\right)+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})\right)\right)\right) = 0 \quad \text{, by Lemma 2 (equal)}$$

(A.101)
$$\Big(((S_{A_0A_2A_1A_3}\cdot S_{A_0XA_1}) + (S_{A_0A_2A_1A_3}\cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3}\cdot S_{A_2XA_3}))$$
$$+\Big((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1}) + \Big(\Big(S_{A_0A_2A_1A_3}\cdot \frac{(0+(-1\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_0})))}{S_{A_0A_2A_1A_3}}\Big) + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})\Big)\Big)\Big)$$
$$= 0 \quad , \quad \text{by multiplication by 0}$$

(A.102)
$$\Big(((S_{A_0A_2A_1A_3}\cdot S_{A_0XA_1}) + (S_{A_0A_2A_1A_3}\cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3}\cdot S_{A_2XA_3})) + \Big((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})$$
$$+\Big(\Big(S_{A_0A_2A_1A_3}\cdot \frac{(-1\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_0}))}{S_{A_0A_2A_1A_3}}\Big) + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})\Big)\Big)\Big) = 0, \quad \text{by addition with 0}$$

(A.103)
$$\Big(((S_{A_0A_2A_1A_3}\cdot S_{A_0XA_1}) + (S_{A_0A_2A_1A_3}\cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3}\cdot S_{A_2XA_3})) + \Big((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})$$
$$+\Big(\frac{(S_{A_0A_2A_1A_3}\cdot(-1\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_0})))}{S_{A_0A_2A_1A_3}} + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})\Big)\Big)\Big) = 0, \quad \text{by multiplication of fractions}$$

(A.104)
$$\Big(((S_{A_0A_2A_1A_3}\cdot S_{A_0XA_1}) + (S_{A_0A_2A_1A_3}\cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3}\cdot S_{A_2XA_3})) + \Big((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})$$
$$+\Big(\frac{(1\cdot(-1\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_0})))}{1} + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})\Big)\Big)\Big) = 0 \quad , \quad \text{by ratio cancellation}$$

(A.105)
$$\Big(((S_{A_0A_2A_1A_3}\cdot S_{A_0XA_1}) + (S_{A_0A_2A_1A_3}\cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3}\cdot S_{A_2XA_3})) + \Big((S_{A_0A_2A_3}\cdot S_{A_3A_0A_1})$$
$$+\Big(\frac{(-1\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_0}))}{1} + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2Y})\Big)\Big)\Big) = 0 \quad , \quad \text{by multiplication by 1}$$

$$\begin{aligned}&((((S_{A_0A_2A_1A_3} \cdot S_{A_0XA_1}) + (S_{A_0A_2A_1A_3} \cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3} \cdot S_{A_2XA_3})) + ((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1})\\ &\quad + ((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2Y})))) = 0 \qquad , \text{ by fraction with number denominator}\end{aligned} \tag{A.106}$$

$$\begin{aligned}&\Bigg((((S_{A_0A_2A_1A_3} \cdot S_{A_0XA_1}) + (S_{A_0A_2A_1A_3} \cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3} \cdot S_{A_2XA_3}))\\ &+ \Bigg((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + \Bigg((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_0A_2A_3} \cdot S_{A_3A_2A_1}) + (-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_3A_2A_0})))}{S_{A_0A_2A_1A_3}} \Bigg)\Bigg)\Bigg)\Bigg)\\ &= 0 \qquad , \text{ by Lemma 30 (point } Y \text{ eliminated)}\end{aligned} \tag{A.107}$$

$$\begin{aligned}&\Bigg((((S_{A_0A_2A_1A_3} \cdot S_{A_0XA_1}) + (S_{A_0A_2A_1A_3} \cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3} \cdot S_{A_2XA_3})) + \Bigg((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1})\\ &+ \Bigg((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_0A_2A_3} \cdot S_{A_3A_2A_1}) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_0A_2A_3}))))}{S_{A_0A_2A_1A_3}} \Bigg)\Bigg)\Bigg)\Bigg)\\ &= 0 \qquad , \text{ by Lemma 1}\end{aligned} \tag{A.108}$$

$$\begin{aligned}&\Bigg((((S_{A_0A_2A_1A_3} \cdot S_{A_0XA_1}) + (S_{A_0A_2A_1A_3} \cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3} \cdot S_{A_2XA_3})) + \Bigg((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1})\\ &+ \Bigg((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_0A_2A_3} \cdot (-1 \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_0A_2A_3}))))}{S_{A_0A_2A_1A_3}} \Bigg)\Bigg)\Bigg)\Bigg)\\ &= 0 \qquad , \text{ by Lemma 1}\end{aligned} \tag{A.109}$$

$$\begin{aligned}&\Bigg((((S_{A_0A_2A_1A_3} \cdot S_{A_1A_0X}) + (S_{A_0A_2A_1A_3} \cdot S_{A_0XA_3})) + (S_{A_0A_2A_1A_3} \cdot S_{A_2XA_3})) + \Bigg((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1})\\ &+ \Bigg((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_0A_2A_3} \cdot (-1 \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_0A_2A_3}))))}{S_{A_0A_2A_1A_3}} \Bigg)\Bigg)\Bigg)\Bigg)\\ &= 0 \qquad , \text{ by Lemma 1}\end{aligned} \tag{A.110}$$

$$\begin{aligned}\Bigg(&(((S_{A_0A_2A_1A_3} \cdot S_{A_1A_0X}) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})) + (S_{A_0A_2A_1A_3} \cdot S_{A_2XA_3})) + \Bigg((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) \\ &+ \Bigg((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_0A_2A_3} \cdot (-1 \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_0A_2A_3}))))}{S_{A_0A_2A_1A_3}}\Bigg)\Bigg)\Bigg)\Bigg) \\ &= 0 \qquad , \quad \text{by Lemma 1}\end{aligned} \tag{A.111}$$

$$\begin{aligned}\Bigg(&(((S_{A_0A_2A_1A_3} \cdot S_{A_1A_0X}) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})) + \Bigg((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) \\ &+ \Bigg((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_0A_2A_3} \cdot (-1 \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_0A_2A_3}))))}{S_{A_0A_2A_1A_3}}\Bigg)\Bigg)\Bigg)\Bigg) \\ &= 0 \qquad , \quad \text{by Lemma 1}\end{aligned} \tag{A.112}$$

$$\begin{aligned}\Bigg(&(((S_{A_0A_2A_1A_3} \cdot S_{A_1A_0X}) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})) \\ &+ \Bigg((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + \Bigg((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((0 \cdot (S_{A_0A_2A_3} \cdot (-1 \cdot S_{A_1A_2A_3}))) + 0)}{S_{A_0A_2A_1A_3}}\Bigg)\Bigg)\Bigg)\Bigg) \\ &= 0 \qquad , \quad \text{by similar summands}\end{aligned} \tag{A.113}$$

$$\begin{aligned}\Bigg(&(((S_{A_0A_2A_1A_3} \cdot S_{A_1A_0X}) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})) + \Bigg((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) \\ &+ \Bigg((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{(0 + 0)}{S_{A_0A_2A_1A_3}}\Bigg)\Bigg)\Bigg)\Bigg) = 0 \qquad , \quad \text{by multiplication by } 0\end{aligned} \tag{A.114}$$

$$\begin{aligned}\Bigg(&(((S_{A_0A_2A_1A_3} \cdot S_{A_1A_0X}) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})) + \Bigg((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) \\ &+ \Bigg((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{0}{S_{A_0A_2A_1A_3}}\Bigg)\Bigg)\Bigg)\Bigg) = 0 \qquad , \quad \text{by addition with } 0\end{aligned} \tag{A.115}$$

$$((((S_{A_0A_2A_1A_3} \cdot S_{A_1A_0X}) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})) + ((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + ((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + (S_{A_0A_2A_1A_3} \cdot 0)))) = 0, \quad \text{by 0 numerator in fraction} \tag{A.116}$$

$$((((S_{A_0A_2A_1A_3} \cdot S_{A_1A_0X}) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})) + ((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + ((-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})) + 0))) = 0 \quad , \quad \text{by multiplication by 0} \tag{A.117}$$

$$((((S_{A_0A_2A_1A_3} \cdot S_{A_1A_0X}) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})) + ((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + (-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})))) = 0 \quad , \quad \text{by addition with 0} \tag{A.118}$$

$$\Bigg(\Bigg(\Bigg(\Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_1A_0A_3} \cdot S_{A_1A_0A_2}) + (-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_1})))}{S_{A_1A_0A_2A_3}}\Bigg) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})\Bigg) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})\Bigg) + ((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + (-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})))\Bigg) = 0 \quad , \quad \text{by Lemma 30 (point } X \text{ eliminated)} \tag{A.119}$$

$$\Bigg(\Bigg(\Bigg(\Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_1A_0A_3} \cdot S_{A_1A_0A_2}) + (-1 \cdot (S_{A_2A_0A_3} \cdot 0)))}{S_{A_1A_0A_2A_3}}\Bigg) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})\Bigg) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})\Bigg) + ((S_{A_0A_2A_3} \cdot S_{A_3A_0A_1}) + (-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})))\Bigg) = 0 \quad , \quad \text{by Lemma 2 (equal)} \tag{A.120}$$

$$\Bigg(\Bigg(\Bigg(\Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_1A_0A_3} \cdot S_{A_1A_0A_2}) + (-1 \cdot (S_{A_2A_0A_3} \cdot 0)))}{S_{A_1A_0A_2A_3}}\Bigg) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})\Bigg) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})\Bigg) + ((S_{A_0A_2A_3} \cdot (-1 \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_2A_0})))\Bigg) = 0 \quad , \quad \text{by Lemma 1} \tag{A.121}$$

$$\Bigg(\Bigg(\Bigg(\Bigg(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_1A_0A_3} \cdot S_{A_1A_0A_2}) + (-1 \cdot (S_{A_2A_0A_3} \cdot 0)))}{S_{A_1A_0A_2A_3}}\Bigg) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})\Bigg) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})\Bigg) + ((S_{A_0A_2A_3} \cdot (-1 \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_1A_0A_2}))))\Bigg) = 0 \quad , \quad \text{by Lemma 1} \tag{A.122}$$

$$\left(\left(\left(\left(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_1A_0A_3} \cdot S_{A_1A_0A_2}) + (-1 \cdot (S_{A_2A_0A_3} \cdot 0)))}{S_{A_1A_0A_2A_3}}\right) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})\right) + (((-1 \cdot S_{A_2A_0A_3}) \cdot (-1 \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_1A_0A_2}))))\right) = 0, \quad \text{by Lemma 1} \tag{A.123}$$

$$\left(\left(\left(\left(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_1A_0A_3} \cdot S_{A_1A_0A_2}) + (-1 \cdot 0))}{S_{A_1A_0A_2A_3}}\right) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})\right) + (((-1 \cdot S_{A_2A_0A_3}) \cdot (-1 \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_1A_0A_2}))))\right) = 0, \quad \text{by multiplication by 0} \tag{A.124}$$

$$\left(\left(\left(\left(S_{A_0A_2A_1A_3} \cdot \frac{((S_{A_1A_0A_3} \cdot S_{A_1A_0A_2}) + 0)}{S_{A_1A_0A_2A_3}}\right) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})\right) + (((-1 \cdot S_{A_2A_0A_3}) \cdot (-1 \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_1A_0A_2}))))\right) = 0, \quad \text{by multiplication by 0} \tag{A.125}$$

$$\left(\left(\left(\left(S_{A_0A_2A_1A_3} \cdot \frac{(S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})}{S_{A_1A_0A_2A_3}}\right) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})\right) + (((-1 \cdot S_{A_2A_0A_3}) \cdot (-1 \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_1A_0A_2}))))\right) = 0, \quad \text{by addition with 0} \tag{A.126}$$

$$\left(\left(\left(\frac{(S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2}))}{S_{A_1A_0A_2A_3}} + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3} \cdot S_{A_3A_2X})\right) + (((-1 \cdot S_{A_2A_0A_3}) \cdot (-1 \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_2A_3} \cdot (-1 \cdot S_{A_1A_0A_2}))))\right) = 0, \quad \text{by multiplication of fractions} \tag{A.127}$$

$$\left(\left(\left(\frac{(S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))}{S_{A_1A_0A_2A_3}} + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\right) + ((-1\cdot((-1\cdot S_{A_2A_0A_3})\cdot S_{A_1A_0A_3})) + (-1\cdot(S_{A_1A_2A_3}\cdot(-1\cdot S_{A_1A_0A_2}))))\right) = 0, \quad \text{by associativity and commutativity} \tag{A.128}$$

$$\left(\left(\left(\frac{(S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))}{S_{A_1A_0A_2A_3}} + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\right) + ((-1\cdot((-1\cdot S_{A_2A_0A_3})\cdot S_{A_1A_0A_3})) + (-1\cdot(-1\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))))\right) = 0, \quad \text{by associativity and commutativity} \tag{A.129}$$

$$\left(\left(\left(\frac{(S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))}{S_{A_1A_0A_2A_3}} + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\right) + ((-1\cdot((-1\cdot S_{A_2A_0A_3})\cdot S_{A_1A_0A_3})) + (1\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2})))\right) = 0 \quad , \quad \text{by multiplication of constants} \tag{A.130}$$

$$\left(\left(\left(\frac{(S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))}{S_{A_1A_0A_2A_3}} + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\right) + ((-1\cdot((-1\cdot S_{A_2A_0A_3})\cdot S_{A_1A_0A_3})) + (S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\right) = 0 \quad , \quad \text{by multiplication by 1} \tag{A.131}$$

$$\left(\left(\left(\frac{(S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))}{S_{A_1A_0A_2A_3}} + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_0X})\right) + (S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\right) + ((-1\cdot(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_0A_3}))) + (S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\right) = 0 \quad , \quad \text{by right association} \tag{A.132}$$

$$\left(\left(\left(\frac{(S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))}{S_{A_1A_0A_2A_3}}+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_0X})\right)+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\right)+((1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_0A_3}))+(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\right)=0 \quad \text{, by multiplication of constants} \tag{A.133}$$

$$\left(\left(\left(\frac{(S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))}{S_{A_1A_0A_2A_3}}+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_0X})\right)+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\right)+((S_{A_2A_0A_3}\cdot S_{A_1A_0A_3})+(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\right)=0 \quad \text{, by multiplication by 1} \tag{A.134}$$

$$\left(\left(\frac{((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+((S_{A_0A_2A_1A_3}\cdot S_{A_3A_0X})\cdot S_{A_1A_0A_2A_3}))}{S_{A_1A_0A_2A_3}}+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\right)+((S_{A_2A_0A_3}\cdot S_{A_1A_0A_3})+(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\right)=0 \quad \text{, by sum of fractions} \tag{A.135}$$

$$\left(\left(\frac{((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_0X}\cdot S_{A_1A_0A_2A_3})))}{S_{A_1A_0A_2A_3}}+(S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\right)+((S_{A_2A_0A_3}\cdot S_{A_1A_0A_3})+(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\right)=0 \quad \text{, by right association} \tag{A.136}$$

$$\left(\frac{(((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_0X}\cdot S_{A_1A_0A_2A_3})))+((S_{A_0A_2A_1A_3}\cdot S_{A_3A_2X})\cdot S_{A_1A_0A_2A_3}))}{S_{A_1A_0A_2A_3}}+((S_{A_2A_0A_3}\cdot S_{A_1A_0A_3})+(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\right)=0 \quad \text{, by sum of fractions} \tag{A.137}$$

$$\left(\frac{(((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_0X}\cdot S_{A_1A_0A_2A_3})))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3})))}{S_{A_1A_0A_2A_3}} + ((S_{A_2A_0A_3}\cdot S_{A_1A_0A_3})+(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\right)=0 \quad \text{, by right association} \tag{A.138}$$

$$\frac{((((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_0X}\cdot S_{A_1A_0A_2A_3})))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3})))+(((S_{A_2A_0A_3}\cdot S_{A_1A_0A_3})+(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\cdot \cdots}{S_{A_1A_0A_2A_3}} = 0 \quad \text{, by sum of fractions} \tag{A.139}$$

$$((((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_0X}\cdot S_{A_1A_0A_2A_3})))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3}))) + (((S_{A_2A_0A_3}\cdot S_{A_1A_0A_3})+(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))\cdot S_{A_1A_0A_2A_3}))=0 \quad \text{, by fraction equal 0} \tag{A.140}$$

$$((((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_0X}\cdot S_{A_1A_0A_2A_3})))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3}))) + ((S_{A_1A_0A_2A_3}\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_0A_3}))+(S_{A_1A_0A_2A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))))=0, \quad \text{by distribution of multiplication over addition} \tag{A.141}$$

$$\left(\left(\left((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+\left(S_{A_0A_2A_1A_3}\cdot\left(\frac{((S_{A_1A_0A_3}\cdot S_{A_3A_0A_2})+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_3A_0A_1})))}{S_{A_1A_0A_2A_3}}\cdot S_{A_1A_0A_2A_3}\right)\right)\right) + (S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3}))\right) + ((S_{A_1A_0A_2A_3}\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_0A_3}))+(S_{A_1A_0A_2A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2})))\right)=0, \quad \text{by Lemma 30 (point } X \text{ eliminated)} \tag{A.142}$$

$$\left(\left(\left((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+\left(S_{A_0A_2A_1A_3}\cdot\left(\frac{((S_{A_1A_0A_3}\cdot S_{A_3A_0A_2})+(-1\cdot(S_{A_2A_0A_3}\cdot(-1\cdot S_{A_1A_0A_3}))))}{S_{A_1A_0A_2A_3}}\cdot S_{A_1A_0A_2A_3}\right)\right)\right) + (S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3}))\right) + ((S_{A_1A_0A_2A_3}\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_0A_3}))+(S_{A_1A_0A_2A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2})))\right)=0, \quad \text{by Lemma 1} \tag{A.143}$$

(A.144)
$$\Bigg(\Bigg(\Bigg((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \Bigg(\frac{((S_{A_1A_0A_3} \cdot S_{A_3A_0A_2}) + (-1 \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot (-1 \cdot S_{A_1A_0A_3}))))}{S_{A_1A_0A_2A_3}} \cdot S_{A_1A_0A_2A_3}\Bigg)\Bigg)\Bigg)$$
$$+ (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3}))\Bigg) + ((S_{A_1A_0A_2A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})) + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))\Bigg)$$
$$= 0$$
, by Lemma 1

(A.145)
$$\Bigg(\Bigg(\Bigg((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \Bigg(\frac{((S_{A_1A_0A_3} \cdot S_{A_3A_0A_2}) + (-1 \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot (-1 \cdot S_{A_1A_0A_3}))))}{S_{A_1A_0A_2A_3}} \cdot S_{A_1A_0A_2A_3}\Bigg)\Bigg)\Bigg)$$
$$+ (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3}))\Bigg) + ((S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot S_{A_1A_0A_3})) + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))\Bigg)$$
$$= 0$$
, by Lemma 1

(A.146)
$$\Bigg(\Bigg(\Bigg((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{(((S_{A_1A_0A_3} \cdot S_{A_3A_0A_2}) + (-1 \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot (-1 \cdot S_{A_1A_0A_3})))) \cdot S_{A_1A_0A_2A_3})}{S_{A_1A_0A_2A_3}}\Bigg)\Bigg)$$
$$+ (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3}))\Bigg) + ((S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot S_{A_1A_0A_3})) + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))\Bigg)$$
$$= 0$$
, by multiplication of fractions

$$\Bigg(\Bigg(\Bigg((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \frac{(((S_{A_1A_0A_3} \cdot S_{A_3A_0A_2}) + (-1 \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot (-1 \cdot S_{A_1A_0A_3})))) \cdot 1)}{1}\Bigg)\Bigg)$$
(A.147)
$$+ (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3}))\Bigg) + ((S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot S_{A_1A_0A_3}))$$
$$+ (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))\Bigg) = 0$$
, by ratio cancellation

$$\Bigg(\Bigg(\Bigg((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+\Bigg(S_{A_0A_2A_1A_3}\cdot\frac{((S_{A_1A_0A_3}\cdot S_{A_3A_0A_2})+(-1\cdot((-1\cdot S_{A_3A_0A_2})\cdot(-1\cdot S_{A_1A_0A_3}))))}{1}\Bigg)\Bigg)$$
$$+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3}))\Bigg)+((S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_0A_2})\cdot S_{A_1A_0A_3}))$$
$$+(S_{A_1A_0A_2A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2})))\Bigg)=0 \quad , \text{ by multiplication by 1} \tag{A.148}$$

$$((((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot((S_{A_1A_0A_3}\cdot S_{A_3A_0A_2})+(-1\cdot((-1\cdot S_{A_3A_0A_2})\cdot(-1\cdot S_{A_1A_0A_3}))))))$$
$$+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3})))+((S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_0A_2})\cdot S_{A_1A_0A_3}))$$
$$+(S_{A_1A_0A_2A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))))=0 \quad , \text{ by fraction with number denominator} \tag{A.149}$$

$$((((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot((S_{A_1A_0A_3}\cdot S_{A_3A_0A_2})+(-1\cdot(-1\cdot((-1\cdot S_{A_3A_0A_2})\cdot S_{A_1A_0A_3}))))))$$
$$+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3})))+((S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_0A_2})\cdot S_{A_1A_0A_3}))$$
$$+(S_{A_1A_0A_2A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))))=0 \quad , \text{ by associativity and commutativity} \tag{A.150}$$

$$((((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot((S_{A_1A_0A_3}\cdot S_{A_3A_0A_2})+(1\cdot((-1\cdot S_{A_3A_0A_2})\cdot S_{A_1A_0A_3})))))$$
$$+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3})))+((S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_0A_2})\cdot S_{A_1A_0A_3}))$$
$$+(S_{A_1A_0A_2A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))))=0 \quad , \text{ by multiplication of constants} \tag{A.151}$$

$$((((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot((S_{A_1A_0A_3}\cdot S_{A_3A_0A_2})+((-1\cdot S_{A_3A_0A_2})\cdot S_{A_1A_0A_3}))))$$
$$+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3})))+((S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_0A_2})\cdot S_{A_1A_0A_3}))$$
$$+(S_{A_1A_0A_2A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))))=0 \quad , \text{ by multiplication by 1} \tag{A.152}$$

$$((((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot((0\cdot(S_{A_1A_0A_3}\cdot S_{A_3A_0A_2}))+0)))+(S_{A_0A_2A_1A_3}\cdot(S_{A_3A_2X}\cdot S_{A_1A_0A_2A_3})))$$
$$+((S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_0A_2})\cdot S_{A_1A_0A_3}))+(S_{A_1A_0A_2A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_0A_2}))))=0, \text{ by similar summands} \tag{A.153}$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (S_{A_0A_2A_1A_3} \cdot (0+0))) + (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3})))\\
&\quad + ((S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot S_{A_1A_0A_3})) + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2}))))\\
&\quad = 0 \qquad , \quad \text{by multiplication by 0}
\end{aligned} \tag{A.154}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (S_{A_0A_2A_1A_3} \cdot 0)) + (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3})))\\
&\quad + ((S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot S_{A_1A_0A_3}))\\
&\quad\quad + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))) = 0 \qquad , \quad \text{by addition with 0}
\end{aligned} \tag{A.155}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + 0) + (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3})))\\
&\quad + ((S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot S_{A_1A_0A_3})) + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))) = 0, \quad \text{by multiplication by 0}
\end{aligned} \tag{A.156}
$$

$$
\begin{aligned}
&(((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3}))) + ((S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_0A_2}) \cdot S_{A_1A_0A_3}))\\
&\quad + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))) = 0 \qquad , \quad \text{by addition with 0}
\end{aligned} \tag{A.157}
$$

$$
\begin{aligned}
&(((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3}))) + ((S_{A_1A_0A_2A_3} \cdot (-1 \cdot (S_{A_3A_0A_2} \cdot S_{A_1A_0A_3})))\\
&\quad + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))) = 0 \qquad , \quad \text{by right association}
\end{aligned} \tag{A.158}
$$

$$
\begin{aligned}
&(((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (S_{A_0A_2A_1A_3} \cdot (S_{A_3A_2X} \cdot S_{A_1A_0A_2A_3}))) + ((-1 \cdot (S_{A_1A_0A_2A_3} \cdot (S_{A_3A_0A_2} \cdot S_{A_1A_0A_3})))\\
&\quad + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))) = 0 \qquad , \quad \text{by associativity and commutativity}
\end{aligned} \tag{A.159}
$$

$$
\begin{aligned}
&\left(\left((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \left(S_{A_0A_2A_1A_3} \cdot \left(\frac{((S_{A_1A_0A_3} \cdot S_{A_3A_2A_2}) + (-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))}{S_{A_1A_0A_2A_3}} \cdot S_{A_1A_0A_2A_3}\right)\right)\right)\right.\\
&\quad \left. + ((-1 \cdot (S_{A_1A_0A_2A_3} \cdot (S_{A_3A_0A_2} \cdot S_{A_1A_0A_3}))) + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))\right) = 0, \quad \text{by Lemma 30 (point } X \text{ eliminated)}
\end{aligned} \tag{A.160}
$$

$$
\begin{aligned}
&\Bigg(\Bigg((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \Bigg(\frac{((S_{A_1A_0A_3} \cdot 0) + (-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))}{S_{A_1A_0A_2A_3}} \cdot S_{A_1A_0A_2A_3}\Bigg)\Bigg)\Bigg) \\
&+ ((-1 \cdot (S_{A_1A_0A_2A_3} \cdot (S_{A_3A_0A_2} \cdot S_{A_1A_0A_3}))) + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))\Bigg) \\
&= 0 \qquad , \quad \text{by Lemma 2 (equal)}
\end{aligned} \tag{A.161}
$$

$$
\begin{aligned}
&\Bigg(\Bigg((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \Bigg(\frac{((S_{A_1A_0A_3} \cdot 0) + (-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))}{S_{A_1A_0A_2A_3}} \cdot S_{A_1A_0A_2A_3}\Bigg)\Bigg)\Bigg) \\
&+ ((-1 \cdot (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_2A_0A_3}) \cdot S_{A_1A_0A_3}))) + (S_{A_1A_0A_2A_3} \cdot (S_{A_1A_2A_3} \cdot S_{A_1A_0A_2})))\Bigg) \\
&= 0 \qquad , \quad \text{by Lemma 1}
\end{aligned} \tag{A.162}
$$

$$
\begin{aligned}
&\Bigg(\Bigg((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \Bigg(\frac{((S_{A_1A_0A_3} \cdot 0) + (-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))}{S_{A_1A_0A_2A_3}} \cdot S_{A_1A_0A_2A_3}\Bigg)\Bigg)\Bigg) \\
&+ ((-1 \cdot (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_2A_0A_3}) \cdot S_{A_1A_0A_3}))) + (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_2A_1}) \cdot S_{A_1A_0A_2})))\Bigg) \\
&= 0 \qquad , \quad \text{by Lemma 1}
\end{aligned} \tag{A.163}
$$

$$
\begin{aligned}
&\Bigg(\Bigg((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \Bigg(\frac{(0 + (-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))}{S_{A_1A_0A_2A_3}} \cdot S_{A_1A_0A_2A_3}\Bigg)\Bigg)\Bigg) \\
&+ ((-1 \cdot (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_2A_0A_3}) \cdot S_{A_1A_0A_3}))) \\
&+ (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_2A_1}) \cdot S_{A_1A_0A_2})))\Bigg) = 0 \qquad , \quad \text{by multiplication by 0}
\end{aligned} \tag{A.164}
$$

$$
\begin{aligned}
&\Bigg(\Bigg((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + \Bigg(S_{A_0A_2A_1A_3} \cdot \Bigg(\frac{(-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))}{S_{A_1A_0A_2A_3}} \cdot S_{A_1A_0A_2A_3}\Bigg)\Bigg)\Bigg) \\
&+ ((-1 \cdot (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_2A_0A_3}) \cdot S_{A_1A_0A_3}))) \\
&+ (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_2A_1}) \cdot S_{A_1A_0A_2})))\Bigg) = 0 \qquad , \quad \text{by addition with 0}
\end{aligned} \tag{A.165}
$$

$$\left(\left((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+\left(S_{A_0A_2A_1A_3}\cdot\frac{((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_3A_2A_1}))\cdot S_{A_1A_0A_2A_3})}{S_{A_1A_0A_2A_3}}\right)\right)\right.$$
$$+((-1\cdot(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_2A_0A_3})\cdot S_{A_1A_0A_3})))$$
$$\left.+(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_2A_1})\cdot S_{A_1A_0A_2})))\right)=0 \quad \text{, by multiplication of fractions} \tag{A.166}$$

$$\left(\left((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+\left(S_{A_0A_2A_1A_3}\cdot\frac{((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_3A_2A_1}))\cdot 1)}{1}\right)\right)\right.$$
$$+((-1\cdot(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_2A_0A_3})\cdot S_{A_1A_0A_3})))$$
$$\left.+(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_2A_1})\cdot S_{A_1A_0A_2})))\right)=0 \quad \text{, by ratio cancellation} \tag{A.167}$$

$$\left(\left((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+\left(S_{A_0A_2A_1A_3}\cdot\frac{(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_3A_2A_1}))}{1}\right)\right)\right.$$
$$+((-1\cdot(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_2A_0A_3})\cdot S_{A_1A_0A_3})))$$
$$\left.+(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_2A_1})\cdot S_{A_1A_0A_2})))\right)=0 \quad \text{, by multiplication by 1} \tag{A.168}$$

$$(((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(S_{A_0A_2A_1A_3}\cdot(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_3A_2A_1}))))+((-1\cdot(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_2A_0A_3})\cdot S_{A_1A_0A_3})))$$
$$+(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_2A_1})\cdot S_{A_1A_0A_2}))))=0 \quad \text{, by fraction with number denominator} \tag{A.169}$$

$$(((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(-1\cdot(S_{A_0A_2A_1A_3}\cdot(S_{A_2A_0A_3}\cdot S_{A_3A_2A_1}))))+((-1\cdot(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_2A_0A_3})\cdot S_{A_1A_0A_3})))$$
$$+(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_2A_1})\cdot S_{A_1A_0A_2}))))=0 \quad \text{, by associativity and commutativity} \tag{A.170}$$

$$(((S_{A_0A_2A_1A_3}\cdot(S_{A_1A_0A_3}\cdot S_{A_1A_0A_2}))+(-1\cdot(S_{A_0A_2A_1A_3}\cdot(S_{A_2A_0A_3}\cdot S_{A_3A_2A_1}))))+((-1\cdot(S_{A_1A_0A_2A_3}\cdot(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_0A_3}))))$$
$$+(S_{A_1A_0A_2A_3}\cdot((-1\cdot S_{A_3A_2A_1})\cdot S_{A_1A_0A_2}))))=0 \quad \text{, by right association} \tag{A.171}$$

$$(((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot (S_{A_0A_2A_1A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))) + ((-1 \cdot (-1 \cdot (S_{A_1A_0A_2A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})))) + (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_2A_1}) \cdot S_{A_1A_0A_2})))) = 0 \quad \text{, by associativity and commutativity} \tag{A.172}$$

$$(((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot (S_{A_0A_2A_1A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))) + ((1 \cdot (S_{A_1A_0A_2A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3}))) + (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_2A_1}) \cdot S_{A_1A_0A_2})))) = 0 \quad \text{, by multiplication of constants} \tag{A.173}$$

$$(((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot (S_{A_0A_2A_1A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))) + ((S_{A_1A_0A_2A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})) + (S_{A_1A_0A_2A_3} \cdot ((-1 \cdot S_{A_3A_2A_1}) \cdot S_{A_1A_0A_2})))) = 0 \quad \text{, by multiplication by 1} \tag{A.174}$$

$$(((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot (S_{A_0A_2A_1A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))) + ((S_{A_1A_0A_2A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})) + (S_{A_1A_0A_2A_3} \cdot (-1 \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \quad \text{, by right association} \tag{A.175}$$

$$(((S_{A_0A_2A_1A_3} \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot (S_{A_0A_2A_1A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))) + ((S_{A_1A_0A_2A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_0A_2A_3} \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \quad \text{, by associativity and commutativity} \tag{A.176}$$

$$((((S_{A_0A_2A_1} + S_{A_0A_1A_3}) \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot (S_{A_0A_2A_1A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))) + ((S_{A_1A_0A_2A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_0A_2A_3} \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \quad \text{, by Definition 4} \tag{A.177}$$

$$((((S_{A_0A_2A_1} + S_{A_0A_1A_3}) \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_0A_1A_3}) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))) + ((S_{A_1A_0A_2A_3} \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_0A_2A_3} \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \quad \text{, by Definition 4} \tag{A.178}$$

$$((((S_{A_0A_2A_1} + S_{A_0A_1A_3}) \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_0A_1A_3}) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1})))) + ((((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})) + (-1 \cdot (S_{A_1A_0A_2A_3} \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \quad \text{, by Definition 4} \tag{A.179}$$

$$\begin{aligned}&((((S_{A_0A_2A_1} + S_{A_0A_1A_3}) \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_0A_1A_3}) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\ &\quad + (((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3}))\\ &\qquad + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \qquad , \quad \text{by Definition 4}\end{aligned} \tag{A.180}$$

$$\begin{aligned}&((((S_{A_0A_2A_1} + (-1 \cdot S_{A_1A_0A_3})) \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_0A_1A_3}) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\ &\quad + (((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})) + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2})))))\\ &\qquad = 0 \qquad , \quad \text{by Lemma 1}\end{aligned} \tag{A.181}$$

$$\begin{aligned}&((((S_{A_0A_2A_1} + (-1 \cdot S_{A_1A_0A_3})) \cdot (S_{A_1A_0A_3} \cdot S_{A_1A_0A_2})) + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot S_{A_1A_0A_3})) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\ &\quad + (((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_0A_3})) + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2})))))\\ &\qquad = 0 \qquad , \quad \text{by Lemma 1}\end{aligned} \tag{A.182}$$

$$\begin{aligned}&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_1A_0A_2}))\\ &\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\ &\quad + (((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\ &\qquad + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \qquad , \quad \text{by Lemma 4}\end{aligned} \tag{A.183}$$

$$\begin{aligned}&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_1A_0A_2}))\\ &\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\ &\quad + (((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\ &\qquad + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \qquad , \quad \text{by Lemma 1}\end{aligned} \tag{A.184}$$

$$\begin{aligned}&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_1A_0A_2}))\\ &\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\ &\quad + (((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\ &\qquad + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \qquad , \quad \text{by Lemma 1}\end{aligned} \tag{A.185}$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\
&\quad + (((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \qquad , \quad \text{by Lemma 1}
\end{aligned}
\tag{A.186}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\
&\quad + (((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \qquad , \quad \text{by Lemma 1}
\end{aligned}
\tag{A.187}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_1A_0A_2} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \qquad , \quad \text{by Lemma 1}
\end{aligned}
\tag{A.188}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + (-1 \cdot ((S_{A_1A_0A_2} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \qquad , \quad \text{by Lemma 1}
\end{aligned}
\tag{A.189}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_1A_0A_2}))))) = 0 \qquad , \quad \text{by Lemma 1}
\end{aligned}
\tag{A.190}
$$

$$\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_3A_2A_1}))))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_0A_2A_1}))))) = 0 \qquad , \quad \text{by Lemma 1}
\end{aligned} \tag{A.191}$$

$$\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot (-1 \cdot S_{A_1A_2A_3})))))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_3A_2A_1} \cdot S_{A_0A_2A_1}))))) = 0 \qquad , \quad \text{by Lemma 1}
\end{aligned} \tag{A.192}$$

$$\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot (-1 \cdot S_{A_1A_2A_3})))))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot ((-1 \cdot S_{A_1A_2A_3}) \cdot S_{A_0A_2A_1}))))) = 0 \qquad , \quad \text{by Lemma 1}
\end{aligned} \tag{A.193}$$

$$\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot ((-1 \cdot S_{A_1A_2A_3}) \cdot S_{A_0A_2A_1}))))) = 0 \qquad , \quad \text{by associativity and commutativity}
\end{aligned} \tag{A.194}$$

$$\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (-1 \cdot (-1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot ((-1 \cdot S_{A_1A_2A_3}) \cdot S_{A_0A_2A_1}))))) = 0 \qquad , \quad \text{by associativity and commutativity}
\end{aligned} \tag{A.195}$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + (1 \cdot ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot ((-1 \cdot S_{A_1A_2A_3}) \cdot S_{A_0A_2A_1}))))) = 0 \qquad , \quad \text{by multiplication of constants}
\end{aligned}
\tag{A.196}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot ((-1 \cdot S_{A_1A_2A_3}) \cdot S_{A_0A_2A_1}))))) = 0 \qquad , \quad \text{by multiplication by 1}
\end{aligned}
\tag{A.197}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (-1 \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))))) = 0 \qquad , \quad \text{by right association}
\end{aligned}
\tag{A.198}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (-1 \cdot (-1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))))) = 0 \qquad , \quad \text{by associativity and commutativity}
\end{aligned}
\tag{A.199}
$$

$$
\begin{aligned}
&((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))\\
&\quad + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad\quad + (1 \cdot ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1}))))) = 0 \qquad , \quad \text{by multiplication of constants}
\end{aligned}
\tag{A.200}
$$

$$((((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1})) + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \tag{A.201}$$

, by multiplication by 1

$$((((((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_0A_2A_1})\cdot S_{A_0A_2A_1})+(((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_0A_2A_1})\cdot(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))) + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \tag{A.202}$$

, by distribution of multiplication over addition

$$((((((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_0A_2A_1})\cdot S_{A_0A_2A_1})+(-1\cdot(((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_0A_2A_1})\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))) + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \tag{A.203}$$

, by associativity and commutativity

$$(((((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_0A_2A_1})\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))) + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \tag{A.204}$$

, by right association

$$(((((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))) + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \tag{A.205}$$

, by right association

$$
\begin{aligned}
&((((((S_{A_0A_2A_1} \cdot S_{A_0A_2A_1}) \cdot S_{A_0A_2A_1}) + ((S_{A_0A_2A_1} \cdot S_{A_0A_2A_1}) \cdot (S_{A_1A_2A_3} + S_{A_2A_0A_3}))) + (-1 \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))))\\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1}))))\\
&= 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.206}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((S_{A_0A_2A_1} \cdot S_{A_0A_2A_1}) \cdot (S_{A_1A_2A_3} + S_{A_2A_0A_3}))) + (-1 \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))))\\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1}))))\\
&= 0 \qquad , \quad \text{by right association}
\end{aligned}
\tag{A.207}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + (-1 \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))))\\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1}))))\\
&= 0 \qquad , \quad \text{by right association}
\end{aligned}
\tag{A.208}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + (S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} \cdot S_{A_1A_2A_3}) + (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3})))) + (-1 \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S\\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1}))))\\
&= 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.209}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3})))) + (-1 \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} + (S\\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1}))))\\
&= 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.210}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3})))) + (-1 \cdot (((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))) \cdot S_{A_0A_2A_1}) + \\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) \\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) \\
&= 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned} \tag{A.211}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3})))) + (-1 \cdot ((S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1})) + \\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) \\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) \\
&= 0 \qquad , \quad \text{by right association}
\end{aligned} \tag{A.212}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3})))) + (-1 \cdot ((S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1})) + \\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) \\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) \\
&= 0 \qquad , \quad \text{by right association}
\end{aligned} \tag{A.213}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3})))) + ((-1 \cdot (S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_0A_2A_1}))) + \\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) \\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) \\
&= 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned} \tag{A.214}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (S_{A_0A_2A_1} \cdot (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3})))) + ((-1 \cdot (S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} \cdot S_{A_0A_2A_1}) + (S_{A_0A_2A_1} \cdot (S_{A_1A_2A_3} + S \\
&+ ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3}))) \\
&+ (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) \\
&= 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned} \tag{A.215}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+((-1\cdot((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot(S_{A}\\
&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \qquad\qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.216}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+(((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A}\\
&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \qquad\qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.217}
$$

$$
\begin{aligned}
&(((((0\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1})))+((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+((0+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))+(-\\
&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \qquad\qquad , \quad \text{by similar summands}
\end{aligned}
\tag{A.218}
$$

$$
\begin{aligned}
&((((0+((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+((0+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}\\
&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \qquad\qquad , \quad \text{by multiplication by 0}
\end{aligned}
\tag{A.219}
$$

$$
\begin{aligned}
&(((((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+((0+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A}\\
&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \qquad\qquad , \quad \text{by addition with 0}
\end{aligned}
\tag{A.220}
$$

$$(((((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A}$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by addition with 0} \tag{A.221}$$

$$(((((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+((-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})+(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A}$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by distribution of multiplication over addition} \tag{A.222}$$

$$(((((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+((-1\cdot((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+(-1\cdot(S_{A_0A_2A_1}\cdot(((S_{A_0A_2A}$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by distribution of multiplication over addition} \tag{A.223}$$

$$(((((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+(((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by distribution of multiplication over addition} \tag{A.224}$$

$$(((((0\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+((0+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by similar summands} \tag{A.225}$$

$$\begin{aligned}&((((0+(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+((0+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))))\\&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\&=0 \qquad ,\quad \text{by multiplication by 0}\end{aligned}\tag{A.226}$$

$$\begin{aligned}&((((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))+((0+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))))\\&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\&=0 \qquad ,\quad \text{by addition with 0}\end{aligned}\tag{A.227}$$

$$\begin{aligned}&((((S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))))\\&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\&=0 \qquad ,\quad \text{by addition with 0}\end{aligned}\tag{A.228}$$

$$\begin{aligned}&((((0\cdot(S_{A_0A_2A_1}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+(0+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))))\\&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\\&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \qquad ,\quad \text{by similar summands}\end{aligned}\tag{A.229}$$

$$\begin{aligned}&(((0+(0+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))))\\&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\\&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \qquad ,\quad \text{by multiplication by 0}\end{aligned}\tag{A.230}$$

$$\begin{aligned}
&(((0 + (-1 \cdot (S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot (S_{A_1A_2A_3} + S_{A_2A_0A_3})))))\\
&\quad + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by addition with 0}
\end{aligned} \tag{A.231}$$

$$\begin{aligned}
&(((-1 \cdot (S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\quad + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by addition with 0}
\end{aligned} \tag{A.232}$$

$$\begin{aligned}
&(((-1 \cdot (S_{A_0A_2A_1} \cdot (((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_1A_2A_3}) + ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_2A_0A_3}))))\\
&\quad + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned} \tag{A.233}$$

$$\begin{aligned}
&(((-1 \cdot ((S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_1A_2A_3})) + (S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_2A_0A_3}))))\\
&\quad + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned} \tag{A.234}$$

$$\begin{aligned}
&((((-1 \cdot (S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_1A_2A_3}))) + (-1 \cdot (S_{A_0A_2A_1} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot S_{A_2A_0A_3}))))\\
&\quad + ((S_{A_0A_2A_1} + (-1 \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})))) \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})))\\
&\quad + (((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_2A_0A_3} \cdot (S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3}))))\\
&\qquad + ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (S_{A_1A_2A_3} \cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned} \tag{A.235}$$

$$((((-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})+(S_{A_1A_2A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_2A_0A_3}))))$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by distribution of multiplication over addition} \tag{A.236}$$

$$((((-1\cdot((S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))+(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_2A_0A_3}))))$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by distribution of multiplication over addition} \tag{A.237}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_2A_0A_3}))))$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by distribution of multiplication over addition} \tag{A.238}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})+(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_2A_0A_3}))))$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by distribution of multiplication over addition} \tag{A.239}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot((S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad \text{, by distribution of multiplication over addition} \tag{A.240}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A}$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad , \quad \text{by distribution of multiplication over addition} \tag{A.241}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot((S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})+(S_{A_2A_0A}$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad , \quad \text{by distribution of multiplication over addition} \tag{A.242}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot((S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1}))+(S_{A_0A_2A}$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad , \quad \text{by distribution of multiplication over addition} \tag{A.243}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad , \quad \text{by distribution of multiplication over addition} \tag{A.244}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \quad , \quad \text{by distribution of multiplication over addition} \tag{A.245}$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot((\\
&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \quad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.246}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(\\
&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \quad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.247}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(0+(-\\
&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \quad , \quad \text{by similar summands}
\end{aligned}
\tag{A.248}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S\\
&+((S_{A_0A_2A_1}+(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \quad , \quad \text{by addition with 0}
\end{aligned}
\tag{A.249}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S\\
&+(((S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})+((S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})\cdot(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))))\\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\
&=0 \quad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.250}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S \\
&+(0+((S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})\cdot(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))) \\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) \\
&=0 \qquad , \quad \text{by similar summands}
\end{aligned}
\tag{A.251}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S \\
&+((S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})\cdot(-1\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))) \\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) \\
&=0 \qquad , \quad \text{by addition with 0}
\end{aligned}
\tag{A.252}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S \\
&+(-1\cdot((S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))) \\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) \\
&=0 \qquad , \quad \text{by associativity and commutativity}
\end{aligned}
\tag{A.253}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S \\
&+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))) \\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) \\
&=0 \qquad , \quad \text{by right association}
\end{aligned}
\tag{A.254}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S \\
&+(-1\cdot(S_{A_2A_0A_3}\cdot((S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})+(S_{A_1A_2A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))) \\
&+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) \\
&=0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.255}
$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+(-1\cdot((S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))+(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \qquad , \quad \text{by distribution of multiplication over addition} \tag{A.256}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \qquad , \quad \text{by distribution of multiplication over addition} \tag{A.257}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+(0+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \qquad , \quad \text{by similar summands} \tag{A.258}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \qquad , \quad \text{by addition with } 0 \tag{A.259}$$

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+(-1\cdot(S_{A_2A_0A_3}\cdot((S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})+(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0 \qquad , \quad \text{by distribution of multiplication over addition} \tag{A.260}$$

(A.261)

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+(-1\cdot((S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3}))+(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0$$
, by distribution of multiplication over addition

(A.262)

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))$$
$$+(((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))$$
$$=0$$
, by distribution of multiplication over addition

(A.263)

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))$$
$$+((((S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3})))\cdot S_{A_0A_2A_1})+((S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3})))\cdot S_{A_1A_2A_3}))$$
$$+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0$$
, by distribution of multiplication over addition

(A.264)

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))$$
$$+(((S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_0A_2A_1}))+((S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3})))\cdot S_{A_1A_2A_3}))$$
$$+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0$$
, by right association

(A.265)

$$(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S$$
$$+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))$$
$$+(((S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_0A_2A_1}))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))$$
$$+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0$$
, by right association

$$\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+(((S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}\cdot S_{A_0A_2A_1})+(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \quad , \text{ by distribution of multiplication over addition}
\end{aligned} \tag{A.266}$$

$$\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+((((S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \quad , \text{ by distribution of multiplication over addition}
\end{aligned} \tag{A.267}$$

$$\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+((0\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})))+(-1\cdot(S_A\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+(((0+(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \quad , \text{ by similar summands}
\end{aligned} \tag{A.268}$$

$$\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(0+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3})))))\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+(((0+(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \quad , \text{ by multiplication by 0}
\end{aligned} \tag{A.269}$$

$$\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+(((0+(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \quad , \text{ by addition with 0}
\end{aligned} \tag{A.270}$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+(((S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by addition with } 0
\end{aligned}
\tag{A.271}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+(((S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})+(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.272}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-2\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+((((S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by distribution of multiplication over addition}
\end{aligned}
\tag{A.273}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+(((0+(S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3})))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by similar summands}
\end{aligned}
\tag{A.274}
$$

$$
\begin{aligned}
&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\
&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\
&+(((S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\
&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))) = 0 \qquad , \quad \text{by addition with } 0
\end{aligned}
\tag{A.275}
$$

$$\begin{aligned}&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+(0\cdot(S_{A_0A_2A_1}\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\&+((0+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))\\&=0 \qquad , \quad \text{by similar summands}\end{aligned} \tag{A.276}$$

$$\begin{aligned}&(((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))+0)\\&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\&+((0+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\&\quad+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \qquad , \quad \text{by multiplication by 0}\end{aligned} \tag{A.277}$$

$$\begin{aligned}&((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\&+((0+(S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3})))\\&\quad+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \qquad , \quad \text{by addition with 0}\end{aligned} \tag{A.278}$$

$$\begin{aligned}&((((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1})))+((-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_0A_2A_1}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\&+((-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}\cdot S_{A_2A_0A_3})))))\\&+((S_{A_2A_0A_3}\cdot((S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3}))\cdot S_{A_1A_2A_3}))\\&\quad+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(S_{A_1A_2A_3}\cdot S_{A_0A_2A_1}))))=0 \qquad , \quad \text{by addition with 0}\end{aligned} \tag{A.279}$$

Proving lemma...(level 1)

$$S_{A_1 A_2 A_3} = 0 \quad , \quad \text{by the statement} \tag{A.280}$$

Failed to prove or disprove the conjecture.

$$\begin{aligned}&((((-1 \cdot (S_{A_0A_2A_1} \cdot (1 \cdot S_{A_0A_2A_1}))) + ((-1 \cdot (S_{A_0A_2A_1} \cdot (1 \cdot S_{A_1A_2A_3}))) + (-1 \cdot (S_{A_0A_2A_1} \cdot (1 \cdot S_{A_2A_0A_3})))))\\ &+ ((-1 \cdot (S_{A_2A_0A_3} \cdot (1 \cdot S_{A_1A_2A_3}))) + (-1 \cdot (S_{A_2A_0A_3} \cdot (1 \cdot S_{A_2A_0A_3}))))) + ((S_{A_2A_0A_3} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot 1))\\ &+ ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (1 \cdot S_{A_0A_2A_1})))) = 0 \qquad , \text{ by cancellation rule (assuming } S_{A_1A_2A_3} \neq 0)\end{aligned} \tag{A.281}$$

$$\begin{aligned}&((((-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((-1 \cdot (S_{A_0A_2A_1} \cdot (1 \cdot S_{A_1A_2A_3}))) + (-1 \cdot (S_{A_0A_2A_1} \cdot (1 \cdot S_{A_2A_0A_3})))))\\ &+ ((-1 \cdot (S_{A_2A_0A_3} \cdot (1 \cdot S_{A_1A_2A_3}))) + (-1 \cdot (S_{A_2A_0A_3} \cdot (1 \cdot S_{A_2A_0A_3}))))) + ((S_{A_2A_0A_3} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot 1))\\ &+ ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (1 \cdot S_{A_0A_2A_1})))) = 0 \qquad , \text{ by multiplication by 1}\end{aligned} \tag{A.282}$$

$$\begin{aligned}&((((-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_0A_2A_1} \cdot (1 \cdot S_{A_2A_0A_3})))))\\ &+ ((-1 \cdot (S_{A_2A_0A_3} \cdot (1 \cdot S_{A_1A_2A_3}))) + (-1 \cdot (S_{A_2A_0A_3} \cdot (1 \cdot S_{A_2A_0A_3}))))) + ((S_{A_2A_0A_3} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot 1))\\ &+ ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (1 \cdot S_{A_0A_2A_1})))) = 0 \qquad , \text{ by multiplication by 1}\end{aligned} \tag{A.283}$$

$$\begin{aligned}&((((-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3}))))\\ &+ ((-1 \cdot (S_{A_2A_0A_3} \cdot (1 \cdot S_{A_1A_2A_3}))) + (-1 \cdot (S_{A_2A_0A_3} \cdot (1 \cdot S_{A_2A_0A_3}))))) + ((S_{A_2A_0A_3} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot 1))\\ &+ ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (1 \cdot S_{A_0A_2A_1})))) = 0 \qquad , \text{ by multiplication by 1}\end{aligned} \tag{A.284}$$

$$\begin{aligned}&((((-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3}))))\\ &+ ((-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_2A_0A_3} \cdot (1 \cdot S_{A_2A_0A_3}))))) + ((S_{A_2A_0A_3} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot 1))\\ &+ ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (1 \cdot S_{A_0A_2A_1})))) = 0 \qquad , \text{ by multiplication by 1}\end{aligned} \tag{A.285}$$

$$\begin{aligned}&((((-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_0A_2A_1})) + ((-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_0A_2A_1} \cdot S_{A_2A_0A_3}))))\\ &+ ((-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_1A_2A_3})) + (-1 \cdot (S_{A_2A_0A_3} \cdot S_{A_2A_0A_3})))) + ((S_{A_2A_0A_3} \cdot ((S_{A_0A_2A_1} + (S_{A_1A_2A_3} + S_{A_2A_0A_3})) \cdot 1))\\ &+ ((S_{A_0A_2A_1} + S_{A_1A_2A_3}) \cdot (1 \cdot S_{A_0A_2A_1})))) = 0 \qquad , \text{ by multiplication by 1}\end{aligned} \tag{A.286}$$

$$\begin{aligned}&((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))\\&+((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))+((S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3})))\\&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot(1\cdot S_{A_0A_2A_1}))))=0 \quad , \text{ by multiplication by 1}\end{aligned}\tag{A.287}$$

$$\begin{aligned}&((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))\\&+((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))+((S_{A_2A_0A_3}\cdot(S_{A_0A_2A_1}+(S_{A_1A_2A_3}+S_{A_2A_0A_3})))\\&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad , \text{ by multiplication by 1}\end{aligned}\tag{A.288}$$

$$\begin{aligned}&((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))\\&+((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))+(((S_{A_2A_0A_3}\cdot S_{A_0A_2A_1})+(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))\\&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad , \text{ by distribution of multiplication over addition}\end{aligned}\tag{A.289}$$

$$\begin{aligned}&((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+(0\cdot(S_{A_0A_2A_1}\cdot S_{A_2A_0A_3}))))\\&+((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))+((0+(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))\\&+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad , \text{ by similar summands}\end{aligned}\tag{A.290}$$

$$\begin{aligned}&((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3}))+0))+((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\&+((0+(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0, \quad \text{by multiplication by 0}\end{aligned}\tag{A.291}$$

$$\begin{aligned}&((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\&+((0+(S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3})))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0, \quad \text{by addition with 0}\end{aligned}\tag{A.292}$$

$$\begin{aligned}&((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))\\&+((S_{A_2A_0A_3}\cdot(S_{A_1A_2A_3}+S_{A_2A_0A_3}))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0, \quad \text{by addition with 0}\end{aligned}\tag{A.293}$$

$$((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+((-1\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))$$
$$+(((S_{A_2A_0A_3}\cdot S_{A_1A_2A_3})+(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))$$
$$=0 \quad \text{, by distribution of multiplication over addition} \tag{A.294}$$

$$((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+((0\cdot(S_{A_2A_0A_3}\cdot S_{A_1A_2A_3}))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))$$
$$+((0+(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad \text{, by similar summands} \tag{A.295}$$

$$((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+(0+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))))$$
$$+((0+(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad \text{, by multiplication by 0} \tag{A.296}$$

$$((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3})))$$
$$+((0+(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3}))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad \text{, by addition with 0} \tag{A.297}$$

$$((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+(-1\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3})))$$
$$+((S_{A_2A_0A_3}\cdot S_{A_2A_0A_3})+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad \text{, by addition with 0} \tag{A.298}$$

$$((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+(0\cdot(S_{A_2A_0A_3}\cdot S_{A_2A_0A_3})))$$
$$+(0+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad \text{, by similar summands} \tag{A.299}$$

$$((((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))+0)$$
$$+(0+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad \text{, by multiplication by 0} \tag{A.300}$$

$$(((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))$$
$$+(0+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1})))=0 \quad \text{, by addition with 0} \tag{A.301}$$

$$(((-1\cdot(S_{A_0A_2A_1}\cdot S_{A_0A_2A_1}))+(-1\cdot(S_{A_0A_2A_1}\cdot S_{A_1A_2A_3})))$$
$$+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot S_{A_0A_2A_1}))=0 \quad \text{, by addition with 0} \tag{A.302}$$

Proving lemma...(level 1)

(A.303) $$S_{A_0 A_2 A_1} = 0$$, by the statement

Failed to prove or disprove the conjecture.

$$(A.304)\quad (((-1\cdot(1\cdot S_{A_0A_2A_1}))+(-1\cdot(1\cdot S_{A_1A_2A_3})))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot 1))=0 \quad \text{, by cancellation rule (assuming } S_{A_0A_2A_1}\neq 0)$$

$$(A.305)\quad (((-1\cdot S_{A_0A_2A_1})+(-1\cdot(1\cdot S_{A_1A_2A_3})))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot 1))=0 \quad \text{, by multiplication by 1}$$

$$(A.306)\quad (((-1\cdot S_{A_0A_2A_1})+(-1\cdot S_{A_1A_2A_3}))+((S_{A_0A_2A_1}+S_{A_1A_2A_3})\cdot 1))=0 \quad \text{, by multiplication by 1}$$

$$(A.307)\quad (((-1\cdot S_{A_0A_2A_1})+(-1\cdot S_{A_1A_2A_3}))+(S_{A_0A_2A_1}+S_{A_1A_2A_3}))=0 \quad \text{, by multiplication by 1}$$

$$(A.308)\quad (((0\cdot S_{A_0A_2A_1})+(-1\cdot S_{A_1A_2A_3}))+(0+S_{A_1A_2A_3}))=0 \quad \text{, by similar summands}$$

$$(A.309)\quad ((0+(-1\cdot S_{A_1A_2A_3}))+(0+S_{A_1A_2A_3}))=0 \quad \text{, by multiplication by 0}$$

$$(A.310)\quad ((-1\cdot S_{A_1A_2A_3})+(0+S_{A_1A_2A_3}))=0 \quad \text{, by addition with 0}$$

$$(A.311)\quad ((-1\cdot S_{A_1A_2A_3})+S_{A_1A_2A_3})=0 \quad \text{, by addition with 0}$$

$$(A.312)\quad ((0\cdot S_{A_1A_2A_3})+0)=0 \quad \text{, by similar summands}$$

$$(A.313)\quad (0+0)=0 \quad \text{, by multiplication by 0}$$

$$(A.314)\quad 0=0 \quad \text{, by addition with 0}$$

Q.E.D.

NDG conditions are:

$S_{A_1A_0A_3} \neq S_{A_2A_0A_3}$ i.e., lines A_1A_2 and A_0A_3 are not parallel (construction based assumption)

$S_{A_0A_2A_3} \neq S_{A_1A_2A_3}$ i.e., lines A_0A_1 and A_2A_3 are not parallel (construction based assumption)

$S_{A_1A_2A_3} \neq 0$ i.e., points A_1, A_2 and A_3 are not collinear (cancellation assumption)

$S_{A_0A_2A_1} \neq 0$ i.e., points A_0, A_2 and A_1 are not collinear (cancellation assumption)

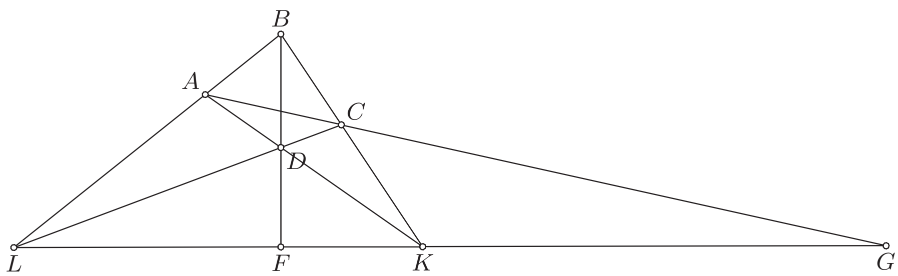
Number of elimination proof steps: 14
Number of geometric proof steps: 51
Number of algebraic proof steps: 234
Total number of proof steps: 299
Time spent by the prover: 0.029 seconds

A.3 GEO0003 — Harmonic Set

The Theorem Statement [CGZ96]

Theorem 3 (Harmonic Set) *Let L be the intersection of AB and CD, K the intersection of AD and BC, F the intersection of BD and KL, and G the intersection of AC and KL. Then* $\frac{\overline{LF}}{\overline{KF}} = \frac{\overline{LG}}{\overline{GK}}$.

The Image – GCLC 5.0



Prover's Code

```
point A 30 32
point B 40 40
point C 48 28
point D 40 25

cmark_lt A
cmark_t B
cmark_rt C
cmark_rb D

% point L
line AB A B
```

```
line CD C D
intersec L AB CD
cmark_b L

% point K
line AD A D
line BC B C
intersec K AD BC
cmark_b K

% point F
line BD B D
line KL K L
intersec F BD KL
cmark_b F

% point G
line AC A C
intersec G AC KL
cmark_b G

drawsegment B L
drawsegment C L
drawsegment A K
drawsegment B K
drawsegment B F
drawsegment A G
drawsegment L G

prove { equal { sratio L F K F } { sratio L G G K }  }
```

Proved — Proof, made with GCLC, v1.0

(A.315) $$\frac{\overrightarrow{LF}}{\overrightarrow{KF}} = \frac{\overrightarrow{LG}}{\overrightarrow{GK}}$$, by the statement

(A.316) $$\frac{\overrightarrow{LF}}{\overrightarrow{KF}} = \left(-1 \cdot \frac{\overrightarrow{LG}}{\overrightarrow{KG}}\right)$$, by geometric simplifications

(A.317) $$\frac{\overrightarrow{LF}}{\overrightarrow{KF}} = \left(-1 \cdot \frac{S_{LAC}}{S_{KAC}}\right)$$, by Lemma 8 (point G eliminated)

(A.318) $$\frac{\overrightarrow{LF}}{\overrightarrow{KF}} = \frac{(-1 \cdot S_{LAC})}{S_{KAC}}$$, by algebraic simplifications

(A.319) $$\frac{S_{LBD}}{S_{KBD}} = \frac{(-1 \cdot S_{LAC})}{S_{KAC}}$$, by Lemma 8 (point F eliminated)

(A.320) $$(S_{LBD} \cdot S_{KAC}) = (-1 \cdot (S_{LAC} \cdot S_{KBD}))$$, by algebraic simplifications

(A.321) $$(S_{LBD} \cdot S_{ACK}) = (-1 \cdot (S_{LAC} \cdot S_{BDK}))$$, by geometric simplifications

(A.322) $$\left(S_{LBD} \cdot \frac{((S_{ABC} \cdot S_{ACD}) + (-1 \cdot (S_{DBC} \cdot S_{ACA})))}{S_{ABDC}}\right) = (-1 \cdot (S_{LAC} \cdot S_{BDK}))$$, by Lemma 30 (point K eliminated)

(A.323) $$\left(S_{LBD} \cdot \frac{((S_{ABC} \cdot S_{ACD}) + (-1 \cdot (S_{DBC} \cdot 0)))}{S_{ABDC}}\right) = (-1 \cdot (S_{LAC} \cdot S_{BDK}))$$, by geometric simplifications

$$\frac{(S_{LBD} \cdot (S_{ABC} \cdot S_{ACD}))}{S_{ABDC}} = (-1 \cdot (S_{LAC} \cdot S_{BDK})) \quad \text{, by algebraic simplifications} \tag{A.324}$$

$$\frac{(S_{LBD} \cdot (S_{ABC} \cdot S_{ACD}))}{S_{ABDC}} = \Bigg(-1 \cdot \left(S_{LAC} \cdot \frac{((S_{ABC} \cdot S_{BDD}) + (-1 \cdot (S_{DBC} \cdot S_{BDA})))}{S_{ABDC}}\right)\Bigg), \quad \text{by Lemma 30 (point } K \text{ eliminated)} \tag{A.325}$$

$$\frac{(S_{LBD} \cdot (S_{ABC} \cdot S_{ACD}))}{S_{ABDC}} = \Bigg(-1 \cdot \left(S_{LAC} \cdot \frac{((S_{ABC} \cdot 0) + (-1 \cdot (S_{DBC} \cdot S_{BDA})))}{S_{ABDC}}\right)\Bigg), \quad \text{by geometric simplifications} \tag{A.326}$$

$$(S_{LBD} \cdot (S_{ABC} \cdot S_{ACD})) = (S_{LAC} \cdot (S_{DBC} \cdot S_{BDA})) \quad \text{, by algebraic simplifications} \tag{A.327}$$

$$(S_{BDL} \cdot (S_{ABC} \cdot S_{ACD})) = (S_{ACL} \cdot (S_{DBC} \cdot S_{BDA})) \quad \text{, by geometric simplifications} \tag{A.328}$$

$$\left(\frac{((S_{ACD} \cdot S_{BDB}) + (-1 \cdot (S_{BCD} \cdot S_{BDA})))}{S_{ACBD}} \cdot (S_{ABC} \cdot S_{ACD})\right) = (S_{ACL} \cdot (S_{DBC} \cdot S_{BDA})), \quad \text{by Lemma 30 (point } L \text{ eliminated)} \tag{A.329}$$

$$\left(\frac{((S_{ACD} \cdot 0) + (-1 \cdot (S_{BCD} \cdot S_{BDA})))}{S_{ACBD}} \cdot (S_{ABC} \cdot S_{ACD})\right) = (S_{ACL} \cdot (S_{BCD} \cdot S_{BDA})), \quad \text{by geometric simplifications} \tag{A.330}$$

$$\frac{(-1 \cdot (S_{BCD} \cdot (S_{BDA} \cdot (S_{ABC} \cdot S_{ACD}))))}{S_{ACBD}} = (S_{ACL} \cdot (S_{BCD} \cdot S_{BDA})) \quad \text{, by algebraic simplifications} \tag{A.331}$$

$$\frac{(-1 \cdot (S_{BCD} \cdot (S_{BDA} \cdot (S_{ABC} \cdot S_{ACD}))))}{S_{ACBD}} = \left(\frac{((S_{ACD} \cdot S_{ACB}) + (-1 \cdot (S_{BCD} \cdot S_{ACA})))}{S_{ACBD}} \cdot (S_{BCD} \cdot S_{BDA})\right) \quad \text{, by Lemma 30 (point } L \text{ eliminated)} \tag{A.332}$$

$$\frac{(-1 \cdot (S_{BCD} \cdot (S_{BDA} \cdot (S_{ABC} \cdot S_{ACD}))))}{S_{ACBD}} = \left(\frac{((S_{ACD} \cdot (-1 \cdot S_{ABC})) + (-1 \cdot (S_{BCD} \cdot 0)))}{S_{ACBD}} \cdot (S_{BCD} \cdot S_{BDA})\right) \quad \text{, by geometric simplifications} \tag{A.333}$$

$$0 = 0 \quad \text{, by algebraic simplifications} \tag{A.334}$$

Q.E.D.

NDG conditions are:

$S_{ACD} \neq S_{BCD}$ i.e., lines AB and CD are not parallel (construction based assumption)

$S_{ABC} \neq S_{DBC}$ i.e., lines AD and BC are not parallel (construction based assumption)

$S_{BKL} \neq S_{DKL}$ i.e., lines BD and KL are not parallel (construction based assumption)

$S_{AKL} \neq S_{CKL}$ i.e., lines AC and KL are not parallel (construction based assumption)

$P_{KFK} \neq 0$ i.e., points K and F are not identical (conjecture based assumption)

$P_{GKG} \neq 0$ i.e., points G and K are not identical (conjecture based assumption)

Number of elimination proof steps: 6

Number of geometric proof steps: 11

Number of algebraic proof steps: 34

Total number of proof steps: 51

Time spent by the prover: 0.003 seconds

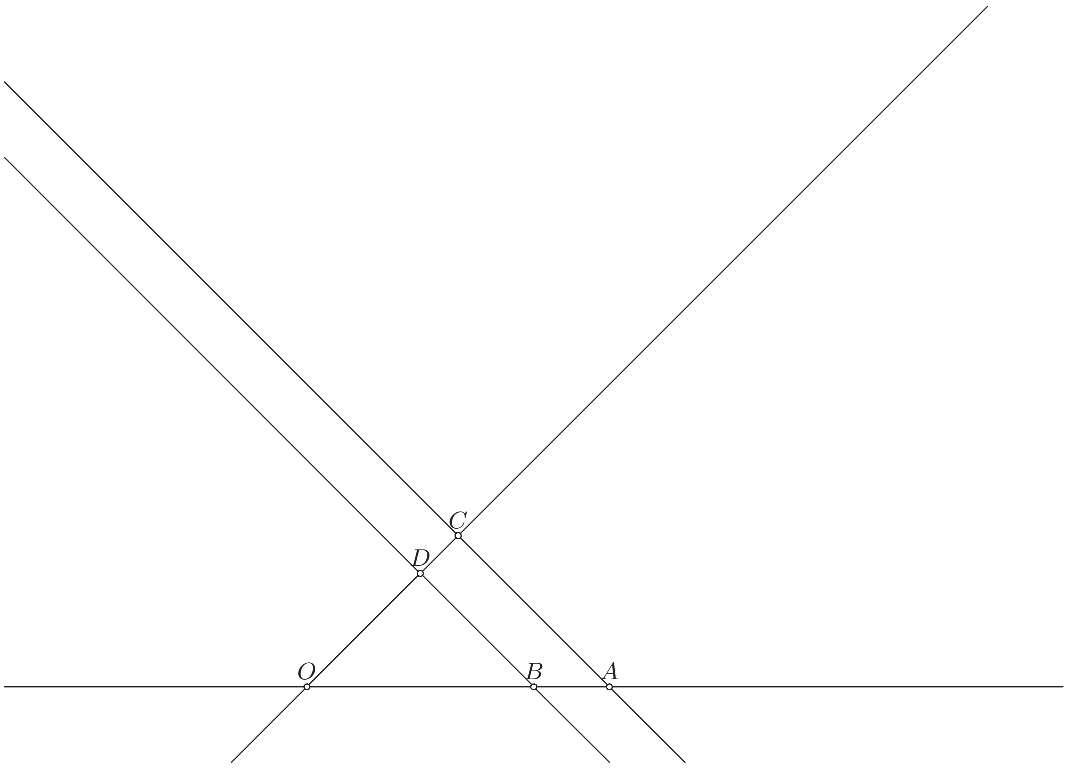
A.4 GEO0004 — Thales' Theorem

The Theorem Statment

Theorem 4 (Thales' Theorem) *At intersecting sides of an angle $\angle BOD$ by parallel lines BD and AC, the angle sides are divided into the proportional segments:*

$$\frac{\overline{OA}}{\overline{OB}} = \frac{\overline{OC}}{\overline{OD}}$$

The Image – GCLC 5.0



Prover's Code

```
point O 40 10
point A 80 10
point C 60 30

online B O A

line a A C
line c O C
parallel b B a
intersec D c b
```

```
drawline O A
drawline O C
drawline A C
drawline B D

cmark_t O
cmark_t A
cmark_t B
cmark_t C
cmark_t D

% parallel BC' and B'C
prove { equal { sratio O A O B } { sratio O C O D  } }
```

Proved — Proof, made with GCLC, v1.0 Let r_0 be the number such that PRATIO B O O A r_0 (for a concrete example r_0=-0.553045).

Let P_b^1 be the point such that lines P_b^1B and AC are parallel (and PRATIO P_b^1 B A C 1).

(A.335) $$\frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{\overrightarrow{OC}}{\overrightarrow{OD}}$$, by the statement

(A.336) $$\frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{S_{OBCP_b^1}}{S_{OBP_b^1}}$$, by Lemma 37 (reciprocial , second case — points O, O, and C are collinear) (point D eliminated)

(A.337) $$\frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{(S_{OBCB} + (1 \cdot (S_{OBCC} + (-1 \cdot S_{OBCA}))))}{S_{OBP_b^1}}$$, by Lemma 29 (point P_b^1 eliminated)

(A.338) $$\frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{(S_{OBCB} + (S_{OBCC} + (-1 \cdot S_{OBCA})))}{S_{OBP_b^1}}$$, by algebraic simplifications

(A.339) $$\frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{(S_{OBCB} + (S_{OBCC} + (-1 \cdot S_{OBCA})))}{(S_{OBB} + (1 \cdot (S_{OBC} + (-1 \cdot S_{OBA}))))}$$, by Lemma 29 (point P_b^1 eliminated)

(A.340) $$\frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{(S_{OBCB} + (S_{OBCC} + (-1 \cdot S_{OBCA})))}{(0 + (1 \cdot (S_{OBC} + (-1 \cdot S_{OBA}))))}$$, by geometric simplifications

(A.341) $$\frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{(S_{OBCB} + (S_{OBCC} + (-1 \cdot S_{OBCA})))}{(S_{OBC} + (-1 \cdot S_{OBA}))}$$, by algebraic simplifications

(A.342) $$\frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{((S_{COB} + S_{OCB}) + ((S_{COB} + S_{OCC}) + (-1 \cdot (S_{COB} + S_{OCA}))))}{(S_{COB} + (-1 \cdot S_{AOB}))}$$, by geometric simplifications

$$\frac{\overrightarrow{OA}}{\overrightarrow{OB}} = \frac{((S_{COB} + S_{OCB}) + (S_{OCC} + (-1 \cdot S_{OCA})))}{(S_{COB} + (-1 \cdot S_{AOB}))} \tag{A.343}$$
, by algebraic simplifications

$$\frac{\frac{\overrightarrow{OA}}{\overrightarrow{OA}}}{\left(\frac{\overrightarrow{OO}}{\overrightarrow{OA}} + r_0\right)} = \frac{((S_{COB} + S_{OCB}) + (S_{OCC} + (-1 \cdot S_{OCA})))}{(S_{COB} + (-1 \cdot S_{AOB}))} \tag{A.344}$$
, by Lemma 39 (reciprocial) (point B eliminated)

$$\frac{\frac{\overrightarrow{OA}}{\overrightarrow{OA}}}{(0 + r_0)} = \frac{((S_{COB} + (-1 \cdot S_{COB})) + (0 + (-1 \cdot S_{OCA})))}{(S_{COB} + (-1 \cdot S_{AOB}))} \tag{A.345}$$
, by geometric simplifications

$$(S_{COB} + (-1 \cdot S_{AOB})) = (-1 \cdot (S_{OCA} \cdot r_0)) \tag{A.346}$$
, by algebraic simplifications

$$((S_{COO} + (r_0 \cdot (S_{COA} + (-1 \cdot S_{COO})))) + (-1 \cdot S_{AOB})) = (-1 \cdot (S_{OCA} \cdot r_0)) \tag{A.347}$$
, by Lemma 29 (point B eliminated)

$$((0 + (r_0 \cdot (S_{COA} + (-1 \cdot 0)))) + (-1 \cdot S_{AOB})) = (-1 \cdot ((-1 \cdot S_{COA}) \cdot r_0)) \tag{A.348}$$
, by geometric simplifications

$$S_{AOB} = 0 \tag{A.349}$$
, by algebraic simplifications

$$(S_{AOO} + (r_0 \cdot (S_{AOA} + (-1 \cdot S_{AOO})))) = 0 \tag{A.350}$$
, by Lemma 29 (point B eliminated)

$$(0 + (r_0 \cdot (0 + (-1 \cdot 0)))) = 0 \tag{A.351}$$
, by geometric simplifications

$$0 = 0 \tag{A.352}$$
, by algebraic simplifications

Q.E.D.

NDG conditions are:

$S_{OBP_b^1} \neq S_{CBP_b^1}$ i.e., lines OC and BP_b^1 are not parallel (construction based assumption)

$P_{OBO} \neq 0$ i.e., points O and B are not identical (conjecture based assumption)

$P_{ODO} \neq 0$ i.e., points O and D are not identical (conjecture based assumption)

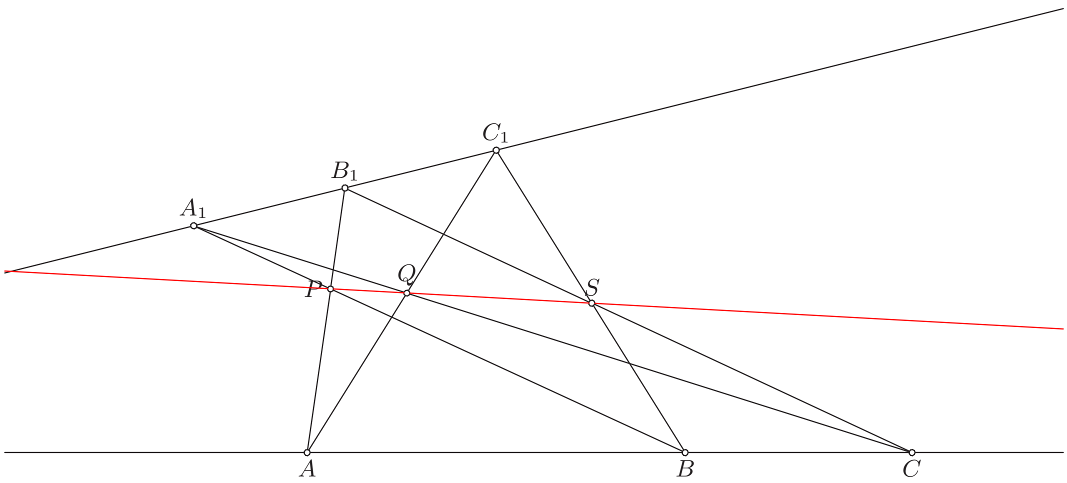
Number of elimination proof steps: 6
Number of geometric proof steps: 18
Number of algebraic proof steps: 34
Total number of proof steps: 58
Time spent by the prover: 0.001 seconds

A.5 GEO0005 — Pappus' Hexagon Theorem

The Theorem Statement [ZCG95]

Theorem 5 (Pappus's Hexagon Theorem) *If A, B, and C are three points on one line, A_1, B_1, and C_1 are three points on another line, and AB_1 meets BA_1 at P, AC_1 meets CA_1 at Q, and BC_1 meets CD_1 at S, then the three points P, Q, and S are collinear.*

The Image – GCLC 5.0



Prover's Code

```
point A 40 10
point B 90 10
%point C 120 10
online C A B

drawline A C

cmark_b A
cmark_b B
cmark_b C

point A_1 25 40
```

```
point B_1 45 45
%point C_1 65 50
online C_1 A_1 B_1


line A_1B_1 A_1 B_1

drawline A_1 C_1
cmark_t A_1
cmark_t B_1
cmark_t C_1

line AB_1 A B_1
line AC_1 A C_1
line BA_1 B A_1
line BC_1 B C_1
line CA_1 C A_1
line CB_1 C B_1

drawsegment A B_1
drawsegment A C_1
drawsegment B A_1
drawsegment B C_1
drawsegment C A_1
drawsegment C B_1


intersec P AB_1 BA_1
intersec Q AC_1 CA_1
intersec S BC_1 CB_1
cmark_l P
cmark_t Q
cmark_t S

color 255 0 0
drawline P S

prove { equal { signed_area3 P Q S } 0 }
```

Proved — Proof, made with GCLC, v1.0 Let r_0 be the number such that PRATIO C A A B r_0 (for a concrete example r_0=-0.553045).

Let r_1 be the number such that PRATIO C_1 A_1 A_1 B_1 r_1 (for a concrete example r_1=1.951010).

(A.353) $$S_{PQS} = 0$$, by the statement

(A.354) $$\frac{((S_{BCB_1} \cdot S_{PQC_1}) + (-1 \cdot (S_{C_1CB_1} \cdot S_{PQB})))}{S_{BCC_1B_1}} = 0$$, by Lemma 30 (point S eliminated)

(A.355) $$\frac{((S_{BCB_1} \cdot S_{C_1PQ}) + (-1 \cdot (S_{C_1CB_1} \cdot S_{BPQ})))}{S_{BCC_1B_1}} = 0$$, by geometric simplifications

(A.356) $$((S_{BCB_1} \cdot S_{C_1PQ}) + (-1 \cdot (S_{C_1CB_1} \cdot S_{BPQ}))) = 0$$, by algebraic simplifications

(A.357)
$$\left(\left(S_{BCB_1} \cdot \frac{((S_{ACA_1} \cdot S_{C_1PC_1}) + (-1 \cdot (S_{C_1CA_1} \cdot S_{C_1PA})))}{S_{ACC_1A_1}}\right) + (-1 \cdot (S_{C_1CB_1} \cdot S_{BPQ}))\right) = 0,$$ by Lemma 30 (point Q eliminated)

(A.358)
$$\left(\left(S_{BCB_1} \cdot \frac{((S_{ACA_1} \cdot 0) + (-1 \cdot (S_{C_1CA_1} \cdot S_{C_1PA})))}{S_{ACC_1A_1}}\right) + (-1 \cdot (S_{C_1CB_1} \cdot S_{BPQ}))\right) = 0$$, by geometric simplifications

(A.359) $$((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot S_{C_1PA})) + (S_{C_1CB_1} \cdot (S_{BPQ} \cdot S_{ACC_1A_1}))) = 0$$, by algebraic simplifications

(A.360) $$\left((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot S_{C_1PA})) + \left(S_{C_1CB_1} \cdot \left(\frac{((S_{ACA_1} \cdot S_{BPC_1}) + (-1 \cdot (S_{C_1CA_1} \cdot S_{BPA})))}{S_{ACC_1A_1}} \cdot S_{ACC_1A_1}\right)\right)\right)$$
$$= 0$$, by Lemma 30 (point Q eliminated)

(A.361) $$\left((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot S_{AC_1P})) + \left(S_{C_1CB_1} \cdot \left(\frac{((S_{ACA_1} \cdot S_{C_1BP}) + (-1 \cdot (S_{C_1CA_1} \cdot S_{ABP})))}{S_{ACC_1A_1}} \cdot S_{ACC_1A_1}\right)\right)\right)$$
$$= 0$$, by geometric simplifications

$$((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot S_{AC_1P})) + ((S_{C_1CB_1} \cdot (S_{ACA_1} \cdot S_{C_1BP})) + (-1 \cdot (S_{C_1CB_1} \cdot (S_{C_1CA_1} \cdot S_{ABP}))))) = 0, \text{ by algebraic simplifications} \tag{A.362}$$

$$\left(\left(S_{BCB_1} \cdot \left(S_{C_1CA_1} \cdot \frac{((S_{ABA_1} \cdot S_{AC_1B_1}) + (-1 \cdot (S_{B_1BA_1} \cdot S_{AC_1A})))}{S_{ABB_1A_1}}\right)\right) + ((S_{C_1CB_1} \cdot (S_{ACA_1} \cdot S_{C_1BP})) + (-1 \cdot (S_{C_1CB_1} \cdot (S_{C_1CA_1} \cdot S_{ABP}))))\right) = 0 \quad , \text{ by Lemma 30 (point } P \text{ eliminated)} \tag{A.363}$$

$$\left(\left(S_{BCB_1} \cdot \left(S_{C_1CA_1} \cdot \frac{((S_{ABA_1} \cdot S_{AC_1B_1}) + (-1 \cdot (S_{B_1BA_1} \cdot 0)))}{S_{ABB_1A_1}}\right)\right) + ((S_{C_1CB_1} \cdot (S_{ACA_1} \cdot S_{C_1BP})) + (-1 \cdot (S_{C_1CB_1} \cdot (S_{C_1CA_1} \cdot S_{ABP}))))\right) = 0 \quad , \text{ by geometric simplifications} \tag{A.364}$$

$$((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot (S_{ABA_1} \cdot S_{AC_1B_1}))) + ((S_{ABB_1A_1} \cdot (S_{C_1CB_1} \cdot (S_{ACA_1} \cdot S_{C_1BP}))) + (-1 \cdot (S_{ABB_1A_1} \cdot (S_{C_1CB_1} \cdot (S_{C_1CA_1} \cdot S_{ABP})))))) = 0, \text{ by algebraic simplifications} \tag{A.365}$$

$$\left((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot (S_{ABA_1} \cdot S_{AC_1B_1}))) + \left(\left(S_{ABB_1A_1} \cdot \left(S_{C_1CB_1} \cdot \left(S_{ACA_1} \cdot \frac{((S_{ABA_1} \cdot S_{C_1BB_1}) + (-1 \cdot (S_{B_1BA_1} \cdot S_{C_1BA})))}{S_{ABB_1A_1}}\right)\right)\right) + (-1 \cdot (S_{ABB_1A_1} \cdot (S_{C_1CB_1} \cdot (S_{C_1CA_1} \cdot S_{ABP}))))\right)\right) = 0 \quad , \text{ by Lemma 30 (point } P \text{ eliminated)} \tag{A.366}$$

$$((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot (S_{ABA_1} \cdot S_{AC_1B_1}))) + (((S_{C_1CB_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{C_1BB_1}))) + (-1 \cdot (S_{C_1CB_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{C_1BA}))))) + (-1 \cdot (S_{ABB_1A_1} \cdot (S_{C_1CB_1} \cdot (S_{C_1CA_1} \cdot S_{ABP})))))) = 0 \quad , \text{ by algebraic simplifications} \tag{A.367}$$

$$\begin{aligned}&\left((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot (S_{ABA_1} \cdot S_{AC_1B_1}))) + \left(((S_{C_1CB_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{C_1BB_1}))) + (-1 \cdot (S_{C_1CB_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{C_1BA})))))\right.\right.\\ &\left.\left.+ \left(-1 \cdot \left(S_{ABB_1A_1} \cdot \left(S_{C_1CB_1} \cdot \left(S_{C_1CA_1} \cdot \frac{((S_{ABA_1} \cdot S_{ABB_1}) + (-1 \cdot (S_{B_1BA_1} \cdot S_{ABA})))}{S_{ABB_1A_1}}\right)\right)\right)\right)\right)\right)\\ &= 0 \qquad \text{, by Lemma 30 (point } P \text{ eliminated)}\end{aligned} \tag{A.368}$$

$$\begin{aligned}&\left((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot (S_{ABA_1} \cdot S_{AC_1B_1}))) + \left(((S_{C_1CB_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{C_1BB_1}))) + (-1 \cdot (S_{C_1CB_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{C_1BA})))))\right.\right.\\ &\left.\left.+ \left(-1 \cdot \left(S_{ABB_1A_1} \cdot \left(S_{C_1CB_1} \cdot \left(S_{C_1CA_1} \cdot \frac{((S_{ABA_1} \cdot S_{ABB_1}) + (-1 \cdot (S_{B_1BA_1} \cdot 0)))}{S_{ABB_1A_1}}\right)\right)\right)\right)\right)\right)\\ &= 0 \qquad \text{, by geometric simplifications}\end{aligned} \tag{A.369}$$

$$\begin{aligned}&((S_{BCB_1} \cdot (S_{C_1CA_1} \cdot (S_{ABA_1} \cdot S_{AC_1B_1}))) + (((S_{C_1CB_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{C_1BB_1}))) + (-1 \cdot (S_{C_1CB_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{C_1BA})))))\\ &\quad + (-1 \cdot (S_{C_1CB_1} \cdot (S_{C_1CA_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \qquad \text{, by algebraic simplifications}\end{aligned} \tag{A.370}$$

$$\begin{aligned}&((S_{BCB_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{B_1AC_1}))) + (((S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{BB_1C_1}))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))\\ &\quad + (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \qquad \text{, by geometric simplifications}\end{aligned} \tag{A.371}$$

$$\begin{aligned}&((S_{BCB_1} \cdot ((S_{CA_1A_1} + (r_1 \cdot (S_{CA_1B_1} + (-1 \cdot S_{CA_1A_1})))) \cdot (S_{ABA_1} \cdot S_{B_1AC_1})))\\ &\quad + (((S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{BB_1C_1}))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))\\ &\qquad + (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \qquad \text{, by Lemma 29 (point } C_1 \text{ eliminated)}\end{aligned} \tag{A.372}$$

$$\begin{aligned}&((S_{BCB_1} \cdot ((0 + (r_1 \cdot (S_{CA_1B_1} + (-1 \cdot 0)))) \cdot (S_{ABA_1} \cdot S_{B_1AC_1})))\\ &\quad + (((S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{BB_1C_1}))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))\\ &\qquad + (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \qquad \text{, by geometric simplifications}\end{aligned} \tag{A.373}$$

$$\text{(A.374)}\quad ((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AC_1})))) + (((S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{BB_1C_1}))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \quad , \text{ by algebraic simplifications}$$

$$((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} + (r_1 \cdot (S_{B_1AB_1} + (-1 \cdot S_{B_1AA_1}))))))))$$
$$\text{(A.375)}\quad + (((S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{BB_1C_1}))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \quad , \text{ by Lemma 29 (point } C_1 \text{ eliminated)}$$

$$((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} + (r_1 \cdot (0 + (-1 \cdot S_{B_1AA_1}))))))))$$
$$\text{(A.376)}\quad + (((S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{BB_1C_1}))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \quad , \text{ by geometric simplifications}$$

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$\text{(A.377)}\quad + (((S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{BB_1C_1}))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \quad , \text{ by algebraic simplifications}$$

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$\text{(A.378)}\quad + ((((S_{CB_1A_1} + (r_1 \cdot (S_{CB_1B_1} + (-1 \cdot S_{CB_1A_1})))) \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{BB_1C_1}))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \quad , \text{ by Lemma 29 (point } C_1 \text{ eliminated)}$$

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$\text{(A.379)}\quad + (((((-1 \cdot S_{CA_1B_1}) + (r_1 \cdot (0 + (-1 \cdot (-1 \cdot S_{CA_1B_1}))))) \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot S_{BB_1C_1}))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \quad , \text{ by geometric simplifications}$$

(A.380)
$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+ ((((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{BB_1C_1} \cdot S_{CA_1B_1})))) + (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{BB_1C_1} \cdot (r_1 \cdot S_{CA_1B_1}))))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \quad , \text{ by algebraic simplifications}$$

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot ((S_{BB_1A_1} + (r_1 \cdot (S_{BB_1B_1} + (-1 \cdot S_{BB_1A_1})))) \cdot S_{CA_1B_1})))) + (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{BB_1C_1} \cdot (r_1 \cdot S_{CA_1B_1}))))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{B_1BA_1} \cdot S_{BAC_1})))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \tag{A.381}$$
, by Lemma 29 (point C_1 eliminated)

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot ((S_{BB_1A_1} + (r_1 \cdot (0 + (-1 \cdot S_{BB_1A_1})))) \cdot S_{CA_1B_1})))) + (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{BB_1C_1} \cdot (r_1 \cdot S_{CA_1B_1}))))) + (-1 \cdot (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot ((-1 \cdot S_{BB_1A_1}) \cdot S_{BAC_1})))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \tag{A.382}$$
, by geometric simplifications

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot S_{BB_1A_1})))) + (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))) + (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{BB_1C_1} \cdot (r_1 \cdot S_{CA_1B_1}))))) + (S_{CB_1C_1} \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot S$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \tag{A.383}$$
, by algebraic simplifications

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot S_{BB_1A_1})))) + (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))) + (S_{ACA_1} \cdot (S_{ABA_1} \cdot ((S_{BB_1A_1} + (r_1 \cdot (S_{BB_1B_1} + (-1 \cdot S_{BB_1A_1})))) \cdot (r_1 \cdot S_{CA_1B_1}))$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \tag{A.384}$$
, by Lemma 29 (point C_1 eliminated)

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot S_{BB_1A_1})))) + (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))) + (S_{ACA_1} \cdot (S_{ABA_1} \cdot ((S_{BB_1A_1} + (r_1 \cdot (0 + (-1 \cdot S_{BB_1A_1})))) \cdot (r_1 \cdot S_{CA_1B_1}))))) + (S$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \tag{A.385}$$
, by geometric simplifications

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot S_{BB_1A_1})))) + (2 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))))) + (-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))))) + (S_{CB_1C_1} \cdot (S_{AC}$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0 \tag{A.386}$$
, by algebraic simplifications

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+ ((((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot S_{BB_1A_1})))) + (2 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))))) + (-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))))) + ((S_{CB_1A_1} + (r_1 \quad \text{(A.387)}$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0$$
, by Lemma 29 (point C_1 eliminated)

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+ ((((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot S_{BB_1A_1})))) + (2 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))))) + (-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))))) + (((-1 \cdot S_{CA_1B_1} \quad \text{(A.388)}$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0$$
, by geometric simplifications

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+ ((((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot S_{BB_1A_1})))) + (2 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))))) + (-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))))) + ((-1 \cdot (S_{ACA_1} \cdot \quad \text{(A.389)}$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0$$
, by algebraic simplifications

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+ ((((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot S_{BB_1A_1})))) + (2 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))))) + (-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))))) + ((-1 \cdot (S_{ACA_1} \cdot \quad \text{(A.390)}$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{ABB_1})))))) = 0$$
, by Lemma 29 (point C_1 eliminated)

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+ ((((-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot S_{BB_1A_1})))) + (2 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))))) + (-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))))) + ((-1 \cdot (S_{ACA_1} \cdot \quad \text{(A.391)}$$
$$+ (-1 \cdot (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot (-1 \cdot S_{BAB_1}))))))) = 0$$
, by geometric simplifications

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+ (((2 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))) + (-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))))) + ((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BAB_1}))))) + (S_{ACA_1} \cdot (S_{BB_1} \quad \text{(A.392)}$$
$$+ (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))))) = 0$$
, by algebraic simplifications

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+(((S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))) + (-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))))) + ((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BAB_1}))))) + (S_{ACA_1} \cdot (S_{BB_1} \quad \text{(A.393)}$$
$$+ (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))))) = 0 \quad , \text{ by Lemma 29 (point } C_1 \text{ eliminated)}$$

$$(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))$$
$$+(((S_{ACA_1} \cdot (S_{ABA_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1})))) + (-1 \cdot (S_{ACA_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BB_1A_1}))))))) + ((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BAB_1}))))) + (S_{ACA_1} \cdot (S_{BB_1} \quad \text{(A.394)}$$
$$+ (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))))) = 0 \quad , \text{ by geometric simplifications}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))\\ &+ (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BAB_1}))))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BAB_1}))))))\\ &+ (S_{CB_1C_1} \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))))) = 0 \quad , \text{ by algebraic simplifications}\end{aligned} \tag{A.395}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))\\ &+ (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BAB_1}))))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BAB_1}))))))\\ &+ ((S_{CB_1A_1} + (r_1 \cdot (S_{CB_1B_1} + (-1 \cdot S_{CB_1A_1})))) \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))))) = 0, \text{ by Lemma 29 (point } C_1 \text{ eliminated)}\end{aligned} \tag{A.396}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))))\\ &+ (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BAB_1}))))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot (S_{CA_1B_1} \cdot (r_1 \cdot S_{BAB_1}))))))\\ &+ (((-1 \cdot S_{CA_1B_1}) + (r_1 \cdot (0 + (-1 \cdot (-1 \cdot S_{CA_1B_1}))))) \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{BAB_1})))))\\ &= 0 \quad , \text{ by geometric simplifications}\end{aligned} \tag{A.397}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (r_1 \cdot (S_{ABA_1} \cdot S_{B_1AA_1}))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1}))))))\\ &+ (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{BAB_1})))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot (r_1 \cdot S_{BAB_1})))))\\ &+ ((-1 \cdot (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))) + (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot (S_{BAB_1} \cdot r_1)))))) = 0, \text{ by algebraic simplifications}\end{aligned} \tag{A.398}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (r_1 \cdot (S_{ABA_1} \cdot S_{B_1AA_1}))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1}))))))\\ &\quad + (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{BAB_1})))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot (r_1 \cdot S_{BAB_1})))))\\ &\quad + ((-1 \cdot ((S_{CA_1A_1} + (r_1 \cdot (S_{CA_1B_1} + (-1 \cdot S_{CA_1A_1})))) \cdot (S_{ABA_1} \cdot S_{BAB_1}))) + (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot (S_{BAB_1} \cdot r_1))))))\\ &\quad = 0 \qquad , \quad \text{by Lemma 29 (point } C_1 \text{ eliminated)}\end{aligned} \tag{A.399}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (r_1 \cdot (S_{ABA_1} \cdot S_{B_1AA_1}))) + (-1 \cdot (S_{BCB_1} \cdot (r_1 \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1}))))))\\ &\quad + (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{BAB_1})))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot (r_1 \cdot S_{BAB_1})))))\\ &\quad + ((-1 \cdot ((0 + (r_1 \cdot (S_{CA_1B_1} + (-1 \cdot 0)))) \cdot (S_{ABA_1} \cdot S_{BAB_1}))) + (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot (S_{BAB_1} \cdot r_1))))))\\ &\quad = 0 \qquad , \quad \text{by geometric simplifications}\end{aligned} \tag{A.400}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})) + (-1 \cdot (S_{BCB_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\ &\quad + (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot S_{BAB_1}))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{BAB_1}))))\\ &\quad + ((-1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))) + (S_{CA_1C_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))))) = 0 \qquad , \quad \text{by algebraic simplifications}\end{aligned} \tag{A.401}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})) + (-1 \cdot (S_{BCB_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\ &\quad + (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot S_{BAB_1}))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{BAB_1}))))\\ &\quad + ((-1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))) + ((S_{CA_1A_1} + (r_1 \cdot (S_{CA_1B_1} + (-1 \cdot S_{CA_1A_1})))) \cdot (S_{ABA_1} \cdot S_{BAB_1})))))\\ &\quad = 0 \qquad , \quad \text{by Lemma 29 (point } C_1 \text{ eliminated)}\end{aligned} \tag{A.402}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})) + (-1 \cdot (S_{BCB_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\ &\quad + (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot S_{BAB_1}))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{BAB_1}))))\\ &\quad + ((-1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))) + ((0 + (r_1 \cdot (S_{CA_1B_1} + (-1 \cdot 0)))) \cdot (S_{ABA_1} \cdot S_{BAB_1})))))\\ &\quad = 0 \qquad , \quad \text{by geometric simplifications}\end{aligned} \tag{A.403}$$

$$\begin{aligned}&(((S_{BCB_1} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})) + (-1 \cdot (S_{BCB_1} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\&\quad + (((-1 \cdot (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot S_{BAB_1}))) + (S_{ACA_1} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{BAB_1}))))\\&\quad\quad + ((-1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{BAB_1}))) + (r_1 \cdot (S_{CA_1B_1} \cdot (S_{ABA_1} \cdot S_{BAB_1})))))) = 0, \text{ by algebraic simplifications}\end{aligned} \tag{A.404}$$

$$\begin{aligned}&(((S_{B_1BC} \cdot (S_{ABA_1} \cdot S_{B_1AA_1})) + (-1 \cdot (S_{B_1BC} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\&\quad + (((-1 \cdot (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot S_{BAB_1}))) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{BAB_1}))))\\&\quad\quad + ((-1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{BAB_1}))) + (r_1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{BAB_1})))))) = 0, \text{ by geometric simplifications}\end{aligned} \tag{A.405}$$

$$\begin{aligned}&((((S_{B_1BA} + (r_0 \cdot (S_{B_1BB} + (-1 \cdot S_{B_1BA})))) \cdot (S_{ABA_1} \cdot S_{B_1AA_1})) + (-1 \cdot (S_{B_1BC} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\&+ (((-1 \cdot (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot S_{BAB_1}))) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{BAB_1}))))\\&\quad + ((-1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{BAB_1}))) + (r_1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{BAB_1})))))) = 0, \text{ by Lemma 29 (point } C \text{ eliminated)}\end{aligned} \tag{A.406}$$

$$\begin{aligned}&((((S_{B_1BA} + (r_0 \cdot (0 + (-1 \cdot S_{B_1BA})))) \cdot (S_{ABA_1} \cdot S_{B_1AA_1})) + (-1 \cdot (S_{B_1BC} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\&+ (((-1 \cdot (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot S_{B_1BA}))) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{B_1BA}))))\\&\quad + ((-1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{B_1BA}))) + (r_1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{B_1BA})))))) = 0, \text{ by geometric simplifications}\end{aligned} \tag{A.407}$$

$$\begin{aligned}&((((S_{ABA_1} \cdot (S_{B_1AA_1} \cdot S_{B_1BA})) + (-1 \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} \cdot (r_0 \cdot S_{B_1BA}))))) + (-1 \cdot (S_{B_1BC} \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\&+ (((-1 \cdot (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot S_{B_1BA}))) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{B_1BA}))))\\&\quad + ((-1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{B_1BA}))) + (r_1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{B_1BA})))))) = 0, \text{ by algebraic simplifications}\end{aligned} \tag{A.408}$$

$$\begin{aligned}&((((S_{ABA_1} \cdot (S_{B_1AA_1} \cdot S_{B_1BA})) + (-1 \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} \cdot (r_0 \cdot S_{B_1BA})))))\\&+ (-1 \cdot ((S_{B_1BA} + (r_0 \cdot (S_{B_1BB} + (-1 \cdot S_{B_1BA})))) \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\&+ (((-1 \cdot (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot S_{B_1BA}))) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{B_1BA}))))\\&\quad + ((-1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{B_1BA}))) + (r_1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{B_1BA})))))) = 0, \text{ by Lemma 29 (point } C \text{ eliminated)}\end{aligned} \tag{A.409}$$

$$
\begin{aligned}
&((((S_{ABA_1} \cdot (S_{B_1AA_1} \cdot S_{B_1BA})) + (-1 \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} \cdot (r_0 \cdot S_{B_1BA})))))\\
&+ (-1 \cdot ((S_{B_1BA} + (r_0 \cdot (0 + (-1 \cdot S_{B_1BA})))) \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1})))))\\
&+ (((-1 \cdot (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot S_{B_1BA}))) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot (r_1 \cdot S_{B_1BA}))))\\
&+ ((-1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{B_1BA}))) + (r_1 \cdot (S_{A_1B_1C} \cdot (S_{ABA_1} \cdot S_{B_1BA})))))) = 0, \quad \text{by geometric simplifications}
\end{aligned} \tag{A.410}
$$

$$
\begin{aligned}
&((((S_{ABA_1} \cdot S_{B_1AA_1}) + (-1 \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} \cdot r_0)))) + ((-1 \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1}))) + (S_{ABA_1} \cdot (r_1 \cdot (S_{B_1AA_1} \cdot r_0)))))\\
&+ (((-1 \cdot (S_{A_1AC} \cdot S_{BB_1A_1})) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot r_1))) + ((-1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1})) + (r_1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1})))))\\
&= 0 \quad , \quad \text{by algebraic simplifications}
\end{aligned} \tag{A.411}
$$

$$
\begin{aligned}
&((((S_{ABA_1} \cdot S_{B_1AA_1}) + (-1 \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} \cdot r_0)))) + ((-1 \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1}))) + (S_{ABA_1} \cdot (r_1 \cdot (S_{B_1AA_1} \cdot r_0)))))\\
&+ (((-1 \cdot ((S_{A_1AA} + (r_0 \cdot (S_{A_1AB} + (-1 \cdot S_{A_1AA})))) \cdot S_{BB_1A_1})) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot r_1)))\\
&+ ((-1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1})) + (r_1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1}))))) = 0 \quad , \quad \text{by Lemma 29 (point } C \text{ eliminated)}
\end{aligned} \tag{A.412}
$$

$$
\begin{aligned}
&((((S_{ABA_1} \cdot S_{B_1AA_1}) + (-1 \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} \cdot r_0)))) + ((-1 \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1}))) + (S_{ABA_1} \cdot (r_1 \cdot (S_{B_1AA_1} \cdot r_0)))))\\
&+ (((-1 \cdot ((0 + (r_0 \cdot (S_{ABA_1} + (-1 \cdot 0)))) \cdot S_{BB_1A_1})) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot r_1)))\\
&+ ((-1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1})) + (r_1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1}))))) = 0 \quad , \quad \text{by geometric simplifications}
\end{aligned} \tag{A.413}
$$

$$
\begin{aligned}
&((((S_{ABA_1} \cdot S_{B_1AA_1}) + (-1 \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} \cdot r_0)))) + ((-1 \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1}))) + (S_{ABA_1} \cdot (r_1 \cdot (S_{B_1AA_1} \cdot r_0)))))\\
&+ (((-1 \cdot (r_0 \cdot (S_{ABA_1} \cdot S_{BB_1A_1}))) + (S_{A_1AC} \cdot (S_{BB_1A_1} \cdot r_1)))\\
&+ ((-1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1})) + (r_1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1}))))) = 0 \quad , \quad \text{by algebraic simplifications}
\end{aligned} \tag{A.414}
$$

$$
\begin{aligned}
&((((S_{ABA_1} \cdot S_{B_1AA_1}) + (-1 \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} \cdot r_0)))) + ((-1 \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1}))) + (S_{ABA_1} \cdot (r_1 \cdot (S_{B_1AA_1} \cdot r_0)))))\\
&+ (((-1 \cdot (r_0 \cdot (S_{ABA_1} \cdot S_{BB_1A_1}))) + ((S_{A_1AA} + (r_0 \cdot (S_{A_1AB} + (-1 \cdot S_{A_1AA})))) \cdot (S_{BB_1A_1} \cdot r_1)))\\
&+ ((-1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1})) + (r_1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1}))))) = 0 \quad , \quad \text{by Lemma 29 (point } C \text{ eliminated)}
\end{aligned} \tag{A.415}
$$

$$\begin{aligned}((((S_{ABA_1} \cdot S_{B_1AA_1}) + (-1 \cdot (S_{ABA_1} \cdot (S_{B_1AA_1} \cdot r_0)))) + ((-1 \cdot (S_{ABA_1} \cdot (r_1 \cdot S_{B_1AA_1}))) + (S_{ABA_1} \cdot (r_1 \cdot (S_{B_1AA_1} \cdot r_0)))))\\ + (((-1 \cdot (r_0 \cdot (S_{ABA_1} \cdot S_{BB_1A_1}))) + ((0 + (r_0 \cdot (S_{ABA_1} + (-1 \cdot 0)))) \cdot (S_{BB_1A_1} \cdot r_1)))\\ + ((-1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1})) + (r_1 \cdot (S_{A_1B_1C} \cdot S_{ABA_1}))))) = 0 \quad , \text{ by geometric simplifications}\end{aligned} \tag{A.416}$$

$$\begin{aligned}(((S_{B_1AA_1} + (-1 \cdot (S_{B_1AA_1} \cdot r_0))) + ((-1 \cdot (r_1 \cdot S_{B_1AA_1})) + (r_1 \cdot (S_{B_1AA_1} \cdot r_0))))\\ + (((-1 \cdot (r_0 \cdot S_{BB_1A_1})) + (r_0 \cdot (S_{BB_1A_1} \cdot r_1))) + ((-1 \cdot S_{A_1B_1C}) + (r_1 \cdot S_{A_1B_1C})))) = 0, \text{ by algebraic simplifications}\end{aligned} \tag{A.417}$$

$$\begin{aligned}(((S_{B_1AA_1} + (-1 \cdot (S_{B_1AA_1} \cdot r_0))) + ((-1 \cdot (r_1 \cdot S_{B_1AA_1})) + (r_1 \cdot (S_{B_1AA_1} \cdot r_0)))) + (((-1 \cdot (r_0 \cdot S_{BB_1A_1})) + (r_0 \cdot (S_{BB_1A_1} \cdot r_1)))\\ + ((-1 \cdot (S_{A_1B_1A} + (r_0 \cdot (S_{A_1B_1B} + (-1 \cdot S_{A_1B_1A}))))) + (r_1 \cdot S_{A_1B_1C})))) = 0 \quad , \text{ by Lemma 29 (point } C \text{ eliminated)}\end{aligned} \tag{A.418}$$

$$\begin{aligned}(((S_{B_1AA_1} + (-1 \cdot (S_{B_1AA_1} \cdot r_0))) + ((-1 \cdot (r_1 \cdot S_{B_1AA_1})) + (r_1 \cdot (S_{B_1AA_1} \cdot r_0)))) + (((-1 \cdot (r_0 \cdot S_{BB_1A_1})) + (r_0 \cdot (S_{BB_1A_1} \cdot r_1)))\\ + ((-1 \cdot (S_{B_1AA_1} + (r_0 \cdot ((-1 \cdot S_{BB_1A_1}) + (-1 \cdot S_{B_1AA_1}))))) + (r_1 \cdot S_{A_1B_1C})))) = 0, \text{ by geometric simplifications}\end{aligned} \tag{A.419}$$

$$(((-1 \cdot S_{B_1AA_1}) + (S_{B_1AA_1} \cdot r_0)) + ((r_0 \cdot S_{BB_1A_1}) + S_{A_1B_1C})) = 0 \quad , \text{ by algebraic simplifications} \tag{A.420}$$

$$\begin{aligned}(((-1 \cdot S_{B_1AA_1}) + (S_{B_1AA_1} \cdot r_0))\\ + ((r_0 \cdot S_{BB_1A_1}) + (S_{A_1B_1A} + (r_0 \cdot (S_{A_1B_1B} + (-1 \cdot S_{A_1B_1A})))))) = 0, \text{ by Lemma 29 (point } C \text{ eliminated)}\end{aligned} \tag{A.421}$$

$$\begin{aligned}(((-1 \cdot S_{B_1AA_1}) + (S_{B_1AA_1} \cdot r_0))\\ + ((r_0 \cdot S_{BB_1A_1}) + (S_{B_1AA_1} + (r_0 \cdot ((-1 \cdot S_{BB_1A_1}) + (-1 \cdot S_{B_1AA_1})))))) = 0, \text{ by geometric simplifications}\end{aligned} \tag{A.422}$$

$$0 = 0 \quad , \text{ by algebraic simplifications} \tag{A.423}$$

Q.E.D.

NDG conditions are:

$S_{ABA_1} \neq S_{B_1BA_1}$ i.e., lines AB_1 and BA_1 are not parallel (construction based assumption)

$S_{ACA_1} \neq S_{C_1CA_1}$ i.e., lines AC_1 and CA_1 are not parallel (construction based assumption)

$S_{BCB_1} \neq S_{C_1CB_1}$ i.e., lines BC_1 and CB_1 are not parallel (construction based assumption)

$S_{CA_1B_1} \neq 0$ i.e., points C, A_1 and B_1 are not collinear (cancellation assumption)

$r_1 \neq 0$ (cancellation assumption)

$S_{B_1BA} \neq 0$ i.e., points B_1, B and A are not collinear (cancellation assumption)

$S_{ABA_1} \neq 0$ i.e., points A, B and A_1 are not collinear (cancellation assumption)

$r_0 \neq 0$ (cancellation assumption)

Number of elimination proof steps: 24
Number of geometric proof steps: 65
Number of algebraic proof steps: 269
Total number of proof steps: 358
Time spent by the prover: 0.039 seconds

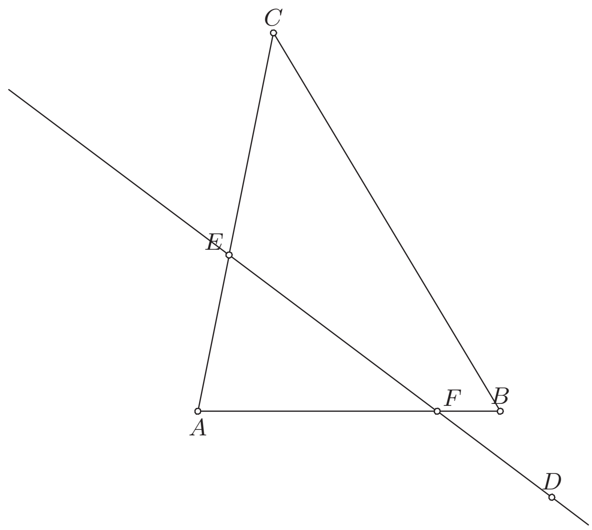
A.6 GEO0006 — Menelaus' Theorem

The Theorem Statement [CGZ96]

Theorem 6 (Menelaus' Theorem) *A transversal meets the three sides AB, BC, and CA of a triangle ABC in F, D, and E respectively. Then $\frac{\overline{AF}}{\overline{FB}} = -\frac{\overline{DC}}{\overline{BD}}\frac{\overline{EA}}{\overline{CE}}$.*

The non-degenerate condition is that A, B, and C are not on line EF.

The Image – GCLC 5.0



Prover's Code

```
dim 100 100
area 5 5 90 90

point A 30 20
point B 70 20
point C 40 70

point X 15 55
point Y 75 10

line a B C
line b A C
line c A B
```

```
line p X Y

intersec D a p
intersec E b p
intersec F c p

drawsegment A B
drawsegment A C
drawsegment B C

drawline p

cmark_b A
cmark_t B
cmark_t C

cmark_t D
cmark_lt E
cmark_rt F

prove { equal { signed_area3 D E F } { 0 } }
```

Proved — Proof, made with GCLC, v1.0

(A.424) $$S_{DEF} = 0$$, by the statement

(A.425) $$\frac{((S_{AXY} \cdot S_{DEB}) + (-1 \cdot (S_{BXY} \cdot S_{DEA})))}{S_{AXBY}} = 0$$, by Lemma 30 (point F eliminated)

(A.426) $$\frac{((S_{AXY} \cdot S_{BDE}) + (-1 \cdot (S_{BXY} \cdot S_{ADE})))}{S_{AXBY}} = 0$$, by geometric simplifications

(A.427) $$((S_{AXY} \cdot S_{BDE}) + (-1 \cdot (S_{BXY} \cdot S_{ADE}))) = 0$$, by algebraic simplifications

(A.428)
$$\left(\left(S_{AXY} \cdot \frac{((S_{AXY} \cdot S_{BDC}) + (-1 \cdot (S_{CXY} \cdot S_{BDA})))}{S_{AXCY}}\right) + (-1 \cdot (S_{BXY} \cdot S_{ADE}))\right) = 0$$, by Lemma 30 (point E eliminated)

(A.429)
$$\left(\left(S_{AXY} \cdot \frac{((S_{AXY} \cdot 0) + (-1 \cdot (S_{CXY} \cdot S_{BDA})))}{S_{AXCY}}\right) + (-1 \cdot (S_{BXY} \cdot S_{ADE}))\right) = 0,$$ by geometric simplifications

(A.430) $$((S_{AXY} \cdot (S_{CXY} \cdot S_{BDA})) + (S_{BXY} \cdot (S_{ADE} \cdot S_{AXCY}))) = 0$$, by algebraic simplifications

(A.431)
$$\left((S_{AXY} \cdot (S_{CXY} \cdot S_{BDA})) + \left(S_{BXY} \cdot \left(\frac{((S_{AXY} \cdot S_{ADC}) + (-1 \cdot (S_{CXY} \cdot S_{ADA})))}{S_{AXCY}} \cdot S_{AXCY}\right)\right)\right)$$
$$= 0$$, by Lemma 30 (point E eliminated)

(A.432)
$$\Big((S_{AXY}\cdot(S_{CXY}\cdot S_{ABD}))\\ +\Big(S_{BXY}\cdot\Big(\frac{((S_{AXY}\cdot S_{CAD})+(-1\cdot(S_{CXY}\cdot 0)))}{S_{AXCY}}\cdot S_{AXCY}\Big)\Big)\Big)\\ =0$$
, by geometric simplifications

(A.433)
$$((S_{CXY}\cdot S_{ABD})+(S_{BXY}\cdot S_{CAD}))=0$$
, by algebraic simplifications

(A.434)
$$\Big(\Big(S_{CXY}\cdot\frac{((S_{BXY}\cdot S_{ABC})+(-1\cdot(S_{CXY}\cdot S_{ABB})))}{S_{BXCY}}\Big)+(S_{BXY}\cdot S_{CAD})\Big)=0,$$
by Lemma 30 (point D eliminated)

(A.435)
$$\Big(\Big(S_{CXY}\cdot\frac{((S_{BXY}\cdot S_{ABC})+(-1\cdot(S_{CXY}\cdot 0)))}{S_{BXCY}}\Big)+(S_{BXY}\cdot S_{CAD})\Big)=0$$
, by geometric simplifications

(A.436)
$$((S_{CXY}\cdot S_{ABC})+(S_{CAD}\cdot S_{BXCY}))=0$$
, by algebraic simplifications

(A.437)
$$\Big((S_{CXY}\cdot S_{ABC})+\Big(\frac{((S_{BXY}\cdot S_{CAC})+(-1\cdot(S_{CXY}\cdot S_{CAB})))}{S_{BXCY}}\cdot S_{BXCY}\Big)\Big)=0,$$
by Lemma 30 (point D eliminated)

(A.438)
$$\Big((S_{CXY}\cdot S_{ABC})+\Big(\frac{((S_{BXY}\cdot 0)+(-1\cdot(S_{CXY}\cdot S_{ABC})))}{S_{BXCY}}\cdot S_{BXCY}\Big)\Big)=0$$
, by geometric simplifications

(A.439)
$$0=0$$
, by algebraic simplifications

Q.E.D.

NDG conditions are:

$S_{BXY} \neq S_{CXY}$ i.e., lines BC and XY are not parallel (construction based assumption)

$S_{AXY} \neq S_{CXY}$ i.e., lines AC and XY are not parallel (construction based assumption)

$S_{AXY} \neq S_{BXY}$ i.e., lines AB and XY are not parallel (construction based assumption)

$S_{AXY} \neq 0$ i.e., points A, X and Y are not collinear (cancellation assumption)

$S_{BXY} \neq 0$ i.e., points B, X and Y are not collinear (cancellation assumption)

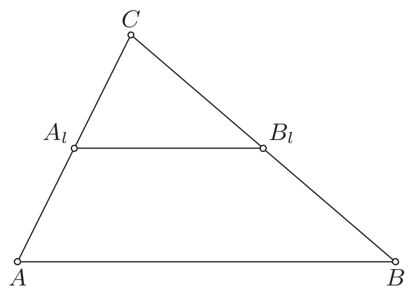
Number of elimination proof steps: 5
Number of geometric proof steps: 9
Number of algebraic proof steps: 39
Total number of proof steps: 53
Time spent by the prover: 0.002 seconds

A.7 GEO0007 — Midpoint Theorem

The Theorem Statement [Nar04]

Theorem 7 (Midpoint Theorem) *Let ABC be a triangle, and let A_l and B_l be the midpoints of BC and AC respectively. Then the line A_lB_l is parallel to the base AB.*

The Image – GCLC 5.0



Prover's Code

```
point A 20 10
point B 70 10
point C 35 40

drawsegment A B
drawsegment A C
drawsegment B C

midpoint B_l B C
midpoint A_l A C

drawsegment A_l B_l
```

```
cmark_b A
cmark_b B
cmark_t C

cmark_lt A_l
cmark_rt B_l

prove { equal { signed_area3 A_l B_l A } { signed_area3 A_l B_l B } }
```

Proved — Proof, made with GCLC, v1.0

$$S_{A_l B_l A} = S_{A_l B_l B} \quad \text{(A.440)}$$
, by the statement

$$S_{B_l A A_l} = S_{B_l B A_l} \quad \text{(A.441)}$$
, by geometric simplifications

$$\left(S_{B_l A A} + \left(\frac{1}{2} \cdot (S_{B_l A C} + (-1 \cdot S_{B_l A A}))\right)\right) = S_{B_l B A_l} \quad \text{(A.442)}$$
, by Lemma 29 (point A_l eliminated)

$$\left(0 + \left(\frac{1}{2} \cdot (S_{B_l A C} + (-1 \cdot 0))\right)\right) = S_{B_l B A_l} \quad \text{(A.443)}$$
, by geometric simplifications

$$\left(\frac{1}{2} \cdot S_{B_l A C}\right) = S_{B_l B A_l} \quad \text{(A.444)}$$
, by algebraic simplifications

$$\left(\frac{1}{2} \cdot S_{B_l A C}\right) = \left(S_{B_l B A} + \left(\frac{1}{2} \cdot (S_{B_l B C} + (-1 \cdot S_{B_l B A}))\right)\right) \quad \text{(A.445)}$$
, by Lemma 29 (point A_l eliminated)

$$\left(\frac{1}{2} \cdot S_{A C B_l}\right) = \left(S_{B A B_l} + \left(\frac{1}{2} \cdot (S_{B C B_l} + (-1 \cdot S_{B A B_l}))\right)\right) \quad \text{(A.446)}$$
, by geometric simplifications

$$S_{A C B_l} = (S_{B A B_l} + S_{B C B_l}) \quad \text{(A.447)}$$
, by algebraic simplifications

$$\left(S_{A C B} + \left(\frac{1}{2} \cdot (S_{A C C} + (-1 \cdot S_{A C B}))\right)\right) = (S_{B A B_l} + S_{B C B_l}) \quad \text{(A.448)}$$
, by Lemma 29 (point B_l eliminated)

$$\left(S_{A C B} + \left(\frac{1}{2} \cdot (0 + (-1 \cdot S_{A C B}))\right)\right) = (S_{B A B_l} + S_{B C B_l}) \quad \text{(A.449)}$$
, by geometric simplifications

(A.450) $$\left(\frac{1}{2} \cdot S_{ACB}\right) = (S_{BAB_l} + S_{BCB_l})$$, by algebraic simplifications

(A.451) $$\left(\frac{1}{2} \cdot S_{ACB}\right) = \left(\left(S_{BAB} + \left(\frac{1}{2} \cdot (S_{BAC} + (-1 \cdot S_{BAB}))\right)\right) + S_{BCB_l}\right)$$, by Lemma 29 (point B_l eliminated)

(A.452) $$\left(\frac{1}{2} \cdot S_{ACB}\right) = \left(\left(0 + \left(\frac{1}{2} \cdot (S_{ACB} + (-1 \cdot 0))\right)\right) + S_{BCB_l}\right)$$, by geometric simplifications

(A.453) $$0 = S_{BCB_l}$$, by algebraic simplifications

(A.454) $$0 = \left(S_{BCB} + \left(\frac{1}{2} \cdot (S_{BCC} + (-1 \cdot S_{BCB}))\right)\right)$$, by Lemma 29 (point B_l eliminated)

(A.455) $$0 = \left(0 + \left(\frac{1}{2} \cdot (0 + (-1 \cdot 0))\right)\right)$$, by geometric simplifications

(A.456) $$0 = 0$$, by algebraic simplifications

Q.E.D.

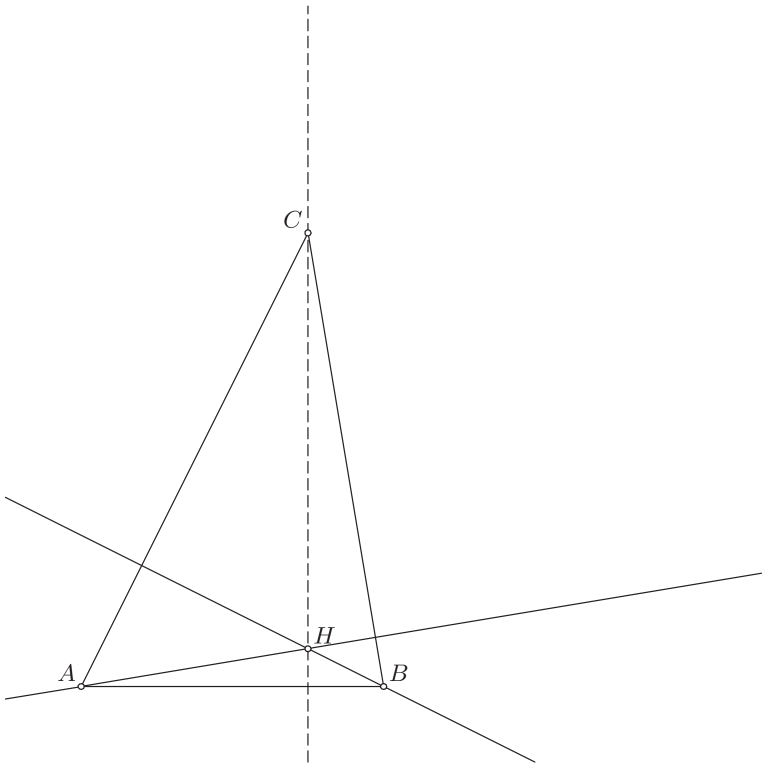
There are no ndg conditions.
Number of elimination proof steps: 5
Number of geometric proof steps: 15
Number of algebraic proof steps: 25
Total number of proof steps: 45
Time spent by the prover: 0.001 seconds

A.8 GEO0008 — Orthocenter Theorem

The Theorem Statement [CGZ93]

Theorem 8 (Orthocenter Theorem) *Given a triangle ABC, the three altitudes are concurrent in a point H.*

The Image – GCLC 5.0



Prover's Code

```
dim 100 100

point A 10 10
point B 50 10
point C 40 70

line a B C
line b A C
line c B A

perp hA A a
perp hB B b

intersec H hA hB
```

```
drawsegment A B
drawsegment A C
drawsegment B C

drawline A H
drawline B H

cmark_lt A
cmark_rt B
cmark_lt C

cmark_rt H

drawdashline C H

prove { equal { pythagoras_difference3 A C H  } { pythagoras_difference3 B C H } }
```

Proved — Proof, made with GCLC, v1.0 Let F^0_{hA} be the foot of the normal from the point A on the line a.

Let F^1_{hB} be the foot of the normal from the point B on the line b.

(A.457) $$P_{ACH} = P_{BCH}$$, by the statement

(A.458) $$\frac{\left(\left(S_{ABF^1_{hB}} \cdot P_{ACF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{ACA}\right)\right)\right)}{S_{ABF^0_{hA}F^1_{hB}}} = P_{BCH}$$, by Lemma 30 (point H eliminated)

(A.459) $$\frac{\left(\left(S_{ABF^1_{hB}} \cdot P_{ACF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{ACA}\right)\right)\right)}{S_{ABF^0_{hA}F^1_{hB}}} = \frac{\left(\left(S_{ABF^1_{hB}} \cdot P_{BCF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{BCA}\right)\right)\right)}{S_{ABF^0_{hA}F^1_{hB}}}$$, by Lemma 30 (point H eliminated)

(A.460) $$\left(\left(S_{ABF^1_{hB}} \cdot P_{ACF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{ACA}\right)\right)\right) = \left(\left(S_{ABF^1_{hB}} \cdot P_{BCF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{BCA}\right)\right)\right)$$, by algebraic simplifications

(A.461) $$\left(\left(\frac{((P_{BAC} \cdot S_{ABC}) + (P_{BCA} \cdot S_{ABA}))}{P_{ACA}} \cdot P_{ACF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{ACA}\right)\right)\right) = \left(\left(S_{ABF^1_{hB}} \cdot P_{BCF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{BCA}\right)\right)\right)$$, by Lemma 31 (point F^1_{hB} eliminated)

$$\text{(A.462)}\quad \left(\left(\frac{((P_{BAC} \cdot S_{ABC}) + (P_{BCA} \cdot 0))}{P_{ACA}} \cdot P_{ACF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{ACA}\right)\right)\right)$$

$$= \left(\left(S_{ABF^1_{hB}} \cdot P_{BCF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{BCA}\right)\right)\right) \quad \text{, by geometric simplifications}$$

$$\text{(A.463)}\quad \frac{\left(\left(P_{BAC} \cdot \left(S_{ABC} \cdot P_{ACF^0_{hA}}\right)\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot (P_{ACA} \cdot P_{ACA})\right)\right)\right)}{P_{ACA}}$$

$$= \left(\left(S_{ABF^1_{hB}} \cdot P_{BCF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{BCA}\right)\right)\right) \quad \text{, by algebraic simplifications}$$

$$\text{(A.464)}\quad \frac{\left(\left(P_{BAC} \cdot \left(S_{ABC} \cdot P_{ACF^0_{hA}}\right)\right) + \left(-1 \cdot \left(\frac{\left(\left(P_{BAC} \cdot S_{F^0_{hA}BC}\right) + \left(P_{BCA} \cdot S_{F^0_{hA}BA}\right)\right)}{P_{ACA}} \cdot (P_{ACA} \cdot P_{ACA})\right)\right)\right)}{P_{ACA}} = \left(\left(S_{ABF^1_{hB}} \cdot P_{BCF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{BCA}\right)\right)\right)$$

, by Lemma 31 (point F^1_{hB} eliminated)

$$\text{(A.465)}\quad \frac{\left(\left(P_{BAC} \cdot \left(S_{ABC} \cdot P_{ACF^0_{hA}}\right)\right) + \left(\left(-1 \cdot \left(P_{ACA} \cdot \left(P_{BAC} \cdot S_{F^0_{hA}BC}\right)\right)\right) + \left(-1 \cdot \left(P_{ACA} \cdot \left(P_{BCA} \cdot S_{F^0_{hA}BA}\right)\right)\right)\right)\right)}{P_{ACA}}$$

$$= \left(\left(S_{ABF^1_{hB}} \cdot P_{BCF^0_{hA}}\right) + \left(-1 \cdot \left(S_{F^0_{hA}BF^1_{hB}} \cdot P_{BCA}\right)\right)\right) \quad \text{, by algebraic simplifications}$$

$$\frac{\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{ACF_{hA}^{0}}\right)\right)+\left(\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot S_{F_{hA}^{0}BC}\right)\right)\right)+\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BCA}\cdot S_{F_{hA}^{0}BA}\right)\right)\right)\right)\right)}{P_{ACA}}$$
$$= \left(\left(\frac{((P_{BAC}\cdot S_{ABC})+(P_{BCA}\cdot S_{ABA}))}{P_{ACA}}\cdot P_{BCF_{hA}^{0}}\right) + \left(-1\cdot\left(S_{F_{hA}^{0}BF_{hB}^{1}}\cdot P_{BCA}\right)\right)\right) \quad \text{, by Lemma 31 (point } F_{hB}^{1} \text{ eliminated)} \tag{A.466}$$

$$\frac{\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{ACF_{hA}^{0}}\right)\right)+\left(\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot S_{F_{hA}^{0}BC}\right)\right)\right)+\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BCA}\cdot S_{F_{hA}^{0}BA}\right)\right)\right)\right)\right)}{P_{ACA}}$$
$$= \left(\left(\frac{((P_{BAC}\cdot S_{ABC})+(P_{BCA}\cdot 0))}{P_{ACA}}\cdot P_{BCF_{hA}^{0}}\right) + \left(-1\cdot\left(S_{F_{hA}^{0}BF_{hB}^{1}}\cdot P_{BCA}\right)\right)\right) \quad \text{, by geometric simplifications} \tag{A.467}$$

$$\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{ACF_{hA}^{0}}\right)\right) + \left(\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot S_{F_{hA}^{0}BC}\right)\right)\right)+\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BCA}\cdot S_{F_{hA}^{0}BA}\right)\right)\right)\right)\right) = \left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{BCF_{hA}^{0}}\right)\right) + \left(-1\cdot\left(S_{F_{hA}^{0}BF_{hB}^{1}}\cdot(P_{BCA}\cdot P_{ACA})\right)\right)\right) \quad \text{, by algebraic simplifications} \tag{A.468}$$

$$\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{ACF_{hA}^{0}}\right)\right) + \left(\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot S_{F_{hA}^{0}BC}\right)\right)\right)+\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BCA}\cdot S_{F_{hA}^{0}BA}\right)\right)\right)\right)\right) = \left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{BCF_{hA}^{0}}\right)\right)+\left(-1\cdot\left(\frac{\left(\left(P_{BAC}\cdot S_{F_{hA}^{0}BC}\right)+\left(P_{BCA}\cdot S_{F_{hA}^{0}BA}\right)\right)}{P_{ACA}}\cdot(P_{BCA}\cdot P_{ACA})\right)\right)\right) \text{, by Lemma 31 (point } F_{hB}^{1} \text{ eliminated)} \tag{A.469}$$

$$\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{ACF^0_{hA}}\right)\right) + \left(\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot S_{F^0_{hA}BC}\right)\right)\right) + \left(-1\cdot\left(P_{ACA}\cdot\left(P_{BCA}\cdot S_{F^0_{hA}BA}\right)\right)\right)\right)\right) = \left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{BCF^0_{hA}}\right)\right) + \left(\left(-1\cdot\left(P_{BCA}\cdot\left(P_{BAC}\cdot S_{F^0_{hA}BC}\right)\right)\right) + \left(-1\cdot\left(P_{BCA}\cdot\left(P_{BCA}\cdot S_{F^0_{hA}BA}\right)\right)\right)\right)\right) \quad \text{, by algebraic simplifications} \tag{A.470}$$

$$\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{ACF^0_{hA}}\right)\right) + \left(\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right) + \left(-1\cdot\left(P_{ACA}\cdot\left(P_{BCA}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right) = \left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{BCF^0_{hA}}\right)\right) + \left(\left(-1\cdot\left(P_{BCA}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right) + \left(-1\cdot\left(P_{BCA}\cdot\left(P_{BCA}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right) \quad \text{, by geometric simplifications} \tag{A.471}$$

$$\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot\frac{((P_{ABC}\cdot P_{ACC}) + (P_{ACB}\cdot P_{ACB}))}{P_{BCB}}\right)\right) + \left(\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right) + \left(-1\cdot\left(P_{ACA}\cdot\left(P_{BCA}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right) = \left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{BCF^0_{hA}}\right)\right) + \left(\left(-1\cdot\left(P_{BCA}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right) + \left(-1\cdot\left(P_{BCA}\cdot\left(P_{BCA}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right) \quad \text{, by Lemma 31 (point } F^0_{hA} \text{ eliminated)} \tag{A.472}$$

$$\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot\frac{((P_{ABC}\cdot 0)+(P_{ACB}\cdot P_{ACB}))}{P_{BCB}}\right)\right)+\left(\left(-1\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right)+\left(-1\cdot\left(P_{ACA}\cdot\left(P_{ACB}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right)=\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{BCF^0_{hA}}\right)\right)+\left(\left(-1\cdot\left(P_{ACB}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right)+\left(-1\cdot\left(P_{ACB}\cdot\left(P_{ACB}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right) \quad \text{, by geometric simplifications} \tag{A.473}$$

$$\frac{\left((P_{BAC}\cdot(S_{ABC}\cdot(P_{ACB}\cdot P_{ACB})))+\left(\left(-1\cdot\left(P_{BCB}\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right)\right)+\left(-1\cdot\left(P_{BCB}\cdot\left(P_{ACA}\cdot\left(P_{ACB}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right)\right)}{P_{BCB}}=\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{BCF^0_{hA}}\right)\right)+\left(\left(-1\cdot\left(P_{ACB}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right)+\left(-1\cdot\left(P_{ACB}\cdot\left(P_{ACB}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right) \quad \text{, by algebraic simplifications} \tag{A.474}$$

$$\frac{\left((P_{BAC}\cdot(S_{ABC}\cdot(P_{ACB}\cdot P_{ACB})))+\left(\left(-1\cdot\left(P_{BCB}\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot\frac{((P_{ABC}\cdot S_{BCC})+(P_{ACB}\cdot S_{BCB}))}{P_{BCB}}\right)\right)\right)\right)+\left(-1\cdot\left(P_{BCB}\cdot\left(P_{ACA}\cdot\left(P_{ACB}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right)\right)}{P_{BCB}}=\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{BCF^0_{hA}}\right)\right)+\left(\left(-1\cdot\left(P_{ACB}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right)+\left(-1\cdot\left(P_{ACB}\cdot\left(P_{ACB}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right) \quad \text{, by Lemma 31 (point } F^0_{hA} \text{ eliminated)} \tag{A.475}$$

$$\frac{\left((P_{BAC}\cdot(S_{ABC}\cdot(P_{ACB}\cdot P_{ACB})))+\left(\left(-1\cdot\left(P_{BCB}\cdot\left(P_{ACA}\cdot\left(P_{BAC}\cdot\frac{((P_{ABC}\cdot 0)+(P_{ACB}\cdot 0))}{P_{BCB}}\right)\right)\right)\right)+\left(-1\cdot\left(P_{BCB}\cdot\left(P_{ACA}\cdot\left(P_{ACB}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right)\right)}{P_{BCB}}=\left(\left(P_{BAC}\cdot\left(S_{ABC}\cdot P_{BCF^0_{hA}}\right)\right)+\left(\left(-1\cdot\left(P_{ACB}\cdot\left(P_{BAC}\cdot S_{BCF^0_{hA}}\right)\right)\right)+\left(-1\cdot\left(P_{ACB}\cdot\left(P_{ACB}\cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right) \quad \text{, by geometric simplifications} \tag{A.476}$$

(A.477)

$$\frac{\left((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + \left(-1 \cdot \left(P_{BCB} \cdot \left(P_{ACA} \cdot \left(P_{ACB} \cdot S_{BAF_{hA}^0}\right)\right)\right)\right)\right)}{P_{BCB}} = \left(\left(P_{BAC} \cdot \left(S_{ABC} \cdot P_{BCF_{hA}^0}\right)\right) + \left(\left(-1 \cdot \left(P_{ACB} \cdot \left(P_{BAC} \cdot S_{BCF_{hA}^0}\right)\right)\right) + \left(-1 \cdot \left(P_{ACB} \cdot \left(P_{ACB} \cdot S_{BAF_{hA}^0}\right)\right)\right)\right)\right), \quad \text{by algebraic simplifications}$$

(A.478)

$$\frac{\left((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + \left(-1 \cdot \left(P_{BCB} \cdot \left(P_{ACA} \cdot \left(P_{ACB} \cdot \frac{((P_{ABC} \cdot S_{BAC}) + (P_{ACB} \cdot S_{BAB}))}{P_{BCB}}\right)\right)\right)\right)\right)}{P_{BCB}} = \left(\left(P_{BAC} \cdot \left(S_{ABC} \cdot P_{BCF_{hA}^0}\right)\right) + \left(\left(-1 \cdot \left(P_{ACB} \cdot \left(P_{BAC} \cdot S_{BCF_{hA}^0}\right)\right)\right) + \left(-1 \cdot \left(P_{ACB} \cdot \left(P_{ACB} \cdot S_{BAF_{hA}^0}\right)\right)\right)\right)\right), \quad \text{by Lemma 31 (point } F_{hA}^0 \text{ eliminated)}$$

(A.479)

$$\frac{\left((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + \left(-1 \cdot \left(P_{BCB} \cdot \left(P_{ACA} \cdot \left(P_{ACB} \cdot \frac{((P_{ABC} \cdot (-1 \cdot S_{ABC})) + (P_{ACB} \cdot 0))}{P_{BCB}}\right)\right)\right)\right)\right)}{P_{BCB}} = \left(\left(P_{BAC} \cdot \left(S_{ABC} \cdot P_{BCF_{hA}^0}\right)\right) + \left(\left(-1 \cdot \left(P_{ACB} \cdot \left(P_{BAC} \cdot S_{BCF_{hA}^0}\right)\right)\right) + \left(-1 \cdot \left(P_{ACB} \cdot \left(P_{ACB} \cdot S_{BAF_{hA}^0}\right)\right)\right)\right)\right), \quad \text{by geometric simplifications}$$

(A.480)

$$\frac{((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + (P_{ACA} \cdot (P_{ACB} \cdot (P_{ABC} \cdot S_{ABC}))))}{P_{BCB}} = \left(\left(P_{BAC} \cdot \left(S_{ABC} \cdot P_{BCF_{hA}^0}\right)\right) + \left(\left(-1 \cdot \left(P_{ACB} \cdot \left(P_{BAC} \cdot S_{BCF_{hA}^0}\right)\right)\right) + \left(-1 \cdot \left(P_{ACB} \cdot \left(P_{ACB} \cdot S_{BAF_{hA}^0}\right)\right)\right)\right)\right), \quad \text{by algebraic simplifications}$$

(A.481)

$$\frac{((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + (P_{ACA} \cdot (P_{ACB} \cdot (P_{ABC} \cdot S_{ABC}))))}{P_{BCB}} = \left(\left(P_{BAC} \cdot \left(S_{ABC} \cdot \frac{((P_{ABC} \cdot P_{BCC}) + (P_{ACB} \cdot P_{BCB}))}{P_{BCB}}\right)\right) + \left(\left(-1 \cdot \left(P_{ACB} \cdot \left(P_{BAC} \cdot S_{BCF^0_{hA}}\right)\right)\right) + \left(-1 \cdot \left(P_{ACB} \cdot \left(P_{ACB} \cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right)$$

, by Lemma 31 (point F^0_{hA} eliminated)

(A.482)

$$\frac{((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + (P_{ACA} \cdot (P_{ACB} \cdot (P_{ABC} \cdot S_{ABC}))))}{P_{BCB}} = \left(\left(P_{BAC} \cdot \left(S_{ABC} \cdot \frac{((P_{ABC} \cdot 0) + (P_{ACB} \cdot P_{BCB}))}{P_{BCB}}\right)\right) + \left(\left(-1 \cdot \left(P_{ACB} \cdot \left(P_{BAC} \cdot S_{BCF^0_{hA}}\right)\right)\right) + \left(-1 \cdot \left(P_{ACB} \cdot \left(P_{ACB} \cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right)$$

, by geometric simplifications

(A.483)

$$\frac{((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + (P_{ACA} \cdot (P_{ACB} \cdot (P_{ABC} \cdot S_{ABC}))))}{P_{BCB}} = \left((P_{BAC} \cdot (S_{ABC} \cdot P_{ACB})) + \left(\left(-1 \cdot \left(P_{ACB} \cdot \left(P_{BAC} \cdot S_{BCF^0_{hA}}\right)\right)\right) + \left(-1 \cdot \left(P_{ACB} \cdot \left(P_{ACB} \cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right)$$

, by algebraic simplifications

(A.484)

$$\frac{((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + (P_{ACA} \cdot (P_{ACB} \cdot (P_{ABC} \cdot S_{ABC}))))}{P_{BCB}} = \left((P_{BAC} \cdot (S_{ABC} \cdot P_{ACB})) + \left(\left(-1 \cdot \left(P_{ACB} \cdot \left(P_{BAC} \cdot \frac{((P_{ABC} \cdot S_{BCC}) + (P_{ACB} \cdot S_{BCB}))}{P_{BCB}}\right)\right)\right) + \left(-1 \cdot \left(P_{ACB} \cdot \left(P_{ACB} \cdot S_{BAF^0_{hA}}\right)\right)\right)\right)\right)$$

, by Lemma 31 (point F^0_{hA} eliminated)

$$\frac{((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + (P_{ACA} \cdot (P_{ACB} \cdot (P_{ABC} \cdot S_{ABC}))))}{P_{BCB}} = \Bigg((P_{BAC} \cdot (S_{ABC} \cdot P_{ACB})) + \Bigg(\Bigg(-1 \cdot \Bigg(P_{ACB} \cdot \Bigg(P_{BAC} \cdot \frac{((P_{ABC} \cdot 0) + (P_{ACB} \cdot 0))}{P_{BCB}} \Bigg) \Bigg) \Bigg) + \Big(-1 \cdot \Big(P_{ACB} \cdot \Big(P_{ACB} \cdot S_{BAF^0_{hA}} \Big) \Big) \Big) \Bigg) \Bigg) \quad \text{, by geometric simplifications} \tag{A.485}$$

$$\frac{((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + (P_{ACA} \cdot (P_{ACB} \cdot (P_{ABC} \cdot S_{ABC}))))}{P_{BCB}} = \Big((P_{BAC} \cdot (S_{ABC} \cdot P_{ACB})) + \Big(-1 \cdot \Big(P_{ACB} \cdot \Big(P_{ACB} \cdot S_{BAF^0_{hA}} \Big) \Big) \Big) \Big) \quad \text{, by algebraic simplifications} \tag{A.486}$$

$$\frac{((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + (P_{ACA} \cdot (P_{ACB} \cdot (P_{ABC} \cdot S_{ABC}))))}{P_{BCB}} = \Bigg((P_{BAC} \cdot (S_{ABC} \cdot P_{ACB})) + \Bigg(-1 \cdot \Bigg(P_{ACB} \cdot \Bigg(P_{ACB} \cdot \frac{((P_{ABC} \cdot S_{BAC}) + (P_{ACB} \cdot S_{BAB}))}{P_{BCB}} \Bigg) \Bigg) \Bigg) \Bigg) \quad \text{, by Lemma 31 (point } F^0_{hA} \text{ eliminated)} \tag{A.487}$$

$$\frac{((P_{BAC} \cdot (S_{ABC} \cdot (P_{ACB} \cdot P_{ACB}))) + (P_{ACA} \cdot (P_{ACB} \cdot (P_{ABC} \cdot S_{ABC}))))}{P_{BCB}} = \Bigg((P_{BAC} \cdot (S_{ABC} \cdot P_{ACB})) + \Bigg(-1 \cdot \Bigg(P_{ACB} \cdot \Bigg(P_{ACB} \cdot \frac{((P_{ABC} \cdot (-1 \cdot S_{ABC})) + (P_{ACB} \cdot 0))}{P_{BCB}} \Bigg) \Bigg) \Bigg) \Bigg) \quad \text{, by geometric simplifications} \tag{A.488}$$

$$((P_{BAC} \cdot P_{ACB}) + (P_{ACA} \cdot P_{ABC})) = ((P_{BAC} \cdot P_{BCB}) + (P_{ACB} \cdot P_{ABC})) \quad \text{, by algebraic simplifications} \tag{A.489}$$

$$
\begin{aligned}
&(((((BA \cdot BA) + (CA \cdot CA)) + (-1 \cdot (BC \cdot BC))) \cdot (((CA \cdot CA) + (BC \cdot BC)) + (-1 \cdot (BA \cdot BA)))) \\
&+((2 \cdot (CA \cdot CA)) \cdot (((BA \cdot BA)+(BC \cdot BC))+(-1 \cdot (CA \cdot CA))))) = (((((BA \cdot BA)+(CA \cdot CA))+(-1 \cdot (BC \cdot BC))) \cdot (2 \cdot (BC \cdot BC))) \\
&+ ((((CA \cdot CA) + (BC \cdot BC)) + (-1 \cdot (BA \cdot BA))) \\
&\cdot (((BA \cdot BA) + (BC \cdot BC)) + (-1 \cdot (CA \cdot CA))))) \quad \text{, by geometric simplifications}
\end{aligned}
\tag{A.490}
$$

$$
0 = 0 \quad \text{, by algebraic simplifications} \tag{A.491}
$$

Q.E.D.

NDG conditions are:

$S_{ABC} \neq 0$ i.e., points A, B and C are not collinear (foot is not the point itself; construction based assumption)

$S_{BAC} \neq 0$ i.e., points B, A and C are not collinear (foot is not the point itself; construction based assumption)

$S_{ABF^1_{hB}} \neq S_{F^0_{hA}BF^1_{hB}}$ i.e., lines AF^0_{hA} and BF^1_{hB} are not parallel (construction based assumption)

$P_{ACB} \neq 0$ i.e., angle ACB is not right angle (cancellation assumption)

$CA \neq 0$ (cancellation assumption)

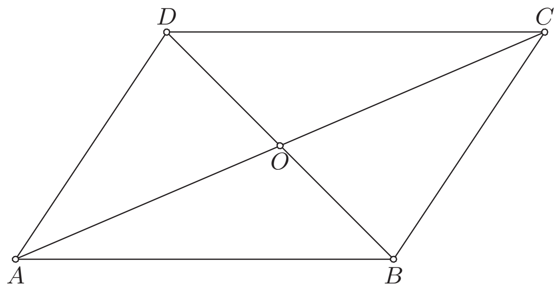
Number of elimination proof steps: 12
Number of geometric proof steps: 51
Number of algebraic proof steps: 212
Total number of proof steps: 275
Time spent by the prover: 0.019 seconds

A.9 GEO0009 — Midpoint of a Parallelogram

The Theorem Statement [ZCG95]

Theorem 9 *Let O be the intersection of the two diagonals AC and BD of a parallelogram $ABCD$. Then O is the midpoint of AC.*

The Image – GCLC 5.0



Prover's Code

```
point A 20 10
point B 70 10
point C 90 40

line ab A B
parallel cd C ab

line bc B C
parallel ad A bc

intersec D ad cd

line a A C
line b D B
```

```
intersec O a b

drawsegment A B
drawsegment A D
drawsegment D C
drawsegment B C
drawsegment A C
drawsegment B D


cmark_b A
cmark_b B
cmark_t C
cmark_t D
cmark_b O

prove { equal { sratio A O O C } { 1 }  }
```

Proved — Proof, made with GCLC, v1.0 Let P^0_{cd} be the point such that lines $P^0_{cd}C$ and AB are parallel (and `PRATIO` P^0_{cd} C A B 1).

Let P^1_{ad} be the point such that lines $P^1_{ad}A$ and BC are parallel (and `PRATIO` P^1_{ad} A B C 1).

(A.492) $$\frac{\overrightarrow{AO}}{\overrightarrow{OC}} = 1$$, by the statement

(A.493) $$\left(-1 \cdot \frac{\overrightarrow{AO}}{\overrightarrow{CO}}\right) = 1$$, by geometric simplifications

(A.494) $$\left(-1 \cdot \frac{S_{ADB}}{S_{CDB}}\right) = 1$$, by Lemma 8 (point O eliminated)

(A.495) $$\frac{(-1 \cdot S_{ADB})}{S_{CDB}} = 1$$, by algebraic simplifications

(A.496) $$\frac{(-1 \cdot S_{BAD})}{S_{BCD}} = 1$$, by geometric simplifications

(A.497) $$\frac{\left(-1 \cdot \frac{\left(\left(S_{ACP^0_{cd}} \cdot S_{BAP^1_{ad}}\right) + \left(-1 \cdot \left(S_{P^1_{ad}CP^0_{cd}} \cdot S_{BAA}\right)\right)\right)}{S_{ACP^1_{ad}P^0_{cd}}}\right)}{S_{BCD}} = 1$$, by Lemma 30 (point D eliminated)

(A.498) $$\frac{\left(-1 \cdot \frac{\left(\left(S_{ACP^0_{cd}} \cdot S_{BAP^1_{ad}}\right) + \left(-1 \cdot \left(S_{P^1_{ad}CP^0_{cd}} \cdot 0\right)\right)\right)}{S_{ACP^1_{ad}P^0_{cd}}}\right)}{S_{BCD}} = 1$$, by geometric simplifications

$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAP_{ad}^{1}}\right)\right)}{\left(S_{ACP_{ad}^{1}P_{cd}^{0}} \cdot S_{BCD}\right)} = 1 \tag{A.499}$$
, by algebraic simplifications

$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAP_{ad}^{1}}\right)\right)}{\left(S_{ACP_{ad}^{1}P_{cd}^{0}} \cdot \frac{\left(\left(S_{ACP_{cd}^{0}} \cdot S_{BCP_{ad}^{1}}\right) + \left(-1 \cdot \left(S_{P_{ad}^{1}CP_{cd}^{0}} \cdot S_{BCA}\right)\right)\right)}{S_{ACP_{ad}^{1}P_{cd}^{0}}}\right)} = 1 \tag{A.500}$$
, by Lemma 30 (point D eliminated)

$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAP_{ad}^{1}}\right)\right)}{\left(\left(S_{ACP_{cd}^{0}} \cdot S_{BCP_{ad}^{1}}\right) + \left(-1 \cdot \left(S_{P_{ad}^{1}CP_{cd}^{0}} \cdot S_{BCA}\right)\right)\right)} = 1 \tag{A.501}$$
, by algebraic simplifications

$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAP_{ad}^{1}}\right)\right)}{\left(\left(S_{ACP_{cd}^{0}} \cdot S_{BCP_{ad}^{1}}\right) + \left(-1 \cdot \left(S_{CP_{cd}^{0}P_{ad}^{1}} \cdot S_{BCA}\right)\right)\right)} = 1 \tag{A.502}$$
, by geometric simplifications

$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot (S_{BAA} + (1 \cdot (S_{BAC} + (-1 \cdot S_{BAB}))))\right)\right)}{\left(\left(S_{ACP_{cd}^{0}} \cdot S_{BCP_{ad}^{1}}\right) + \left(-1 \cdot \left(S_{CP_{cd}^{0}P_{ad}^{1}} \cdot S_{BCA}\right)\right)\right)} = 1 \tag{A.503}$$
, by Lemma 29 (point P_{ad}^{1} eliminated)

$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot (0 + (1 \cdot (S_{BAC} + (-1 \cdot 0))))\right)\right)}{\left(\left(S_{ACP_{cd}^{0}} \cdot S_{BCP_{ad}^{1}}\right) + \left(-1 \cdot \left(S_{CP_{cd}^{0}P_{ad}^{1}} \cdot (-1 \cdot S_{BAC})\right)\right)\right)} = 1 \tag{A.504}$$
, by geometric simplifications

(A.505)
$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAC}\right)\right)}{\left(\left(S_{ACP_{cd}^{0}} \cdot S_{BCP_{ad}^{1}}\right) + \left(S_{CP_{cd}^{0}P_{ad}^{1}} \cdot S_{BAC}\right)\right)} = 1 \quad \text{, by algebraic simplifications}$$

(A.506)
$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAC}\right)\right)}{\left(\left(S_{ACP_{cd}^{0}} \cdot (S_{BCA} + (1 \cdot (S_{BCC} + (-1 \cdot S_{BCB}))))\right) + \left(S_{CP_{cd}^{0}P_{ad}^{1}} \cdot S_{BAC}\right)\right)} = 1, \quad \text{by Lemma 29 (point } P_{ad}^{1} \text{ eliminated)}$$

(A.507)
$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAC}\right)\right)}{\left(\left(S_{ACP_{cd}^{0}} \cdot ((-1 \cdot S_{BAC}) + (1 \cdot (0 + (-1 \cdot 0))))\right) + \left(S_{CP_{cd}^{0}P_{ad}^{1}} \cdot S_{BAC}\right)\right)} = 1 \quad \text{, by geometric simplifications}$$

(A.508)
$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAC}\right)\right)}{\left(\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAC}\right)\right) + \left(S_{CP_{cd}^{0}P_{ad}^{1}} \cdot S_{BAC}\right)\right)} = 1 \quad \text{, by algebraic simplifications}$$

(A.509)
$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAC}\right)\right)}{\left(\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAC}\right)\right) + \left(\left(S_{CP_{cd}^{0}A} + \left(1 \cdot \left(S_{CP_{cd}^{0}C} + \left(-1 \cdot S_{CP_{cd}^{0}B}\right)\right)\right)\right) \cdot S_{BAC}\right)\right)}$$
$$= 1 \quad \text{, by Lemma 29 (point } P_{ad}^{1} \text{ eliminated)}$$

(A.510)
$$\frac{\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAC}\right)\right)}{\left(\left(-1 \cdot \left(S_{ACP_{cd}^{0}} \cdot S_{BAC}\right)\right) + \left(\left(S_{ACP_{cd}^{0}} + \left(1 \cdot \left(0 + \left(-1 \cdot S_{CP_{cd}^{0}B}\right)\right)\right)\right) \cdot S_{BAC}\right)\right)} = 1 \quad \text{, by geometric simplifications}$$

(A.511) $$\frac{S_{ACP^0_{cd}}}{S_{CP^0_{cd}B}} = 1$$, by algebraic simplifications

(A.512) $$\frac{S_{ACP^0_{cd}}}{S_{BCP^0_{cd}}} = 1$$, by geometric simplifications

(A.513) $$\frac{(S_{ACC} + (1 \cdot (S_{ACB} + (-1 \cdot S_{ACA}))))}{S_{BCP^0_{cd}}} = 1$$, by Lemma 29 (point P^0_{cd} eliminated)

(A.514) $$\frac{(0 + (1 \cdot (S_{ACB} + (-1 \cdot 0))))}{S_{BCP^0_{cd}}} = 1$$, by geometric simplifications

(A.515) $$\frac{S_{ACB}}{S_{BCP^0_{cd}}} = 1$$, by algebraic simplifications

(A.516) $$\frac{S_{ACB}}{(S_{BCC} + (1 \cdot (S_{BCB} + (-1 \cdot S_{BCA}))))} = 1$$, by Lemma 29 (point P^0_{cd} eliminated)

(A.517) $$\frac{S_{ACB}}{(0 + (1 \cdot (0 + (-1 \cdot (-1 \cdot S_{ACB})))))} = 1$$, by geometric simplifications

(A.518) $$1 = 1$$, by algebraic simplifications

Q.E.D.

NDG conditions are:

$S_{ACP^0_{cd}} \neq S_{P^1_{ad}CP^0_{cd}}$ i.e., lines AP^1_{ad} and CP^0_{cd} are not parallel (construction based assumption)

$S_{ADB} \neq S_{CDB}$ i.e., lines AC and DB are not parallel (construction based assumption)

$P_{OCO} \neq 0$ i.e., points O and C are not identical (conjecture based assumption)

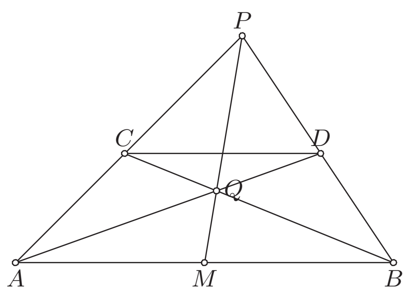
Number of elimination proof steps: 8
Number of geometric proof steps: 19
Number of algebraic proof steps: 45
Total number of proof steps: 72
Time spent by the prover: 0.002 seconds

A.10 GEO0010 — The fundamental principle of affine geometry

The Theorem Statement [ZCG95]

Theorem 10 (The Fundamental Principle of Affine Geometry) *Let A, B, and P be three points on a plane, and C be a point on line PA. The line passing through C and parallel to AB intersects PB in D. Q is the intersection of AD and BC. M is the intersection of AB and PQ. Show that M is the midpoint of AB.*

The Image – GCLC 5.0



Prover's Code

```
point A 20 10
point B 70 10
point P 50 40

line ab A B
%line pa P A
line pb P B

online C P A
parallel pab C ab
```

```
intersec D pab pb

line ad A D
line bc B C
intersec Q ad bc

line pq P Q
intersec M ab pq

cmark_b A
cmark_b B
cmark_t P
cmark_t C
cmark_t D
cmark_r Q
cmark_b M

drawsegment A B
drawsegment A P
drawsegment A D
drawsegment B C
drawsegment B P
drawsegment C D
drawsegment P M

prove { equal { sratio A M M B } { 1 }  }
```

Proved — Proof, made with GCLC, v1.0 Let r_0 be the number such that `PRATIO` C P P A r_0 (for a concrete example r_0=-0.553045).

Let P^1_{pab} be the point such that lines $P^1_{pab}C$ and AB are parallel (and `PRATIO` P^1_{pab} C A B 1).

(A.519) $$\frac{\overrightarrow{AM}}{\overrightarrow{MB}} = 1$$, by the statement

(A.520) $$\left(-1 \cdot \frac{\overrightarrow{AM}}{\overrightarrow{BM}}\right) = 1$$, by geometric simplifications

(A.521) $$\left(-1 \cdot \frac{S_{APQ}}{S_{BPQ}}\right) = 1$$, by Lemma 8 (point M eliminated)

(A.522) $$\frac{(-1 \cdot S_{APQ})}{S_{BPQ}} = 1$$, by algebraic simplifications

(A.523) $$\frac{\left(-1 \cdot \frac{((S_{ABC} \cdot S_{APD})+(-1 \cdot (S_{DBC} \cdot S_{APA})))}{S_{ABDC}}\right)}{S_{BPQ}} = 1$$, by Lemma 30 (point Q eliminated)

(A.524) $$\frac{\left(-1 \cdot \frac{((S_{ABC} \cdot S_{APD})+(-1 \cdot (S_{DBC} \cdot 0)))}{S_{ABDC}}\right)}{S_{BPQ}} = 1$$, by geometric simplifications

(A.525) $$\frac{(-1 \cdot (S_{ABC} \cdot S_{APD}))}{(S_{ABDC} \cdot S_{BPQ})} = 1$$, by algebraic simplifications

(A.526) $$\frac{(-1 \cdot (S_{ABC} \cdot S_{APD}))}{\left(S_{ABDC} \cdot \frac{((S_{ABC} \cdot S_{BPD})+(-1 \cdot (S_{DBC} \cdot S_{BPA})))}{S_{ABDC}}\right)} = 1$$, by Lemma 30 (point Q eliminated)

$$\frac{\left(-1\cdot\left(S_{ABC}\cdot S_{APD}\right)\right)}{\left(\left(S_{ABC}\cdot S_{BPD}\right)+\left(-1\cdot\left(S_{DBC}\cdot S_{BPA}\right)\right)\right)}=1 \qquad \text{, by algebraic simplifications} \tag{A.527}$$

$$\frac{\left(-1\cdot\left(S_{ABC}\cdot S_{APD}\right)\right)}{\left(\left(S_{ABC}\cdot S_{BPD}\right)+\left(-1\cdot\left(S_{BCD}\cdot S_{BPA}\right)\right)\right)}=1 \qquad \text{, by geometric simplifications} \tag{A.528}$$

$$\frac{\left(-1\cdot\left(S_{ABC}\cdot\frac{\left(\left(S_{CPB}\cdot S_{APP^1_{pab}}\right)+\left(-1\cdot\left(S_{P^1_{pab}PB}\cdot S_{APC}\right)\right)\right)}{S_{CPP^1_{pab}B}}\right)\right)}{\left(\left(S_{ABC}\cdot S_{BPD}\right)+\left(-1\cdot\left(S_{BCD}\cdot S_{BPA}\right)\right)\right)}=1 \qquad \text{, by Lemma 30 (point } D \text{ eliminated)} \tag{A.529}$$

$$\frac{\left(\left(-1\cdot\left(S_{ABC}\cdot\left(S_{CPB}\cdot S_{APP^1_{pab}}\right)\right)\right)+\left(S_{ABC}\cdot\left(S_{P^1_{pab}PB}\cdot S_{APC}\right)\right)\right)}{\left(\left(S_{CPP^1_{pab}B}\cdot\left(S_{ABC}\cdot S_{BPD}\right)\right)+\left(-1\cdot\left(S_{CPP^1_{pab}B}\cdot\left(S_{BCD}\cdot S_{BPA}\right)\right)\right)\right)}=1 \qquad \text{, by algebraic simplifications} \tag{A.530}$$

$$\frac{\left(\left(-1\cdot\left(S_{ABC}\cdot\left(S_{CPB}\cdot S_{APP^1_{pab}}\right)\right)\right)+\left(S_{ABC}\cdot\left(S_{P^1_{pab}PB}\cdot S_{APC}\right)\right)\right)}{\left(\left(S_{CPP^1_{pab}B}\cdot\left(S_{ABC}\cdot\frac{\left(\left(S_{CPB}\cdot S_{BPP^1_{pab}}\right)+\left(-1\cdot\left(S_{P^1_{pab}PB}\cdot S_{BPC}\right)\right)\right)}{S_{CPP^1_{pab}B}}\right)\right)+\left(-1\cdot\left(S_{CPP^1_{pab}B}\cdot\left(S_{BCD}\cdot S_{BPA}\right)\right)\right)\right)}$$
$$=1 \qquad \text{, by Lemma 30 (point } D \text{ eliminated)} \tag{A.531}$$

(A.532)

$$\frac{\left(\left(-1\cdot\left(S_{ABC}\cdot\left(S_{CPB}\cdot S_{APP^1_{pab}}\right)\right)\right)+\left(S_{ABC}\cdot\left(S_{P^1_{pab}PB}\cdot S_{APC}\right)\right)\right)}{\left(\left(S_{CPP^1_{pab}B}\cdot\left(S_{ABC}\cdot\frac{\left(\left(S_{CPB}\cdot\left(-1\cdot S_{P^1_{pab}PB}\right)\right)+\left(-1\cdot\left(S_{P^1_{pab}PB}\cdot\left(-1\cdot S_{CPB}\right)\right)\right)\right)}{S_{CPP^1_{pab}B}}\right)\right)+\left(-1\cdot\left(S_{CPP^1_{pab}B}\cdot\left(S_{BCD}\cdot S_{BPA}\right)\right)\right)\right)}$$
$$=1 \qquad \text{, by geometric simplifications}$$

(A.533)
$$\frac{\left(\left(-1 \cdot \left(S_{ABC} \cdot \left(S_{CPB} \cdot S_{APP^1_{pab}}\right)\right)\right) + \left(S_{ABC} \cdot \left(S_{P^1_{pab}PB} \cdot S_{APC}\right)\right)\right)}{\left(-1 \cdot \left(S_{CPP^1_{pab}B} \cdot (S_{BCD} \cdot S_{BPA})\right)\right)} = 1$$
, by algebraic simplifications

(A.534)
$$\frac{\left(\left(-1 \cdot \left(S_{ABC} \cdot \left(S_{CPB} \cdot S_{APP^1_{pab}}\right)\right)\right) + \left(S_{ABC} \cdot \left(S_{P^1_{pab}PB} \cdot S_{APC}\right)\right)\right)}{\left(-1 \cdot \left(S_{CPP^1_{pab}B} \cdot \left(\frac{\left(\left(S_{CPB} \cdot S_{BCP^1_{pab}}\right) + \left(-1 \cdot \left(S_{P^1_{pab}PB} \cdot S_{BCC}\right)\right)\right)}{S_{CPP^1_{pab}B}} \cdot S_{BPA}\right)\right)\right)} = 1$$
, by Lemma 30 (point D eliminated)

(A.535)
$$\frac{\left(\left(-1 \cdot \left(S_{ABC} \cdot \left(S_{CPB} \cdot S_{APP^1_{pab}}\right)\right)\right) + \left(S_{ABC} \cdot \left(S_{P^1_{pab}PB} \cdot S_{APC}\right)\right)\right)}{\left(-1 \cdot \left(S_{CPP^1_{pab}B} \cdot \left(\frac{\left(\left(S_{CPB} \cdot S_{BCP^1_{pab}}\right) + \left(-1 \cdot \left(S_{P^1_{pab}PB} \cdot 0\right)\right)\right)}{S_{CPP^1_{pab}B}} \cdot S_{BPA}\right)\right)\right)} = 1$$
, by geometric simplifications

(A.536)
$$\frac{\left(\left(-1 \cdot \left(S_{ABC} \cdot \left(S_{CPB} \cdot S_{APP^1_{pab}}\right)\right)\right) + \left(S_{ABC} \cdot \left(S_{P^1_{pab}PB} \cdot S_{APC}\right)\right)\right)}{\left(-1 \cdot \left(S_{CPB} \cdot \left(S_{BCP^1_{pab}} \cdot S_{BPA}\right)\right)\right)} = 1$$
, by algebraic simplifications

(A.537)
$$\frac{\left(\left(-1 \cdot \left(S_{ABC} \cdot \left(S_{CPB} \cdot S_{APP^1_{pab}}\right)\right)\right) + \left(S_{ABC} \cdot \left(S_{PBP^1_{pab}} \cdot S_{APC}\right)\right)\right)}{\left(-1 \cdot \left(S_{CPB} \cdot \left(S_{BCP^1_{pab}} \cdot S_{BPA}\right)\right)\right)} = 1$$
, by geometric simplifications

(A.538)
$$\frac{\left((-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot (S_{APC} + (1 \cdot (S_{APB} + (-1 \cdot S_{APA}))))))) + \left(S_{ABC} \cdot \left(S_{PBP^1_{pab}} \cdot S_{APC}\right)\right)\right)}{\left(-1 \cdot \left(S_{CPB} \cdot \left(S_{BCP^1_{pab}} \cdot S_{BPA}\right)\right)\right)}$$
$$= 1$$
, by Lemma 29 (point P^1_{pab} eliminated)

$$\frac{\left((-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot (S_{APC} + (1 \cdot (S_{APB} + (-1 \cdot 0))))))) + \left(S_{ABC} \cdot \left(S_{PBP^1_{pab}} \cdot S_{APC}\right)\right)\right)}{\left(-1 \cdot \left(S_{CPB} \cdot \left(S_{BCP^1_{pab}} \cdot (-1 \cdot S_{APB})\right)\right)\right)} = 1 \quad \text{, by geometric simplifications} \tag{A.539}$$

$$\frac{\left(((-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot S_{APC}))) + (-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot S_{APB})))) + \left(S_{ABC} \cdot \left(S_{PBP^1_{pab}} \cdot S_{APC}\right)\right)\right)}{\left(S_{CPB} \cdot \left(S_{BCP^1_{pab}} \cdot S_{APB}\right)\right)} = 1 \quad \text{, by algebraic simplifications} \tag{A.540}$$

$$\frac{(((-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot S_{APC}))) + (-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot S_{APB})))) + (S_{ABC} \cdot ((S_{PBC} + (1 \cdot (S_{PBB} + (-1 \cdot S_{PBA})))) \cdot S_{APC})))}{\left(S_{CPB} \cdot \left(S_{BCP^1_{pab}} \cdot S_{APB}\right)\right)} = 1 \quad \text{, by Lemma 29 (point } P^1_{pab} \text{ eliminated)} \tag{A.541}$$

$$\frac{(((-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot S_{APC}))) + (-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot S_{APB})))) + (S_{ABC} \cdot ((S_{CPB} + (1 \cdot (0 + (-1 \cdot S_{APB})))) \cdot S_{APC})))}{\left(S_{CPB} \cdot \left(S_{BCP^1_{pab}} \cdot S_{APB}\right)\right)} = 1 \quad \text{, by geometric simplifications} \tag{A.542}$$

$$\frac{((-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot S_{APB}))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot S_{APB}))))}{\left(S_{CPB} \cdot \left(S_{BCP^1_{pab}} \cdot S_{APB}\right)\right)} = 1 \quad \text{, by algebraic simplifications} \tag{A.543}$$

$$\frac{((-1 \cdot (S_{ABC} \cdot (S_{CPB} \cdot S_{APB}))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot S_{APB}))))}{(S_{CPB} \cdot ((S_{BCC} + (1 \cdot (S_{BCB} + (-1 \cdot S_{BCA})))) \cdot S_{APB}))} = 1 \quad \text{, by Lemma 29 (point } P^1_{pab} \text{ eliminated)} \tag{A.544}$$

$$\frac{((-1 \cdot (S_{ABC} \cdot (S_{PBC} \cdot S_{APB}))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot S_{APB}))))}{(S_{PBC} \cdot ((0 + (1 \cdot (0 + (-1 \cdot S_{ABC})))) \cdot S_{APB}))} = 1 \quad \text{, by geometric simplifications} \tag{A.545}$$

$$\frac{((-1 \cdot (S_{ABC} \cdot (S_{PBC} \cdot S_{APB}))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot S_{APB}))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{APB})))} = 1 \quad \text{, by algebraic simplifications} \tag{A.546}$$

$$\frac{((-1 \cdot ((S_{ABP} + (r_0 \cdot (S_{ABA} + (-1 \cdot S_{ABP})))) \cdot (S_{PBC} \cdot S_{APB}))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot S_{APB}))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{APB})))} = 1 \quad \text{, by Lemma 29 (point } C \text{ eliminated)} \tag{A.547}$$

$$\frac{((-1 \cdot ((S_{ABP} + (r_0 \cdot (0 + (-1 \cdot S_{ABP})))) \cdot (S_{PBC} \cdot (-1 \cdot S_{ABP})))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot (-1 \cdot S_{ABP})))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot (-1 \cdot S_{ABP}))))} = 1 \quad \text{, by geometric simplifications} \tag{A.548}$$

$$\frac{(((S_{PBC} \cdot (S_{ABP} \cdot S_{ABP})) + (-1 \cdot (S_{PBC} \cdot (S_{ABP} \cdot (r_0 \cdot S_{ABP}))))) + (S_{ABC} \cdot (S_{APC} \cdot S_{ABP})))}{(S_{PBC} \cdot (S_{ABC} \cdot S_{ABP}))} = 1 \quad \text{, by algebraic simplifications} \tag{A.549}$$

$$\frac{((((S_{PBP} + (r_0 \cdot (S_{PBA} + (-1 \cdot S_{PBP})))) \cdot (S_{ABP} \cdot S_{ABP})) + (-1 \cdot (S_{PBC} \cdot (S_{ABP} \cdot (r_0 \cdot S_{ABP}))))) + (S_{ABC} \cdot (S_{APC} \cdot S_{ABP})))}{(S_{PBC} \cdot (S_{ABC} \cdot S_{ABP}))} = 1 \quad \text{, by Lemma 29 (point } C \text{ eliminated)} \tag{A.550}$$

$$\frac{((((0 + (r_0 \cdot (S_{PBA} + (-1 \cdot 0)))) \cdot ((-1 \cdot S_{PBA}) \cdot (-1 \cdot S_{PBA}))) + (-1 \cdot (S_{PBC} \cdot ((-1 \cdot S_{PBA}) \cdot (r_0 \cdot (-1 \cdot S_{PBA})))))) + (S_{ABC} \cdot (S_{APC} \cdot (-1 \cdot S_{PBA}))))}{(S_{PBC} \cdot (S_{ABC} \cdot (-1 \cdot S_{PBA})))} = 1 \quad \text{, by geometric simplifications} \tag{A.551}$$

$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (S_{PBC} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot S_{PBA}))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))} = 1 \quad \text{, by algebraic simplifications} \tag{A.552}$$

$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot ((S_{PBP} + (r_0 \cdot (S_{PBA} + (-1 \cdot S_{PBP})))) \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot S_{PBA}))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))} = 1 \quad \text{, by Lemma 29 (point } C \text{ eliminated)} \tag{A.553}$$

$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot ((0 + (r_0 \cdot (S_{PBA} + (-1 \cdot 0)))) \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot S_{PBA}))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))} = 1 \quad \text{, by geometric simplifications} \tag{A.554}$$

$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))) + (-1 \cdot (S_{ABC} \cdot (S_{APC} \cdot S_{PBA}))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))} = 1 \quad \text{, by algebraic simplifications} \tag{A.555}$$

$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))) + (-1 \cdot ((S_{ABP} + (r_0 \cdot (S_{ABA} + (-1 \cdot S_{ABP})))) \cdot (S_{APC} \cdot S_{PBA}))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))} = 1 \quad \text{, by Lemma 29 (point } C \text{ eliminated)} \tag{A.556}$$

$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))) + (-1 \cdot (((-1 \cdot S_{PBA}) + (r_0 \cdot (0 + (-1 \cdot (-1 \cdot S_{PBA}))))) \cdot (S_{APC} \cdot S_{PBA}))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))} = 1 \quad \text{, by geometric simplifications} \tag{A.557}$$

$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))) + ((S_{APC} \cdot (S_{PBA} \cdot S_{PBA})) + (-1 \cdot (S_{APC} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))} = 1 \quad \text{, by algebraic simplifications} \tag{A.558}$$

(A.559)
$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))) + (((S_{APP} + (r_0 \cdot (S_{APA} + (-1 \cdot S_{APP})))) \cdot (S_{PBA} \cdot S_{PBA})) + (-1 \cdot (S_{APC} \cdot (S_{PBA} \cdot (r_0}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))}$$
$= 1$, by Lemma 29 (point C eliminated)

(A.560)
$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))) + (((0 + (r_0 \cdot (0 + (-1 \cdot 0)))) \cdot (S_{PBA} \cdot S_{PBA})) + (-1 \cdot (S_{APC} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))}$$
$= 1$, by geometric simplifications

(A.561)
$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))) + (-1 \cdot (S_{APC} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))}$$
$= 1$, by algebraic simplifications

(A.562)
$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))) + (-1 \cdot ((S_{APP} + (r_0 \cdot (S_{APA} + (-1 \cdot S_{APP})))) \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))}$$
$= 1$, by Lemma 29 (point C eliminated)

(A.563)
$$\frac{(((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))) + (-1 \cdot ((0 + (r_0 \cdot (0 + (-1 \cdot 0)))) \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA})))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))}$$
$= 1$, by geometric simplifications

(A.564)
$$\frac{((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))))}{(-1 \cdot (S_{PBC} \cdot (S_{ABC} \cdot S_{PBA})))} = 1$$
, by algebraic simplifications

(A.565)
$$\frac{((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))))}{(-1 \cdot ((S_{PBP} + (r_0 \cdot (S_{PBA} + (-1 \cdot S_{PBP})))) \cdot (S_{ABC} \cdot S_{PBA})))} = 1$$
, by Lemma 29 (point C eliminated)

(A.566)
$$\frac{((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))))}{(-1 \cdot ((0 + (r_0 \cdot (S_{PBA} + (-1 \cdot 0)))) \cdot (S_{ABC} \cdot S_{PBA})))} = 1$$
, by geometric simplifications

(A.567)
$$\frac{((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))))}{(-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{ABC} \cdot S_{PBA}))))} = 1$$
, by algebraic simplifications

(A.568)
$$\frac{((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))))}{(-1 \cdot (r_0 \cdot (S_{PBA} \cdot ((S_{ABP} + (r_0 \cdot (S_{ABA} + (-1 \cdot S_{ABP})))) \cdot S_{PBA}))))} = 1$$
, by Lemma 29 (point C eliminated)

(A.569)
$$\frac{((r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot S_{PBA}))) + (-1 \cdot (r_0 \cdot (S_{PBA} \cdot (S_{PBA} \cdot (r_0 \cdot S_{PBA}))))))}{(-1 \cdot (r_0 \cdot (S_{PBA} \cdot (((-1 \cdot S_{PBA}) + (r_0 \cdot (0 + (-1 \cdot (-1 \cdot S_{PBA}))))) \cdot S_{PBA}))))} = 1$$
, by geometric simplifications

(A.570) $1 = 1$, by algebraic simplifications

Q.E.D.

NDG conditions are:

$S_{CPB} \neq S_{P^1_{pab}PB}$ i.e., lines CP^1_{pab} and PB are not parallel (construction based assumption)

$S_{ABC} \neq S_{DBC}$ i.e., lines AD and BC are not parallel (construction based assumption)

$S_{APQ} \neq S_{BPQ}$ i.e., lines AB and PQ are not parallel (construction based assumption)

$P_{MBM} \neq 0$ i.e., points M and B are not identical (conjecture based assumption)

Number of elimination proof steps: 17
Number of geometric proof steps: 45
Number of algebraic proof steps: 145
Total number of proof steps: 207
Time spent by the prover: 0.010 seconds

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