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Notes on the product of locales. (English summary)
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The notion of localic product is rather elusive. On the one hand, its categorical description is as straightforward as that of a product in any other category and its construction as a coproduct of frames is very similar to many others. Yet the localic product is still a very opaque object when compared to, say, its spatial counterpart, the product of topological spaces, or even to many other products constructed algebraically, such as the tensor product of abelian groups or of commutative unitary rings. There always seems to be some subtle difference, amidst so much similarity, and possibly it is this similarity and the difference, side by side, that make it more subtle and elusive! One of the very early papers [D. Wigner, J. Austral. Math. Soc. Ser. A **28** (1979), no. 3, 257–268; MR0557275 (81d:06016)] took note of this and then the same issue was discussed and conclusively dealt with in Part C, Propositions 1.1.5–1.1.10 (pp. 475–478) of [P. T. Johnstone, *Sketches of an elephant: a topos theory compendium. Vol. 2*, Oxford Logic Guides, 44, Oxford Univ. Press, Oxford, 2002; MR2063092 (2005g:18007)]. The latter reference is directed toward readers familiar with sophisticated methods of category theory, while the authors of the present paper adopt a point of view suitable for a wider audience.

Section 1 briefly describes the Isbell adjunction between the category **Top** of topological spaces with continuous functions and the category **Loc** of locales and their maps. Then the biproduct and homomorphisms in the category **SLat** of (bounded) semilattices are described, leading to the construction of the free frame over a semilattice.

In Section 2 the construction of a binary coproduct of frames is described, and then its similarities vis-à-vis differences with the binary tensor product of two abelian groups are described in detail. Next, the construction of a binary coproduct of commutative unitary rings via the free construction over commutative semigroups is described and the similarities with the case for frames are analysed.

Section 3 discusses the case of tensor products in the category **SupLat** of complete lattices with morphisms preserving arbitrary suprema. In the process it is observed that the quotient of a frame L by a relation R which respects the meet is similar to the case of sup-lattices (Corollary 3.4.1). Thereafter the tensor product in **SupLat** is constructed and it turns out that the tensor product in **SupLat** of two frames is exactly their binary coproduct. Finally, in this new light, the connection between the binary coproduct of frames and a tensor product is meticulously analysed.

Considering the topology on spaces as a frame, it is natural to expect some connection between the topology of the binary product of spaces and the binary coproduct of the topologies of the two spaces via the Isbell adjunction. In Section 4, it is shown that the topology of the binary product is a dense sublocale of the localic binary product of the topologies, thereby demonstrating the subtle difference between the two notions. The rest of this section is aimed at clarifying this subtlety and describing situations when these products coincide.

In the concluding section the authors argue that, in many respects, locales are better behaved objects than topological spaces. *Partha Pratim Ghosh*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.