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On the Dedekind completion of function rings. (English summary)

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Let $(X, \mathcal{O}X)$ be a topological space and $\mathcal{C}(X)$ the ring of real-valued continuous functions on X. The main goal of this paper is to construct the Dedekind completion of the ring $\mathcal{C}(X)$ in a direct and transparent way, avoiding the approach of R. Anguelov, who used Hausdorff continuous functions [see Quaest. Math. **27** (2004), no. 2, 153–169; MR2091694]. The authors approach the problem from a pointfree viewpoint by replacing topological spaces by an abstraction of their lattices of open sets, namely *frames* (or *locales*). A frame is a complete lattice L with the property that for all $a \in L$ and $B \subseteq L$, $a \wedge \bigvee B = \bigvee\{a \wedge b \mid b \in B\}$. The lattice of open sets $\mathcal{O}X$ is a frame and the correspondence $X \mapsto \mathcal{O}X$ is functorial.

This paper introduces the frame of partially defined real numbers and the lattice-ordered ring of partial real functions on a frame. This is used to construct the order completion of rings of pointfree continuous real functions. The bounded and integer-valued cases are also analyzed. This pointfree approach to the classical case of the ring $\mathcal{C}(X)$ of continuous real-valued functions on a topological space X yields a new construction for the Dedekind completion of $\mathcal{C}(X)$ that differs from the approach of Anguelov.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.