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**Uniform continuity of pointfree real functions via farness and related Galois connections.** (English) Zbl 07604512

Algebra Univers. 83, No. 4, Paper No. 39, 27 p. (2022).

The present paper under review is an elegantly composed paper characterizing uniform continuity of real valued functions defined on a (pre-)uniform frame. Uniform continuity of such frame homomorphisms is first characterised in terms of farness relation on the underlying set of a frame and then derived from it are a separation and an extension theorem for real-valued uniform maps on pre-uniform frames.

Research on insertion and extension of (semi-)continuous or measurable real valued functions in pointfree topology was vibrant as can be viewed from [*I. Arrieta* et al., Quaest. Math. 46, No. 2, 207–242 (2023; Zbl 1533.06005); *J. Gutiérrez García* et al., Algebra Univers. 81, No. 3, Paper No. 32, 18 p. (2020; Zbl 1453.06010); J. Pure Appl. Algebra 223, No. 6, 2345–2370 (2019; Zbl 1471.06005); Quaest. Math. 40, No. 4, 507–518 (2017; Zbl 1436.06022); *J. Gutiérrez García* and *T. Kubiak*, Czech. Math. J. 64, No. 3, 743–749 (2014; Zbl 1349.28002); *J. Gutiérrez García* and *J. Picado*, J. Pure Appl. Algebra 218, No. 5, 784–803 (2014; Zbl 1296.06006); *J. Gutiérrez García* and *T. Kubiak*, Commentat. Math. 53, No. 2, 413–419 (2013; Zbl 1294.06010); J. Pure Appl. Algebra 215, No. 6, 1198–1204 (2011; Zbl 1217.06003); *J. Gutiérrez García* et al., Algebra Univers. 60, No. 2, 169–184 (2009; Zbl 1181.06003); *J. Gutiérrez García* et al., J. Pure Appl. Algebra 213, No. 6, 1064–1074 (2009; Zbl 1187.06005); J. Pure Appl. Algebra 213, No. 1, 98–108 (2009; Zbl 1154.06006); Houston J. Math. 34, No. 1, 123–144 (2008; Zbl 1160.54012); J. Pure Appl. Algebra 212, No. 5, 955–968 (2008; Zbl 1133.06008); Topology Appl. 153, No. 9, 1458–1475 (2006; Zbl 1094.54009)]. The present paper is aimed at developing pointfree counterparts of the insertion theorems in [*D. Preiss* and *J. Vilimovsky*, Trans. Am. Math. Soc. 261, 483–501 (1980; Zbl 0388.54019)].

The notion of *farness* on the underlying set of a space in [*Yu. M. Smirnov*, Transl., Ser. 2, Am. Math. Soc. 38, 5–35 (1964; Zbl 0152.20903); translation from Mat. Sb., N. Ser. 31, (73), 543–574 (1952)] is extended to pointfree setting in §3 leading to an elegant characterisation of uniform continuous real valued functions on a (pre-)uniform frame in Theorem 4.1: given any pre-uniform frame $(L,\mathcal{U})$, $f \in \mathcal{R}(L)$ is uniformly continuous if and only if for every positive rational number $r$ there exists a $U \in \mathcal{U}$ such that:

$$U \leq \left\{ f(p,q) : q-p = \frac{1}{r} \right\},$$

where $\leq$ is the *refinement order* on covers.

It is known from [*J. Picado* and *A. Pultr*, Frames and locales. Topology without points. Berlin: Springer (2012; Zbl 1231.06018), XIV.5.2.2] that ascending/descending scales determine real valued functions on a frame $L$. The characterisation of uniformly continuous real functions on a frame help to extend the notion of scales to *uniform scales* in §5 – gadgets that generate uniformly continuous real functions on a pre-uniform frame.

The notion of farness on a pre-uniform frame induce a self-duality (see §6) leading to a method of constructing uniformly continuous real functions separating far elements of a pre-uniform frame (Theorem 7.5) extending Smirnov functional separation in [*M. Hušek*, Extr. Math. 25, No. 3, 277–308 (2010; Zbl 1232.54004)]: given any pre-uniform frame $(L,\mathcal{U})$, $a, b \in L$ are $U$-far for some $U \in \mathcal{U}$ implies the existence of a $f \in \mathcal{R}(L)$ with $\mathbf{0} \leq f \leq \mathbf{1}$, $f(0,-) \leq a^*$ and $f(-,1) \leq b^*$.

The paper ends with a uniform counterpart of Tietze-extension theorem for closed sublocales [*J. Picado*, Topology Appl. 153, No. 16, 3203–3218 (2006; Zbl 1104.06007); *J. Walters-Wayland*, Completeness and nearly fine uniform frames. University of Cape Town (PhD Thesis) (1996)] in Theorem 8.3: given any pre-uniform frame $(L,\mathcal{U})$ and a uniformly continuous $\mathcal{L}(\mathbb{R}) \xrightarrow{f} (S,\mathcal{U}_S)$, where $S$ is a dense sublocale of $L$, there exists a uniformly continuous extension $\mathcal{L}(\mathbb{R}) \xrightarrow{\bar{f}} (L,\mathcal{U})$.

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**MSC:**

18F70 Frames and locales, pointfree topology, Stone duality

06D22 Frames, locales

06A15 Galois correspondences, closure operators (in relation to ordered sets)

54C30 Real-valued functions in general topology

54E15 Uniform structures and generalizations



**Keywords:**

uniform frame; locale; sublocale; Galois connection; frame; uniform extension; uniformly continuous real function; uniform cover; uniform homomorphism; proximally far elements

**Full Text:** DOI

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