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Extension theory and the calculus of butterflies

Let $\mathcal{C}$ be a semi-abelian category satisfying the condition (SH) (i.e. where two equivalence relation centralize each other as soon as their normalizations commute). We give a cohomological classification of the extensions of an internal crossed module in $\mathcal{C}$ via a given object. More precisely, given an internal crossed module $(\partial\colon K \to K_0, \xi)$ and a morphism $\phi\colon Y \to \pi_0(\partial) = \mathrm{Coker}(\partial)$, we show that the set $\mathrm{Ext}_\phi(Y, \partial)$ of extensions (i.e. short exact sequences) (f, k) filling the following diagram (with $(1_K, \alpha)$ a crossed module morphism)

$$\begin{array}{ccc} K & = & K \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \partial} \\ X & \overset{}{\underset{\alpha}{\dashrightarrow}} & K_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{coker}(\partial)} \\ Y & \underset{\phi}{\longrightarrow} & \pi_0(\partial) \end{array}$$

either is empty, or it is a simply transitive $\mathrm{H}^2_{\overline{\phi}}(Y, \pi_1(\partial))$-set, where $\pi_1(\partial) = \mathrm{Ker}(\partial)$ is a Y-module with the action $\overline{\phi}$ induced by ξ.

The main tool we use is the calculus of *butterflies*, introduced by B. Noohi [5] to deal with monoidal functors between 2-groups and further developed in the semi-abelian context in [1], where the authors show that they are the bicategory of fractions of internal crossed modules with respect to weak equivalences.

The present result is an intrinsic version of a theorem by P. Dedecker [4] (stated in the category of groups) and extends, in the semi-abelian setting, the intrinsic version (developed in [2] and [3]) of the classical Schreier-Mac Lane Theorem on the classification of extensions.

References:

[1] O. Abbad, S. Mantovani, G. Metere and E. M. Vitale, Butterflies in a semi-abelian context, *Adv. Math.* 238 (2013) 140–183.

[2] D. Bourn, Commutator theory, action groupoids, and an intrinsic Schreier-Mac Lane extension theorem, *Adv. Math.* 217 (2008), 2700–2735.

[3] D. Bourn, A. Montoli, Intrinsic Schreier-Mac Lane extension theorem II: the case of action accessible categories, *J. Pure and Appl. Algebra* 216 (2012), 1757–1767.

[4] P. Dedecker, Cohomologie de dimension 2 à coefficients non abéliens, *C. R. Acad. Sci. Paris* 247 (1958) 1160–1163.

[5] B. Noohi, On weak maps between 2-groups (2008) arXiv:math/0506313v3.

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