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Commutative orders in semigroups

We consider commutative orders, that is, commutative semigroups having a semigroup of fractions in a local sense defined as follows. An element $a \in S$ is *square-cancellable* if for all $x, y \in S^1$ we have that $xa^2 = ya^2$ implies $xa = ya$ and also $a^2x = a^2y$ implies $ax = ay$. It is clear that being square-cancellable is a necessary condition for an element to lie in a subgroup of an oversemigroup. In a commutative semigroup S, the square-cancellable elements constitute a subsemigroup $\mathcal{S}(S)$. Let S be a subsemigroup of a semigroup Q. Then S is a *left order* in Q and Q is a *semigroup of left fractions* of S if every $q \in Q$ can be written as $q = a^\sharp b$ where $a \in \mathcal{S}(S)$, $b \in S$ and $a^\sharp$ is the inverse of a in a subgroup of Q and if, in addition, every square-cancellable element of S lies in a subgroup of Q. *Right orders* and *semigroups of right fractions* are defined dually. If S is both a left order and a right order in Q, then S is an *order* in Q and Q is a *semigroup of fractions* of S. We remark that if a commutative semigroup is a left order in Q, then Q is commutative so that S is an order in Q. A given commutative order S may have more than one semigroup of fractions. The semigroups of fractions of S are pre-ordered by the relation $Q \geq P$ if and only if there exists an onto homomorphism $\phi : Q \to P$ which restricts to the identity on S. Such a ϕ is referred to as an S-*homomorphism*; the classes of the associated equivalence relation are the S-isomorphism classes of orders, giving us a partially ordered set $\mathcal{Q}(S)$. In the best case, $\mathcal{Q}(S)$ contains maximum and minimum elements. In a commutative order S, $\mathcal{S}(S)$ is also an order and has a maximum semigroup of fractions R, which is a Clifford semigroup. We investigate how much of the relation between $\mathcal{S}(S)$ and its semigroups of fractions can be lifted to S and its semigroups of fractions.

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