Enriched Morita equivalence for S-sorted theories

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WCMAT 2014

Outline

- History of Morita equivalence results
- Basic notions and our setting
- Our general result
- Examples: sorted Morita equivalence

History of Morita equivalence results

- Original motivation: module theory (Morita, 1950s)
- Two rings R and S are called Morita equivalent if ${}_R\mathrm{Mod}$ is categorically equivalent to ${}_S\mathrm{Mod}$
- Result: $R \simeq_M S$ iff S is an idempotent modification of a matrix ring $R^{[n]}$ for some natural n
- Non-additive version – Banaschewski, Knauer: For monoids M and N, it holds that M-$\mathrm{Act} \simeq N$-Act iff N is an idempotent modification of M.

History of Morita equivalence results

- Dukarm, 1980s: Morita-type result for (one-sorted) Lawvere theories.
  Again using the notion of a pseudoinvertible idempotent
- Adámek, Sobral, Sousa, 2006: many-sorted generalisation of Dukarm's result

Our aim:

- Generalise the 2006 result to the enriched setting
- Make the result modular: other notions of an algebraic theory

Our setting

- We work with categories enriched over $\mathcal{V}$, $\mathcal{V}$ being a symmetric monoidal closed category
- Algebraic theory: a Ψ-theory is a category with Ψ-colimits
- Ψ-theory morphism: a Ψ-cocontinuous functor between Ψ-theories
- Algebras for a Ψ-theory $\mathcal{T}$: a subcategory $\Psi\text{-}\mathrm{Alg}(\mathcal{T})$ of Ψ-limit-preserving functors from $[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]$

What can Ψ be

Ψ is a locally small, sound class of weights.

Local smallness – Kelly, Schmitt

Ψ is a locally small class of weights if for any small $\mathcal{D}$ its free cocompletion $\Psi(\mathcal{D})$ under Ψ-colimits is again small.

Notation: Ψ^+ is a class of weights such that Ψ^+-colimits commute with Ψ-limits (Ψ-flat weights).
Example ($\mathcal{V} = \mathrm{Set}$, Ψ …finite limits): Ψ^+ are weights for filtered colimits.

Soundness – Adámek, Borceux, Lack, Rosický

Ψ is a sound class of weights if for any Ψ-theory $\mathcal{T}$ it holds that

$$\Psi\text{-}\mathrm{Alg}(\mathcal{T}) \simeq \Psi^+(\mathcal{T}).$$

$\mathcal{S}$-sorted theories

$\mathcal{S}$-sorted theory

Fix a discrete category $\mathcal{S}$ of sorts. A Ψ-theory is $\mathcal{S}$-sorted if it is equipped with a theory morphism $\Psi(\mathcal{S}) \to \mathcal{T}$ that is an identity on objects.

Example

Let $\mathcal{V} = \mathsf{Set}$, Ψ be weights for finite coproducts, $\mathrm{ob}(\mathcal{S}) = S$. Then $\mathcal{S}$-sorted Ψ-theories are exactly the S-sorted algebraic theories of [Adámek,Sobral,Sousa].

Idempotent completion vs. $\mathcal{Q}$

- $\mathcal{V} = \mathsf{Set}$: idempotent completion $\mathrm{Idem}(\mathcal{T})$ for a theory $\mathcal{T}$.
- For general $\mathcal{V}$, the role of the idempotent completion is taken by the Cauchy completion $\mathcal{Q}(\mathcal{T})$.

Basic Morita theorem

Two Ψ-theories $\mathcal{S}$ and $\mathcal{T}$ are Morita equivalent iff $\mathcal{Q}(\mathcal{S}) \simeq \mathcal{Q}(\mathcal{T})$. (In the case of $\mathcal{V} = \mathsf{Set}$, $\mathrm{Idem}(\mathcal{S}) \simeq \mathrm{Idem}(\mathcal{T})$.)

Cauchy completion: absolute colimit cocompletion of a theory $\mathcal{T}$.

Idempotent modification of a theory

Given an $\mathcal{S}$-sorted theory $\mathcal{T}$, a collection of idempotents u is

Adámek,Sobral,Sousa:

a choice of an idempotent $u_s : t_s \to t_s$ from $\mathcal{T}$ for every sort $s \in \mathrm{ob}(\mathcal{S})$.

Our approach:

a functor $u : \mathcal{S} \to \mathcal{Q}(\mathcal{T})$.

An idempotent modification $u\mathcal{T}u$ of $\mathcal{T}$ is the closure of $\langle u_s \mid s \in \mathrm{ob}(\mathcal{T})\rangle$ (of the image of $u : \mathcal{S} \to \mathcal{Q}(\mathcal{T})$) under

coproducts.

Ψ-colimits.

Pseudoinvertible idempotent

Given a collection of idempotents u for an $\mathcal{S}$-sorted theory $\mathcal{T}$, we say that u is pseudoinvertible if

Adámek,Sobral,Sousa:

for every sort s from $\mathcal{S}$ there is an idempotent $u_s : t \to t$ from $u\mathcal{T}u$ and morphisms $m : s \to t$ and $e : t \to s$ s.t.

$$\begin{array}{ccc} t & \xrightarrow{u_s} & t \\ {\scriptstyle m}\uparrow & & \downarrow{\scriptstyle e} \\ s & \xrightarrow{\mathrm{id}_s} & s \end{array}$$

Our approach:

the following equivalence

$$\mathcal{Q}(u\mathcal{T}u) \simeq \mathcal{Q}(\mathcal{T}).$$

holds.

Note: the definitions coincide for $\mathcal{V} = \mathrm{Set}$.

Theorem

Morita theorem

Two $\mathcal{S}$-sorted Ψ-theories $\mathcal{T}_1$ and $\mathcal{T}_2$ are Morita equivalent if and only if $\mathcal{T}_2 \simeq u\mathcal{T}_1 u$ for some pseudoinvertible u.

Proof:

- That $\mathcal{T}_1 \simeq u\mathcal{T}_1 u$ is easy; one direction follows directly from this observation.
  If $\mathcal{T}_2$ and $\mathcal{T}_1$ are Morita equivalent, then $\mathcal{Q}(\mathcal{T}_2) \simeq \mathcal{Q}(\mathcal{T}_1)$.
- Use the above equivalence and the sorting functor $\Psi(\mathcal{S}) \to \mathcal{T}_2$ to construct a pseudoinvertible idempotent $u : \mathcal{S} \to \mathcal{Q}(\mathcal{T}_1)$.
- The idempotent gives rise to $u\mathcal{T}_1 u$ which is easily shown to be equivalent to $\mathcal{T}_2$.

Examples, Ψ weights for coproducts

- $\mathcal{V} = \mathrm{Set}$: We get a short and compact proof of the characterisation of Morita equivalent algebraic ($\mathcal{S}$-sorted) theories. Thus we reprove and generalise the results of Dukarm and Adámek, Sobral, Sousa.
- $\mathcal{V} = \mathrm{Pos}$, $\mathcal{V} = \mathrm{Cat}$: Two $\mathcal{S}$-sorted theories $\mathcal{T}_1$ and $\mathcal{T}_2$ are Morita equivalent if and only if $\mathcal{T}_2 \simeq u\mathcal{T}_1 u$. Pseudoinvertible idempotent: for each sort s in $\mathcal{S}$ there needs to be an idempotent $u_s : t \to t$ from $u\mathcal{T}_1 u$ and morphisms $m : s \to t$ and $e : t \to s$ such that

$$\begin{array}{ccc} t & \xrightarrow{u_s} & t \\ {\scriptstyle m}\uparrow & & \downarrow{\scriptstyle e} \\ s & \xrightarrow{\mathrm{id}_s} & s \end{array}$$

Examples, Ψ empty class of weights

- $\mathcal{V} = \mathsf{Ab}$: We get the standard Morita result.
- $\mathcal{V} = \mathsf{Set}$: For one-object $\mathcal{T}$, we recreate the results of Banaschewski and Knauer. This generalises straightforwardly for many-sorted $\mathcal{T}$.
- $\mathcal{V} = \mathsf{Pos}$: Morita equivalence for partially ordered monoids. We get the result of Laan and generalise it to the many-sorted case.
- $\mathcal{V} = \mathsf{Cat}$: Morita equivalence for Cat-enriched monoids and categories.

Future work

- What happens when we enrich over $\mathcal{V}$ being simplicial sets?
- Study the enrichment which yields probabilistic metric spaces as enriched categories.
- Let $\mathcal{V} = [\mathsf{Set}_{fp}, \mathsf{Set}]$ with composition. Then a monoid in $\mathcal{V}$ is a finitary monad. Do we get an interesting Morita theorem for finitary monads?

References

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