A Galois-theoretic approach to the covering theory of quandles

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Denote Qnd the corresponding category.
It is a variety of universal algebras.

Examples

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- Let G be a group, define $g \lhd h = hg^{-1}h$. This defines the *core quandle*.

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Definition

A *connected component* of A is an orbit under the action of $\mathrm{Inn}(A)$.
Two elements a and b of a quandle A are in the same connected component if there exist $a_1, a_2, \ldots a_n \in A$ and $\lhd^{\alpha_i} \in \{\lhd, \lhd^{-1}\}$ such that $(\ldots(((a \lhd^{\alpha_1} a_1) \lhd^{\alpha_2} a_2) \ldots) \lhd^{\alpha_n} a_n = b$.

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Remark

The category Qnd is not Mal'tsev neither Goursat ($R \circ S = S \circ R$ or $R \circ S \circ R = S \circ R \circ S$ for any congruences R, S on an object A).

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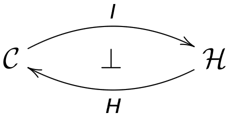

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- $\mathcal{C}$ is an exact category ;
- $\mathcal{H}$ is a Birkhoff subcategory of $\mathcal{C}$ (i.e. closed under quotients and subobjects) ;
- the functor I is left adjoint to the inclusion functor H.

Definition

A Birkhoff subcategory $\mathcal{H}$ is *admissible* with respect to $\mathcal{C}$ when the functor I preserves a certain type of pullbacks :

$$\begin{array}{ccc} B \times_{HIB} HX & \xrightarrow{\pi_2} & HX \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle H\phi} \\ B & \xrightarrow[\eta_B]{} & HIB \end{array}$$

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where

- $X \in \mathcal{H}$;
- $\phi\colon X \to HIB$ is a regular epimorphism.
- $\eta_B\colon B \to HIB$ is the unit of the adjunction at object B.

Definition

A regular epimorphism $f\colon A \to B$ is a *trivial extension* when

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HIA \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle HIf} \\ B & \xrightarrow[\eta_B]{} & HIB. \end{array}$$

is a pullback.

Definition

A regular epimorphism $f: A \to B$ is a *central extension* if there exists a regular epimorphism $p: E \to B$ such that the pullback π_1 of f along p is a trivial extension.

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

Definition (M. Eisermann)

A quandle homomorphism $f\colon A \to B$ is a *covering in the sense of Eisermann* if it is surjective and $f(a) = f(b)$ implies $c \triangleleft a = c \triangleleft b$ for all $a,\ b,\ c \in A$.

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$$\mathsf{Qnd} \underset{U}{\overset{\pi_0}{\rightleftarrows}} \mathsf{Qnd}^* \quad (\pi_0 \dashv U) \ .$$

Lemma

Given a quandle A, there exists a class of congruences $\sim_N$, where N is a normal subgroup of $\mathrm{Inn}(A)$, such that

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Remark

The kernel pair of $\eta_A \colon A \to \pi_0(A)$ is such a congruence.

Proposition

The adjunction

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Remark

The reflection of Qnd onto Qnd^* is not semi-left-exact. (Cassidy-Hébert-Kelly, 1985)

$$\begin{array}{ccc} B \times_{HIB} HX & \xrightarrow{\pi_2} & HX \\ \big\downarrow{\scriptstyle \pi_1} & \lrcorner & \big\downarrow{\scriptstyle H\phi} \\ B & \xrightarrow[\eta_B]{} & HIB \end{array}$$

Proposition

A surjective homomorphism $f : A \to B$ is a trivial extension if and only if the condition (T) is verified :

$$(T) : \qquad \forall a, a' \in A, \text{ if } f(a) = f(a') \text{ and } [a] = [a'], \text{ then } a = a'.$$

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Lemma

Given the pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B, \end{array}$$

where p is a surjective homomorphism, then :

f is an E-covering *if and only if p_1 is an* E-covering.

Corollary

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Theorem

$f : A \to B$ is an E-covering if and only if it is a central extension.

References

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