Semi-localizations of semi-abelian categories

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Work in collaboration with Stephen Lack

Workshop on Categorical Methods in Algebra and in Topology
In honour of Professor Manuela Sobral

Outline

Motivation

Semi-localizations of exact categories

An easy application : the Mal’tsev case

Semi-localizations of exact protomodular categories

The semi-abelian case

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Motivation

Let $\mathcal{C}$ be an abelian category, $(\mathcal{T}, \mathcal{X})$ a torsion theory in $\mathcal{C}$.

This means :

1. $\mathcal{T}$ and $\mathcal{X}$ are full replete subcategories of $\mathcal{C}$;
2. if $T \in \mathcal{T}$ and $X \in \mathcal{X}$ then the only morphism from T to X is

$$T \to 0 \to X;$$

3. for every object $C \in \mathcal{C}$ there is a short exact sequence

$$0 \longrightarrow T(C) \longrightarrow C \longrightarrow F(C) \longrightarrow 0$$

with $T(C) \in \mathcal{T}$ and $F(C) \in \mathcal{X}$.

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Torsion-free subcategories of an abelian category $\mathcal{C}$ correspond to full epireflective subcategories $\mathcal{X}$ of $\mathcal{C}$

$$\mathcal{X} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{C} \quad (F \dashv U)$$

such that $F\colon \mathcal{C} \to \mathcal{X}$ is semi-left-exact (Cassidy-Hébert-Kelly, 1985) :

$F\colon \mathcal{C} \to \mathcal{X}$ preserves all pullbacks of the form

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & U(X) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle U(x)} \\ C & \xrightarrow[\eta_C]{} & UF(C) \end{array}$$

where $x\colon X \to F(C)$ lies in $\mathcal{X}$, and η is the unit of the adjunction.

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A torsion-free subcategory $\mathcal{X}$ of an abelian category $\mathcal{C}$

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For a category $\mathcal{X}$ the following conditions are equivalent :

1. $\mathcal{X}$ is a torsion-free subcategory of an abelian category $\mathcal{C}$;
2. (a) $\mathcal{X}$ is additive ;
   (b) any morphism $f\colon A \to D$ in $\mathcal{X}$ has a factorization $f = kgq$

$$\begin{array}{ccc} A & \xrightarrow{f} & D \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle k} \\ B & \xrightarrow[g]{} & C \end{array}$$

   with q a normal epi, g a bimorphism, k a normal mono ;
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New examples of torsion theories have been studied in the semi-abelian categories Grp, CRng, VNRegRng, XMod, Grp(Comp).

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Can one find an intrinsic characterisation of torsion-free subcategories of a semi-abelian category ?

Let us make the question more precise : observe that *any* (normal epi)-reflective subcategory $\mathcal{X}$ of a semi-abelian category $\mathcal{C}$ is :

- pointed,
- regular,
- protomodular,
- with binary coproducts.

Question
Can one find an additional property on a homological category $\mathcal{X}$ with binary coproducts making $\mathcal{X}$ a torsion-free subcategory of a semi-abelian category $\mathcal{C}$?

More generally : call $\mathcal{X}$ a semi-localizations of $\mathcal{C}$ if it is a full reflective subcategory $\mathcal{X}$ of $\mathcal{C}$ whose reflector is semi-left-exact.

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Semi-localizations of exact categories

A remarkable result in this direction has been discovered by S. Mantovani (1998, Cah. Topol. Géom. Différ. Catég.) :

Theorem (Mantovani)

For a category $\mathcal{X}$ the following conditions are equivalent :

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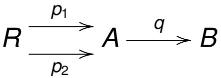

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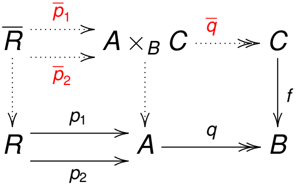

$\mathcal{X}$ has stable coequalizers $\quad\Leftrightarrow\quad \overline{q} = \mathrm{coeq}(\overline{p}_1, \overline{p}_2)$

This theorem uses the exact completion $\mathcal{X}_{\text{ex/reg}}$ of a regular category $\mathcal{X}$.

There is a fully faithful functor $\Gamma\colon \mathcal{X} \to \mathcal{X}_{\text{ex/reg}}$ such that for any regular functor $F\colon \mathcal{X} \to \mathcal{D}$ to an exact category $\mathcal{D}$

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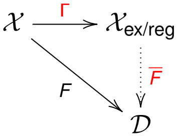

there exists an essentially unique regular functor $\overline{F}\colon \mathcal{X}_{\text{ex/reg}} \to \mathcal{D}$ with

$$\overline{F} \circ \Gamma \cong F.$$

Outline

An easy application : the Mal'tsev case

Any semi-localization $\mathcal{X}$ of a Mal'tsev category $\mathcal{C}$ is a Mal'tsev category :

given a reflexive relation R on X in $\mathcal{X}$

$$R \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} X\ ,$$

it is also a reflexive relation in the Mal'tsev category $\mathcal{C}$.

Then R is an equivalence relation in $\mathcal{C}$

$$R \times_X R \underset{p_2}{\overset{p_1}{\rightrightarrows}} R \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} X\ ,\quad s : R \to R$$

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Conversely, given a regular Mal'tsev category $\mathcal{X}$, its exact completion $\mathcal{X}_{\text{ex/reg}}$ is again a Mal'tsev category :

given a reflexive relation

$$R \underset{c}{\overset{d}{\rightrightarrows}} A \quad (e\colon A \to R)$$

in $\mathcal{X}_{\text{ex/reg}}$,

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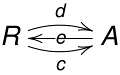

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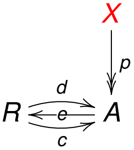

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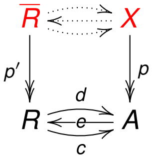

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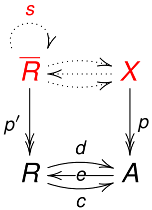

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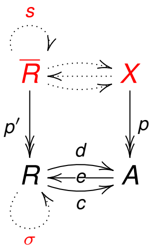

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Since $\mathcal{X}$ is a Mal'tsev category, $\overline{R}$ is a symmetric relation in $\mathcal{X}$.

This implies that R is a symmetric relation in $\mathcal{X}_{\text{ex/reg}}$ as well.

Conclusion :
A regular $\mathcal{X}$ is a Mal'tsev category $\Leftrightarrow$ if $\mathcal{X}_{\text{ex/reg}}$ is a Mal'tsev category.

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For a category $\mathcal{X}$, the following conditions are equivalent :

(a) $\mathcal{X}$ is a semi-localization of an exact Mal'tsev category $\mathcal{C}$;

(b) $\mathcal{X}$ is a regular Mal'tsev category, and has stable coequalizers of equivalence relations ;

(c) $\mathcal{X}$ is a regular Mal'tsev category, and is a semi-localization of its exact completion $\mathcal{X}_{\text{ex/reg}}$ as a regular category.

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For a category $\mathcal{X}$, the following conditions are equivalent :

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Example

Let $\mathbb{T}$ be a Mal'tsev algebraic theory : $\exists\, p(x, y, z)$ such that

$$p(x, y, y) = x \qquad \textit{and} \qquad p(x, x, y) = y.$$

Then the category $\mathbb{T}(\mathrm{Top})$ of topological Mal'tsev algebras is

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Outline

How to deal with protomodularity ?

In order to imitate the Mal'tsev case one needs a new characterization of protomodular categories in terms of internal relations.

Definition

A relation R on X is left pseudoreflexive if xRy implies xRx.

A relation R on X is right pseudoreflexive if xRy implies yRy.

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A relation R on X is left pseudosymmetric if there is a morphism $f\colon Z \to X$ with the property that

$$(fz)Ry \text{ implies } yR(fz)$$

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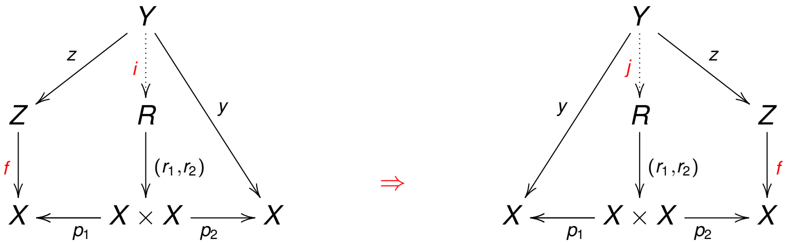

Remark

In the pointed case, R is left pseudosymmetric $\Leftrightarrow$ $0Ry$ implies $yR0$ (Z. Janelidze, Appl. Categ. Structures, 2007).

Theorem

For a finitely complete category $\mathcal{C}$ the following are equivalent :

(a) $\mathcal{C}$ is protomodular ;

(b) every relation in $\mathcal{C}$ which is left pseudoreflexive and left pseudosymmetric is symmetric ;

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Remark

This extends an important result due to Z. Janelidze in the pointed context (Appl. Categ. Structures, 2007).

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The semi-abelian case

In order to give the characterization of torsion-free subcategories we recall the following

Definition

In a protomodular category, a morphism $m \colon X \to Y$ is Bourn-normal when there exists an equivalence relation R and a discrete fibration

$$\begin{array}{ccc} X \times X & \longrightarrow & R \\ p_1 \downarrow \downarrow p_2 & & r_1 \downarrow \downarrow r_2 \\ X & \xrightarrow[m]{} & Y \end{array}$$

Such a morphism $m \colon X \to Y$ is necessarily a monomorphism.

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In a pointed category any normal monomorphism is Bourn-normal. The converse is not true, in general.

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A cokernel $q\colon B \to Q$ of a monomorphism $m\colon A \to B$ is stable if, given any $f\colon D \to Q$ as in

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 & & & & \downarrow{\scriptstyle f} \\
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\end{array}
$$

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$$
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 & & B \times_Q D & \xrightarrow{\pi_2} & D \\
 & {\scriptstyle m'} \nearrow & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f} \\
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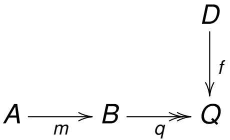

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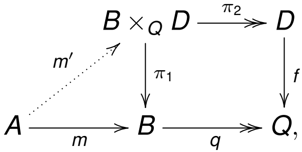

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(b) $\mathcal{X}$ is a torsion-free subcategory a semi-abelian category $\mathcal{C}$;

(c) $\mathcal{X}$ is homological, and has binary coproducts and stable coequalizers of equivalence relations ;

(d) $\mathcal{X}$ is homological, has binary coproducts, and stable cokernels of Bourn-normal monomorphisms ;

(e) $\mathcal{X}$ is homological, has binary coproducts, and every Bourn-normal monomorphism factorizes as a monomorphism with trivial cokernel followed by a normal monomorphism.

Example

Let $\mathbb{T}$ be a semi-abelian algebraic theory. This means that there is a unique constant 0, binary terms $\alpha_i(x,y)$ (for $i \in \{1, \cdots, n\}$) and an $(n+1)$-ary term β such that :

$$\alpha_i(x,x) = 0$$

and

$$\beta(\alpha_1(x,y), \cdots, \alpha_n(x,y), y) = x.$$

As shown by F. Borceux and M.M. Clementino (Adv. Math, 2005), the category $\mathbb{T}(\mathrm{Top})$ is homological, and it has binary coproducts.

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The category $\mathsf{NormMono}(\mathcal{C})$ of normal monomorphisms in a semi-abelian category $\mathcal{C}$ is a semi-localization of a semi-abelian category.

It turns out that

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The category RedRng of reduced rings ($x^n = 0 \Rightarrow x = 0$) is a semi-localization of a semi-abelian category.

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Our results can be used to characterize the hereditarily-torsion-free subcategories of semi-abelian categories.

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References

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Thank you for your attention !