Push forwards of crossed squares

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26th January 2014[1]

[1] Joint work with L. Pizzamiglio

Crossed modules [Whitehead 48]

Definition

A *crossed module* consists of a group homomorphism $\partial : G_1 \to G_0$, endowed with a left action of G_0 on G_1, satisfying:

$$\text{(i) } \partial({}^{g}\alpha) = g\,\partial(\alpha)\,g^{-1} \quad \text{and} \quad \text{(ii) } {}^{\partial\alpha_1}\alpha_2 = \alpha_1\,\alpha_2\,\alpha_1^{-1}.$$

$$\begin{tikzcd}
G_1 \times G_1 \arrow[r, "(\partial, id_{G_1})"] \arrow[d, "\chi"'] & G_0 \times G_1 \arrow[r, "(id_{G_0}, \partial)"] \arrow[d, "\xi"'] & G_0 \times G_0 \arrow[d, "\chi"] \\
G_1 \arrow[r, "id_{G_1}"] & G_1 \arrow[r, "\partial"] & G_0
\end{tikzcd}$$

Definition

A *morphism between crossed modules* $\partial : G_1 \to G_0$ and $\partial' : \Gamma_1 \to \Gamma_0$ consists of homomorphisms $\varphi : G_1 \to \Gamma_1$ and $\psi : G_0 \to \Gamma_0$ such that

$$\text{(i) } \partial'\,\varphi = \psi\,\partial \quad \text{and} \quad \text{(ii) } \varphi({}^{g}\alpha) = {}^{\psi(g)}\varphi(\alpha).$$

Crossed modules and their morphisms form a category $\mathcal{CM}$.

Kernels and cokernels of crossed modules

Let $\partial : G_1 \to G_0$ be a crossed module, then:

- $\ker \partial$ is G_0-invariant;

- there is an action of $\operatorname{coker} \partial$ on the abelian group $\ker \partial$ such that the following composition

$$\begin{tikzcd} \ker \partial \arrow[r, hook] \arrow[rr, bend right, "1"'] & G_1 \arrow[r, "\partial"] \arrow[rr, bend left, "1"] & G_0 \arrow[r, two heads] & \operatorname{coker} \partial \end{tikzcd} \tag{1}$$

is a crossed module.

We are going to show that these properties hold, in a 2-dimensional form, provided we change the notions of kernels and cokernels by the homotopical versions.

Crossed squares [Guin-Waléry, Loday 80]

Definition

A crossed square is a commutative diagram of groups

$$\begin{tikzcd} G_1 \arrow[r, "p_1"] \arrow[d, "\partial"'] & \Gamma_1 \arrow[d, "\partial'"] \\ G_0 \arrow[r, "p_0"] & \Gamma_0 \end{tikzcd}$$

with actions of the group Γ_0 on G_1, Γ_1 and G_0 and a function $h : \Gamma_1 \times G_0 \to G_1$, such that the following axioms are satisfied:

(i) the maps p_1, ∂ preserve the actions of Γ_0. Furthermore, with the given actions the maps ∂', p_0 and $\partial' p_1 = p_0 \partial$ are crossed modules;

(ii) $p_1\, h(\beta, g) = \beta\, {}^{g}\beta^{-1}$, $\partial\, h(\beta, g) = {}^{\beta}g\, g^{-1}$;

(iii) $h(p_1(\alpha), g) = \alpha\, {}^{g}\alpha^{-1}$, $h(\beta, \partial(\alpha)) = {}^{\beta}\alpha\, \alpha^{-1}$;

(iv) $h(\beta_1 \beta_2, g) = {}^{\beta_1}h(\beta_2, g)\, h(\beta_1, g)$, $h(\beta, g_1 g_2) = h(\beta, g_1)\, {}^{g_1}h(\beta, g_2)$;

(v) $h({}^{\sigma}\beta, {}^{\sigma}g) = {}^{\sigma}h(\beta, g)$;

for all $\alpha \in G_1$, β, β_1, $\beta_2 \in \Gamma_1$, g, g_1, $g_2 \in G_0$ and $\sigma \in \Gamma_0$.

An easy example of a crossed square is given by two normal subgroups N and M of P and their intersection $N \cap M$:

$$\begin{tikzcd} N \cap M \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & P \end{tikzcd}$$

In this case P acts by conjugation and the function $h : M \times N \to N \cap M$ is given by $h(m, n) = [m, n]$.
More in general, any pullback of a crossed module along a crossed module gives an example of a crossed square.

Crossed squares and their morphisms form a category, which is equivalent to the category of **internal crossed modules in the category of crossed modules.**

This category can be described also by using the notion of **strict** categorical crossed modules, as shown in [Carrasco, Cegarra, Garzón 2010]. This last notion was introduced in [Carrasco, Garzón, Vitale 2006] as a 2-dimensional analog of a crossed module in the category of categorical groups.

Given a crossed square

$$\begin{tikzcd} G_1 \arrow[r, "p_1"] \arrow[d, "\partial"'] & \Gamma_1 \arrow[d, "\partial'"] \\ G_0 \arrow[r, "p_0"] & \Gamma_0, \end{tikzcd}$$

we can form the pullback of p_0 along ∂'

$$\begin{tikzcd} G_1 \arrow[rr, "p_1"] \arrow[dd, "\partial"'] & & \Gamma_1 \arrow[dd, "\partial'"] \\ & G_0 \times_{\Gamma_0} \Gamma_1 \arrow[ur] \arrow[dl, "p_{G_0}"'] & \\ G_0 \arrow[rr, "p_0"] & & \Gamma_0, \end{tikzcd} \tag{2}$$

Homotopical kernels of crossed squares

Given a crossed square

$$\begin{tikzcd} G_1 \arrow[r, "p_1"] \arrow[d, "\partial"'] & \Gamma_1 \arrow[d, "\partial'"] \\ G_0 \arrow[r, "p_0"] & \Gamma_0, \end{tikzcd}$$

we can form the pullback of p_0 along ∂'

$$\begin{tikzcd} G_1 \arrow[rr, "p_1"] \arrow[dr, "\overline{\partial}"] \arrow[dd, "\partial"'] & & \Gamma_1 \arrow[dd, "\partial'"] \\ & G_0 \times_{\Gamma_0} \Gamma_1 \arrow[ur] \arrow[dl, "p_{G_0}"'] & \\ G_0 \arrow[rr, "p_0"] & & \Gamma_0, \end{tikzcd} \tag{2}$$

It turns out that the induced morphism $\overline{\partial} : G_1 \to G_0 \times_{\Gamma_0} \Gamma_1$ gives rise to a crossed module, where the action of $G_0 \times_{\Gamma_0} \Gamma_1$ on G_1 is given by ${}^{(g,\theta)}\alpha = {}^{g}\alpha$.

Remark

- *If* $< p_1, p_0 >$ *is just a morphism of crossed modules then* $\overline{\partial} : G_1 \to G_0 \times_{\Gamma_0} \Gamma_1$ *is still a crossed module.*
- *The previous result is true also in the internal version, where* $< p_1, p_0 >$ *is a morphism of crossed modules in a semi-abelian category (with Huq=Smith).*
- *More, if in this context we take a weak morphism, i.e. a butterfly:*

$$\begin{tikzcd}
G_1 \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & \Gamma_1 \arrow[dl, "\iota"'] \arrow[dd, "\partial'"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
G_0 & & \Gamma_0
\end{tikzcd}$$

Remark

- *If* $< p_1, p_0 >$ *is just a morphism of crossed modules then* $\overline{\partial} : G_1 \to G_0 \times_{\Gamma_0} \Gamma_1$ *is still a crossed module.*
- *The previous result is true also in the internal version, where* $< p_1, p_0 >$ *is a morphism of crossed modules in a semi-abelian category (with Huq=Smith).*
- *More, if in this context we take a weak morphism, i.e. a butterfly:*

$$
\begin{tikzcd}
G_1 \arrow[rr, equal] \arrow[ddd, "\overline{\partial}"] & & G_1 \arrow[ddd, "\partial"'] \arrow[dr, "\kappa"] & & \Gamma_1 \arrow[dl, "\iota"'] \arrow[ddd, "\partial'"] \\
 & & & E \arrow[ddl, "\sigma"] \arrow[ddr, "\rho"'] & \\
 & & & & \\
K \arrow[r] & E \arrow[r, "\sigma"] & G_0 & & \Gamma_0
\end{tikzcd}
$$

we can still construct an homotopical kernel by taking the kernel $\ker\rho : K \to E$ *of* ρ *and considering the induced arrow* $\overline{\partial} : G_1 \to K$*, which is still a crossed module in* $\mathbb{C}$*.*

If we call $\mathbf{G}$ the strict categorical group associated with $\partial : G_1 \to G_0$ and $\mathbf{\Gamma}$ the strict categorical group associated with $\partial' : \Gamma_1 \to \Gamma_0$, there is an associated strict categorical crossed module $\mathbf{T} : \mathbf{G} \to \mathbf{\Gamma}$. The homotopical kernel $\ker \mathbf{T}$ of $\mathbf{T} : \mathbf{G} \to \mathbf{\Gamma}$ of [Carrasco, Garzón, Vitale 2006] is a strict categorical group that corresponds to the crossed module

$$\overline{\partial} : G_1 \quad \to \quad G_0 \times_{\Gamma_0} \Gamma_1 .$$

Homotopical kernels of crossed squares

We prove the following Propositions.

Proposition

The diagram

$$
\begin{tikzcd}
G_1 \arrow[r, equal] \arrow[d, "\overline{\partial}"'] & G_1 \arrow[d, "\partial"] \\
G_0 \times_{\Gamma_0} \Gamma_1 \arrow[r, "p_{G_0}"] & G_0
\end{tikzcd} \tag{3}
$$

gives rise to a crossed square.

Remark

If $< p_1, p_0 >$ *is just a morphism of crossed modules then (3) is still a crossed square.*

ker $\mathbf{T}$ as a strict categorical $\mathbf{\Gamma}$-crossed module

Proposition

The outer diagram

$$\begin{tikzcd}
G_1 \arrow[r, equal] \arrow[d, "\overline{\partial}"'] \arrow[rr, bend left, "p_1"] & G_1 \arrow[r, "p_1"] \arrow[d, "\partial"'] & \Gamma_1 \arrow[d, "\partial'"] \\
G_0 \times_{\Gamma_0} \Gamma_1 \arrow[r, "p_{G_0}"] \arrow[rr, bend right, "\overline{p}_0"'] & G_0 \arrow[r, "p_0"] & \Gamma_0
\end{tikzcd} \tag{4}$$

gives rise to a crossed square.

Homotopical cokernels of crossed squares

For cokernels, in the abelian context, we can work dually by using pushouts instead of pullbacks. But in a semi-abelian situation, we have to work with the push forward construction, introduced in [Noohi 2008] for groups and extended for semi-abelian categories in [Cigoli,M.,Metere]. If we start with just a morphism of crossed modules:

$$\begin{tikzcd} G_1 \arrow[r, "p_1"] \arrow[d, "\partial"'] & \Gamma_1 \arrow[d, "\partial'"] \\ G_0 \arrow[r, "p_0"] & \Gamma_0, \end{tikzcd}$$

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$$\begin{tikzcd}
G_1 \arrow[rr, "p_1"] \arrow[dr, "<-p_1{,}\partial>"] \arrow[dd, "\partial"'] & & \Gamma_1 \arrow[dd, "\partial'"] \\
& \Gamma_1 \rtimes G_0 \arrow[dr, "\varphi"] & \\
G_0 \arrow[rr, "p_0"] & & \Gamma_0,
\end{tikzcd}$$

we take its "mapping cone", which is $< -p_1, \partial >$: $G_1 \to \Gamma_1 \rtimes G_0$ followed by $\varphi : \Gamma_1 \rtimes G_0 \to \Gamma_0$ given by the product $\partial'(\gamma) \cdot p_0(g)$, which is a morphism (a sort of twisted cooperator).

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$$\begin{tikzcd}
G_1 \arrow[rrr, "p_1"] \arrow[dr, "<-p_1{,}\partial>"] \arrow[dd, "\partial"'] & & & \Gamma_1 \arrow[dd, "\partial'"] \\
& \Gamma_1 \rtimes G_0 \arrow[r, "q"] \arrow[drr, "\varphi"] & \Gamma_1 \rtimes^{G_1} G_0 & \\
G_0 \arrow[rrr, "p_0"] & & & \Gamma_0,
\end{tikzcd}$$

we take its "mapping cone", which is $< -p_1, \partial >$: $G_1 \to \Gamma_1 \rtimes G_0$ followed by $\varphi : \Gamma_1 \rtimes G_0 \to \Gamma_0$ given by the product $\partial'(\gamma) \cdot p_0(g)$, which is a morphism (a sort of twisted cooperator). If we take the cokernel $q : \Gamma_1 \rtimes G_0 \to \Gamma_1 \rtimes^{G_1} G_0$ of $< -p_1, \partial >$, we get

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$$
\begin{tikzcd}
G_1 \arrow[rrr, "p_1"] \arrow[dr, "<-p_1{,}\partial>"] \arrow[dd, "\partial"'] & & & \Gamma_1 \arrow[dd, "\partial'"] \\
& \Gamma_1 \rtimes G_0 \arrow[r, "q"] \arrow[drr, "\varphi"] & \Gamma_1 \rtimes^{G_1} G_0 & \\
G_0 \arrow[rrr, "p_0"] & & & \Gamma_0,
\end{tikzcd}
$$

we take its "mapping cone", which is $< -p_1, \partial >$: $G_1 \to \Gamma_1 \rtimes G_0$ followed by $\varphi : \Gamma_1 \rtimes G_0 \to \Gamma_0$ given by the product $\partial'(\gamma) \cdot p_0(g)$, which is a morphism (a sort of twisted cooperator). If we take the cokernel $q : \Gamma_1 \rtimes G_0 \to \Gamma_1 \rtimes^{G_1} G_0$ of $< -p_1, \partial >$, we get

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Homotopical cokernels of crossed squares

For cokernels, in the abelian context, we can work dually by using pushouts instead of pullbacks. But in a semi-abelian situation, we have to work with the push forward construction, introduced in [Noohi 2008] for groups and extended for semi-abelian categories in [Cigoli,M.,Metere]. If we start with just a morphism of crossed modules:

$$\begin{tikzcd}
G_1 \arrow[rrr, "p_1"] \arrow[dr, "<-p_1{,}\partial>"] \arrow[dd, "\partial"'] & & & \Gamma_1 \arrow[dd, "\partial'"] \\
& \Gamma_1 \rtimes G_0 \arrow[r, "q"] \arrow[drr, "\varphi"] & \Gamma_1 \rtimes^{G_1} G_0 \arrow[dr, "d"] & \\
G_0 \arrow[rrr, "p_0"] & & & \Gamma_0,
\end{tikzcd}$$

we take its "mapping cone", which is $< -p_1, \partial >: G_1 \to \Gamma_1 \rtimes G_0$ followed by $\varphi : \Gamma_1 \rtimes G_0 \to \Gamma_0$ given by the product $\partial'(\gamma) \cdot p_0(g)$, which is a morphism (a sort of twisted cooperator). If we take the cokernel $q : \Gamma_1 \rtimes G_0 \to \Gamma_1 \rtimes^{G_1} G_0$ of $< -p_1, \partial >$, we get

- a comparison morphism $d : \Gamma_1 \rtimes^{G_1} G_0 \to \Gamma_0$
-

Homotopical cokernels of crossed squares

For cokernels, in the abelian context, we can work dually by using pushouts instead of pullbacks. But in a semi-abelian situation, we have to work with the push forward construction, introduced in [Noohi 2008] for groups and extended for semi-abelian categories in [Cigoli,M.,Metere]. If we start with just a morphism of crossed modules:

$$\begin{tikzcd}
G_1 \arrow[rrrr, "p_1"] \arrow[dr, "<-p_1{,}\partial>"] \arrow[dd, "\partial"'] & & & & \Gamma_1 \arrow[dl, "\partial''"'] \arrow[dd, "\partial'"] \\
& \Gamma_1 \rtimes G_0 \arrow[rr, "q"] \arrow[drrr, "\varphi"] & & \Gamma_1 \rtimes^{G_1} G_0 \arrow[dr, "d"] & \\
G_0 \arrow[rrrr, "p_0"] & & & & \Gamma_0,
\end{tikzcd}$$

we take its "mapping cone", which is $< -p_1, \partial >$: $G_1 \to \Gamma_1 \rtimes G_0$ followed by $\varphi : \Gamma_1 \rtimes G_0 \to \Gamma_0$ given by the product $\partial'(\gamma) \cdot p_0(g)$, which is a morphism (a sort of twisted cooperator). If we take the cokernel $q : \Gamma_1 \rtimes G_0 \to \Gamma_1 \rtimes^{G_1} G_0$ of $< -p_1, \partial >$, we get

- a comparison morphism $d : \Gamma_1 \rtimes^{G_1} G_0 \to \Gamma_0$
- a crossed module ∂'', which is the push forward of ∂ along p_1

But in the case of a crossed square of groups, Conduché observed that then the associated mapping cone

$$G_1 \xrightarrow{<-p_1,\partial>} \Gamma_1 \rtimes G_0 \xrightarrow{\varphi} \Gamma_1$$

has a structure of a 2-crossed module and this ensures that $d : \Gamma_1 \rtimes^{G_1} G_0 \to \Gamma_0$ is actually a crossed module in this case.

We obtained also a direct prove, by showing that in the case of crossed squares, d corresponds to the quotient categorical group $\dfrac{\mathbf{\Gamma}}{< \mathbf{G}, \mathbf{T} >}$ introduced in [Carrasco, Garzón, Vitale 2006], where $\mathbf{T} : \mathbf{G} \to \mathbf{\Gamma}$ is the strict categorical crossed module associated with the crossed square we started with.

Using the previous result, we can then form a new diagram taking into account both the constructions of kernel and cokernel of a crossed square:

Proposition

The outer diagram

$$
\begin{tikzcd}
G_1 \arrow[r, equal] \arrow[d, "\overline{\partial}"'] \arrow[rrr, bend left, "\tilde{p}_1"] & G_1 \arrow[r, "p_1"] \arrow[d, "\partial"'] & \Gamma_1 \arrow[r, "\partial''"] \arrow[d, "\partial'"] & \Gamma_1 \rtimes^{G_1} G_0 \arrow[d, "d"] \\
G_0 \times_{\Gamma_0} \Gamma_1 \arrow[r, "p_{G_0}"] \arrow[rr, bend right, "\overline{\overline{p}}_0"'] \arrow[rrr, bend right=40, "\tilde{p}_0"'] & G_0 \arrow[r, "p_0"] & \Gamma_0 \arrow[r, equal] & \Gamma_0
\end{tikzcd}
\tag{5}
$$

gives rise to a crossed square, where the function $\overline{h} : (\Gamma_1 \rtimes^{G_1} G_0) \times (G_0 \times_{\Gamma_0} \Gamma_1)) \to G_1$ *is given by*

$$\overline{h}((\beta_1, g_1), (\beta_2, g_2)) = h(\beta_1, g_1 g_2 {g_1}^{-1}) h(\beta_2, g_1)^{-1}$$

, where h is in the structure of the original crossed square.

When we apply this construction to the crossed square of the intersection of 2 normal subgroups, which is already a pullback, we get:

$$\begin{tikzcd}
N \cap M \arrow[r, equal] \arrow[d, equal] & N \cap M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \cup M \arrow[d] \\
N \cap M \arrow[r] & N \arrow[r] & P \arrow[r, equal] & P
\end{tikzcd}$$

where we can see that the squares on the right are NOT crossed squares (the function h should be given by taking commutators).

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