A classification theorem for normal extensions

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Outline

Galois structures and extensions

Normalisation functor as a pointwise Kan extension

A classification theorem for normal extensions

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Definition

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Notation

$\mathrm{Ext}_{\mathcal{E}}(\mathscr{C})$: category of extensions.

Galois structures

We will only consider Galois structures of type (A), i.e. which satisfy:

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- I preserves pullbacks

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[h]{} & C \end{array}$$

  with g, h in $\mathcal{E}$ and g split epi;

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Notation

$f \in \mathrm{Split}\mathcal{E}$ if $f \in \mathcal{E}$ and f is a split epimorphism (a split extension).

Monadicity

Definition

An extension $p\colon E \to B$ is monadic if

$$p^*\colon \mathscr{C} \downarrow_{\mathcal{E}} B \to \mathscr{C} \downarrow_{\mathcal{E}} E \text{ is monadic}$$

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$$\begin{array}{ccccc} & & & & A \\ & & & & \downarrow{\scriptstyle f} \\ \mathrm{Eq}(p) & \underset{\pi_2^p}{\overset{\pi_1^p}{\rightrightarrows}} & E & \xrightarrow[p]{} & B. \end{array}$$

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$$\begin{array}{ccccc}
 & & E \times_B A & \xrightarrow{p'} & A \\
 & & {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
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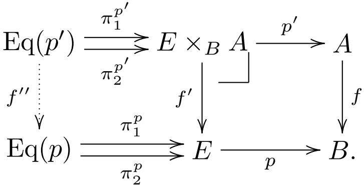

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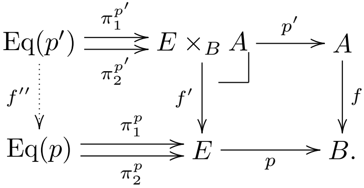

$$K^p(f) = (f', f'')$$

Monadicity

Lemma

For

$$\begin{array}{ccccc} F & \longrightarrow & A & \longrightarrow & D \\ \downarrow & (1) & \downarrow f & (2) & \downarrow h \\ E & \underset{p}{\longrightarrow} & B & \longrightarrow & C \end{array}$$

with $p, f, h \in \mathcal{E}$ *and (1)+(2) pullback:*

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$$\text{(1) pullback} \;\Leftrightarrow\; \text{(2) pullback}$$

Extensions

Definition

An extension $f : A \to B$ is a trivial extension if

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & I(E) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle I(f)} \\ B & \xrightarrow[\eta_B]{} & I(B) \end{array}$$

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Notation

$\mathrm{Triv}_\Gamma(\mathscr{C})$: category of trivial extensions.

Extensions

For an extension $p\colon E \to B$.

Definition

An extension $f : A \to B$ is is split by p if in the pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

p_1 is trivial.

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$\mathrm{Spl}_\Gamma(E, p)$: category of extensions split by p.

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An extension $f\colon A \to B$ is a normal extension if in

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π_1 and π_2 are trivial (i.e. if f is split by itself).

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A regular epimorphism $f\colon A \to B$ is normal iff

$$\mathsf{Ker} f \subseteq Z(A).$$

Extensions

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$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[h]{} & C \end{array}$$

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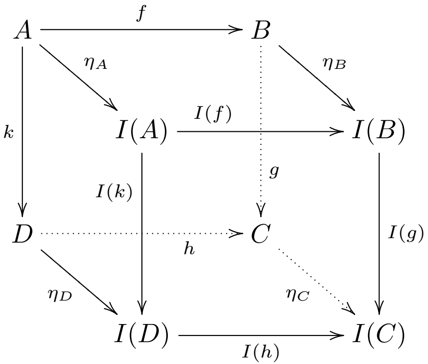
Extensions

Proof: Consider:



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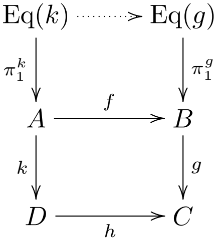

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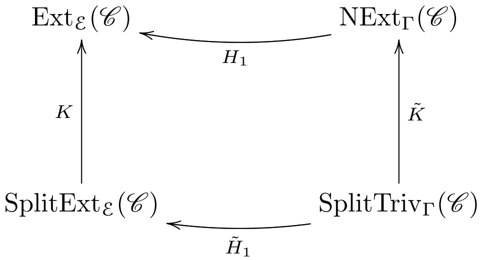
A classification theorem for normal extensions

Trivialisation functor

$$
\begin{array}{ccc}
\mathrm{Ext}_{\mathcal{E}}(\mathscr{C}) & \xleftarrow[H_1]{} & \mathrm{NExt}_{\Gamma}(\mathscr{C}) \\
\uparrow K & & \uparrow \tilde{K} \\
\mathrm{SplitExt}_{\mathcal{E}}(\mathscr{C}) & \xleftarrow[\tilde{H}_1]{} & \mathrm{SplitTriv}_{\Gamma}(\mathscr{C})
\end{array}
$$

Trivialisation functor



Notation

$\mathrm{SplitExt}_{\mathcal{E}}(\mathscr{C})$: category of split extensions.

Trivialisation functor

$$\begin{array}{ccc}
\mathrm{Ext}_{\mathcal{E}}(\mathscr{C}) & \xleftarrow{\quad H_1 \quad} & \mathrm{NExt}_{\Gamma}(\mathscr{C}) \\
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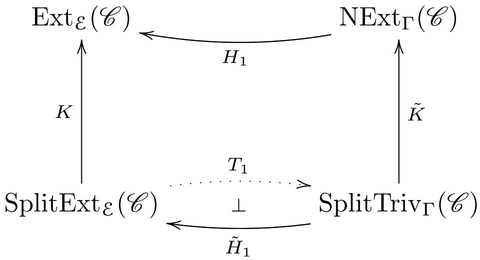

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$$\mathrm{SplitExt}_{\mathcal{E}}(\mathscr{C}) \underset{\tilde{H}_1}{\overset{T_1}{\rightleftarrows}} \mathrm{SplitTriv}_{\Gamma}(\mathscr{C}) \quad (\bot)$$

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For $f\colon A \to B$ in $\mathrm{Split}\mathcal{E}$:

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$$\mathrm{SplitExt}_{\mathcal{E}}(\mathscr{C}) \underset{\tilde{H}_1}{\overset{T_1}{\rightleftarrows}} \mathrm{SplitTriv}_{\Gamma}(\mathscr{C}) \qquad (T_1 \dashv \tilde{H}_1)$$

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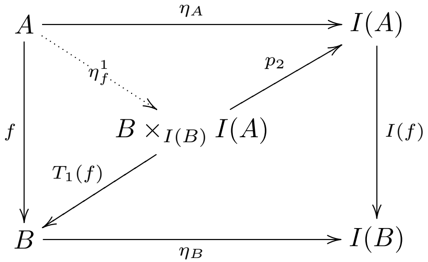
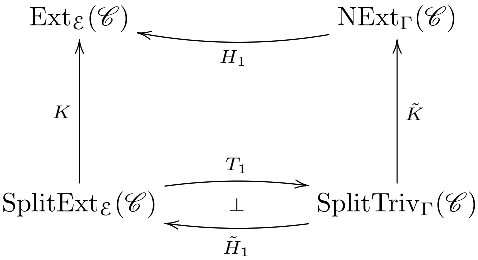


Normalisation functor as a pointwise Kan extension



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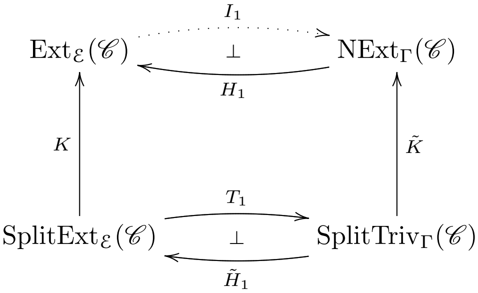

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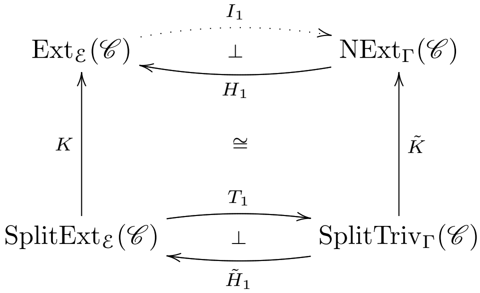

Normalisation functor as a pointwise Kan extension

Proof:

- For all f in $\mathrm{Ext}_{\mathcal{E}}(\mathscr{C})$, the comma category $K \downarrow f$

$$
\begin{array}{ccc}
K \downarrow f & \longrightarrow & 1 \\
{\scriptstyle P^f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\mathrm{SplitExt}_{\mathcal{E}}(\mathscr{C}) & \xrightarrow[K]{} & \mathrm{Ext}_{\mathcal{E}}(\mathscr{C})
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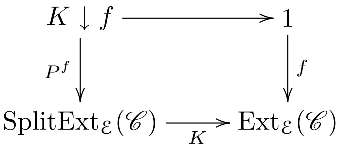
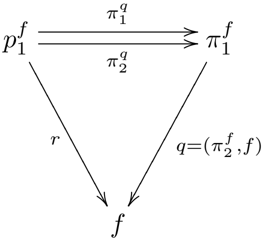


admits J_f



as a final subcategory (the inclusion functor $L_f \colon J_f \to K \downarrow f$ is final).

Normalisation functor as a pointwise Kan extension

$$p_1^f \underset{\pi_2^q}{\overset{\pi_1^q}{\rightrightarrows}} \pi_1^f \xrightarrow{q} f$$

Normalisation functor as a pointwise Kan extension

$$p_1^f \underset{\pi_2^q=(p_2^f,\pi_2^f)}{\overset{\pi_1^q=(\tau,\pi_1^f)}{\rightrightarrows}} \pi_1^f \xrightarrow{q=(\pi_2^f,f)} f$$

$$\begin{array}{ccccc}
\mathrm{Eq}(\pi_2^f) & \underset{p_2^f}{\overset{\tau}{\rightrightarrows}} & A \times_B A & \xrightarrow{\pi_2^f} & A \\
{\scriptstyle p_1^f}\Big\downarrow & & {\scriptstyle \pi_1^f}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\
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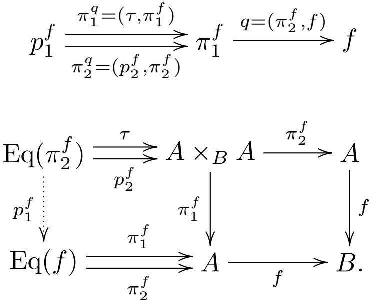

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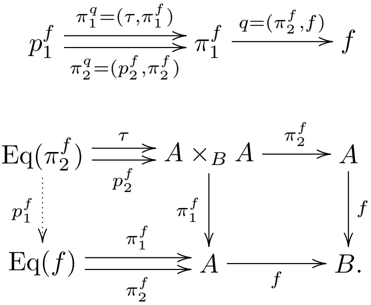

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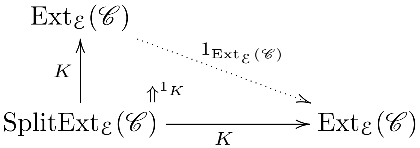
\end{array}$$

$$f = \mathsf{Colim}(K \circ P^f \circ L_f) = \mathsf{Colim}(K \circ P^f)$$

- Consequently, $K\colon \mathrm{SplitExt}_{\mathcal{E}}(\mathscr{C}) \to \mathrm{Ext}_{\mathcal{E}}(\mathscr{C})$ is dense.

Normalisation functor as a pointwise Kan extension

- Equivalently, $1_{\mathrm{Ext}_{\mathcal{E}}(\mathscr{C})} = \mathrm{Lan}_K(K)$:



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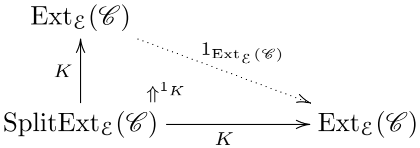

- Consequently, if I_1 exists, $I_1 = \mathrm{Lan}_K(I_1 \circ K) = \mathrm{Lan}_K(\tilde{K} \circ T_1)$:

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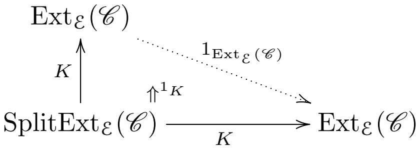
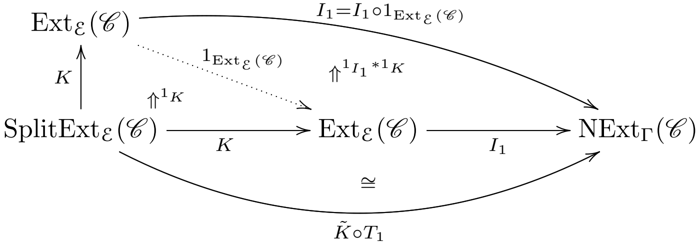


- Consequently, if I_1 exists, $I_1 = \mathrm{Lan}_K(I_1 \circ K) = \mathrm{Lan}_K(\tilde{K} \circ T_1)$:



Normalisation functor as a pointwise Kan extension

- One can show that there is an isomorphism

$$f \downarrow H_1 \cong \mathsf{Cocone}(\tilde{K} \circ T_1 \circ P^f)$$

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I_1 exists if the coequalizer of

$$T_1(p_1^f) \underset{T_1(\pi_2^q)}{\overset{T_1(\pi_1^q)}{\rightrightarrows}} T_1(\pi_1^f),$$

exists in $\mathrm{NExt}_\Gamma(\mathscr{C})$ for every f in $\mathrm{NExt}_\Gamma(\mathscr{C})$

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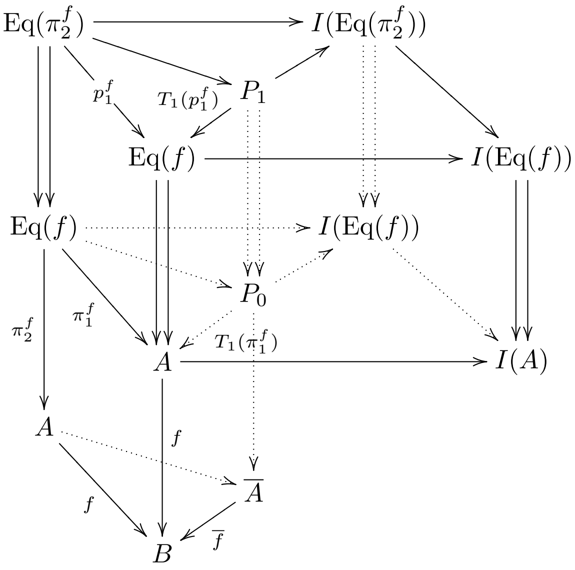
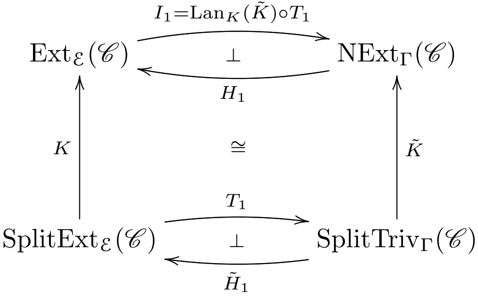


Normalisation functor as a pointwise Kan extension: Summary

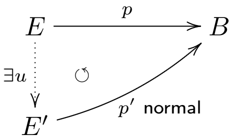


Outline

Weakly universal normal extensions

Definition

A normal extension $p \colon E \to B$ is weakly universal if it factors through every other normal extension with the same codomain:



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For all B *in* $\mathscr{C}$ *one can construct a weakly universal normal extension of* B.

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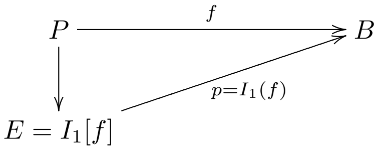

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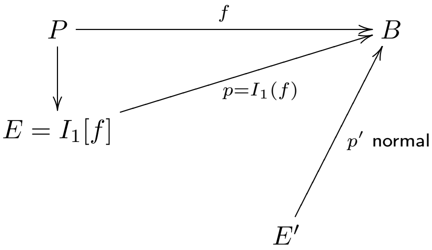

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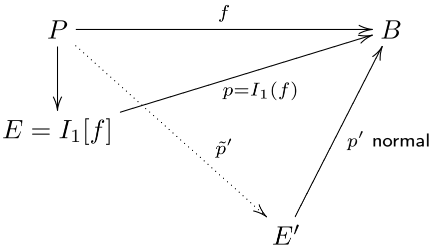

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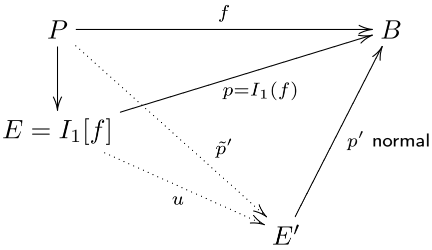


A classification theorem for normal extensions

Theorem

Let Γ be a Galois structure of type (A):

$$\mathscr{X} \underset{\subseteq}{\overset{I}{\rightleftarrows}} \mathscr{C} \quad + \quad \mathcal{E}$$

and

$$p\colon E \to B$$

a weakly universal normal extension of B. Then one has an equivalence of categories

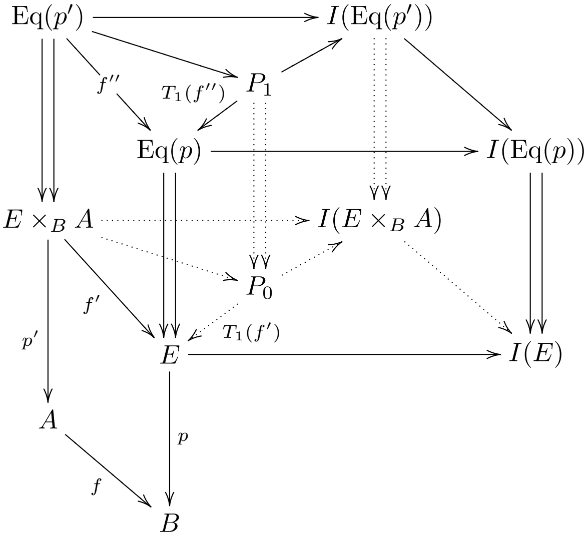
$$\mathrm{NExt}_\Gamma(B) \cong \mathscr{X}^{\downarrow_{\mathrm{Split}\,\mathcal{E}}\mathrm{Gal}(E,p)}$$

where $\mathrm{Gal}(E,p)$ is the internal groupoid in $\mathscr{X}$

$$I(\mathrm{Eq}(p) \times_E \mathrm{Eq}(p)) \overset{I(p_1^p),\ I(\tau),\ I(p_2^p)}{\rightrightarrows} I(\mathrm{Eq}(p)) \overset{I(\pi_1^p),\ I(\pi_2^p)}{\underset{I(\delta)}{\rightleftarrows}} I(E), \quad I(\sigma)\colon I(\mathrm{Eq}(p)) \to I(\mathrm{Eq}(p))$$

A classification theorem for normal extensions

Proof: For (A, f) in $\mathrm{NExt}_\Gamma(B)$:



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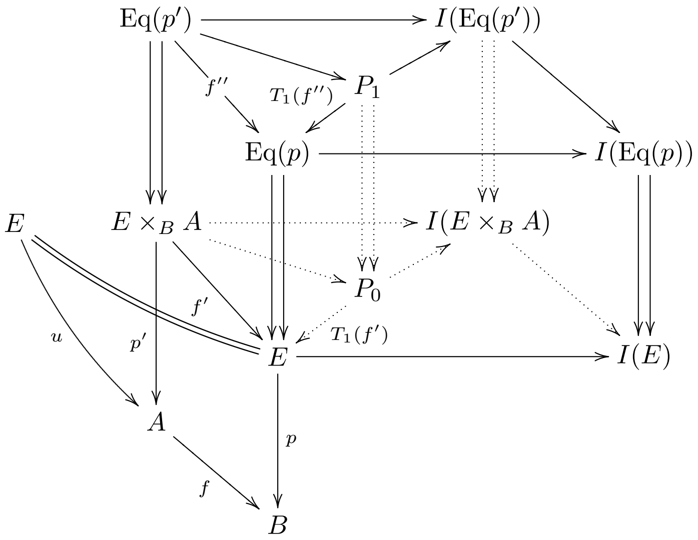

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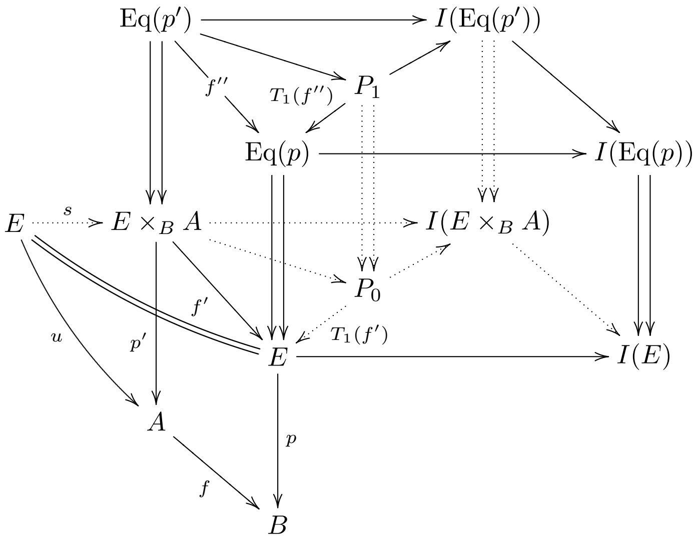

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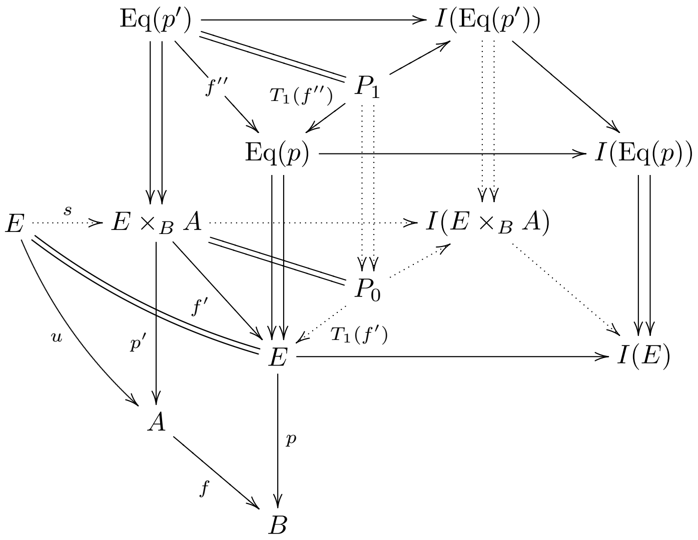

The classical categorical Galois theorem [G. Janelidze]

Theorem

Let Γ be an *admissible* *Galois structure :*

$$\mathscr{C} \underset{\supseteq}{\overset{I}{\rightleftarrows}} \mathscr{X} \quad (\bot) \qquad + \qquad \mathcal{E}$$

and

$$p \colon E \to B$$

a weakly universal normal extension of B. Then one has an equivalence of categories

$$\mathrm{Spl}_{\Gamma}(E,p) \cong \mathscr{X}^{\downarrow_{\mathcal{E}} \mathrm{Gal}(E,p)}$$

A non-classical example

Example

The Galois structure

$$\mathrm{Ab} \underset{\supseteq}{\overset{I}{\rightleftarrows}} \mathrm{Ab}^* \quad (\bot) \qquad + \qquad \mathrm{RegEpi}(\mathrm{Ab})$$

where Ab^* is the full subcategory of Ab whose objects satisfy

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is not admissible but is of type (A).

The end

Thank you for your attention!