Derived categories and Fourier Mukai transforms in Algebraic Geometry

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Triangulated Categories

A triangulated category $\mathcal{D}$ is an additive category with

- an additive equivalence $T : \mathcal{D} \to \mathcal{D}$, called the *shift functor*;
- a set of *distinguished triangles* $A \to B \to C \to T(A)$ subject to axioms TR1-TR4 below.

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Morphisms between triangles:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] := T(A) \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1]:=T(f) \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] := T(A') \end{array}$$

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isomorphisms: if f, g, and h are isomorphisms.

Axioms of triangulated categories

TR1:

i) $A \xrightarrow{id} A \longrightarrow 0 \longrightarrow A[1]$ is distinguished.

ii) Triangles isomorphic to a distinguished triangles are distinguished.

iii) Morphisms $f : A \to B$ can be completed to distinguished triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$.

Axioms of triangulated categories

TR2:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is a distinguished triangle.

Axioms of triangulated categories

TR3: A commutative diagram of distinguished triangles

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Remark

- *TR1 + TR3 give that $A \longrightarrow C$ is zero.*
- *If two among $f, g,$ and h are isos, then so is the third.*

Equivalence of triangulated categories

Definition

An additive functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ *between triangulated categories* $\mathcal{D}$ *and* $\mathcal{D}'$ *is exact if:*

i) *There exists a functor isomorphism* $F \circ T_{\mathcal{D}} \stackrel{\sim}{\longrightarrow} T_{\mathcal{D}'} \circ F$.

ii) *A distinguished triangle* $A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \stackrel{h}{\longrightarrow} A[1]$ *in* $\mathcal{D}$ *is mapped to a distinguished triangle* $F(A) \stackrel{f}{\longrightarrow} F(B) \stackrel{g}{\longrightarrow} F(C) \stackrel{h}{\longrightarrow} F(A)[1]$ *in* $\mathcal{D}'$, *where* $F(A[1])$ *is identified with* $F(A)[1]$ *via the functor isomorphism in i).*

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Definition

Two triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ are equivalent if there exists an exact equivalence $F : \mathcal{D} \longrightarrow \mathcal{D}'$.

If D is triangulated, the set $Aut(D)$ of isomorphism classes of equivalences $F : \mathcal{D} \longrightarrow \mathcal{D}$ is the group of autoequivalences of D.

The category of complexes of an abelian category

Let $\mathcal{A}$ be an abelian category. We define $Kom(\mathcal{A})$:

- Objects are exact sequences

$$\ldots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \ldots$$

  i.e., $d^i \circ d^{i-1} = 0$;

- Morphisms:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \xrightarrow{d_A^{i+1}} & \ldots \\ & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} & & \\ \ldots & \longrightarrow & B^{i-1} & \xrightarrow[d_B^{i-1}]{} & B^i & \xrightarrow[d_B^i]{} & B^{i+1} & \xrightarrow[d_B^{i+1}]{} & \ldots \end{array}$$

If $\mathcal{A}$ is abelian, $Kom(\mathcal{A})$ is abelian again.

There is a shift functor T in $Kom(\mathcal{A})$:
$A^\bullet[1]$ is defined by $(A^\bullet[1])^i := A^{i+1}$ and $d^i_{A[1]} := -d^{i+1}_A$;
$f[1]^i := f^{i+1}$.
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Can define cohomology $H^i(A^\bullet)$ of complexes,
$H^i(A^\bullet) := \frac{Ker(d^i)}{Im(d^{i-1})} \in \mathcal{A}$.

Definition

A morphism of complexes $f : A^\bullet \longrightarrow B^\bullet$ is a quasi-isomorphism if for all $i \in \mathbb{Z}$ the induced map $H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism.

Theorem

Given an abelian category $\mathcal{A}$, there is a category $D(\mathcal{A})$ and a functor

$$Q : Kom(\mathcal{A}) \to D(\mathcal{A})$$

such that

(i) *If $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism, then $Q(f)$ is an isomorphism in $D(\mathcal{A})$.*

(ii) *$D(\mathcal{A})$ is universal for categories endowed with a morphism satisfying (i).*

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- Objects of $Kom(A)$ and $D(\mathcal{A})$ are identified via Q;
- There is a well defined cohomology of objects $H^i(A^\bullet)$ for $A \in D(\mathcal{A})$;
- $\mathcal{A}$ can be seen as the full subcategory of $D(\mathcal{A})$ of complexes such that $H^i(A^\bullet) = 0$ for $i \neq 0$.
- $D(\mathcal{A})$ is in general not abelian, but its triangulated!

Derived categories of coherent sheaves

Let X be a scheme (or algebraic variety).

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Definition

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Definition

The derived category of X is the bounded derived category of the abelian category $Coh(X)$,

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Two k-schemes X and Y are *derived equivalent* if there exists a k-linear exact equivalence $D^b(X) \sim D^b(Y)$.

Bondal-Orlov's result

Theorem (Bondal, Orlov)

Let X and Y be smooth projective varieties and assume that the (anti-)canonical bundle of X is ample. If there exists an exact equivalence $D^b(X) \sim D^b(Y)$, then X and Y are isomorphic.

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Is derived equivalence an interesting geometric notion (at least for smooth projective varieties)?

Fourier-Mukai transforms

Let $\mathcal{P} \in D^b(X \times Y)$. The induced Fourier-Mukai transform is

$$\begin{aligned} \Phi_{\mathcal{P}} : & D^b(X) \to D^b(Y), \\ & E^\bullet \mapsto \pi_{2*}(\pi_1{}^* E^\bullet \otimes \mathcal{P}). \end{aligned}$$

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Examples:

- $id : D^b(X) \to D^b(X)$ is $\Phi_{\mathcal{O}_\Delta}$;
- $f : X \to Y$, $f_* \sim \Phi_{\Gamma_f}$;
- $T : D^b(X) \to D^b(X)$ is $\Phi_{\mathcal{O}_\Delta[1]}$.

Proposition (Bondal, Orlov)

Φ_P is fully faithful if and only if for any two closed points $x, y \in X$

$$Hom(\Phi_P(k(x)), \Phi_P(k(y))[i]) = \begin{cases} k \text{ if } x = y \text{ and } i = 0 \\ 0 \text{ if } x \neq y \text{ or } i < 0 \text{ or } i > dim(X). \end{cases}$$

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Proposition

If $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ is fully faithful, then $\Phi_{\mathcal{P}}$ is an equivalence if and only if $\Phi_{\mathcal{P}}(k(x)) \otimes \omega_Y \cong \Phi_{\mathcal{P}}(k(x))$ for every closed point $x \in X$.

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Theorem (Orlov)

If $F : D^b(X) \to D^b(Y)$ is fully faithful and exact functor admitting right and left adjoint functors, then there exists a unique $\mathcal{P} \in D^b(X \times Y) : F \sim \Phi_{\mathcal{P}}$.

Abelian Varieties

An **abelian variety** A is a projective connected algebraic k-group.

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$$Pic^0 A \cong Hom(\ \ , \hat{A}),$$

where

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$\hat{A}$ is abelian as well.
Let $\mathcal{P} \in Pic(\hat{A})$ be the element corresponding to $id_{\hat{A}} \in Hom(\hat{A}, \hat{A})$: $\mathcal{P}$ is called the *Poincaré bundle*.

Theorem (Mukai)

If $\mathcal{P}$ is the Poincaré bundle on $A \times \hat{A}$, then

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When are two (smooth projective) varieties derived equivalent?

Hitchin fibration

Given a curve X, let $\mathcal{H}iggs$ be the moduli space of Higgs bundles, parametrising pairs (E, θ):

- E vector bundle (G-bundle) on X;
- $\theta : E \to E \otimes \omega_C$ Higgs field.

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$$h : \mathcal{H}iggs \longrightarrow \mathbb{A}^r$$

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Classical limit of Geometric Langlands Conjecture

$\exists D^b(\mathcal{H}iggs) \xrightarrow{\sim} D^b(\mathcal{H}iggs)$ equivalence of triangulated categories:

(i) $D^b(h^{-1}(a)) \xrightarrow{\sim} D^b(h^{-1}(a))$;

(ii) "intertwines" Hecke operators and translation operators.

Beauville-Narasimhan-Ramanan, Schaub correspondence

$h^{-1}(a) \cong \overline{J}_X$, a compactified (Picard) variety of $Pic^0(\tilde{X}_a)$, where $\tilde{X}_a$ is the spectral curve of X (a possibly singular covering of X living in the total space of the Hitchin fibration).

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If $\tilde{X}_a$ is smooth CLGLC(i) follows from Mukai's result.

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If $\tilde{X}_a$ is smooth CLGLC(i) follows from Mukai's result.
If $\tilde{X}_a$ is singular, $Pic^0(\tilde{X}_a)$ is a semiabelian variety and $\overline{J}_X$ is a compactification of it: projective variety but not an algebraic group.

Theorem (M, Rapagnetta, Viviani)

Let X be a reduced curve with planar singularities. Then there is a Poincaré sheaf $\overline{P}$ on $\overline{J}_X \times \overline{J}_X$ such that

$$\Phi^{\overline{P}} : D^b(\overline{J}_X) \to D^b(\overline{J}_X)$$

is an equivalence of categories.

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Applications

- CLGLC(i);
- Study of the Hitchin fibration (e.g the study of the cohomology of the fibers of the Hitchin fibration in the singular locus was fundamental in Ngo's work);
- Kawamata's conjecture on derived equivalence being identified with birationality for CY varieties.

Further directions

- Derived categories seem to be appropriate to study birational aspects of algebro-geometric varieties (Kawamata's conjecture);
- Kontsevich homological mirror symmetry: mirror symmetry can be seen as an equivalence of the derived category of coherent sheaves of certain projective varieties with Fukaya categories associated to symplectic geometry of the mirror.

Thank you!