Torsional Alfvén waves in a rotating spherical shell: transmission and reflection in the Earth's outer core

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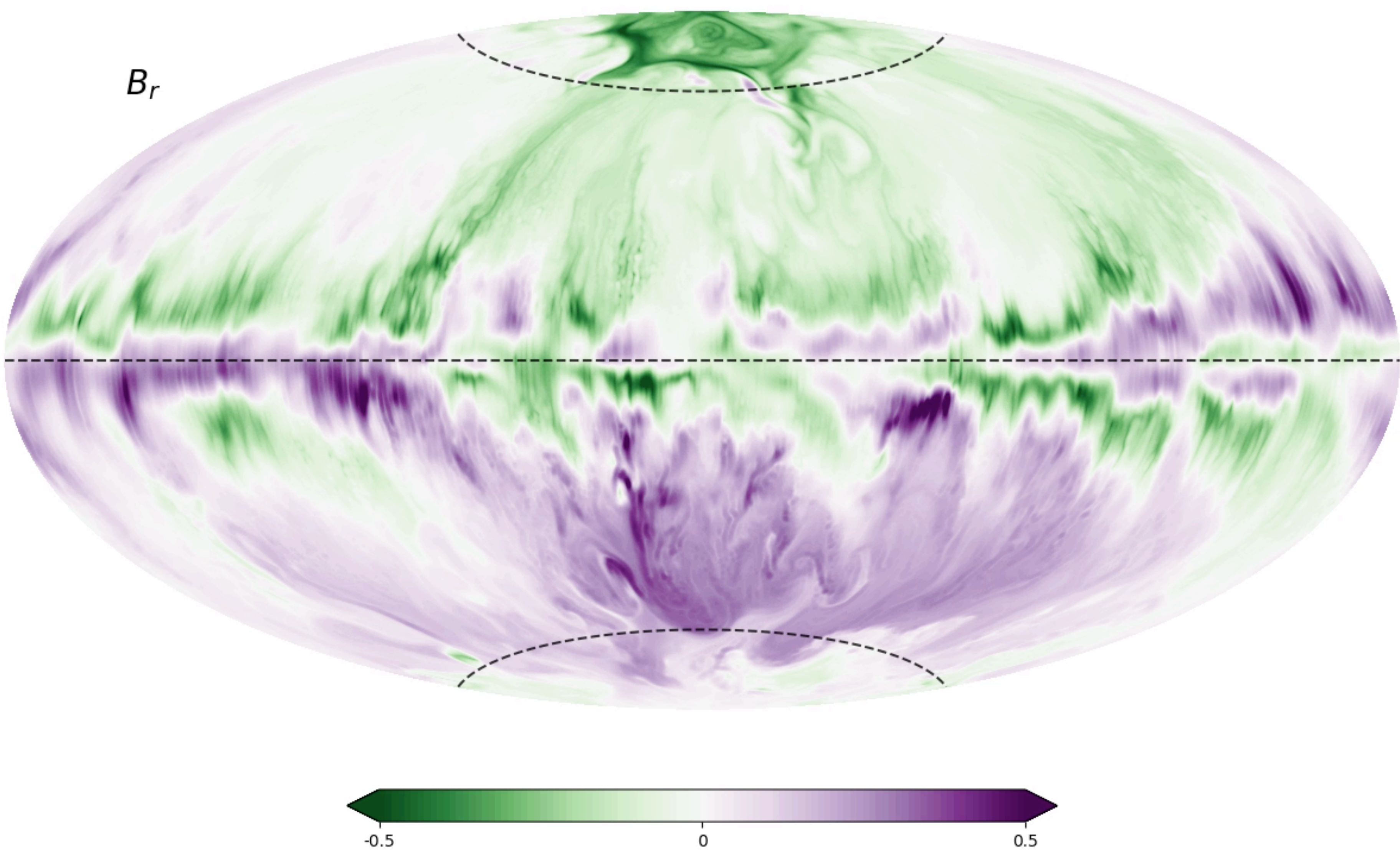
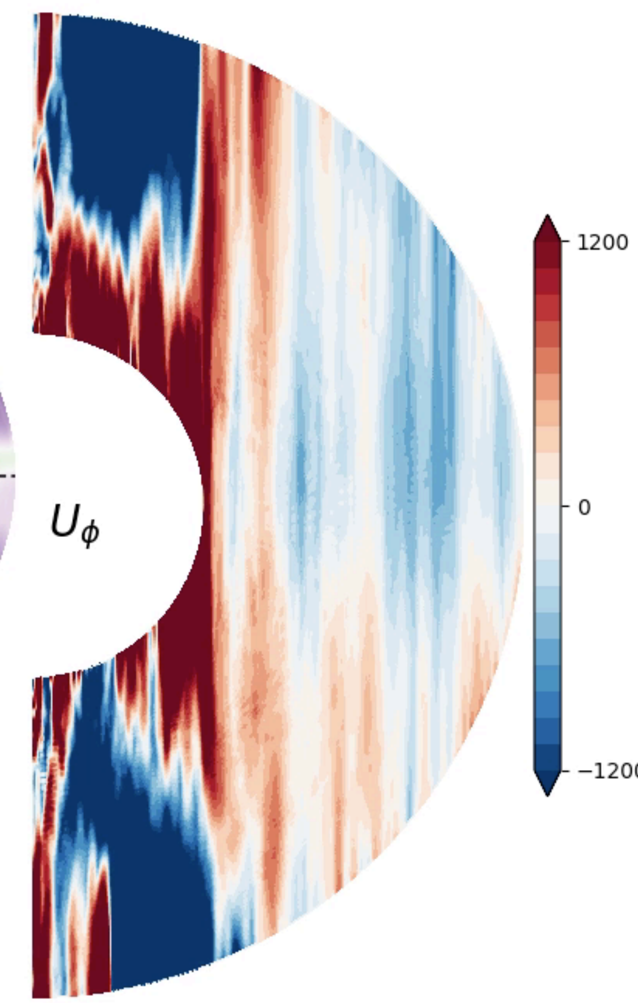
Space Weather: coupling of ionosphere, magnetosphere and Earth's core

Torsional oscillations from numerical simulations of the geodynamo



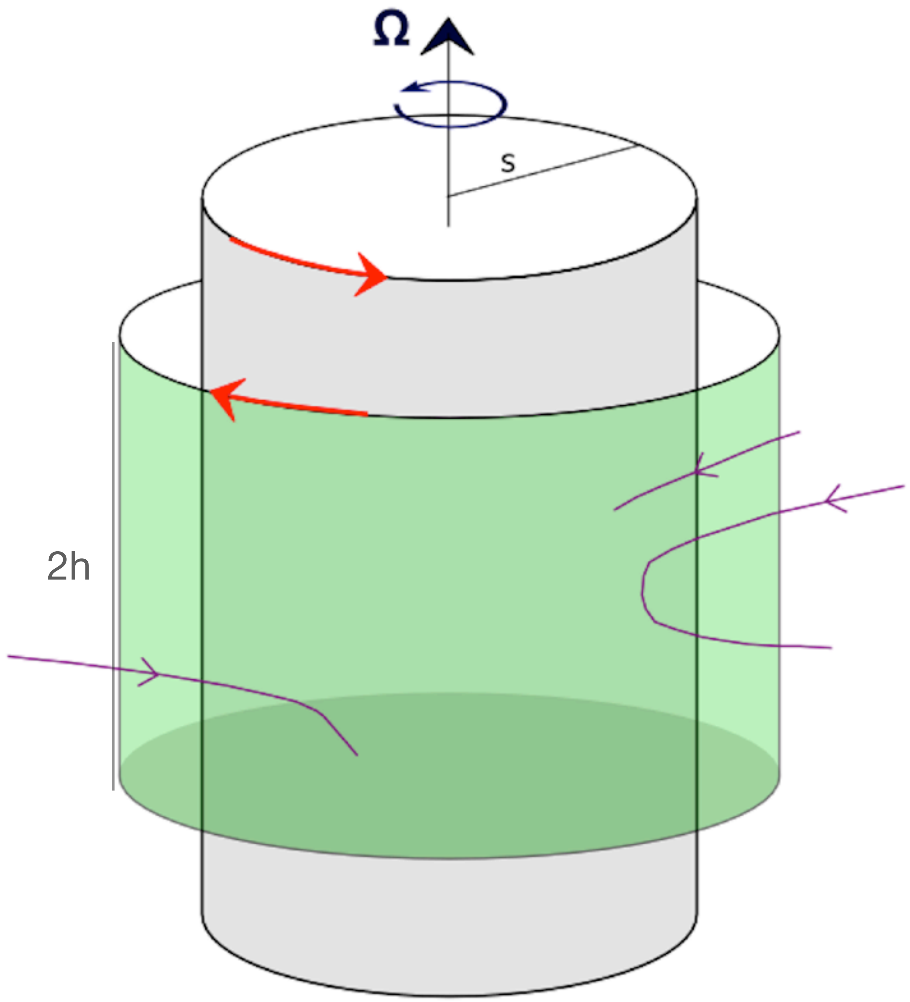


Pm=0.1, Ro=5 10^{-4}, Le=2 10^{-3}, A=0.27

See also *Aubert, 2018*: Pm=, Ro=2.4 10^{-4}, Le=2.2 10^{-3}, A=0.11

Schaeffer & al., 2017

Torsional Alfvén waves



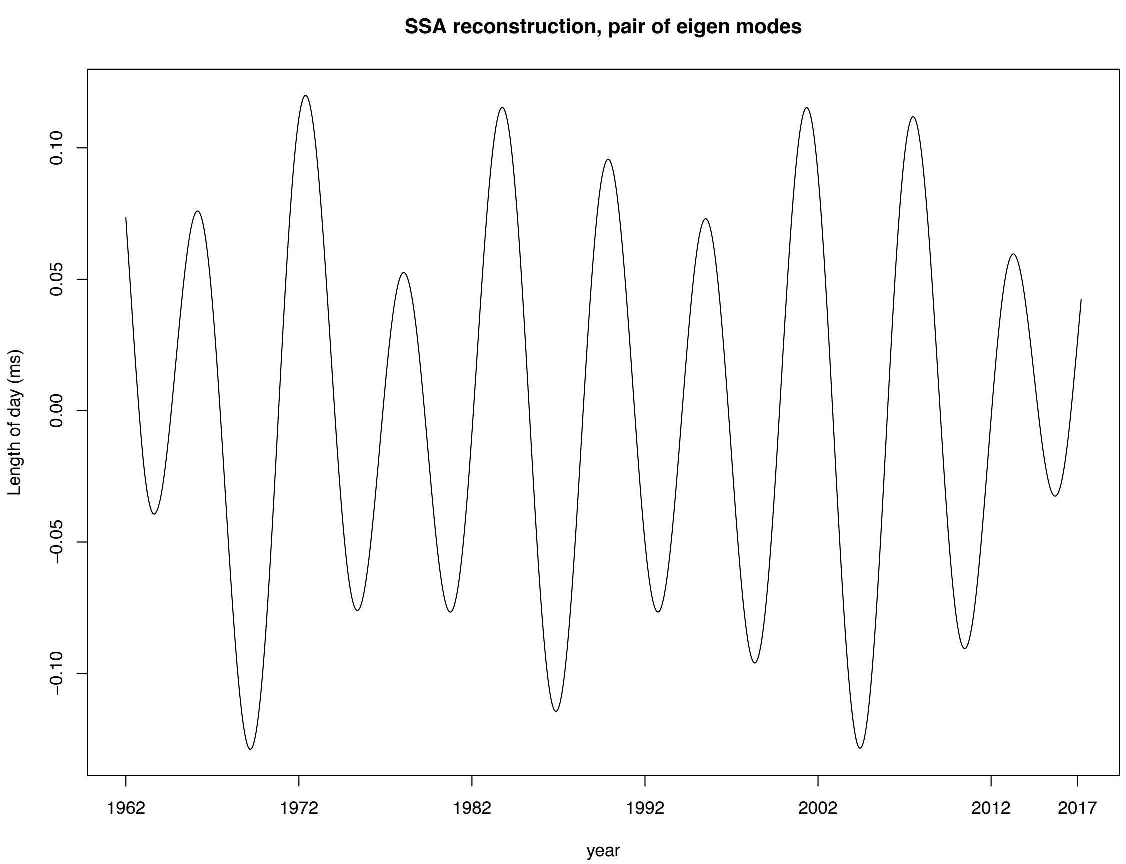
$$\frac{\partial^2 \zeta_G}{\partial t^2} = \frac{1}{m}\frac{\partial}{\partial s}\left(m V_A^2 \frac{\partial \zeta_G}{\partial s}\right),$$

$$m = s^3 h, \qquad V_A^2 = \frac{1}{\Sigma}\iint \frac{B_s^2}{\mu_0 \rho}\mathrm{d}\Sigma$$

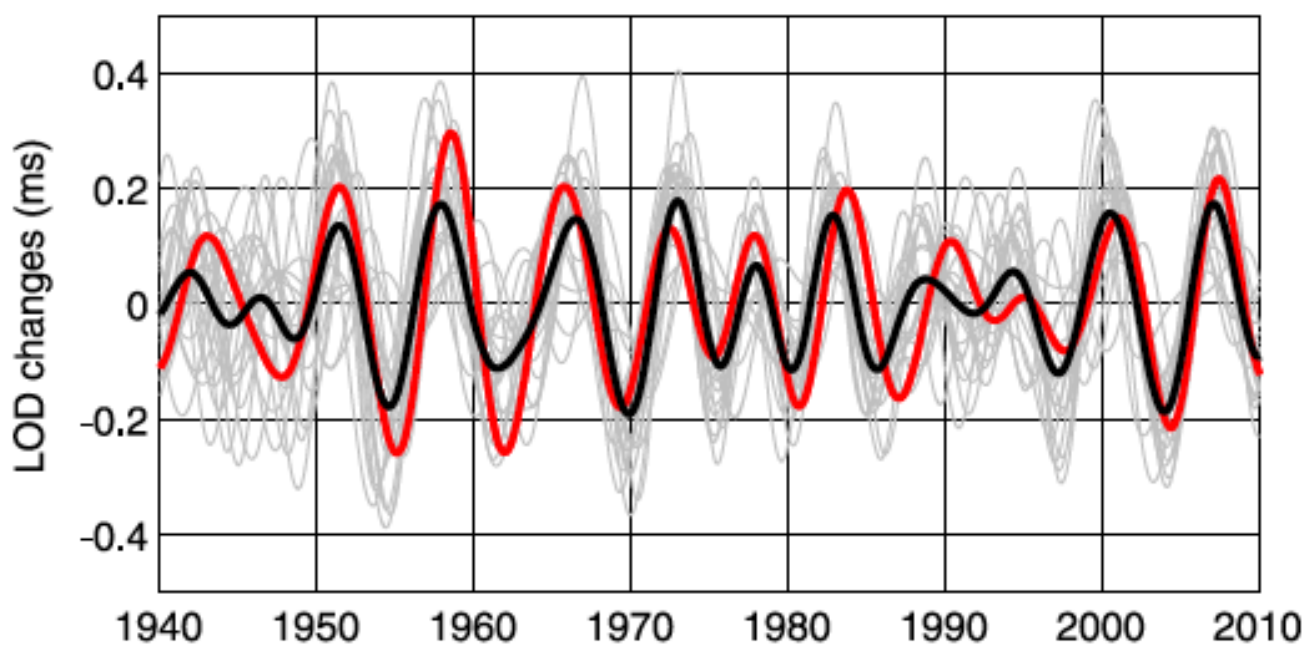
$$s^2 + h^2 = 1, \quad \Sigma = 4\pi s h$$

Torsional waves consist of geostrophic motions coupled by the magnetic field in the Earth's fluid outer core

Six year oscillation in length-of-day



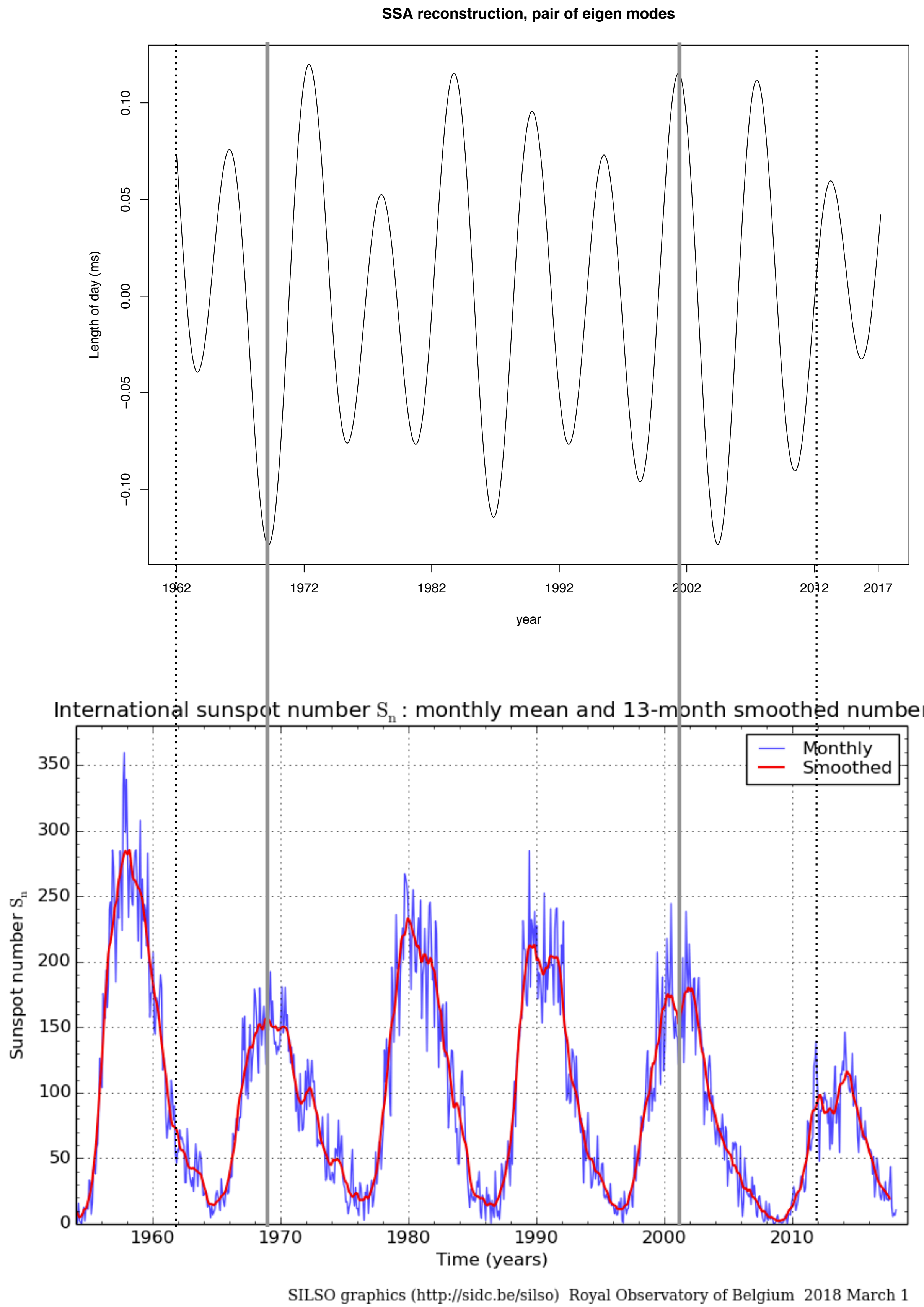
4 - 9.5 year band pass filter



Gillet & al. (2015)

Torsional Alfvén modes of periods ~ 6 years and less

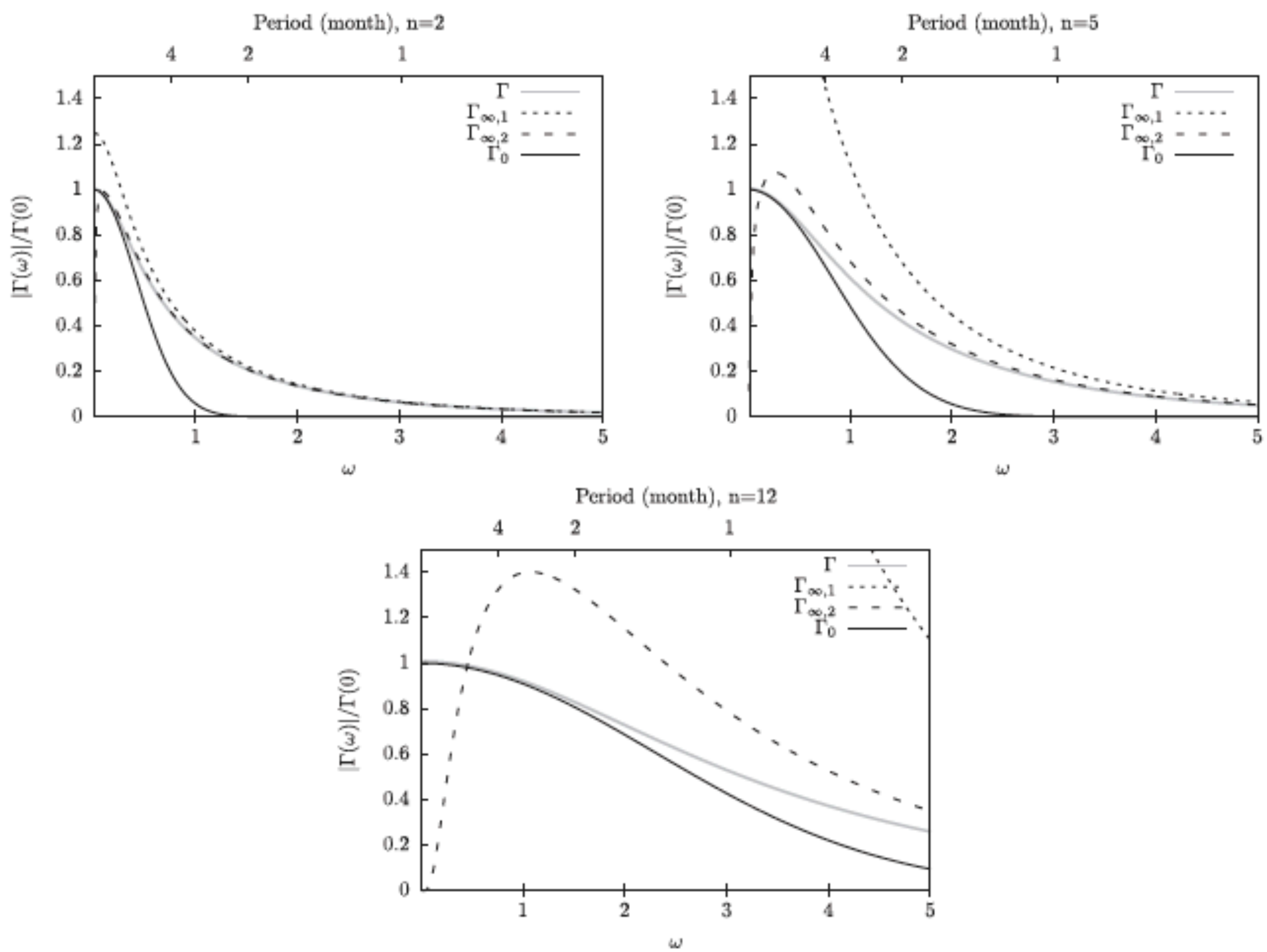
Not an harmonic
of the solar cycle



Coupling between external magnetic fields and torsional waves

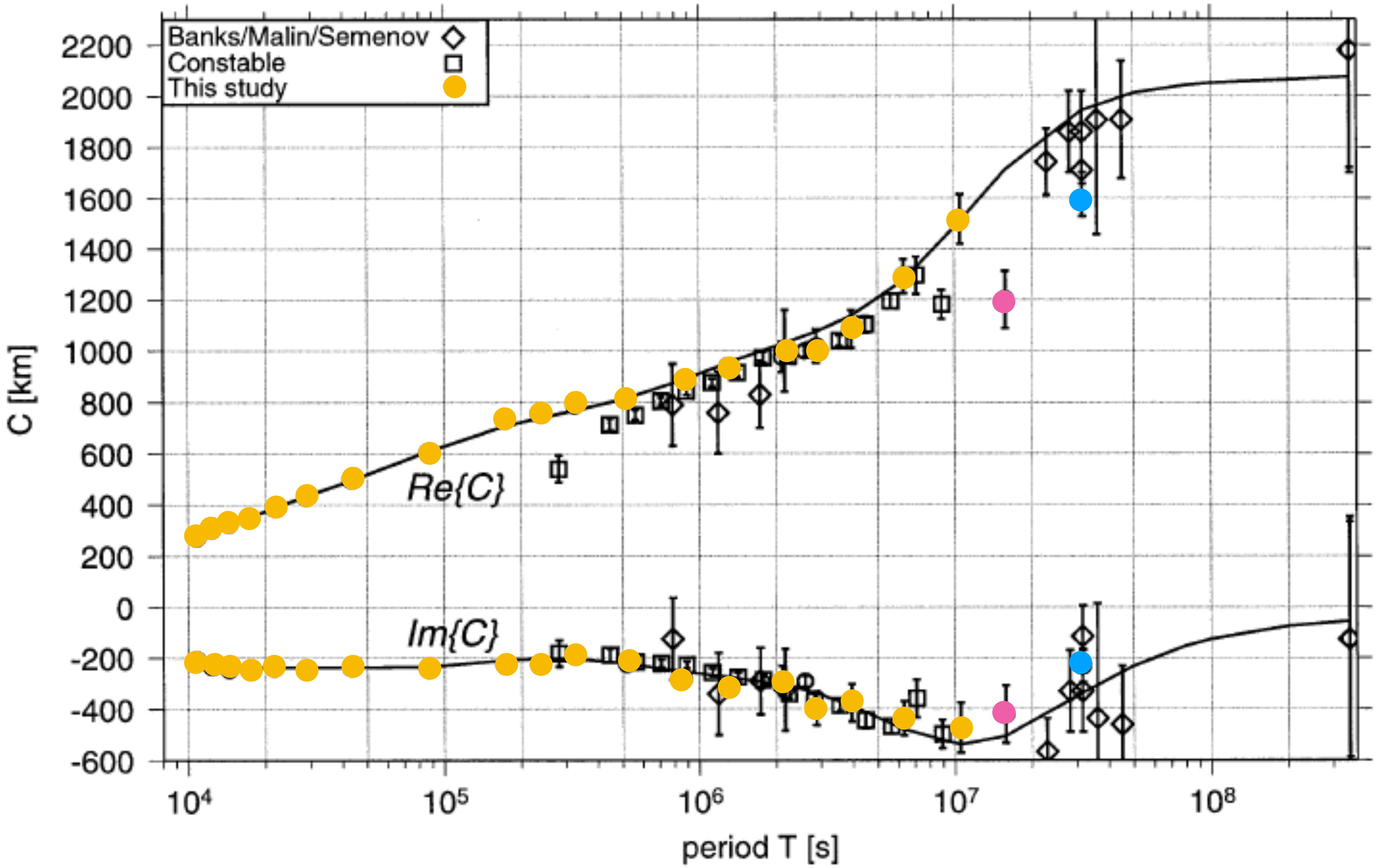
- External fields of periods 11 years (dipole), 1 year (axial quadrupole) and 6 months (axial dipole).
- Induced field, solid core approximation: B_r=0 at the core-mantle boundary (as a result of electrical currents in a magnetic diffusive layer at the core surface).
- Previous studies (Gédéon Légaut thesis, 2005): 11 years component interacting with torsional waves, assuming the period of the main mode is ~60 years.
- Axial part of the dipole: ineffective; weak waves propagating inwards in the fluid core from the equator.
- *Gillet & al., 2010*: revised period of the main mode, 6 years.
- What induced field in the Earth's fluid core with period 1 year, implications for models of mantle electrical conductivity ? Emission of torsional waves with 1 year period? Auspicious quadrupolar geometry.
- But : attenuation across the mantle.

The ‘mantle filter’



Trade-off between geometric attenuation and the impact of the electrical conductivity of the mantle (*Jault, 2015*)

Electromagnetic sounding



Transfer function for European observatories
assuming the Earth's core can be treated as a solid (*Olsen, 1999*)

1 year, 6 months

Domain of existence of torsional Alfvén waves

Torsional waves present when:

$$\boxed{\lambda = \frac{V_A}{\Omega r_c} = \frac{B}{\sqrt{\rho\mu}\Omega r_c} \ll 1}$$

Frequency: $$\omega \sim \frac{V_A}{r_c} = \lambda\Omega$$

Standard equations developed in a full sphere for:

$$\left(\frac{\nu}{\Omega\cos\theta}\right)^{1/2} \ll \left(\frac{\eta}{\omega}\right)^{1/2}, \qquad \boxed{P_m\left(\frac{\omega}{\Omega}\right) \ll \cos\theta}$$

(θ colatitude)

The hydromagnetic assumption

- Stewartson's jump condition through the boundary layer (with $B_{0,r}$ directed towards the fluid interior) for the zonal toroidal components:

$$\left[P_m^{1/2} u_\phi - \frac{b_\phi}{\sqrt{\mu\rho}} \right] = 0, \quad P_m = \frac{\nu}{\eta}$$

- when it is not satisfied: emission of Alfvén waves to erase the discontinuity

- application: boundary condition for torsional waves at the Earth's core equator

- In the $\lambda \ll 1$ framework, assumption: velocity shear in the direction parallel to the rotation axis forbidden ⇒ inhibition of the emission of Alfvén waves and construction of a magnetic diffusive layer **away from the equator**

Torsional wave equation

High frequency approximation

$$\zeta(s,t) = \exp(-\mathrm{i}\omega t)\zeta_G(s), \qquad \boxed{\omega^2\zeta_G(s) = -\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}s}\left(mV_A^2\frac{\mathrm{d}\zeta_G}{\mathrm{d}s}\right)} \qquad m = s^3 h$$

- Same equation for long gravity waves (of small height) in a channel of slowly varying width and depth
- George Green (1837): high frequency approximations to the solution, first instance of the method later known as the WKB method (two-lengthscale expansion)
- approximated solution away from the boundaries:

$$\zeta_G(s) = \frac{C}{\sqrt{mV_A}}\exp\left[\mathrm{i}\int_{s_0}^{s}\frac{\omega}{V_A}\mathrm{d}s - \mathrm{i}\omega t\right]$$

high frequency approximation valid where: $\frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}s} \ll \frac{\omega}{V_A}, \quad \frac{\mathrm{d}V_A}{\mathrm{d}s} \ll \omega$

Equatorial (V_A constant) solution

Limit cases: Neumann or Dirichlet boundary conditions

Boundary condition (at the equator): $\left.\frac{\mathrm{d}\zeta_G}{\mathrm{d}s}\right|_{s=1} = 0 \quad (\frac{\nu}{\eta} = P_m = 0)$ or $\zeta_G(1) = 0 \quad (P_m = \infty)$

Choice of unit: $V_A(1) \equiv 1$

$h = \sqrt{2x}$

$$x = 1 - s, \quad -\omega^2 \zeta_G(x) = \frac{1}{\sqrt{x}} \frac{\mathrm{d}}{\mathrm{d}x} \left(\sqrt{x} \frac{\mathrm{d}\zeta_G(x)}{\mathrm{d}x} \right)$$

Two independent solutions: $x^{1/4} J_{-1/4}(\omega x), \quad x^{1/4} J_{1/4}(\omega x)$

For $\left.\frac{\mathrm{d}\zeta_G}{\mathrm{d}s}\right|_{s=1} = 0$

$$\zeta_G(s) = C_{\mathrm{III}} (1 - s)^{1/4} J_{-1/4}(\omega(1 - s))$$

Matching between the WKB and equatorial solutions

In the limit of high frequency:

$$\zeta_G(s) = C_{\mathrm{III}} \left(\frac{2}{\pi}\right)^{1/2} \omega^{-1/2}(1-s)^{-1/4} \cos\left[\omega(1-s) - \frac{\pi}{8}\right] + O\left(\omega^{-3/2}\right)$$

Matching with the WKB solution in the interior (as in *Maffei & Jackson, 2016*):

$$\zeta_G(s) = \frac{C_{\mathrm{II}}}{\sqrt{m V_A}} \cos(\omega\tau(s) + \phi_0), \qquad \tau(s) = \int_s^1 \frac{\mathrm{d}s}{V_A} = 1 - s, \qquad \phi_0 = -\frac{\pi}{8}$$

On the axis (*Mound & Buffett, 2007*):

$$\left.\frac{\partial \zeta_G}{\partial s}\right|_{s=0} = 0 \qquad \zeta_G(s) = C_{\mathrm{I}} \frac{J_1\left(\omega s / V_A(0)\right)}{s} \qquad J_1(\omega s) \sim \sqrt{\frac{2V_A(0)}{\pi \omega s}} \cos\left[-\frac{\omega s}{V_A(0)} + \frac{3\pi}{4}\right]$$

$\Longrightarrow$ quantification of the eigenvalues:

Example:

$$\forall s, \quad V_A(s) = 1 \qquad \omega(1-s) - \frac{\pi}{8} = -\omega s + \frac{3\pi}{4} + n\pi, \quad \omega_n = \left(n + \frac{7}{8}\right)\pi$$

The second family of normal modes

For $\zeta_G(1) = 0$

$$\zeta_G(s) = C_{\mathrm{III}}(1-s)^{1/4} J_{1/4}(\omega(1-s))$$

In the limit of high frequency:

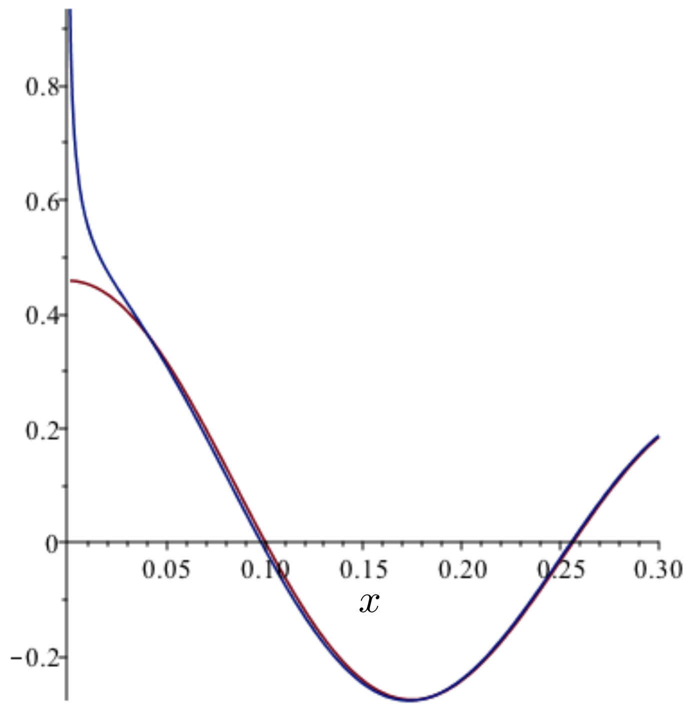
$$\zeta_G(s) \sim C_{\mathrm{III}} \left(\frac{2}{\pi}\right)^{1/2} \omega^{-1/2}(1-s)^{-1/4} \cos\left[\omega(1-s) - \frac{3\pi}{8}\right]$$

$$\omega_n = \left(n + \frac{9}{8}\right)\pi$$

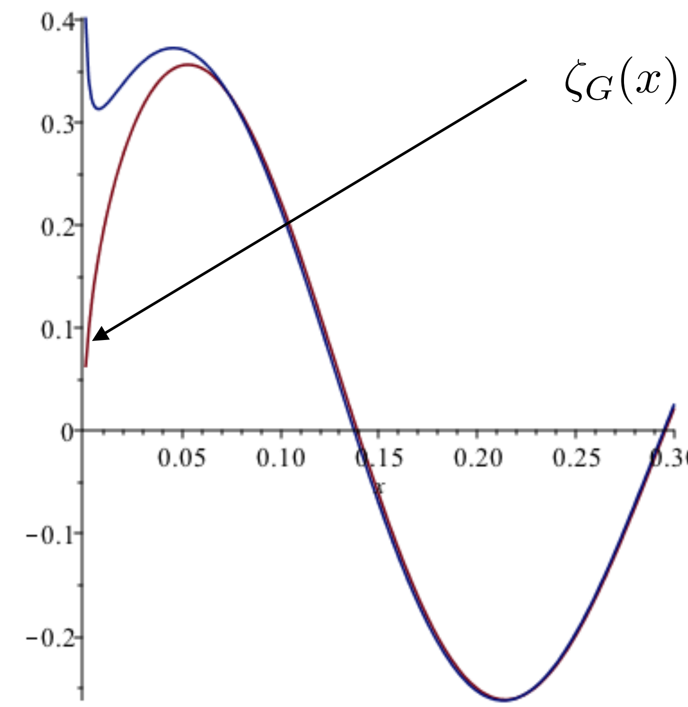
Asymptotic matching in the equatorial region

$$x = 1 - s,$$

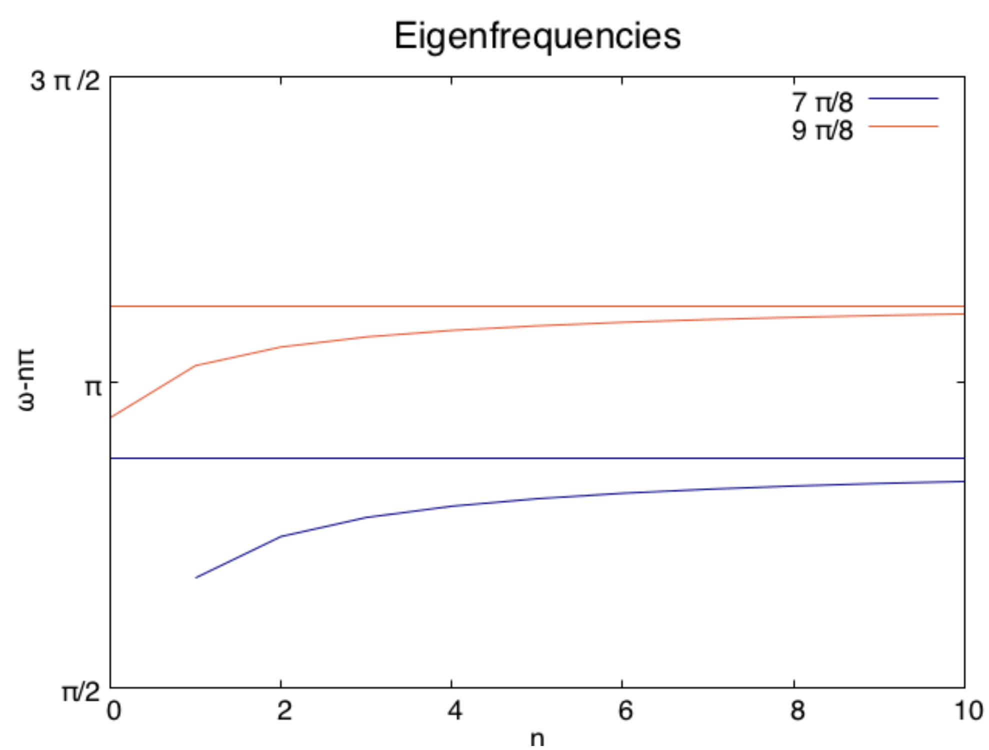
$$\left.\frac{\mathrm{d}\zeta_G}{\mathrm{d}x}\right|_{x=0} = 0$$



$$\zeta_G(0) = 0$$



$$\zeta_G(x) = O\left(\sqrt{x}\right)$$



$$\text{For } \zeta_G(0) = 0, \quad \omega_n = \left(n + \frac{9}{8}\right)\pi$$

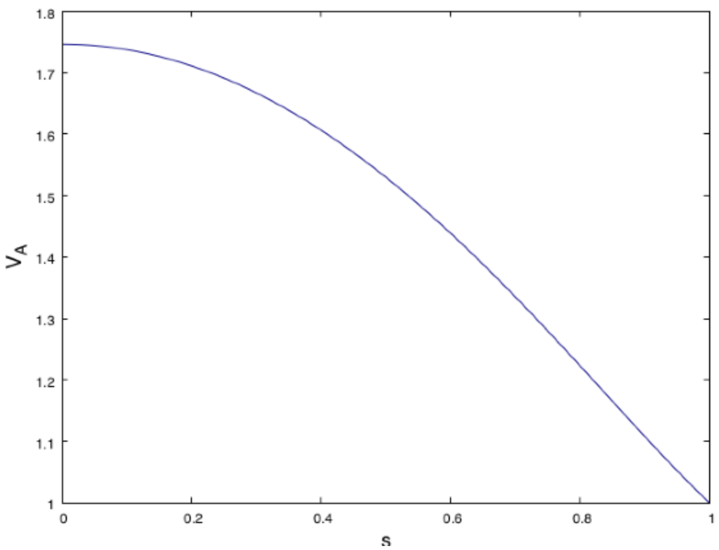
see
Schaeffer & al. (2012)

$$\text{For } \left.\frac{\mathrm{d}\zeta_G}{\mathrm{d}x}\right|_{x=0} = 0, \quad \omega_n = \left(n + \frac{7}{8}\right)\pi$$

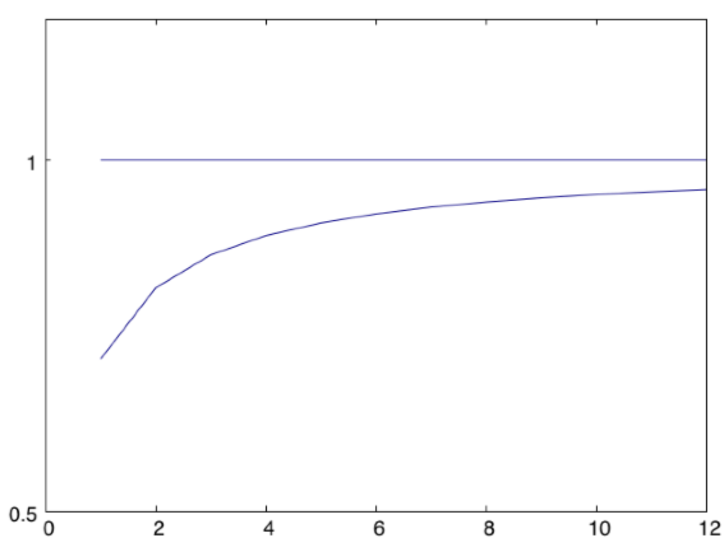
see
Roberts & Aurnou (2012)

Illustration with a non-uniform Alfvén velocity from Roberts & Aurnou (2012)

Alfvén velocity model

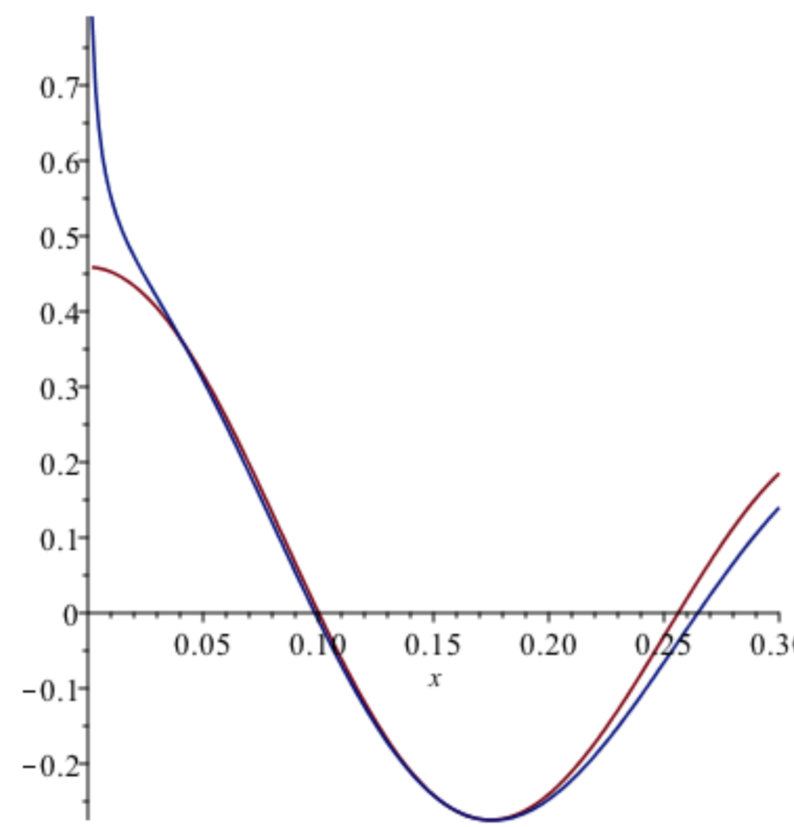


$$(\tau\omega - n\pi)/\frac{7\pi}{8}$$



$$\left.\frac{\partial \zeta_G}{\partial s}\right|_{s=1} = 0$$

Matching between the equatorial solution (V_A=1) and the high frequency interior solution



τ travel time

$$\tau = \int_0^1 \frac{\mathrm{d}s}{V_A},$$

$$\tau\omega_n = \frac{7\pi}{8} + n\pi$$

ω_n depends of the magnetic field model, only through the travel time τ

Finite magnetic Prandtl number

1-D theory: Alfvén waves across a plane slab

$$0 < P_m < \infty$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 \leq x \leq 1$$

Problem set-up such that the boundary condition at one extremity (x=0) is: $\forall t, \quad \frac{\partial u}{\partial x}(0,t) = 0$

⇒ general form of the solution: $u = \exp(\lambda t)\left(\exp(\lambda x) + \exp(-\lambda x)\right)$

From the jump condition $P_m^{1/2} u - b = 0$ and the induction equation $\frac{\partial b}{\partial t} = -\frac{\partial u}{\partial x}$ *($B_{0,x}$ enters the fluid)*

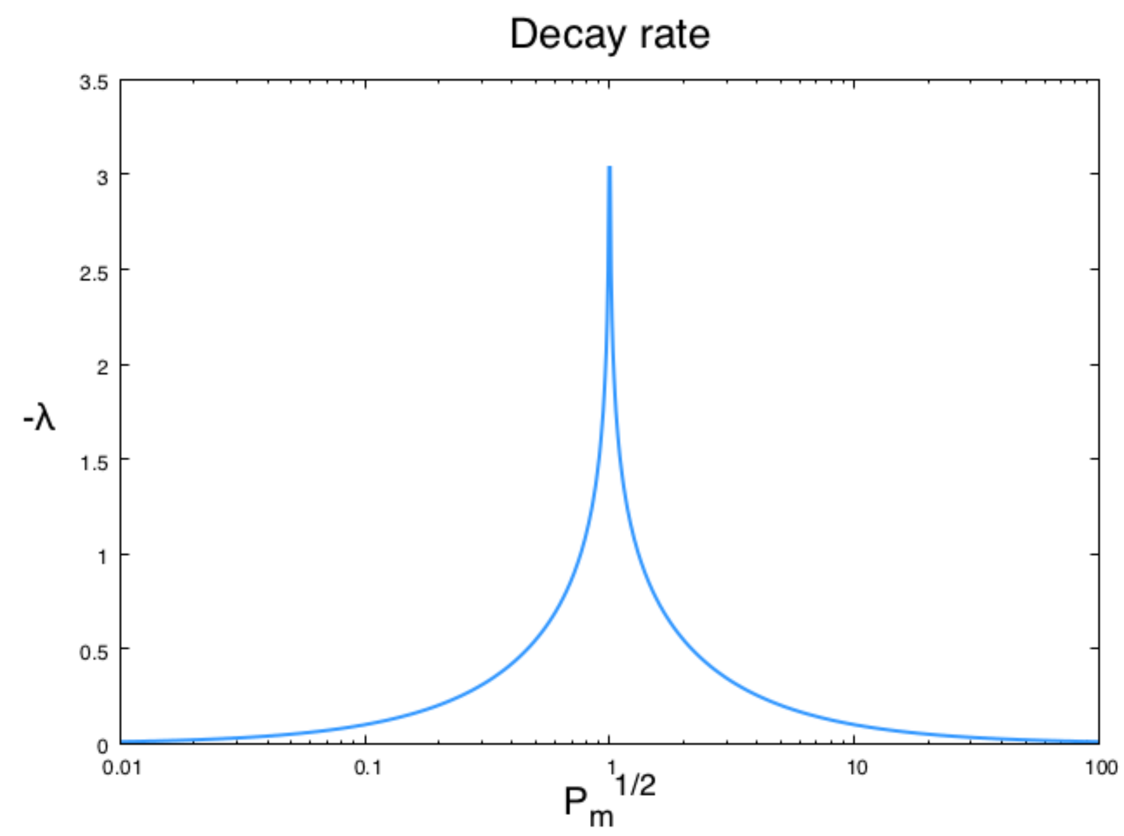
the boundary condition at the other extremity (x=1) is of mixed type: $\forall t, \quad \frac{\partial u}{\partial x} = -P_m^{1/2} \frac{\partial u}{\partial t} = -\lambda P_m^{1/2} u$

$$\exp(2\lambda) = \frac{1 - P_m^{1/2}}{1 + P_m^{1/2}}, \quad \lambda = \frac{1}{2}\left(\log\left[\frac{1 - P_m^{1/2}}{1 + P_m^{1/2}}\right] + 2n\pi \mathrm{i}\right)$$

log, principal branch (in the complex plane) of the logarithm.

Frequency jump from $P_m^{1/2} = 1^-$ to $P_m^{1/2} = 1^+$: $\quad n\pi \to n\pi + \frac{\pi}{2}$

Decay rate: $\Re(\lambda) = \frac{1}{2}\ln\left[\left|\frac{1-P_m^{1/2}}{1+P_m^{1/2}}\right|\right]$



Reflection of an impulsive wave

Solution of the form $f(t-x) + F(t+x)$

At x=1 $\frac{\partial u}{\partial x} = -P_m^{1/2}\frac{\partial u}{\partial t}$

Reflected wave $F = \frac{1-P_m^{1/2}}{1+P_m^{1/2}} f$

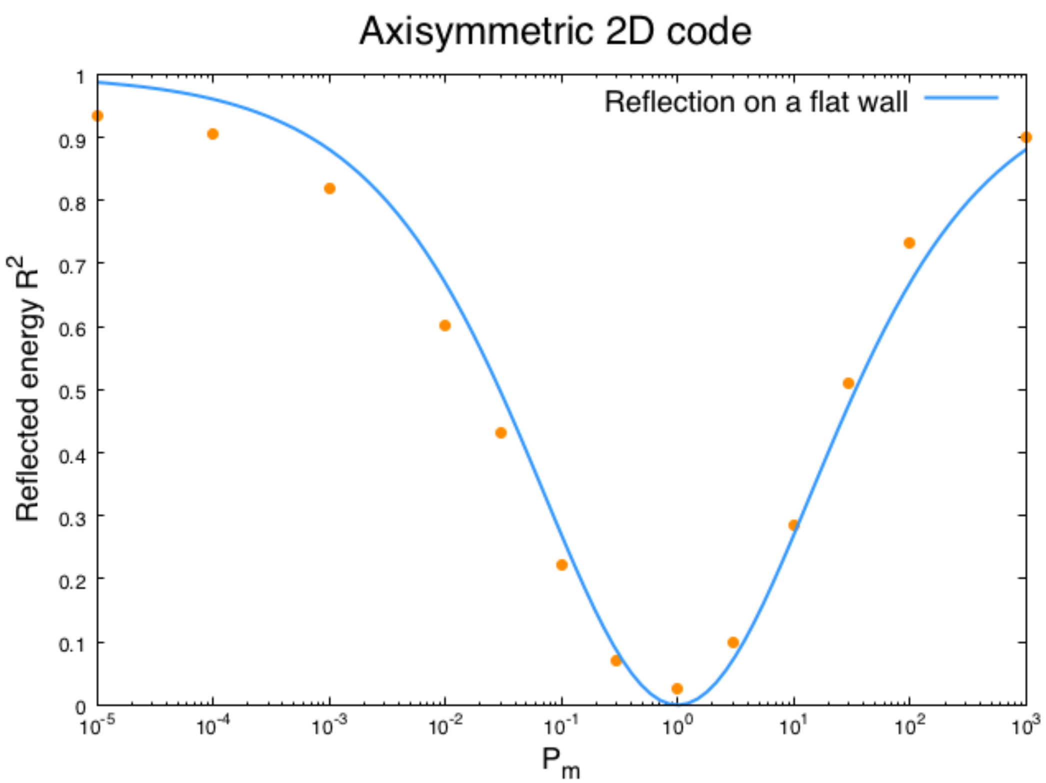
Reflection coefficient: $\boxed{R = \frac{1-P_m^{1/2}}{1+P_m^{1/2}}}$

Reflection at the equator

Comparison between 1D theory and 3D simulations

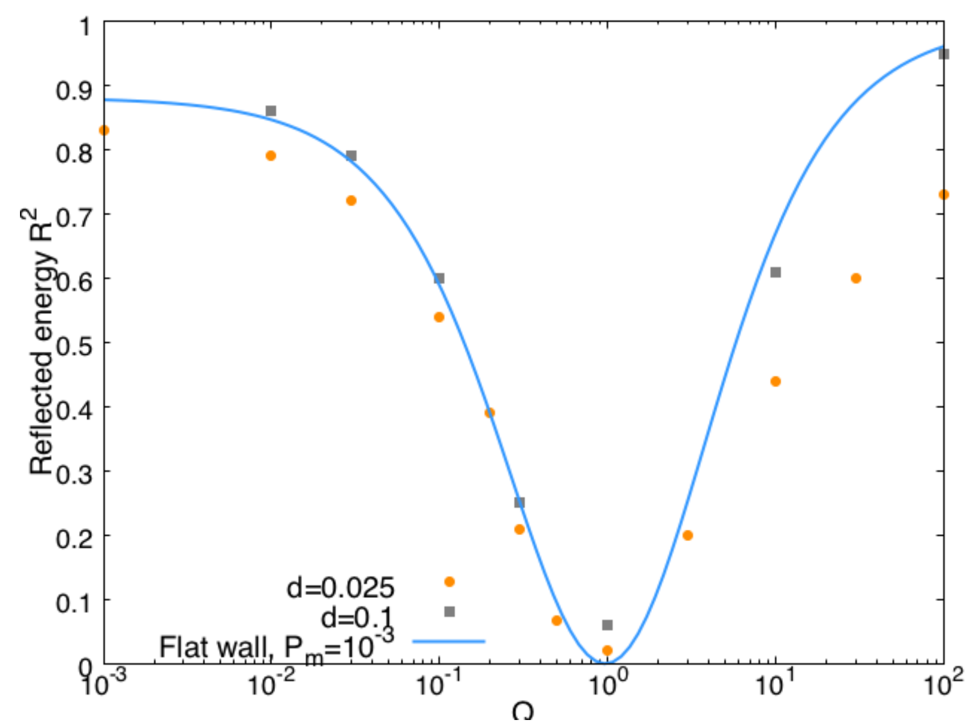
Reflection of an impulsive wave

Magnetic Prandtl number:



Schaeffer & al., 2012

Mantle conductivity:



Schaeffer & Jault, 2016

$$R = \frac{1 - Q - \sqrt{P_m}}{1 + Q + \sqrt{P_m}}$$

$$Q = \sqrt{\frac{\mu_0}{\rho}} B_r \Big|_{s=1} \int_{\text{mantle}} \sigma \mathrm{d}r$$

Dependence on the width of the incoming pulse d and apparent dispersion upon reflection

1D equations for torsional waves in the sphere with magnetic and/or viscous coupling

From the estimation of the magnetic field at the top of the mainstream
(in the presence of a conducting mantle):

Braginsky,1970

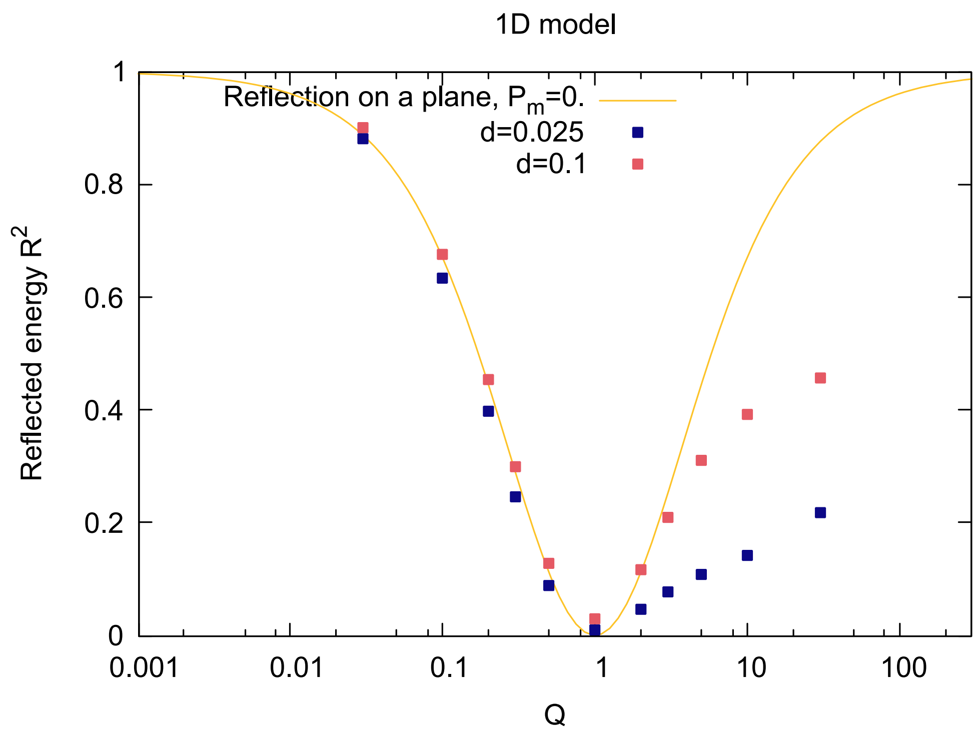
$$b_\phi = -Qs\zeta_G$$

$$\frac{\partial^2 \zeta_G}{\partial t^2} = \frac{1}{m}\frac{\partial}{\partial s}\left(mV_A^2 \frac{\partial \zeta_G}{\partial s}\right) - \frac{Q(\theta)}{h^2}\frac{\partial \zeta_G}{\partial t}$$

$$Q(\theta) = \sqrt{\frac{\mu_0}{\rho}} B_r(\theta) \int_{\text{mantle}} \sigma(r,\theta)\mathrm{d}r$$

Plane wall prediction vs reduced (1D) modelling

Gillet & al., 2017



Transmission, branching and reflection