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TWISTED JACOBI MANIFOLDS, TWISTED DIRAC-JACOBI STRUCTURES AND QUASI-JACOBI BIALGEBROIDS

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Abstract: We study twisted Jacobi manifolds, a concept that we had introduced in a previous Note. Twisted Jacobi manifolds can be characterized using twisted Dirac-Jacobi, which are sub-bundles of Courant-Jacobi algebroids. We show that each twisted Jacobi manifold has an associated Lie algebroid with a 1-cocycle. We introduce the notion of quasi-Jacobi bialgebroid and we prove that each twisted Jacobi manifold has a quasi-Jacobi bialgebroid canonically associated. Moreover, the double of a quasi-Jacobi bialgebroid is a Courant-Jacobi algebroid. Several examples of twisted Jacobi manifolds and twisted Dirac-Jacobi structures are presented.

Keywords: Twisted Jacobi manifold, twisted Dirac-Jacobi structure, Jacobi bialgebroid, Courant-Jacobi algebroid, quasi-Jacobi bialgebroid.
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1. Introduction

Jacobi manifolds were introduced by Lichnerowicz [14] and Kirillov [8] as smooth manifolds endowed with a bivector field Λ and a vector field E satisfying some compatibility conditions. When the vector field E identically vanishes, the Jacobi manifold is just a Poisson manifold. So, Poisson manifolds are particular cases of Jacobi manifolds. But there are other examples of Jacobi structures on manifolds which are not Poisson, such as contact structures and local conformally symplectic structures.

The notion of twisted Poisson manifold (or Poisson manifold with a 3-form background) was introduced by Ševera and Weinstein [24], motivated by the works of Klimčik and Strobl [9] on topological field theory and Park [20] on string theory. Since Jacobi structures on manifolds generalize Poisson structures, the introduction of the concept of a twisted Jacobi manifold seems very natural. This task was achieved in the Note [19] where, besides we have introduced that notion, we briefly presented some of its properties.

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Dirac structures on manifolds were introduced by Courant and Weinstein [1] and developed in detail by Courant [2]. Dirac structures include presymplectic forms, Poisson structures and foliations. The first approach to extend the theory of Dirac structures to Jacobi manifolds was done by Wade [25], who introduced the $\mathcal{E}^1(M)$-Dirac structures as a natural extension of Dirac bundles in the sense of Courant [2]. These $\mathcal{E}^1(M)$-Dirac structures, which we call Dirac-Jacobi structures, include Jacobi manifolds and are sub-bundles of the vector bundle $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ over M, satisfying a certain integrability condition. However, the vector bundle $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ is not a Courant algebroid. This fact motivated a more general treatment, proposed in [4, 18]. The concept of Courant-Jacobi algebroid was introduced, independently, in [4] and [18], and the main example of this structure is the double of a Jacobi bialgebroid [5, 3]. A Dirac structure for a Courant-Jacobi algebroid E is defined as a sub-bundle of the vector bundle E over M satisfying an integrability condition. Dirac-Jacobi bundles arise then as a particular case of these structures.

As we have already mentioned, twisted Poisson manifolds were introduced by Ševera and Weinstein [24] who studied them in the framework of Courant algebroids and Dirac structures. For the case of twisted Jacobi manifolds, we use Dirac-Jacobi structures. More precisely, we use twisted Dirac-Jacobi structures, which are sub-bundles of the Courant-Jacobi algebroid $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ equipped with a "twisted bracket" on its space of sections. These Dirac-Jacobi bundles enable us to characterize twisted Jacobi structures on manifolds.

On the other hand, Roytenberg [22] developed a theory of quasi-Lie bialgebroids and used it to study twisted Poisson manifolds [23]. Namely, with each twisted Poisson structure on a manifold M, a quasi-Lie bialgebroid structure on (TM, T^*M) can be associated. When we try to investigate what happens in the Jacobi framework, we realize that things are different. First of all because, in opposition to the Poisson case, one cannot, in general, define a Lie algebroid structure on the cotangent bundle T^*M of a Jacobi manifold (M, Λ, E). Usually, only the vector bundle $T^*M \times \mathbb{R}$ over M admits such a structure [7]. Furthermore, with each Jacobi manifold, there exists an associated Jacobi bialgebroid [5, 3], while in the case of a Poisson manifold it admits an associated Lie bialgebroid. Motivated by these facts, we introduce the concept of a quasi-Jacobi bialgebroid, which is the one that fits in our theory. We prove that each twisted Jacobi manifold has an associated quasi-Jacobi bialgebroid and that the double of a quasi-Jacobi bialgebroid is a Courant-Jacobi algebroid.

The paper is divided into eight sections. In section 2 we recall some facts on Jacobi manifolds and their relation with Lie algebroid theory. In section 3 we study the main properties of a twisted Jacobi manifold, we present some examples and we show that if M is equipped with a twisted Jacobi structure, then there exists a twisted exact homogeneous Poisson structure on $M \times \mathbb{R}$. Section 4 is devoted to twisted Dirac-Jacobi structures and we characterize twisted Jacobi manifolds using these structures. Several examples of twisted Dirac-Jacobi bundles are presented, including graphs of sections of $\bigwedge^2(T^*M \times \mathbb{R})$ and twisted locally conformal presymplectic structures. We also relate twisted Dirac-Jacobi bundles and Dirac bundles in the sense of Courant. In Section 5 we see how gauge transformations act on twisted Dirac-Jacobi structures. In section 6 we construct a Lie algebroid with a 1-cocycle associated with each twisted Jacobi manifold. The notion of quasi-Jacobi bialgebroid is introduced in section 7 and we prove that its double is a Courant-Jacobi algebroid. In section 8 we show that each twisted Jacobi manifolds admits an associated quasi-Jacobi bialgebroid.

Notation : In this paper, M is a C^∞-differentiable manifold of finite dimension. We denote by TM and T^*M, respectively, the tangent and cotangent bundles over M and by $C^\infty(M, \mathbb{R})$ the space of all real C^∞-differentiable functions on M. For the Schouten bracket and the interior product of a form with a multivector field, we use the convention of sign indicated by Koszul [12], (see also [17]).

2. Jacobi manifolds

A *Jacobi manifold* is a differentiable manifold M equipped with a bivector field Λ and a vector field E such that

$$[\Lambda, \Lambda] = -2E \wedge \Lambda \quad \text{and} \quad [E, \Lambda] = 0, \tag{1}$$

where $[\cdot, \cdot]$ denotes the Schouten bracket [12]. In this case, (Λ, E) defines a bracket on $C^\infty(M, \mathbb{R})$ which is called the *Jacobi bracket* and is given, for all $f, g \in C^\infty(M, \mathbb{R})$, by

$$\{f, g\} = \Lambda(df, dg) + f(E.g) - g(E.f). \tag{2}$$

The Jacobi bracket endows $C^\infty(M,\mathbb{R})$ with a local Lie algebra structure in the sense of Kirillov [8]. Reciprocally, a local Lie algebra structure on $C^\infty(M,\mathbb{R})$ induces on M a Jacobi structure.

When the vector field E identically vanishes on M, the Jacobi structure reduces to a Poisson structure on the manifold. However, there are other examples of Jacobi manifolds either than Poisson manifolds, such as contact and locally conformal symplectic manifolds, [14].

There are some well-known results concerning Jacobi structures on manifolds that we briefly recall.

Let (M,Λ,E) be a Jacobi manifold. Then, the pair (Λ,E) defines the homomorphism of $C^\infty(M,\mathbb{R})$-modules $(\Lambda,E)^\# : \Gamma(T^*M\times\mathbb{R}) \to \Gamma(TM\times\mathbb{R})$ given, for any section (α,f) of $T^*M\times\mathbb{R}$, by

$$(\Lambda,E)^\#(\alpha,f) = (\Lambda^\#(\alpha)+fE, -i_E\alpha), \tag{3}$$

and, with each $f\in C^\infty(M,\mathbb{R})$, we can associate the vector field $X_f = \Lambda^\#(df)+fE$, called the *hamiltonian vector field* of f. We have that

$$X_f = \pi((\Lambda,E)^\#(df,f)),$$

where $\pi : TM\times\mathbb{R}\to TM$ denotes the projection over the first factor. Moreover, for all $f,g\in C^\infty(M,\mathbb{R})$,

$$[X_f,X_g] = X_{\{f,g\}}. \tag{4}$$

Also, the vector bundle $T^*M\times\mathbb{R}$ over M endowed with the anchor map $\pi\circ(\Lambda,E)^\# : T^*M\times\mathbb{R}\to TM$ and the Lie algebra bracket $\{\cdot,\cdot\}$ on the space of its sections, given, for all $(\alpha,f),(\beta,g)\in\Gamma(T^*M\times\mathbb{R})$, by

$$\{(\alpha,f),(\beta,g)\} = (\gamma,r), \tag{5}$$

where

$$\gamma = \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - d(\Lambda(\alpha,\beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha - i_E(\alpha\wedge\beta),$$
$$r = -\Lambda(\alpha,\beta) + \Lambda(\alpha,dg) - \Lambda(\beta,df) + fE(g) - gE(f),$$

is a Lie algebroid over M [7]. The associated exterior derivative d_* on $\Gamma(\bigwedge(TM\times\mathbb{R})) = \oplus_{k\in\mathbb{Z}}\Gamma(\bigwedge^k(TM\times\mathbb{R}))$ is given [13], for all $(P,Q)\in\Gamma(\bigwedge^k(TM\times\mathbb{R}))\cong\Gamma(\bigwedge^k(TM))\oplus\Gamma(\bigwedge^{k-1}(TM))$, by

$$d_*(P,Q) = ([\Lambda,P]+kE\wedge P+\Lambda\wedge Q, -[\Lambda,Q]+(1-k)E\wedge Q+[E,P]). \tag{6}$$

It is well known that, given a Lie algebroid $(A,[\cdot,\cdot],a)$ over a differentiable manifold M with a 1-cocycle $\phi\in\Gamma(A^*)$ in the Lie algebroid cohomology complex with trivial coefficients [16], we can modify the usual representation of the Lie algebra $(\Gamma(A), [\cdot,\cdot])$ on $C^\infty(M,\mathbb{R})$ by defining a new representation $a^\phi : \Gamma(A) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ as

$$a^\phi(X,f) = a(X)f + (i_X\phi)f, \qquad \forall\,(X,f) \in \Gamma(A) \times C^\infty(M,\mathbb{R}). \tag{7}$$

Therefore, we obtain a new cohomology operator d^ϕ on $\Gamma(\bigwedge A^*) = \oplus_{k\in\mathbb{Z}}\Gamma(\bigwedge^k A^*)$ given by

$$d^\phi(\beta) = d\beta + \phi \wedge \beta, \qquad \forall\,\beta \in \Gamma(\bigwedge^k A^*), \tag{8}$$

where d is the cohomology operator defined by $([\cdot,\cdot], a)$ on $\Gamma(\bigwedge A^*)$, and a new Lie derivative operator of forms with respect to $X \in \Gamma(A)$, $\mathcal{L}^\phi_X = d^\phi \circ i_X + i_X \circ d^\phi$, that can be expressed in terms of the usual Lie derivative $\mathcal{L}_X = d \circ i_X + i_X \circ d$, as

$$\mathcal{L}^\phi_X(\beta) = \mathcal{L}_X\beta + (i_X\phi)\beta, \qquad \forall\,\beta \in \Gamma(\bigwedge^k A^*). \tag{9}$$

Using ϕ, it is also possible to modify the Schouten bracket $[\cdot,\cdot]$ on the graded algebra $\Gamma(\bigwedge A) = \oplus_{k\in\mathbb{Z}}\Gamma(\bigwedge^k A)$ to the ϕ-Schouten bracket $[\cdot,\cdot]^\phi$ on $\Gamma(\bigwedge A)$ defined, for any $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge^q A)$, by

$$[P,Q]^\phi = [P,Q] + (p-1)P \wedge (i_\phi Q) + (-1)^p(q-1)(i_\phi P) \wedge Q, \tag{10}$$

where $i_\phi Q$ and $i_\phi P$ can be interpreted as the usual contraction of a multivector field with a 1-form. A differential calculus using a^ϕ, d^ϕ, $\mathcal{L}^\phi$ and $[\cdot,\cdot]^\phi$ can be developed. The formulae obtained are similar, but adapted, to the case of a Lie algebroid [5], [3].

A pair (A,ϕ) formed by a Lie algebroid A and a 1-cocycle ϕ of A, is called a *Jacobi algebroid* in the terminology of [3].

A trivial example of a Jacobi algebroid over M is the vector bundle $TM \times \mathbb{R} \to M$ equipped with the bracket

$$[(X,f),(Y,g)] = ([X,Y], X.g - Y.f), \qquad \forall\,(X,f),(Y,g) \in \Gamma(TM \times \mathbb{R}), \tag{11}$$

the vector bundle map $\pi : TM \times \mathbb{R} \to TM$, that is the projection over the first factor, and the section $(0,1)$ of $T^*M \times \mathbb{R}$. The associated exterior derivative on $\Gamma(\bigwedge(T^*M \times \mathbb{R}))$ is the operator $\mathrm{d} = (d, -d)$ and $(0,1)$ is a 1-cocycle in the cohomology complex with trivial coefficients of $(TM \times \mathbb{R}, [\cdot,\cdot], \pi, \mathrm{d})$. In the sequel, we will denote by $\mathrm{d}^{(0,1)}$ the differential operator on $\Gamma(\bigwedge(T^*M \times \mathbb{R}))$ modified by $(0,1)$, as in (8).

The notion of *generalized Lie bialgebroid* and the equivalent one of *Jacobi bialgebroid* were introduced, respectively, by D. Iglesias and J.C. Marrero in [5] and by J. Grabowski and G. Marmo in [3] in such a way that a Jacobi manifold has a Jacobi bialgebroid canonically associated and conversely. A Jacobi bialgebroid over M is a pair (A, A^*) of Lie algebroids over M, in duality, with differentials d and d_*, respectively, endowed with a 1-cocycle $\phi \in \Gamma(A^*)$ of (A, d) and a 1-cocycle $W \in \Gamma(A)$ of (A^*, d_*), such that, for every $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge A)$, the following condition holds :

$$d_*^W[P,Q]^\phi = [d_*^W P, Q]^\phi + (-1)^{p+1}[P, d_*^W Q]^\phi.$$

The pair formed by the Jacobi algebroid $(TM \times \mathbb{R}, [\cdot,\cdot], \pi, (0,1))$, presented above, together with the Lie algebroid $(T^*M \times \mathbb{R}, \{\cdot,\cdot\}, \pi \circ (\Lambda, E)^\#)$ and the 1-cocycle $(-E, 0) \in \Gamma(TM \times \mathbb{R})$ on it, is a Jacobi bialgebroid over the Jacobi manifold (M, Λ, E), [5].

Finally, let us recall [5] that a section (Λ, E) of $\bigwedge^2(TM \times \mathbb{R})$ defines a Jacobi structure on the manifold M if and only if

$$[(\Lambda, E), (\Lambda, E)]^{(0,1)} = (0, 0). \tag{12}$$

3. Twisted Jacobi manifolds

In [19] we introduced the concept of twisted Jacobi manifold and we presented some of its properties. Now, in this section, we will review and complete the results announced in [19].

We start by recalling that, given a bivector field Λ on a differentiable manifold M, the associated vector bundle map $\Lambda^\# : T^*M \to TM$ induces a homomorphism of $C^\infty(M, \mathbb{R})$-modules $\Lambda^\# : \Gamma(T^*M) \to \Gamma(TM)$,

$$\langle \beta, \Lambda^\#(\alpha) \rangle = \Lambda(\alpha, \beta), \qquad \forall\, \alpha, \beta \in \Gamma(T^*M),$$

that can be extended to a homomorphism, also denoted by $\Lambda^\#$, from $\Gamma(\bigwedge^k(T^*M))$ onto $\Gamma(\bigwedge^k(TM))$, $k \in \mathbb{N}$, as follows:

$$\Lambda^\#(f) = f \quad \text{and} \quad (\Lambda^\#\eta)(\alpha_1, \ldots, \alpha_k) = (-1)^k \eta(\Lambda^\#(\alpha_1), \ldots, \Lambda^\#(\alpha_k)), \tag{13}$$

for all $f \in C^\infty(M, \mathbb{R})$, $\eta \in \Gamma(\bigwedge^k(T^*M))$ and $\alpha_1, \ldots, \alpha_k \in \Gamma(T^*M)$. Analogously, with each section (Λ, E) of $\bigwedge^2(TM \times \mathbb{R})$, we can associate a homomorphism of $C^\infty(M, \mathbb{R})$-modules

$$(\Lambda, E)^\# : \Gamma(\bigwedge^k(T^*M \times \mathbb{R})) \to \Gamma(\bigwedge^k(TM \times \mathbb{R})), \quad k \in \mathbb{N},$$

by setting, for all $f \in C^\infty(M, \mathbb{R})$, $(\eta, \xi) \in \Gamma(\bigwedge^k(T^*M \times \mathbb{R}))$ and $(\alpha_1, f_1), \dots,$ $(\alpha_k, f_k) \in \Gamma(T^*M \times \mathbb{R})$,

$$(\Lambda, E)^{\#}(f) = f$$

and

$$\begin{aligned}(\Lambda, E)^{\#}(\eta, \xi)((\alpha_1, f_1), \dots, (\alpha_k, f_k))\\ = (-1)^k(\eta, \xi)((\Lambda, E)^{\#}(\alpha_1, f_1), \cdots, (\Lambda, E)^{\#}(\alpha_k, f_k)).\end{aligned} \tag{14}$$

We remark that for $k = 1$, we recover (3).

Let us introduce some notation, following [24]. Let Λ be a bivector field on M and φ a 3-form on M. We denote by $(\Lambda^{\#} \otimes 1)(\varphi)$ the section of $(\bigwedge^2 TM) \otimes T^*M$ that acts on multivector fields by contraction with the factor in T^*M. For any $f \in C^\infty(M, \mathbb{R})$, $X \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$,

$$(\Lambda^{\#} \otimes 1)(\varphi)(f) = 0 \quad \text{and} \quad (\Lambda^{\#} \otimes 1)(\varphi)(\alpha, \beta)(X) = -\varphi(\Lambda^{\#}(\alpha), \Lambda^{\#}(\beta), X). \tag{15}$$

Similarly, if ω is a 2-form on M, then, for any $X \in \Gamma(TM)$ and $\alpha \in \Gamma(T^*M)$,

$$(\Lambda^{\#} \otimes 1)(\omega)(\alpha)(X) = \omega(\Lambda^{\#}(\alpha), X).$$

In what follows, we consider the Jacobi algebroid $(TM \times \mathbb{R}, [\cdot, \cdot], \pi, (0, 1))$ and we are mainly interested in the vector bundle map defined by (14) for $k = 3$.

Proposition 3.1. *Let (Λ, E) be a section of $\bigwedge^2(TM \times \mathbb{R})$ and (φ, ω) a section of $\bigwedge^3(T^*M \times \mathbb{R})$. Then,*

$$[(\Lambda, E), (\Lambda, E)]^{(0,1)} = 2(\Lambda, E)^{\#}(\varphi, \omega)$$

if and only if

$$[\Lambda, \Lambda] + 2E \wedge \Lambda = 2\Lambda^{\#}(\varphi) + 2(\Lambda^{\#}\omega) \wedge E \tag{16}$$

and

$$[E, \Lambda] = (\Lambda^{\#} \otimes 1)(\varphi)(E) - ((\Lambda^{\#} \otimes 1)(\omega)(E)) \wedge E. \tag{17}$$

Proof: Let $(\alpha, f), (\beta, g), (\gamma, h)$ be three arbitrary sections of $T^*M \times \mathbb{R}$. We have,

$$\begin{aligned}
&[(\Lambda, E),(\Lambda, E)]^{(0,1)}((\alpha, f),(\beta, g),(\gamma, h)) \\
&\quad = ([\Lambda,\Lambda] + 2E\wedge\Lambda, 2[E,\Lambda])((\alpha, f),(\beta, g),(\gamma, h)) \\
&\quad = ([\Lambda,\Lambda] + 2E\wedge\Lambda)(\alpha,\beta,\gamma) + 2f[E,\Lambda](\beta,\gamma) - 2g[E,\Lambda](\alpha,\gamma) \\
&\qquad +2h[E,\Lambda](\alpha,\beta). \qquad (18)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&2(\Lambda, E)^{\#}(\varphi,\omega)((\alpha, f),(\beta, g),(\gamma, h)) \\
&\quad = 2(\Lambda^{\#}\varphi)(\alpha,\beta,\gamma) + 2((\Lambda^{\#}\omega)\wedge E)(\alpha,\beta,\gamma) \\
&\qquad -2\left(\varphi(\Lambda^{\#}(\beta),\Lambda^{\#}(\gamma), fE) - \varphi(\Lambda^{\#}(\alpha),\Lambda^{\#}(\gamma), gE) + \varphi(\Lambda^{\#}(\alpha),\Lambda^{\#}(\beta), hE)\right) \\
&\qquad -2((i_E\alpha)[\omega(\Lambda^{\#}(\gamma), gE) - \omega(\Lambda^{\#}(\beta), hE)] - (i_E\beta)[\omega(\Lambda^{\#}(\gamma), fE) \\
&\qquad -\omega(\Lambda^{\#}(\alpha), hE)] + (i_E\gamma)[\omega(\Lambda^{\#}(\beta), fE) - \omega(\Lambda^{\#}(\alpha), gE)]) \\
&\quad = 2\left((\Lambda^{\#}\varphi) + ((\Lambda^{\#}\omega)\wedge E), (\Lambda^{\#}\otimes 1)(\varphi)(E)\right. \\
&\qquad \left.-((\Lambda^{\#}\otimes 1)(\omega)(E)\wedge E)\right)((\alpha, f),(\beta, g),(\gamma, h)). \qquad (19)
\end{aligned}$$

Comparing the terms on trivector fields and bivector fields of (18) and (19), we obtain, respectively, the formulæ (16) and (17). ■

The sections of $\bigwedge^3(T^*M\times\mathbb{R})$ that are closed with respect to the differential operator $\mathrm{d}^{(0,1)}$ will have a special role hereafter. We will call them $\mathrm{d}^{(0,1)}$-*closed.*

Lemma 3.2. *A section (φ,ω) of $\bigwedge^3(T^*M\times\mathbb{R})$ is $\mathrm{d}^{(0,1)}$-closed, i.e. $\mathrm{d}^{(0,1)}(\varphi,\omega) = (0,0)$, if and only if $\varphi = d\omega$.*

Thus, we shall denote any $\mathrm{d}^{(0,1)}$-closed section (φ,ω) of $\bigwedge^3(T^*M\times\mathbb{R})$ by $(d\omega,\omega)$, with ω a 2-form on M.

Definition 3.3. *A* twisted Jacobi structure *on a differentiable manifold M is defined by choosing a bivector field Λ, a vector field E and a 2-form ω on M such that*

$$[(\Lambda, E),(\Lambda, E)]^{(0,1)} = 2(\Lambda, E)^{\#}(d\omega,\omega). \qquad (20)$$

A manifold equipped with such a structure is called a twisted Jacobi manifold *or a ω-*Jacobi manifold *and it is denoted by the triple $(M,(\Lambda, E),\omega)$.*

Hence, according to Proposition 3.1, we may define a twisted Jacobi manifold as a manifold M equipped with a section (Λ, E) of $\bigwedge^2(TM \times \mathbb{R})$ and a 2-form ω on M satisfying conditions (16) and (17), for $\varphi = d\omega$.

Examples 3.4.

1. Jacobi manifolds: Any Jacobi manifold (M, Λ, E) endowed with a 2-form ω satisfying $(\Lambda, E)^{\#}(d\omega, \omega) = (0, 0)$ can be viewed as a twisted Jacobi manifold.

2. Twisted locally conformal symplectic manifolds: A *twisted locally conformal symplectic manifold* is a $2n$-dimensional differentiable manifold M equipped with a non-degenerate 2-form Θ, a closed 1-form ϑ, called the *Lee 1-form*, and a 2-form ω such that

$$d(\Theta + \omega) + \vartheta \wedge (\Theta + \omega) = 0.$$

Let E be the unique vector field and Λ the unique bivector field on M which are defined by

$$i(E)\Theta = -\vartheta \quad \text{and} \quad i(\Lambda^{\#}(\alpha))\Theta = -\alpha, \quad \text{for all } \alpha \in \Gamma(T^*M). \tag{21}$$

If we also denote by $\Lambda^{\#}$ the extension (13) of the isomorphism $\Lambda^{\#} : \Gamma(T^*M) \to \Gamma(TM)$ given by (21), we obtain

$$E = \Lambda^{\#}(\vartheta) \;\text{ and }\; \Lambda = \Lambda^{\#}(\Theta).$$

By a simple, but very long computation, we prove that the pair $((\Lambda, E), \omega)$ satisfies the relations (16) and (17), for $\varphi = d\omega$. Whence, $((\Lambda, E), \omega)$ endows M with a twisted Jacobi structure.

3. A trivial example in local coordinates: Let $(x_0, x_1, x_2, x_3, x_4)$ be a system of local coordinates in $\mathbb{R}^5$. Let us consider a bivector field Λ, a vector field E and a 2-form ω on $\mathbb{R}^5$ given, in these coordinates, by

$$\Lambda = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_4}, \quad E = \frac{\partial}{\partial x_0}, \quad \omega = dx_1 \wedge dx_3.$$

A simple computation gives

$$[\Lambda, \Lambda] + 2E \wedge \Lambda = 2\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_0} \qquad \text{and} \qquad [E, \Lambda] = 0.$$

Since

$$\Lambda^{\#}(\omega) = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} \qquad \text{and} \qquad (\Lambda^{\#} \otimes 1)(\omega)(E) = 0,$$

we have

$$[\Lambda,\Lambda]+2E\wedge\Lambda=2\Lambda^{\#}(\omega)\wedge E \qquad \text{and} \qquad [E,\Lambda]=-(\Lambda^{\#}\otimes 1)(\omega)(E)\wedge E.$$

According to Proposition 3.1, with $\varphi=d\omega=0$, $((\Lambda,E),\omega)$ defines a twisted Jacobi structure on the manifold $\mathbb{R}^5$.

Given a twisted Jacobi structure $((\Lambda,E),\omega)$ on M, (Λ,E) defines on $C^\infty(M,\mathbb{R})$ an internal composition law $\{\cdot,\cdot\}$ just as in the case of Jacobi structure: For all $f,g\in C^\infty(M,\mathbb{R})$,

$$\{f,g\}=\Lambda(df,dg)+fE(g)-gE(f). \tag{22}$$

Since (12) does not hold, this bracket fails the Jacobi identity and is no more a Lie bracket.

Proposition 3.5. *Let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold. Then, for all $f,g,h\in C^\infty(M,\mathbb{R})$,*

$$\{f,\{g,h\}\}+c.p.=-(d\omega,\omega)((\Lambda,E)^{\#}(df,f),(\Lambda,E)^{\#}(dg,g),(\Lambda,E)^{\#}(dh,h)),$$

where c.p. denotes sum after circular permutation.

Proof: The result follows directly from (14) for $k=3$ and (20), taking into account that, for any $f,g,h\in C^\infty(M,\mathbb{R})$,

$$\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}((df,f),(dg,g),(dh,h))=\{f,\{g,h\}\}+p.c.$$

■

Let us now examine some relations between twisted Jacobi manifolds and twisted Poisson manifolds.

We recall that a *twisted Poisson manifold* [24] is a differentiable manifold M endowed with a bivector field Λ and a closed 3-form φ on M such that $[\Lambda,\Lambda]=2\Lambda^{\#}(\varphi)$. When φ is exact, i.e. $\varphi=d\omega$ with $\omega\in\Gamma(\bigwedge^2T^*M)$, we say that (M,Λ,φ) is a *twisted exact* Poisson manifold. A twisted Jacobi manifold $(M,(\Lambda,E),\omega)$, with $E=0$, defines a twisted exact Poisson structure on M, since

$$[(\Lambda,0),(\Lambda,0)]^{(0,1)}=2(\Lambda,0)^{\#}(d\omega,\omega) \;\Rightarrow\; [\Lambda,\Lambda]=2\Lambda^{\#}(d\omega).$$

Furthermore, it is well known that there exists a close relationship which links homogeneous Poisson manifolds with Jacobi manifolds [14]. Namely, to each Jacobi manifold (M,Λ,E), we can associate a homogeneous Poisson manifold $(\tilde{M}, \tilde{\Lambda}, \frac{\partial}{\partial t})$, called the *Poissonization* of (M, Λ, E), with $\tilde{M} = M \times \mathbb{R}$ and $\tilde{\Lambda} = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$, t being the canonical coordinate on $\mathbb{R}$. For the twisted exact Poisson structures, we introduce the following definition.

Definition 3.6. *A* homogeneous twisted exact Poisson structure *on a manifold M is defined by a triple (Λ, Z, ω), where Λ is a bivector field on M, Z is a vector field on M and ω is a 2-form on M, such that*

$$[\Lambda, \Lambda] = 2\Lambda^{\#}(d\omega), \quad [Z, \Lambda] = -\Lambda, \quad L_Z\omega = \omega.$$

Proposition 3.7. *Let $(M, (\Lambda, E), \omega)$ be a twisted Jacobi manifold. We set $\tilde{M} = M \times \mathbb{R}$ and we consider on $\tilde{M}$ the tensor fields $\tilde{\Lambda} = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$ and $\tilde{\omega} = e^t\omega$, t being the canonical coordinate on the factor $\mathbb{R}$. Then, the triple $(\tilde{\Lambda}, \frac{\partial}{\partial t}, \tilde{\omega})$ defines an homogeneous twisted exact Poisson structure on $\tilde{M}$.*

Proof: We have $[\frac{\partial}{\partial t}, \tilde{\Lambda}] = -\tilde{\Lambda}$ and $L_{\partial/\partial t}\tilde{\omega} = \tilde{\omega}$. So, according to Definition 3.6, it remains to prove that $[\tilde{\Lambda}, \tilde{\Lambda}] = 2\tilde{\Lambda}^{\#}(d\tilde{\omega})$. From the definition of $\tilde{\Lambda}$, we compute

$$[\tilde{\Lambda}, \tilde{\Lambda}] = e^{-2t}([\Lambda, \Lambda] + 2E \wedge \Lambda) + 2e^{-2t}(\frac{\partial}{\partial t} \wedge [E, \Lambda])$$

and, since $(M, (\Lambda, E), \omega)$ is a twisted Jacobi manifold, from (16) and (17), we can write

$$\begin{aligned} [\tilde{\Lambda}, \tilde{\Lambda}] &= 2e^{-2t}\Big(\Lambda^{\#}(d\omega) + \Lambda^{\#}(\omega) \wedge E \\ &\quad + \frac{\partial}{\partial t} \wedge \big((\Lambda^{\#} \otimes 1)(d\omega)(E) - ((\Lambda^{\#} \otimes 1)(\omega)(E)) \wedge E\big)\Big). \end{aligned} \tag{23}$$

On the other hand,

$$\tilde{\Lambda}^{\#}(d\tilde{\omega}) = e^t\tilde{\Lambda}^{\#}(d\omega + dt \wedge \omega). \tag{24}$$

But,

$$\tilde{\Lambda}^{\#}(d\omega) = e^{-3t}\left(\Lambda^{\#}(d\omega) + \frac{\partial}{\partial t} \wedge (\Lambda^{\#} \otimes 1)(d\omega)\right) \tag{25}$$

and

$$\tilde{\Lambda}^{\#}(dt \wedge \omega) = e^{-3t}\left(\Lambda^{\#}(\omega) - \frac{\partial}{\partial t} \wedge ((\Lambda^{\#} \otimes 1)(\omega)(E))\right) \wedge E. \tag{26}$$

From equations (23)-(26) we obtain $[\tilde{\Lambda}, \tilde{\Lambda}] = 2\tilde{\Lambda}^{\#}(d\tilde{\omega})$. ■

4. Twisted Dirac-Jacobi structures

The notions of Courant-Jacobi algebroid and the equivalent one of generalized Courant algebroid were introduced in [4] and [18], respectively, as a generalization of the definition of Courant algebroid [15, 22].

Definition 4.1. ([18]) *A* generalized Courant algebroid *or a* Courant-Jacobi algebroid *on a differentiable manifold M is a vector bundle E over M equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on the bundle, a skew-symmetric bracket $[\cdot,\cdot]$ on $\Gamma(E)$, a bundle map $\rho^\theta : E \to TM \times \mathbb{R}$ and a section θ of E^* such that, for any $e_1, e_2 \in \Gamma(E)$, the condition $\langle\theta, [e_1, e_2]\rangle = \rho(e_1)\langle\theta, e_2\rangle - \rho(e_2)\langle\theta, e_1\rangle$ holds, ρ being the bundle map from E onto TM induced by ρ^θ, satisfying, for all $e, e_1, e_2, e_3 \in \Gamma(E)$ the following properties:*

i) $[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2] = \mathcal{D}^\theta T(e_1, e_2, e_3)$,
where $T(e_1, e_2, e_3) = \frac{1}{3}([e_1, e_2], e_3) + c.p.$ and $\mathcal{D}^\theta : C^\infty(M, \mathbb{R}) \to \Gamma(E)$ is the first-order differential operator given by $(\mathcal{D}^\theta f, e) = \frac{1}{2}\rho^\theta(e)f$;

ii) $\rho^\theta([e_1, e_2]) = [\rho^\theta(e_1), \rho^\theta(e_2)]$,
where the bracket on the right-hand side is the Lie bracket (11) on $\Gamma(TM \times \mathbb{R})$;

iii) $\rho^\theta(e)(e_1, e_2) = ([e, e_1] + \mathcal{D}^\theta(e, e_1), e_2) + (e_1, [e, e_2] + \mathcal{D}^\theta(e, e_2))$;

iv) *for any $f, g \in C^\infty(M, \mathbb{R})$, $(\mathcal{D}^\theta f, \mathcal{D}^\theta g) = 0$.*

A *Dirac structure* for the generalized Courant algebroid (E, θ) is a subbundle L of E which is closed under the bracket $[\cdot,\cdot]$ and is maximally isotropic with respect to the symmetric bilinear form $(\cdot,\cdot)$. In this case $(L, \rho|_L, [\cdot,\cdot]|_L)$ is a Lie algebroid over M.

An important example of a Courant-Jacobi algebroid is the double $A \oplus A^*$ of a Jacobi bialgebroid $((A, \phi), (A^*, W))$ over M [4, 18]. The bracket on the space $\Gamma(A \oplus A^*)$ of its sections is given, for all $e_1 = X_1 + \alpha_1, e_2 = X_2 + \alpha_2 \in \Gamma(A \oplus A^*)$, by

$$\begin{aligned} [\![X_1 + \alpha_1, X_2 + \alpha_2]\!] &= \left([X_1, X_2]^\phi + \mathcal{L}^W_{*\alpha_1} X_2 - \mathcal{L}^W_{*\alpha_2} X_1 - d^W_*(e_1, e_2)_-\right) \\ &+ \left([\alpha_1, \alpha_2]^W_* + \mathcal{L}^\phi_{X_1}\alpha_2 - \mathcal{L}^\phi_{X_2}\alpha_1 + d^\phi(e_1, e_2)_-\right), \end{aligned} \tag{27}$$

where $(e_1, e_2)_- = \frac{1}{2}(i_{X_2}\alpha_1 - i_{X_1}\alpha_2)$. Moreover, $\theta = \phi + W$, ρ is the sum of the anchor maps of A and A^*, the symmetric bilinear form on $A \oplus A^*$ is the canonical one, i.e. $(e_1,e_2)=(e_1,e_2)_+=\frac{1}{2}(i_{X_2}\alpha_1+i_{X_1}\alpha_2)$, $\mathcal{D}=(d_*+d)_{|C^\infty(M,\mathbb{R})}$ and $\mathcal{D}^\theta=(d_*^W+d^\phi)_{|C^\infty(M,\mathbb{R})}$.

For the case of the Jacobi bialgebroid $((TM\times\mathbb{R},(0,1)),(T^*M\times\mathbb{R},(0,0)))$, where $T^*M\times\mathbb{R}$ is equipped with the null Lie algebroid structure, the Courant-Jacobi structure defined on its double $\mathcal{E}^1(M)=(TM\times\mathbb{R})\oplus(T^*M\times\mathbb{R})$ corresponds to the following bracket on the space $\Gamma(\mathcal{E}^1(M))$, defined in [25] as a direct generalization of the Courant bracket on $\Gamma(TM\oplus T^*M)$ [2], as follows: for all $e_1=(X_1,f_1)+(\alpha_1,g_1)$, $e_2=(X_2,f_2)+(\alpha_2,g_2)\in\Gamma(\mathcal{E}^1(M))$,

$$\begin{aligned}[\![e_1,e_2]\!] &= [\![(X_1,f_1)+(\alpha_1,g_1),(X_2,f_2)+(\alpha_2,g_2)]\!]\\ &= [(X_1,f_1),(X_2,f_2)]^{(0,1)}\\ &\quad+\Big(\mathcal{L}^{(0,1)}_{(X_1,f_1)}(\alpha_2,g_2)-\mathcal{L}^{(0,1)}_{(X_2,f_2)}(\alpha_1,g_1)+\mathrm{d}^{(0,1)}(e_1,e_2)_-\Big)\end{aligned}$$

with

$$(e_1,e_2)_-=\frac{1}{2}(i_{X_2}\alpha_1-i_{X_1}\alpha_2+f_2g_1-f_1g_2).$$

Dirac structures for the Courant-Jacobi algebroid $(\mathcal{E}^1(M),(0,1)+(0,0))$ will be called *Dirac-Jacobi structures*.

Let us now "twist" the bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(\mathcal{E}^1(M))$ with a section (φ,ω) of $\bigwedge^3(T^*M\times\mathbb{R})$ by setting

$$[e_1,e_2]_{(\varphi,\omega)}=[\![e_1,e_2]\!]+(\varphi,\omega)((X_1,f_1),(X_2,f_2),\cdot).$$

Proposition 4.2. *The pair $(\mathcal{E}^1(M),(0,1)+(0,0))$ equipped with the bracket $[\cdot,\cdot]_{(\varphi,\omega)}$ on $\Gamma(\mathcal{E}^1(M))$, the canonical bilinear symmetric form $(\cdot,\cdot)_+$ on the bundle* and the bundle map $\rho=\pi+0$, is a Courant-Jacobi algebroid over M if and only if $\mathrm{d}^{(0,1)}(\varphi,\omega)=0$.*

We denote this new Courant-Jacobi algebroid by $(\mathcal{E}^1(M)_{(d\omega,\omega)},(0,1)+(0,0))$ or simply by $(\mathcal{E}^1(M)_\omega,(0,1)+(0,0))$.

Proof: We know that $(\mathcal{E}^1(M),(0,1)+(0,0))$ equipped with $([\![\cdot,\cdot]\!],\rho,(\cdot,\cdot)_+)$ is a Courant-Jacobi algebroid [18]. Hence, we only have to check the effect of adding the term $(\varphi,\omega)((X_1,f_1),(X_2,f_2),\cdot)$ to the bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(\mathcal{E}^1(M))$. Let us set $\theta=(0,1)+(0,0)$. Then, for any $e_1=(X_1,f_1)+(\alpha_1,g_1),e_2=(X_2,f_2)+(\alpha_2,g_2)\in\Gamma(\mathcal{E}^1(M))$, we compute

$$\begin{aligned}\rho^\theta([e_1,e_2]_{(\varphi,\omega)}) &= \rho^\theta([\![e_1,e_2]\!])+\rho^\theta((\varphi,\omega)((X_1,f_1),(X_2,f_2),\cdot))\\ &= [\rho^\theta(e_1),\rho^\theta(e_2)],\end{aligned}$$

* $(e_1,e_2)_+=\frac{1}{2}(i_{X_2}\alpha_1+i_{X_1}\alpha_2+f_2g_1+f_1g_2)$

and $ii)$ of Definition 4.1 holds. Moreover, for any $e = (X,f) + (\alpha,g) \in \Gamma(\mathcal{E}^1(M))$, condition $iii)$ holds if and only if

$$\begin{aligned}&((\varphi,\omega)((X,f),(X_1,f_1),\cdot),(X_2,f_2)+(\alpha_2,g_2))_+\\&\quad+\ ((X_1,f_1)+(\alpha_1,g_1),(\varphi,\omega)((X,f),(X_2,f_2),\cdot))_+=0,\end{aligned}$$

that is, if and only if

$$\begin{aligned}&((\varphi(X,X_1,\cdot)+\omega(fX_1-f_1X,\cdot),\omega(X,X_1)),(X_2,f_2)+(\alpha_2,g_2))_+\\&\quad+\ ((X_1,f_1)+(\alpha_1,g_1),(\varphi(X,X_2,\cdot)+\omega(fX_2-f_2X,\cdot),\omega(X,X_2))_+=0,\end{aligned}$$

which can be proved by a simple computation. Finally, by a long but straightforward computation, we obtain

$$\begin{aligned}[[e_1,e_2]_{(\varphi,\omega)},e_3]_{(\varphi,\omega)}+c.p. &= \mathrm{d}^{(0,1)}(T_{(\varphi,\omega)}(e_1,e_2,e_3))\\&\quad-\ (\mathrm{d}^{(0,1)}(\varphi,\omega))((X_1,f_1),(X_2,f_2),(X_3,f_3),\cdot)\end{aligned}$$

with $T_{(\varphi,\omega)}(e_1,e_2,e_3)=\frac{1}{3}([e_1,e_2]_{(\varphi,\omega)},e_3)_+ + c.p.$. Thus, condition $i)$ of Definition 4.1 holds if and only if $\mathrm{d}^{(0,1)}(\varphi,\omega)=(0,0)$ and the proof is complete. ∎

Definition 4.3. *A Dirac sub-bundle L for the Courant-Jacobi algebroid $(\mathcal{E}^1(M)_\omega,(0,1)+(0,0))$ over M is called an ω-*Dirac-Jacobi structure *or a* twisted Dirac-Jacobi structure.

Obviously, if L is a twisted Dirac-Jacobi structure, then $(L,[\cdot,\cdot]_{(d\omega,\omega)}|_L,\rho|_L)$ is a Lie algebroid over M.

The next result enables us to characterize twisted Jacobi manifolds in terms of twisted Dirac-Jacobi structures. Hereafter, in order to simplify the notation, we will denote the bracket $[\cdot,\cdot]_{(d\omega,\omega)}$ by $[\cdot,\cdot]_\omega$, whenever is clear to which bracket we refer to.

Proposition 4.4. *Let ω be a 2-form on M and (Λ,E) a section of $\bigwedge^2(TM\times\mathbb{R})$. Then,* $\mathrm{graph}(\Lambda,E)^\#$ *is a ω-Dirac-Jacobi structure if and only if*

$$[(\Lambda,E),(\Lambda,E)]^{(0,1)}=2(\Lambda,E)^\#(d\omega,\omega).$$

Proof: For any $(\alpha,f),(\beta,g)\in\Gamma(T^*M\times\mathbb{R})$, we have

$$\begin{aligned}&[(\Lambda,E)^\#(\alpha,f)+(\alpha,f),(\Lambda,E)^\#(\beta,g)+(\beta,g)]_\omega\\&\quad=[(\Lambda,E)^\#(\alpha,f),(\Lambda,E)^\#(\beta,g)]+\{(\alpha,f),(\beta,g)\}\\&\quad+(d\omega,\omega)((\Lambda,E)^\#(\alpha,f),(\Lambda,E)^\#(\beta,g),\cdot),\end{aligned}$$

where $\{\cdot,\cdot\}$ is the bracket (5). So, $\mathrm{graph}(\Lambda,E)^\#$ is closed under the bracket $[\cdot,\cdot]_\omega$ if and only if

$$\begin{aligned}&[(\Lambda,E)^\#(\alpha,f),(\Lambda,E)^\#(\beta,g)]=\\&\quad=(\Lambda,E)^\#(\{(\alpha,f),(\beta,g)\}+(d\omega,\omega)((\Lambda,E)^\#(\alpha,f),(\Lambda,E)^\#(\beta,g),\cdot)).\end{aligned}\tag{28}$$

But (28) is equivalent to $[(\Lambda,E),(\Lambda,E)]^{(0,1)}=2(\Lambda,E)^\#(d\omega,\omega)$ (see, e.g. [10]). ■

Corollary 4.5. *The triple $(M,(\Lambda,E),\omega)$ is a twisted Jacobi manifold if and only if* $\mathrm{graph}(\Lambda,E)^\#$ *is a ω-Dirac-Jacobi structure.*

Let (η,γ) be a section of $\bigwedge^2(T^*M\times\mathbb{R})$. We denote by $(\eta,\gamma)^\flat:TM\times\mathbb{R}\to T^*M\times\mathbb{R}$ the associated vector bundle morphism that induces on the spaces of sections a map, that we also denote by $(\eta,\gamma)^\flat$, which is given, for any $(X,f)\in\Gamma(TM\times\mathbb{R})$, by

$$(\eta,\gamma)^\flat(X,f)=(i_X\eta+f\gamma,-i_X\gamma).$$

Proposition 4.6. *Let (η,γ) be a section of $\bigwedge^2(T^*M\times\mathbb{R})$. Then,* $\mathrm{graph}(\eta,\gamma)^\flat$ *is a ω-Dirac-Jacobi structure if and only if $\mathrm{d}^{(0,1)}(\eta,\gamma)+(d\omega,\omega)=(0,0)$.*

Proof: We start by remarking that

$$\mathrm{d}^{(0,1)}(\eta,\gamma)+(d\omega,\omega)=(0,0)\ \Leftrightarrow\ \eta=d\gamma-\omega.$$

The vector bundle $\mathrm{graph}(\eta,\gamma)^\flat$ over M, whose space of sections is given by

$$\Gamma(\mathrm{graph}(\eta,\gamma)^\flat)=\{(X,f)+(i_X\eta+f\gamma,-i_X\gamma)\,|\,(X,f)\in\Gamma(TM\times\mathbb{R})\},$$

is a maximally isotropic sub-bundle of $\mathcal{E}^1(M)$ with respect to the symmetric bilinear form $(\cdot,\cdot)_+$. Now, let $e_i=(X_i,f_i)+(i_{X_i}\eta+f_i\gamma,-i_{X_i}\gamma)$, $i=1,2$, be two sections of $\mathrm{graph}(\eta,\gamma)^\flat$. Then,

$$\begin{aligned}[e_1,e_2]_\omega &= ([X_1,X_2],X_1(f_2)-X_2(f_1))\\&\quad+\mathcal{L}^{(0,1)}_{(X_1,f_1)}(i_{X_2}\eta+f_2\gamma,-i_{X_2}\gamma)-i_{(X_2,f_2)}\mathrm{d}^{(0,1)}(i_{X_1}\eta+f_1\gamma,-i_{X_1}\gamma)\\&\quad+(d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)\end{aligned}$$

and $[e_1,e_2]_\omega\in\Gamma(\mathrm{graph}(\eta,\gamma)^\flat)$ if and only if

$$\begin{aligned}&\mathcal{L}^{(0,1)}_{(X_1,f_1)}(i_{X_2}\eta+f_2\gamma,-i_{X_2}\gamma)-i_{(X_2,f_2)}\mathrm{d}^{(0,1)}(i_{X_1}\eta+f_1\gamma,-i_{X_1}\gamma)\\&\qquad+(d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)\\&\quad= (i_{[X_1,X_2]}\eta+(X_1(f_2)-X_2(f_1))\gamma,-i_{(X_1,X_2]}\gamma).\end{aligned}\tag{29}$$

A simple computation shows that (29) is equivalent to $\eta = d\gamma - \omega$. ∎

Let us now look at some other examples of twisted Dirac-Jacobi structures. Recall that a sub-bundle L of the vector bundle $TM \oplus T^*M$ over M is a *Dirac structure in the sense of Courant* [2] if L is maximally isotropic with respect to the symmetric canonical bilinear form on $TM \oplus T^*M$ and $\Gamma(L)$ closes under the Courant bracket, which is given, for any sections $X+\alpha$, $Y+\beta$ of $TM \oplus T^*M$, by

$$[X+\alpha, Y+\beta]_C = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}d(i_Y\alpha - i_X\beta). \tag{30}$$

Example 4.7. Let L be a sub-bundle of $TM \oplus T^*M$, ω a 2-form on M and consider the sub-bundle L_ω of $\mathcal{E}^1_\omega(M)$ whose fiber at a point $x \in M$ is given by

$$L_\omega(x) = \{(X,0)_x + (\alpha - i_X\omega, f)_x \,|\, (X+\alpha)_x \in L_x\}.$$

Then, L_ω is a ω-Dirac-Jacobi structure if and only if L is a Dirac structure in the sense of Courant. It is immediate to verify that L_ω is maximally isotropic with respect to symmetric canonical bilinear form on $\mathcal{E}^1_\omega(M)$ if and only if L is maximally isotropic with respect to symmetric canonical bilinear form on $TM \oplus T^*M$. Moreover, if $(X,0) + (\alpha - i_X\omega, f)$ and $(Y,0) + (\beta - i_Y\omega, g)$ are any two sections of L_ω, then

$$\begin{aligned}&[(X,0) + (\alpha - i_X\omega, f), (Y,0) + (\beta - i_Y\omega, g)]_\omega = ([X,Y],0) + \\ &\quad +(\mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}d(i_Y\alpha - i_X\beta) - i_{[X,Y]}\omega, X.g - Y.f + \frac{1}{2}d(i_Y\alpha - i_X\beta)).\end{aligned}$$

So, the sections of L_ω close under the bracket $[\cdot,\cdot]_\omega$ if and only if the sections of L close under the Courant bracket on $TM \oplus T^*M$.

For the next example we need the following definition.

Definition 4.8. *A* twisted locally conformal presymplectic *structure on a manifold M is a pair $((\Theta, \vartheta), \omega)$, where Θ and ω are two 2-forms on M and ϑ is a closed 1-form on M such that*

$$d(\Theta + \omega) + \vartheta \wedge (\Theta + \omega) = 0.$$

If M is even dimensional and Θ is non-degenerate, $(M, (\Theta, \vartheta), \omega)$ is a twisted locally conformal symplectic manifold (cf. Example 3.4.2).

Example 4.9. Let Θ and ω be two 2-forms on a manifold M and ϑ be a 1-form on M. Consider the sub-bundle $L_{((\Theta,\vartheta),\omega)}$ of $\mathcal{E}^1_\omega(M)$ whose fiber at a point $x \in M$ is given by

$$L_{((\Theta,\vartheta),\omega)}(x) = \{(X, i_X\vartheta)_x + (i_X\Theta - f\vartheta, f)_x \,|\, (X,f)_x \in (TM \times \mathbb{R})_x\}. \quad (31)$$

Then, $L_{((\Theta,\vartheta),\omega)}$ is a twisted Dirac-Jacobi structure if and only if $((\Theta,\vartheta),\omega)$ is a twisted locally conformal presymplectic structure on M. Effectively, it is easy to check that $L_{((\Theta,\vartheta),\omega)}$ is a maximally isotropic sub-bundle of $\mathcal{E}^1_\omega(M)$, with respect to the bilinear symmetric form $(\cdot,\cdot)_+$. Let $(X, i_X\vartheta) + (i_X\Theta - f\vartheta, f)$ and $(Y, i_Y\vartheta) + (i_Y\Theta - g\vartheta, g)$ be two sections of $L_{((\Theta,\vartheta),\omega)}$. We compute

$$\begin{aligned}
&[(X, i_X\vartheta) + (i_X\Theta - f\vartheta, f), (Y, i_Y\vartheta) + (i_Y\Theta - g\vartheta, g)]_\omega \\
&= \; ([X,Y], i_{[X,Y]}\vartheta + d\vartheta(X,Y)) + \\
&\quad +(i_{[X,Y]}\Theta - g i_X d\vartheta + f i_Y d\vartheta + d\Theta(X,Y,\cdot) + d\omega(X,Y,\cdot) \\
&\quad +(\vartheta\wedge\Theta)(X,Y,\cdot) + (\vartheta\wedge\omega)(X,Y,\cdot) \\
&\quad -\{X.g - Y.f - (i_X\vartheta)g + (i_Y\vartheta)f + \Theta(X,Y) + \omega(X,Y)\}\vartheta, \\
&\quad X.g - Y.f - (i_X\vartheta)g + (i_Y\vartheta)f + \Theta(X,Y) + \omega(X,Y));
\end{aligned}$$

so, the space $\Gamma(L_{((\Theta,\vartheta),\omega)})$ is closed under the bracket $[\cdot,\cdot]_\omega$ if and only if $d\vartheta = 0$ and $d(\Theta+\omega) + \vartheta\wedge(\Theta+\omega) = 0$.

Example 4.10. Let Λ be a bivector filed on M, Z a vector field on M and ω a 2-form on M. We denote by $L_{(\Lambda,Z,\omega)}$ the sub-bundle of $\mathcal{E}^1_\omega(M)$ whose fiber at a point $x \in M$ is given by

$$L_{(\Lambda,Z,\omega)}(x) = \{(\Lambda^\#(\alpha) - fZ, f)_x + (\alpha, i_Z\alpha)_x \,|\, (\alpha,f)_x \in (T^*M \times I\!R)_x\}.$$

Then, $L_{(\Lambda,Z,\omega)}$ is a twisted Dirac-Jacobi structure if and only if (Λ, Z, ω) defines an homogeneous twisted exact Poisson structure on M (cf. Definition 3.6). An easy computation shows that $L_{(\Lambda,Z,\omega)}$ is a maximally isotropic sub-bundle of $\mathcal{E}^1(M)$, with respect to symmetric bilinear form $(\cdot,\cdot)_+$. Let $(\Lambda^\#(\alpha) - fZ, f) + (\alpha, i_Z\alpha)$ and $(\Lambda^\#(\beta) - gZ, g) + (\beta, i_Z\beta)$ be two sections of $L_{(\Lambda,Z,\omega)}$. Then, if (Λ, Z, ω) is a twisted exact homogeneous Poisson structure, we compute

$$
\begin{aligned}
&[(\Lambda^{\#}(\alpha)-fZ,f)+(\alpha,i_Z\alpha),(\Lambda^{\#}(\beta)-gZ,g)+(\beta,i_Z\beta)]_\omega\\
&= \Big(\Lambda^{\#}\big(\mathcal{L}_{\Lambda^{\#}(\alpha)}\beta-\mathcal{L}_{\Lambda^{\#}(\beta)}\alpha-d(\Lambda(\alpha,\beta))+g(\mathcal{L}_Z\alpha-\alpha)-f(\mathcal{L}_Z\beta-\beta)\\
&\qquad +d\omega(\Lambda^{\#}(\alpha),\Lambda^{\#}(\beta),\cdot)\big)-\big((\Lambda^{\#}(\alpha)).g-(\Lambda^{\#}(\beta)).f+g(Z.f)-f(Z.g)\big)\,Z\,,\\
&\qquad (\Lambda^{\#}(\alpha)).g-(\Lambda^{\#}(\beta)).f+g(Z.f)-f(Z.g)\Big)+\\
&+\ \Big(\mathcal{L}_{\Lambda^{\#}(\alpha)}\beta-\mathcal{L}_{\Lambda^{\#}(\beta)}\alpha-d(\Lambda(\alpha,\beta))+g(\mathcal{L}_Z\alpha-\alpha)-f(\mathcal{L}_Z\beta-\beta)\\
&\qquad +d\omega(\Lambda^{\#}(\alpha),\Lambda^{\#}(\beta),\cdot)\\
&\qquad \underbrace{-d\omega(\Lambda^{\#}(\alpha),gZ,\cdot)+d\omega(\Lambda^{\#}(\beta),fZ,\cdot)-\omega(g\Lambda^{\#}(\alpha),\cdot)+\omega(f\Lambda^{\#}(\beta),\cdot)}_{=0},\\
&\qquad i_Z\big(\mathcal{L}_{\Lambda^{\#}(\alpha)}\beta-\mathcal{L}_{\Lambda^{\#}(\beta)}\alpha-d(\Lambda(\alpha,\beta))+g(\mathcal{L}_Z\alpha-\alpha)-f(\mathcal{L}_Z\beta-\beta)\\
&\qquad +d\omega(\Lambda^{\#}(\alpha),\Lambda^{\#}(\beta),\cdot)\big)\Big)
\end{aligned}
$$

and we conclude that the space of sections of $L_{(\Lambda,Z,\omega)}$ is closed under the bracket $[\cdot,\cdot]_\omega$. Thus, $L_{(\Lambda,Z,\omega)}$ is a ω-Dirac-Jacobi structure. A similar computation shows that, conversely, if $L_{(\Lambda,Z,\omega)}$ is a ω-Dirac-Jacobi structure, then the triple (Λ,Z,ω) defines an homogeneous twisted exact Poisson structure on M.

Let φ be a closed 3-form on M and L a sub-bundle of $TM\oplus T^*M$. We recall that L is called a *φ-Dirac structure* (in the sense of Courant) [24] if it is maximally isotropic with respect to the canonical bilinear symmetric form on $TM\oplus T^*M$, and its space of sections is closed under the bracket $[\cdot,\cdot]_{C_\omega}$ which is given, for any sections $X+\alpha$ and $Y+\beta$ of $TM\oplus T^*M$, by

$$[X+\alpha,Y+\beta]_{C_\omega}=[X+\alpha,Y+\beta]_C+\varphi(X,Y,\cdot),$$

where $[\cdot,\cdot]_C$ is the Courant bracket given by (30). In [21], we proved that there exists a correspondence between Dirac-Jacobi structures $L\subset\mathcal{E}^1(M)$ and Dirac structures $\tilde{L}\subset T(M\times\mathbb{R})\oplus T^*(M\times\mathbb{R})$ in the sense of Courant (see also [6]).

For twisted Dirac-Jacobi structures we can establish the following.

Proposition 4.11. *Let L be a sub-bundle of $\mathcal{E}^1_\omega(M)$. Then,*

(1) *the sub-bundle $\tilde{L}_\omega\subset T(M\times\mathbb{R})\oplus T^*(M\times\mathbb{R})$ given by*

$$\tilde{L}_\omega=\{(X+f\frac{\partial}{\partial t})+e^t(\alpha+i_X\omega+gdt)\,|\,(X,f)+(\alpha,g)\in L\}$$

is a Dirac structure (in the sense of Courant) if and only if L is an ω-Dirac-Jacobi structure;

(2) *the sub-bundle $\tilde{L} \subset T(M \times I\!R) \oplus T^*(M \times I\!R)$ given by*

$$\tilde{L} = \{(X + f\frac{\partial}{\partial t}) + e^t(\alpha + gdt) \,|\, (X, f) + (\alpha, g) \in L\}$$

is a $d(e^t\omega)$-Dirac structure (in the sense of Courant)if and only if L is an ω-Dirac-Jacobi structure.

Proof: A simple computation proves that each one of the sub-bundles $\tilde{L}_\omega$ and $\tilde{L}$ of $T(M \times \mathbb{R}) \oplus T^*(M \times \mathbb{R})$ is maximally isotropic with respect to the canonical symmetric bilinear form in $T(M \times \mathbb{R}) \oplus T^*(M \times \mathbb{R})$ if and only if $L \subset \mathcal{E}^1_\omega(M)$ is maximally isotropic with respect to the canonical symmetric bilinear form in $\mathcal{E}^1_\omega(M)$. To complete the proof of the first assertion, we take two sections $(X_i + f_i\frac{\partial}{\partial t}) + e^t(\alpha_i + i_{X_i}\omega + g_i dt)$, $i = 1, 2$, of $\tilde{L}_\omega$. Then, denoting by $[\cdot, \cdot]_C$ the Courant bracket on $\Gamma(T(M \times \mathbb{R}) \oplus T^*(M \times \mathbb{R}))$, we compute

$$\begin{aligned}
&[(X_1 + f_1\frac{\partial}{\partial t}) + e^t(\alpha_1 + i_{X_1}\omega + g_1 dt), (X_2 + f_2\frac{\partial}{\partial t}) + e^t(\alpha_2 + i_{X_2}\omega + g_2 dt)]_C \\
&= \left([X_1, X_2] + (X_1.f_2 - X_2.f_1)\frac{\partial}{\partial t}\right) + e^t \big(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 \\
&\quad + \frac{1}{2}d(i_{X_2}\alpha_1 - i_{X_1}\alpha_2) + f_1\alpha_2 - f_2\alpha_1 + \frac{1}{2}(g_2 df_1 - g_1 df_2 - f_1 dg_2 + f_2 dg_1) \\
&\quad + d\omega(X_1, X_2, \cdot) + \omega(f_1X_2 - f_2X_1, \cdot) + i_{[X_1,X_2]}\omega\big) \\
&\quad + e^t\left(X_1.g_2 - X_2.g_1 + \frac{1}{2}(i_{X_2}\alpha_1 - i_{X_1}\alpha_2 - f_2g_1 + f_1g_2) + \omega(X_1, X_2)\right) dt
\end{aligned}$$

and, since

$$\begin{aligned}
&([X_1, X_2], X_1.f_2 - X_2.f_1) \\
&\quad + \Big(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + \frac{1}{2}d(i_{X_2}\alpha_1 - i_{X_1}\alpha_2) + f_1\alpha_2 - f_2\alpha_1 \\
&\quad + \frac{1}{2}(g_2 df_1 - g_1 df_2 - f_1 dg_2 + f_2 dg_1) + d\omega(X_1, X_2, \cdot) + \omega(f_1X_2 - f_2X_1, \cdot), \\
&\quad X_1.g_2 - X_2.g_1 + \frac{1}{2}(i_{X_2}\alpha_1 - i_{X_1}\alpha_2 - f_2g_1 + f_1g_2) + \omega(X_1, X_2)\Big) \\
&= [(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)]_\omega,
\end{aligned}$$

we conclude that the bracket $[\cdot,\cdot]_C$ closes in $\Gamma(\tilde{L}_\omega)$ if and only if the bracket $[\cdot,\cdot]_\omega$ closes in $\Gamma(L)$. The proof of the second assertion is very similar and we omit it. ■

5. Gauge transformations

As in [24], in the case of (twisted) Dirac structures for Courant algebroids, we may define gauge transformations for Dirac-Jacobi sub-bundles. Given a section (η,γ) of $\bigwedge^2(T^*M\times\mathbb{R})$, let us consider the vector bundle map

$$\tau_{(\eta,\gamma)} : \mathcal{E}^1(M) \to \mathcal{E}^1(M)$$

that induces on the spaces of sections a map, that we also denote by $\tau_{(\eta,\gamma)}$, which is defined, for any $(X,f)+(\alpha,g)\in\Gamma(\mathcal{E}^1(M))$, by

$$\tau_{(\eta,\gamma)}((X,f)+(\alpha,g)) = (X,f)+(\alpha,g)+(\eta,\gamma)^\flat(X,f).$$

$\tau_{(\eta,\gamma)}$ is called a *gauge transformation* associated with (η,γ). Let us also consider the Courant-Jacobi algebroids $(\mathcal{E}^1(M)_{(d\omega,\omega)},(0,1)+(0,0))$ and $(\mathcal{E}^1(M)_{(d\omega,\omega)-\mathrm{d}^{(0,1)}(\eta,\gamma)},(0,1)+(0,0))$. Given a $(d\omega,\omega)$-Dirac-Jacobi structure L, its image by $\tau_{(\eta,\gamma)}$ is the vector sub-bundle of $\mathcal{E}^1(M)$,

$$\tau_{(\eta,\gamma)}(L) = \{(X,f)+(\alpha,g)+(\eta,\gamma)^\flat(X,f)\,|\,(X,f)+(\alpha,g)\in L\}.$$

Proposition 5.1. *Let L be a $(d\omega,\omega)$-Dirac-Jacobi structure. Then, for any $(\eta,\gamma)\in\Gamma(\bigwedge^2(T^*M\times\mathbb{R}))$, $\tau_{(\eta,\gamma)}(L)$ is a $((d\omega,\omega)-\mathrm{d}^{(0,1)}(\eta,\gamma))$-Dirac-Jacobi structure. Moreover,*

$$\tau_{(\eta,\gamma)}|_L : (L,[\cdot,\cdot]_{(d\omega,\omega)}|_L,\rho|_L) \to (\tau_{(\eta,\gamma)}(L),[\cdot,\cdot]_{(d\omega,\omega)-\mathrm{d}^{(0,1)}(\eta,\gamma)}|_{\tau_{(\eta,\gamma)}(L)},\rho|_{\tau_{(\eta,\gamma)}(L)})$$

is an isomorphism of Lie algebroids over the identity, with $\rho=\pi+0$.

Proof: Let $e_1=(X_1,f_1)+(\alpha_1,g_1)$ and $e_2=(X_2,f_2)+(\alpha_2,g_2)$ be any two sections of L. Then,

$$\begin{aligned}(\tau_{(\eta,\gamma)}(e_1),\tau_{(\eta,\gamma)}(e_2))_+ &= \underbrace{(e_1,e_2)_+}_{=0}+\frac{1}{2}(\langle(X_1,f_1),(\eta,\gamma)^\flat(X_2,f_2)\rangle \\ &\quad +\langle(X_2,f_2),(\eta,\gamma)^\flat(X_1,f_1)\rangle)=0,\end{aligned}$$

and $\tau_{(\eta,\gamma)}(L)$ is a maximally isotropic sub-bundle of $\mathcal{E}^1(M)$. On the other hand,

$$\begin{aligned}
&[\tau_{(\eta,\gamma)}(e_1), \tau_{(\eta,\gamma)}(e_2)]_{(d\omega,\omega)-\mathrm{d}^{(0,1)}(\eta,\gamma)} = [\![e_1, e_2]\!] \\
&\quad +(d(\omega-\eta), \omega-\eta+d\gamma)((X_1,f_1),(X_2,f_2),\cdot) \\
&\quad +\mathcal{L}^{(0,1)}_{(X_1,f_1)}(i_{X_2}\eta+f_2\gamma, -i_{X_2}\gamma) - \mathcal{L}^{(0,1)}_{(X_2,f_2)}(i_{X_1}\eta+f_1\gamma, -i_{X_1}\gamma) \\
&\quad +\mathrm{d}^{(0,1)}(i_{X_2}(i_{X_1}\eta)+f_1(i_{X_2}\gamma)-f_2(i_{X_1}\gamma)) \\
&= [\![e_1, e_2]\!] + ((d\omega,\omega)-(d\eta,\eta-d\gamma))((X_1,f_1),(X_2,f_2),\cdot) \\
&\quad +(i_{[X_1,X_2]}\eta + i_{X_2}(i_{X_1}d\eta) + (X_1.f_2)\gamma + f_2(\mathcal{L}_{X_1}\gamma) + f_1(i_{X_2}\eta) \\
&\quad -f_2(i_{X_1}\eta) - (X_2.f_1)\gamma - f_1(i_{X_2}d\gamma) - f_2 d(i_{X_1}\gamma), \\
&\quad \eta(X_1,X_2) - i_{X_1}(i_{X_2}\gamma) + i_{X_2}(i_{X_1}\gamma)) \\
&= [\![e_1, e_2]\!] + (d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot) + i_{[(X_1,f_1),(X_2,f_2)]^{(0,1)}}(\eta,\gamma) \\
&= \tau_{(\eta,\gamma)}([e_1,e_2]_{(d\omega,\omega)}),
\end{aligned} \tag{32}$$

which means that $\Gamma(\tau_{(\eta,\gamma)}(L))$ closes under the bracket $[\cdot,\cdot]_{(d\omega,\omega)-\mathrm{d}^{(0,1)}(\eta,\gamma)}$ and we conclude that $\tau_{(\eta,\gamma)}(L)$ is a $((d\omega,\omega)-\mathrm{d}^{(0,1)}(\eta,\gamma))$-Dirac-Jacobi structure.

Moreover, with $\rho = \pi + 0$, we have

$$\rho(\tau_{(\eta,\gamma)}((X,f)+(\alpha,g))) = \rho((X,f)+(\alpha,g)), \tag{33}$$

for any section $(X,f)+(\alpha,g)$ of L. From (32) and (33), we deduce that $\tau_{(\eta,\gamma)}|_L$ is an isomorphism of Lie algebroids over the identity. ∎

The twisted Dirac-Jacobi structures L and $\tau_{(\eta,\gamma)}(L)$ are said to be *gauge-equivalent*.

Corollary 5.2. *Let L be a $(d\omega,\omega)$-Dirac-Jacobi structure and $(\eta,\gamma) \in \Gamma(\bigwedge^2(T^*M \times I\!R))$.*

i) *If $\mathrm{d}^{(0,1)}(\eta,\gamma) = (0,0)$, then $\tau_{(\eta,\gamma)}(L)$ is also a $(d\omega,\omega)$-Dirac-Jacobi structure.*

ii) *If $\mathrm{d}^{(0,1)}(\eta,\gamma) = (d\omega,\omega)$, then $\tau_{(\eta,\gamma)}(L)$ is a Dirac-Jacobi structure.*

Let us denote by Dir_ω the set of all ω-Dirac-Jacobi structures and consider the additive group

$$\mathcal{F} = \{(\eta,\gamma) \in \Gamma(\bigwedge^2(T^*M \times I\!R)) \mid \mathrm{d}^{(0,1)}(\eta,\gamma) = 0\}.$$

Corollary 5.2 i) means that $\mathcal{F}$ acts on Dir_ω with the action,

$$\mathcal{F} \times Dir_\omega \to Dir_\omega, \qquad ((\eta,\gamma),L) \mapsto \tau_{(\eta,\gamma)}(L),$$

and two elements of Dir_ω are gauge equivalent if they lie in the same orbit of the action.

6. The Jacobi algebroid associated to a twisted Jacobi manifold

In this section we will show that we can associate a Jacobi algebroid to each twisted Jacobi manifold.

Proposition 6.1. *Let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold. Then, $(T^*M \times I\!R, \{\cdot,\cdot\}^\omega, \pi \circ (\Lambda,E)^\#)$ is a Lie algebroid over M, where $\{\cdot,\cdot\}^\omega$ is the bracket on $\Gamma(T^*M \times I\!R)$ given, for all $(\alpha,f),(\beta,g) \in \Gamma(T^*M \times I\!R)$, by*

$$\{(\alpha,f),(\beta,g)\}^\omega = \{(\alpha,f),(\beta,g)\} + (d\omega,\omega)((\Lambda,E)^\#(\alpha,f),(\Lambda,E)^\#(\beta,g),\cdot), \tag{34}$$

$\{\cdot,\cdot\}$ being the bracket (5).

Proof: Let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold. From Corollary 4.5, we know that graph$(\Lambda,E)^\#$ is a twisted Dirac-Jacobi sub-bundle of $\mathcal{E}^1_\omega(M)$, hence it is a Lie algebroid over M with the following bracket on the space of its sections,

$$\begin{aligned} &[(\Lambda,E)^\#(\alpha,f)+(\alpha,f),(\Lambda,E)^\#(\beta,g)+(\beta,g)]_{(d\omega,\omega)} \\ &\quad = [(\Lambda,E)^\#(\alpha,f),(\Lambda,E)^\#(\beta,g)] \\ &\quad + \{(\alpha,f),(\beta,g)\} + (d\omega,\omega)((\Lambda,E)^\#(\alpha,f),(\Lambda,E)^\#(\beta,g),\cdot). \end{aligned} \tag{35}$$

Since the bracket (35) splits in the sum $\Gamma(TM \times \mathbb{R}) \oplus \Gamma(T^*M \times \mathbb{R})$, then its projection $\{\cdot,\cdot\}^\omega$ over $\Gamma(T^*M \times \mathbb{R})$ is a Lie bracket. Moreover, for any $h \in C^\infty(M,\mathbb{R})$,

$$\{(\alpha,f),h(\beta,g)\}^\omega = h\{(\alpha,f),(\beta,g)\}^\omega + (((\pi \circ (\Lambda,E)^\#)(\alpha,f)) \cdot h)(\beta,g).$$

So, $(\{\cdot,\cdot\}^\omega, \pi \circ (\Lambda,E)^\#)$ endows $T^*M \times \mathbb{R}$ with a Lie algebroid structure. ■

Corollary 6.2. *Let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold. Then, for any $f,g \in C^\infty(M,I\!R)$,*

$$\{\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g\}^\omega = \mathrm{d}^{(0,1)}\{f,g\} + (d\omega,\omega)((\Lambda,E)^\#(\mathrm{d}^{(0,1)}f),(\Lambda,E)^\#(\mathrm{d}^{(0,1)}g),\cdot).$$

Proof: It is an immediate consequence of Proposition 6.1, taking into account that, for any $f,g\in C^\infty(M,\mathbb{R})$, $\mathrm{d}^{(0,1)}\{f,g\}=\{\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g\}$, with the bracket on the left hand-side given by (22) and the bracket on the right hand-side given by (5). ∎

The differential operator d_*^ω defined on $\Gamma(\bigwedge(TM\times\mathbb{R}))$ by the Lie algebroid structure $(\{\cdot,\cdot\}^\omega,\pi\circ(\Lambda,E)^\#)$ on $T^*M\times\mathbb{R}$ is given,

- for any $f\in C^\infty(M,\mathbb{R})$, by

$$d_*^\omega f=d_*f=-(\Lambda,E)^\#(df,0); \tag{36}$$

- for any $(X,f)\in\Gamma(TM\times\mathbb{R})$, by

$$d_*^\omega(X,f)=d_*(X,f)+((\Lambda,E)^\#\otimes 1)(d\omega,\omega)(X,f), \tag{37}$$

where d_* denotes the operator given by (6) and $(\Lambda,E)^\#\otimes 1$ is defined adapting (15) in the obvious way.

Proposition 6.3. *Let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold. The section $(-E,0)$ of $TM\times\mathbb{R}$ is a 1-cocycle for the Lie algebroid $(T^*M\times\mathbb{R},\{\cdot,\cdot\}^\omega,\pi\circ(\Lambda,E)^\#)$ over M.*

Proof: It suffices to prove that $d_*^\omega(-E,0)=(0,0)$. Let $(\alpha,f),(\beta,g)$ be any sections of $T^*M\times\mathbb{R}$. Then,

$$\begin{aligned}
&d_*^\omega(-E,0)((\alpha,f),(\beta,g))\\
&= d_*(-E,0)((\alpha,f),(\beta,g))+((\Lambda,E)^\#\otimes 1)(d\omega,\omega)(-E,0)((\alpha,f),(\beta,g))\\
&= [E,\Lambda](\alpha,\beta)-(d\omega,\omega)((\Lambda,E)^\#(\alpha,f),(\Lambda,E)^\#(\beta,g),(-E,0))\\
&= ((\Lambda^\#\otimes 1)(\varphi)(E)-((\Lambda^\#\otimes 1)(\omega)(E))\wedge E)(\alpha,\beta)\\
&\quad +d\omega(\Lambda^\#(\alpha),\Lambda^\#(\beta),E)-(i_E\alpha)\omega(\Lambda^\#(\beta),E)+(i_E\beta)\omega(\Lambda^\#(\alpha),E)\\
&= 0,
\end{aligned}$$

and so, $d_*^\omega(-E,0)=(0,0)$. ∎

From Propositions 6.1 and 6.3, we deduce that the twisted Jacobi structure $((\Lambda,E),\omega)$ on M defines a Jacobi algebroid structure on $T^*M\times\mathbb{R}$. Moreover we have, from (8), (36) and (37), that

- for any $f\in C^\infty(M,\mathbb{R})$,

$$(d_*^\omega)^{(-E,0)}f=-(\Lambda,E)^\#(df,f); \tag{38}$$

- for any $(X,f) \in \Gamma(TM \times \mathbb{R})$,

$$(d_*^\omega)^{(-E,0)}(X,f) = [(\Lambda,E),(X,f)]^{(0,1)} + ((\Lambda,E)^\# \otimes 1)(d\omega,\omega)(X,f). \quad (39)$$

The Lie algebra homomorphism, from $C^\infty(M,\mathbb{R})$ to $\Gamma(TM)$, expressed by equation (4) in the case where M is a Jacobi manifold, fails in the case of twisted Jacobi manifolds, as shown in the next proposition.

Proposition 6.4. *Let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold. Then, for any $f,g \in C^\infty(M,\mathbb{R})$,*

$$[X_f,X_g] = X_{\{f,g\}} + (\pi \circ (\Lambda,E)^\#)((d\omega,\omega)((\Lambda,E)^\#(df,f),(\Lambda,E)^\#(dg,g),\cdot)). \quad (40)$$

Proof: From (28) we have, with $(\Lambda,E)^\#(df,f) = (X_f,-E\cdot f)$,

$$\begin{aligned}[(X_f,-E\cdot f),(X_g,-E\cdot g)] &= (X_{\{f,g\}},-E\cdot\{f,g\}) \\ &+(\Lambda,E)^\#((d\omega,\omega)((\Lambda,E)^\#(df,f),(\Lambda,E)^\#(dg,g),\cdot)).\end{aligned}$$

The projection over the first factor gives (40). ∎

7. Quasi-Jacobi bialgebroids and their doubles

The notion of *quasi-Lie bialgebroid* was introduced in [22]. It is a structure on a pair (A,A^*) of vector bundles, in duality, over a differentiable manifold M that is defined by a Lie algebroid structure on A^*, a skew-symmetric bracket on the space of smooth sections of A and a bundle map $a: A \to TM$, satisfying some compatibility conditions. These conditions are expressed in terms of a section of $\bigwedge^3 A^*$, which turns to be an obstruction to the Lie bialgebroid structure on (A,A^*). As in the case of a Lie bialgebroid, the double $A \oplus A^*$ of a quasi-Lie bialgebroid (A,A^*) is endowed with a Courant algebroid structure [22, 11].

In this section, in order to adapt the previous notion to the Jacobi framework, we introduce the concept of *quasi-Jacobi bialgebroid* and we prove that its double is endowed with a Courant-Jacobi algebroid structure [18, 4].

Definition 7.1. *A* quasi-Jacobi bialgebroid *structure on a pair (A,A^*) of dual vector bundles over a differentiable manifold M consists of:*

- *a Lie algebroid structure $([\cdot,\cdot]_*,a_*)$ on A^* with a 1-cocycle W;*
- *a bundle map $a: A \to TM$;*
- *a skew-symmetric operation $[\cdot,\cdot]$ on $\Gamma(A)$;*
- *a section $\phi \in \Gamma(A^*)$;*

- *a section* $\varphi \in \Gamma(\bigwedge^3 A^*)$;

satisfying, for all $X, Y, Z \in \Gamma(A)$ *and* $f \in C^\infty(M, I\!R)$*, the following properties:*

1) $[X, fY] = f[X,Y] + (a(X)f)Y$;
2) $a([X,Y]) = [a(X), a(Y)] - a_*\varphi(X,Y,\cdot)$;
3) $[[X,Y],Z] + c.p. = -d_*^W(\varphi(X,Y,Z)) - ((i_{\varphi(X,Y,\cdot)} d_*^W Z) + c.p.)$*, where* d_*^W *is the modified differential operator on* $\Gamma(\bigwedge A)$ *defined by the Lie algebroid structure of* A^* *and the 1-cocycle* W;
4) $d\phi - \varphi(W,\cdot,\cdot) = 0$*, where* d *is the quasi-differential operator on* $\Gamma(\bigwedge A^*)$ *determined by the structure* $([\cdot,\cdot], a)$ *on* A;
5) $d^\phi \varphi = 0$*, where* d^ϕ *is given, for any* $\beta \in \Gamma(\bigwedge^k A^*)$*, by* $d^\phi(\beta) = d\beta + \phi \wedge \beta$;
6) $d_*^W [P,Q]^\phi = [d_*^W P, Q]^\phi + (-1)^{p+1}[P, d_*^W Q]^\phi$*, with* $P \in \Gamma(\bigwedge^p A)$ *and* $Q \in \Gamma(\bigwedge A)$.

We will denote the quasi-Jacobi bialgebroid by $((A,\phi),(A^*,W),\varphi)$.

Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M, $\mathcal{L}^\phi$ and $\mathcal{L}_*^W$ the quasi-Lie derivative and the Lie derivative operators defined, respectively, by d^ϕ and d_*^W as in (9), a^ϕ and a_*^W the deformed anchor maps according to (7). On the Whitney sum bundle $A \oplus A^*$ we consider the two nondegenerate canonical bilinear forms $(\cdot,\cdot)_\pm$ and, on the space $\Gamma(A \oplus A^*) \cong \Gamma(A) \oplus \Gamma(A^*)$ we define the bracket $[\cdot,\cdot]_\varphi$ by setting, for any $e_1 = X_1 + \alpha_1, e_2 = X_2 + \alpha_2 \in \Gamma(A \oplus A^*)$,

$$[e_1, e_2]_\varphi = [X_1 + \alpha_1, X_2 + \alpha_2]_\varphi = [\![X_1 + \alpha_1, X_2 + \alpha_2]\!] + \varphi(X_1, X_2, \cdot), \quad (41)$$

where $[\![\cdot,\cdot]\!]$ is the bracket (27).

Theorem 7.2. *Let* $((A,\phi),(A^*,W),\varphi)$ *be a quasi-Jacobi bialgebroid over* M*. The vector bundle* $A \oplus A^*$ *over* M *endowed with* $([\cdot,\cdot]_\varphi, (\cdot,\cdot)_+, \rho^\theta, \mathcal{D}^\theta)$*, where* $\theta = \phi + W \in \Gamma(A^* \oplus A)$*,* $\rho^\theta = a^\phi + a_*^W$ *and* $\mathcal{D}^\theta = (d_*^W + d^\phi)|_{C^\infty(M, I\!R)}$*, is a Courant-Jacobi algebroid over* M*.*

For establishing the above theorem, we need the results of the following lemmas. Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M.

Lemma 7.3. *For any* $P \in \Gamma(\bigwedge^k A)$*,* $X, Y \in \Gamma(A)$*,* $\alpha \in \Gamma(A^*)$ *and* $f \in C^\infty(M, I\!R)$*,*

i) $d_*^W[X,Y] = [d_*^W X, Y] + [X, d_*^W Y]$;

ii) $\mathcal{L}^W_{*\phi}P + \mathcal{L}^{\phi}_{W}P = 0$;
iii) $\langle \phi, W\rangle = 0$ *and* $a(W) + a_*(\phi) = 0$;
iv) $\mathcal{L}_{*\phi}X + [W, X] = 0$;
v) $[d^W_* f, X] + \mathcal{L}^W_{*d^\phi f}X = 0$ *and* $[d^\phi f, \alpha]^W_* + \mathcal{L}^\phi_{d^W_* f}\alpha = 0$.

Proof: The proof is based on the facts that d^W_* (resp. d^ϕ) is a derivation of $[\cdot,\cdot]^\phi$ (resp. $[\cdot,\cdot]^W_*$) and it is similar to the case of a Jacobi bialgebroid (see [5, 18]). ∎

On the space $C^\infty(M,\mathbb{R})$ we define the internal composition law $\{\cdot,\cdot\}$ by setting, for any $f, g \in C^\infty(M,\mathbb{R})$,

$$\{f, g\} = \langle d^\phi f, d^W_* g\rangle. \tag{42}$$

Lemma 7.4. *For any $f, g \in C^\infty(M,\mathbb{R})$,*

$$[d^W_* f, d^W_* g] = d^W_*(\{g, f\}). \tag{43}$$

Proof: From the skew-symmetry of the bracket $[\cdot,\cdot]$ on $\Gamma(A)$, from Lemma 7.3 *v*) and because $(d^W_*)^2 = 0$,

$$[d^W_* f, d^W_* g] = -[d^W_* g, d^W_* f] = \mathcal{L}^W_{*d^\phi g}(d^W_* f) = d^W_*(\langle d^\phi g, d^W_* f\rangle) = d^W_*(\{g, f\}).$$

∎

Lemma 7.5. *The bracket (64) is a first-order differential operator on the second argument and it is skew-symmetric.*

Proof: In fact, for any $f, g, h \in C^\infty(M,\mathbb{R})$,

$$\{f, gh\} = g\{f, h\} + h\{f, g\} - gh\{f, 1\} \tag{44}$$

because

$$d^W_*(gh) = g\, d^W_* h + h\, d^W_* g - ghW.$$

In order to establish the skew-symmetry of (64), we will prove that, for any $f \in C^\infty(M,\mathbb{R})$,

$$\{f, f\} = 0. \tag{45}$$

Since $(A^*, [\cdot,\cdot]_*, a_*, W)$ is a Lie algebroid over M with a 1-cocycle, the homomorphism of $C^\infty(M,\mathbb{R})$-modules $a^W_* : \Gamma(A^*) \to \Gamma(TM \times \mathbb{R})$ given by (7), induces a Lie algebroid homomorphism over the identity between the

Lie algebroids with 1-cocycles $(A^*, [\cdot,\cdot]_*, a_*, W)$ and $(TM \times \mathbb{R}, [\cdot,\cdot], \pi, (0,1))$. Hence, for any $f \in C^\infty(M,\mathbb{R})$ †,

$$(a_*^W)^*(0,1) = W, \quad (a_*^W)^*(\delta f, f) = d_*^W f \quad \text{and} \quad (a_*^W)^*(\delta f, 0) = d_* f, \tag{46}$$

where $(a_*^W)^* : \Gamma(T^*M \times \mathbb{R}) \to \Gamma(A)$ denotes the transpose of a_*^W. On the other hand, since the quasi-differential operator d on $\Gamma(A^*)$ is defined by $a : \Gamma(A) \to \Gamma(TM)$ and by the bracket $[\cdot,\cdot]$ on $\Gamma(A)$, we can easily prove that

$$(a^\phi)^*(\delta f, 0) = a^*(\delta f) = df \quad \text{and} \quad (a^\phi)^*(\delta f, f) = d^\phi f, \tag{47}$$

where $(a^\phi)^* : \Gamma(T^*M \times \mathbb{R}) \to \Gamma(A^*)$ denotes the transpose of a^ϕ. So,

$$\begin{aligned} \{f,g\} &= \langle d^\phi f, d_*^W g\rangle \overset{(46),(47)}{=} \langle (a^\phi)^*(\delta f, f), (a_*^W)^*(\delta g, g)\rangle \\ &= \langle (\delta f, f), a^\phi \circ (a_*^W)^*(\delta g, g)\rangle. \end{aligned} \tag{48}$$

When $g = 1$, (48) gives

$$\{f,1\} = \langle (\delta f, f), a^\phi \circ (a_*^W)^*(0,1)\rangle \overset{(46)}{=} \langle (\delta f, f), a^\phi(W)\rangle = -\langle df, a_*(\phi)\rangle, \tag{49}$$

where the last equality follows from Lemma 7.3 *iii*). On the other hand,

$$\begin{aligned} \{1,f\} &= \langle (0,1), a^\phi \circ (a_*^W)^*(\delta f, f)\rangle = \langle (0,1), a^\phi(d_*^W f)\rangle \\ &= \langle \phi, d_* f\rangle = \langle \phi, a_*^*(\delta f)\rangle. \end{aligned} \tag{50}$$

From (49) and (50), we get

$$\{f,1\} = -\{1,f\}. \tag{51}$$

Using Lemma 7.3 *iii*), (46) and (47), we can write

$$\{f,f\} = \langle (\delta f, 0), a^\phi \circ (a_*^W)^*(\delta f, 0)\rangle. \tag{52}$$

From Lemma 7.4 we have,

$$d_*^W(\{f,f\}) = [d_*^W f, d_*^W f] = 0. \tag{53}$$

In particular, for f^2,

$$d_*^W(\{f^2, f^2\}) = 0 \tag{54}$$

†In this section, in order to avoid confusion with the quasi-differential d of A, we will denote by δf the usual de Rham differential of $f \in C^\infty(M,\mathbb{R})$.

and

$$\begin{array}{rcl} 0 = d_*^W(\{f^2,f^2\}) & \stackrel{(52)}{=} & d_*^W(\langle(\delta f^2,0),a^\phi\circ(a_*^W)^*(\delta f^2,0)\rangle \\ & = & 4f^2 d_*^W(\{f,f\}) + 4\{f,f\}d_*^W f^2 \\ & \stackrel{(53)}{=} & 4\{f,f\}d_*^W f^2. \end{array}$$

So, for any $f \in C^\infty(M,\mathbb{R})$,

$$\{f,f\}d_*^W f^2 = 0. \tag{55}$$

Then,

$$\begin{array}{rcl} 0 & \stackrel{(55)}{=} & \langle d^\phi 1, \{f,f\}d_*^W f^2\rangle = \{f,f\}\{1,f^2\} \\ & \stackrel{(44)}{=} & 2f\{f,f\}\{1,f\} - f^2\{f,f\}\underbrace{\{1,1\}}_{=0} \\ & = & 2f\{f,f\}\{1,f\} \end{array} \tag{56}$$

and

$$\begin{array}{rcl} 0 & \stackrel{(55)}{=} & \langle d^\phi f, \{f,f\}d_*^W f^2\rangle = \{f,f\}\{f,f^2\} \\ & \stackrel{(44),(51)}{=} & 2f\{f,f\}^2 + f^2\{f,f\}\{1,f\} \stackrel{(56)}{=} 2f\{f,f\}^2, \end{array}$$

whence we deduce that (45) holds. ■

Remark 7.6. From the skew-symmetry of (42) and the fact that it is first order differential operator on the second argument, we conclude that it is first order differential operator on each argument.

Lemma 7.7. *For any $f \in C^\infty(M,\mathbb{R})$, $X \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$,*

i) $(a\circ d_*^W + a_*\circ d^\phi)f = 0$;

ii) $[a(X),a_*(\alpha)] = a_*(\mathcal{L}_X^\phi\alpha) - a(\mathcal{L}_{*\alpha}^W X) + a(d_*^W\langle\alpha,X\rangle)$.

Proof: For *i*) we have that, for any $g \in C^\infty(M,\mathbb{R})$,

$$\begin{array}{rcl} \langle(a^\phi\circ d_*^W + a_*^W\circ d^\phi)f,(\delta g,g)\rangle & = & \langle d_*^W f,(a^\phi)^*(\delta g,g)\rangle + \langle d^\phi f,(a_*^W)^*(\delta g,g)\rangle \\ & \stackrel{(47),(46)}{=} & \langle d_*^W f, d^\phi g\rangle + \langle d^\phi f, d_*^W g\rangle \\ & \stackrel{(42)}{=} & \{g,f\}+\{f,g\} = 0, \end{array}$$

because $\{\cdot,\cdot\}$ is skew-symmetric. So, $(a^\phi\circ d_*^W + a_*^W\circ d^\phi)f = 0$. But,

$$(a^\phi\circ d_*^W + a_*^W\circ d^\phi)f = (a\circ d_*^W + a_*\circ d^\phi)f + \langle\phi, d_*^W f\rangle + \langle W, d^\phi f\rangle$$

and

$$\langle\phi, d_*^W f\rangle + \langle W, d^\phi f\rangle \overset{(46),(47)}{=} \langle a_*^W(\phi) + a^\phi(W), (\delta f, f)\rangle = 0,$$

where the last equality follows from Lemma 7.3 *iii*). Consequently, for any $f \in C^\infty(M, \mathbb{R})$,

$$(a \circ d_*^W + a_* \circ d^\phi) f = 0.$$

The proof of *ii*) is similar to the case of a Jacobi bialgebroid (see [5, 18]). ■

Lemma 7.8. *Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M. Then, the quasi-Lie derivative operator $\mathcal{L}^\phi$ associated to the quasi-differential operator d^ϕ on $\Gamma(\bigwedge A^*)$ satisfies the following property: For any $X, Y, V_1, \dots, V_p \in \Gamma(A)$ and any $\eta \in \Gamma(\bigwedge^p A^*)$,*

$$\begin{aligned}\mathcal{L}^\phi_{[X,Y]}\eta(V_1,\dots,V_p) &= (\mathcal{L}^\phi_X \circ \mathcal{L}^\phi_Y - \mathcal{L}^\phi_Y \circ \mathcal{L}^\phi_X)\eta(V_1,\dots,V_p) \\ &\quad + \sum_{i=1}^{p}(-1)^i \eta([[X,Y],V_i] + c.p.\,, V_1,\dots,\hat{V}_i,\dots,V_p) \\ &\quad - a_*^W(\varphi(X,Y,\cdot))(\eta(V_1,\dots,V_p)). \qquad (57)\end{aligned}$$

Proof: We prove the above formula by a simple, but long, computation, taking into account the condition 4) of Definition 7.1 of a quasi-Jacobi bialgebroid. ■

Now, we will prove Theorem 7.2.

Proof of Theorem 7.2. We have to check that the conditions $i) - iv)$ of Definition 4.1 hold. In order to establish condition *ii*), we use the results of Lemma 7.7 and the conditions 2) and 4) of Definition 7.1 of a quasi-Jacobi bialgebroid. We obtain that, for any two sections $e_1 = X_1 + \alpha_1$, $e_2 = X_2 + \alpha_2$ of $A \oplus A^*$,

$$\rho^\theta([e_1, e_2]_\varphi) = [\rho^\theta(e_1), \rho^\theta(e_2)].$$

For condition *iii*) we have that, for all $e, e_1, e_2 \in \Gamma(A \oplus A^*)$, $e = X + \alpha$, $e_1 = X_1 + \alpha_1$, $e_2 = X_2 + \alpha_2$,

$$\begin{aligned} & ([e,e_1]_\varphi + \mathcal{D}^\theta(e,e_1)_+, e_2)_+ + (e_1, [e,e_2]_\varphi + \mathcal{D}^\theta(e,e_2)_+)_+ \\ = \;& ([\![e,e_1]\!] + \mathcal{D}^\theta(e,e_1)_+, e_2)_+ + \frac{1}{2}\varphi(X,X_1,X_2) \\ + \;& (e_1, [\![e,e_2]\!] + \mathcal{D}^\theta(e,e_2)_+)_+ + \frac{1}{2}\varphi(X,X_2,X_1) \\ = \;& ([\![e,e_1]\!] + \mathcal{D}^\theta(e,e_1)_+, e_2)_+ + (e_1, [\![e,e_2]\!] + \mathcal{D}^\theta(e,e_2)_+)_+. \end{aligned}$$

But, by doing the same computations as in Proposition 4.1 of [18], we establish the equality

$$([\![e,e_1]\!]+\mathcal{D}^\theta(e,e_1)_+,e_2)_+ + (e_1,[\![e,e_2]\!]+\mathcal{D}^\theta(e,e_2)_+)_+ = \rho^\theta(e)(e_1,e_2)_+.$$

Hence, we conclude

$$\rho^\theta(e)(e_1,e_2)_+ = ([e,e_1]_\varphi+\mathcal{D}^\theta(e,e_1)_+,e_2)_+ + (e_1,[e,e_2]_\varphi+\mathcal{D}^\theta(e,e_2)_+)_+.$$

The condition $iv)$ can be easily proved as follows. For any $f,g\in C^\infty(M,\mathbb{R})$,

$$\begin{aligned}(\mathcal{D}^\theta f,\mathcal{D}^\theta g)_+ &= (d_*^W f+d^\phi f, d_*^W g+d^\phi g)_+ = \frac{1}{2}(\langle d^\phi g, d_*^W f\rangle + \langle d^\phi f, d_*^W g\rangle)\\ &= \frac{1}{2}(\{g,f\}+\{f,g\})=0,\end{aligned}$$

where $\{\cdot,\cdot\}$ is the bracket (42) which, by Lemma 7.5, is skew-symmetric. Finally, it remains to establish condition $i)$ of Definition 4.1, i.e., for any $e_1,e_2,e_3\in\Gamma(A\oplus A^*)$, $e_i=X_i+\alpha_i$, $i=1,2,3$,

$$[[e_1,e_2]_\varphi,e_3]_\varphi+[[e_2,e_3]_\varphi,e_1]_\varphi+[[e_3,e_1]_\varphi,e_2]_\varphi=\mathcal{D}^\theta T_\varphi(e_1,e_2,e_3), \tag{58}$$

where $T_\varphi(e_1,e_2,e_3)=\dfrac{1}{3}(([e_1,e_2]_\varphi,e_3)_+ + c.p.)$. Since the proof involves a very long computation, we only give a short schedule.

First, we note that, if $T(e_1,e_2,e_3)=\dfrac{1}{3}(([\![e_1,e_2]\!],e_3)_+ + c.p.)$, then

$$T_\varphi(e_1,e_2,e_3)=T(e_1,e_2,e_3)+\frac{1}{2}\varphi(X_1,X_2,X_3). \tag{59}$$

Let us set

$$[[e_1,e_2]_\varphi,e_3]_\varphi+[[e_2,e_3]_\varphi,e_1]_\varphi+[[e_3,e_1]_\varphi,e_2]_\varphi=Y+\beta, \tag{60}$$

where Y and β denote the components of $[[e_1,e_2]_\varphi,e_3]_\varphi+c.p.$ on $\Gamma(A)$ and $\Gamma(A^*)$, respectively. We have

$$\begin{aligned}[[e_1,e_2]_\varphi,e_3]_\varphi+c.p. &= [[\![e_1,e_2]\!]+\varphi(X_1,X_2,\cdot),e_3]_\varphi+c.p.\\ &= ([[\![e_1,e_2]\!],e_3]_\varphi+[\varphi(X_1,X_2,\cdot),e_3]_\varphi)+c.p.\\ &= ([\![[\![e_1,e_2]\!],e_3]\!]+\varphi(\widetilde{[\![e_1,e_2]\!]},X_3,\cdot)+[\![\varphi(X_1,X_2,\cdot),e_3]\!])\\ &\quad +c.p.,\end{aligned}$$

where $\widetilde{[\![e_i,e_j]\!]}$, $i,j=1,2,3$, denotes the part of $[\![e_i,e_j]\!]$ that belongs to $\Gamma(A)$. Hence,

$$Y=([\![\widetilde{[\![e_1,e_2]\!]},e_3]\!]+[\![\widetilde{\varphi(X_1,X_2,\cdot)},e_3]\!])+c.p.$$

Taking into account condition 3) of Definition 7.1, the fact that $(A^*, [\cdot,\cdot]_*, a_*)$ is a Lie algebroid over M, so

$$\mathcal{L}^W_{*[\alpha_i,\alpha_j]^W_*} = \mathcal{L}^W_{*\alpha_i} \circ \mathcal{L}^W_{*\alpha_j} - \mathcal{L}^W_{*\alpha_j} \circ \mathcal{L}^W_{*\alpha_i}, \quad \text{for} \quad i,j = 1,2,3,$$

and also (59), we obtain, after a long computation,

$$Y = d^W_*(T_\varphi(e_1,e_2,e_3)). \tag{61}$$

Similarly, for β we have

$$\beta = (\widehat{[\![[\![e_1,e_2]\!],e_3]\!]} + \varphi(\widetilde{[\![e_1,e_2]\!]},X_3,\cdot) + \widehat{[\![\varphi(X_1,X_2,\cdot),e_3]\!]}) + c.p.,$$

where $\widehat{[\![[\![e_i,e_j]\!],e_k]\!]}$ (resp. $\widehat{[\![\varphi(X_i,X_j,\cdot),e_k]\!]}$), $i,j,k = 1,2,3$, denotes the component of $[\![[\![e_i,e_j]\!],e_k]\!]$ (resp. $[\![\varphi(X_i,X_j,\cdot),e_k]\!]$) that is section of A^*. We repeat the computations developed in Proposition 4.1 of [18] for the calculation of the corresponding β and we take into account the conditions 3) and 5) of Definition 7.1, the fact that $(A^*, [\cdot,\cdot]_*, a_*)$ is a Lie algebroid over M, so $[[\alpha_1,\alpha_2]^W_*]^W_* + c.p. = 0$, the result of Lemma 7.8 and (59). After a long calculation we get

$$\beta = d^\phi(T_\varphi(e_1,e_2,e_3)). \tag{62}$$

From (60), (61) and (62) we conclude that (58) holds. □

Remark 7.9. When $\varphi = 0$, the quasi-Jacobi bialgebroid is a Jacobi bialgebroid and we obtain Proposition 4.1 of [18].

8. The quasi-Jacobi bialgebroid of a twisted Jacobi manifold

Let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold. We consider the following skew-symmetric bracket on the space of sections of the vector bundle $TM \times \mathbb{R}$ over M, given, for all $(X,f),(Y,g) \in \Gamma(TM \times \mathbb{R})$, by

$$[(X,f),(Y,g)]' = [(X,f),(Y,g)] - (\Lambda,E)^\#((d\omega,\omega)((X,f),(Y,g),\cdot)), \tag{63}$$

where $[\cdot,\cdot]$ is the bracket (11), and we define an operator d', acting on the space of sections of the exterior algebra $\bigwedge(T^*M \times \mathbb{R})$ as a graduate differential operator, by setting,

- on $f \in C^\infty(M,\mathbb{R})$,

$$\mathrm{d}'f = \mathrm{d}f = (df, 0);$$

- on sections (α, f) of $T^*M \times \mathbb{R}$,

$$\mathrm{d}'(\alpha, f) = \mathrm{d}(\alpha, f) - (d\omega, \omega)((\Lambda, E)^{\#}(\alpha, f), \cdot, \cdot).$$

Then, we extend d', by linearity, to the algebra $(\Gamma(\bigwedge(T^*M \times \mathbb{R})), \wedge)$. The operator d' coincides with the one determined by the structure $([\cdot,\cdot]', \pi)$ on $TM \times \mathbb{R}$.

Now, we use the section $(0, 1) \in \Gamma(T^*M \times \mathbb{R})$ to modify the bracket $[\cdot,\cdot]'$ on $\Gamma(TM \times \mathbb{R})$, according to formula (10), and also the operator d'. The new bracket will be denoted by $[\cdot,\cdot]'^{(0,1)}$ and the resulting operator $\mathrm{d}'^{(0,1)}$ is defined as follows:

- on $f \in C^\infty(M, \mathbb{R})$,

$$\mathrm{d}'^{(0,1)} f = \mathrm{d}^{(0,1)} f = (df, f);$$

- on sections (α, f) of $T^*M \times \mathbb{R}$,

$$\mathrm{d}'^{(0,1)}(\alpha, f) = \mathrm{d}^{(0,1)}(\alpha, f) - (d\omega, \omega)((\Lambda, E)^{\#}(\alpha, f), \cdot, \cdot).$$

Let us extend the bracket $[\cdot,\cdot]'^{(0,1)}$ on $\Gamma(TM \times \mathbb{R})$ to the whole algebra $(\Gamma(\bigwedge(TM \times \mathbb{R})), \wedge)$, as in the case of a Jacobi algebroid. In particular, if $(X, f) \in \Gamma(TM \times \mathbb{R})$ and $(C, Y) \in \Gamma(\bigwedge^2(TM \times \mathbb{R}))$, we have

$$\begin{aligned}[(C, Y), (X, f)]'^{(0,1)} &= [(C, Y), (X, f)]^{(0,1)} - \big(((\Lambda, E)^{\#} \otimes (C, Y)^{\#} \\ &+ (C, Y)^{\#} \otimes (\Lambda, E)^{\#}) \otimes 1\big)\, (d\omega, \omega)(X, f), \end{aligned} \tag{64}$$

where the second term of the right hand-side of (64) is the section of $\bigwedge^2(TM \times \mathbb{R})$ given, for any $(\alpha, g), (\beta, h) \in \Gamma(T^*M \times \mathbb{R})$, by

$$\begin{aligned}&\big(((\Lambda, E)^{\#} \otimes (C, Y)^{\#} + (C, Y)^{\#} \otimes (\Lambda, E)^{\#}) \otimes 1\big)\, (d\omega, \omega)(X, f)((\alpha, g), (\beta, h)) \\ &= (d\omega, \omega)((\Lambda, E)^{\#}(\alpha, g), (C, Y)^{\#}(\beta, h), (X, f)) \\ &\quad + (d\omega, \omega)((C, Y)^{\#}(\alpha, g), (\Lambda, E)^{\#}(\beta, h), (X, f)).\end{aligned}$$

Lemma 8.1. *Let $(M, (\Lambda, E), \omega)$ be a twisted Jacobi manifold. Then, for any $(X, f) \in \Gamma(TM \times \mathbb{R})$, we have*

$$(d_*^{\omega})^{(-E,0)}(X, f) = [(\Lambda, E), (X, f)]'^{(0,1)} - ((\Lambda, E)^{\#} \otimes 1)(d\omega, \omega)(X, f). \tag{65}$$

Proof: It is a direct consequence of (64), (10) and (39). ∎

We remark that if ω is a 2-form on M such that $(\Lambda, E)^{\#}(d\omega, \omega) = (0, 0)$, i.e. when the twisted Jacobi manifold is just a Jacobi manifold, we recover the well-known relation [5], $d_*^{(-E,0)}(X, f) = [(\Lambda, E), (X, f)]^{(0,1)}$.

In the next theorem, which is the main result of this section, we show that one can associate a quasi-Jacobi bialgebroid to each twisted Jacobi manifold.

Theorem 8.2. *Let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold and $(T^*M\times \mathbb{R},\{\cdot,\cdot\}^\omega,\pi\circ(\Lambda,E)^\#)$ its associated Lie algebroid. Consider the vector bundle $TM\times\mathbb{R}$ equipped with the bracket (63) on the space of its sections, the operator d' and the projection $\pi: TM\times\mathbb{R}\to TM$. Then, $((TM\times\mathbb{R},(0,1)),(T^*M\times\mathbb{R},(-E,0)),(d\omega,\omega))$ is a quasi-Jacobi bialgebroid over M.*

Proof: We have to check that all conditions of Definition 7.1 are satisfied. According to Proposition 6.3, the section $(-E,0)$ of $TM\times\mathbb{R}$ is a 1-cocycle for the Lie algebroid $(T^*M\times\mathbb{R},\{\cdot,\cdot\}^\omega,\pi\circ(\Lambda,E)^\#)$.

Let (X,f) and (Y,g) be any two sections of $TM\times\mathbb{R}$ and $h\in C^\infty(M,\mathbb{R})$. Then,

$$[(X,f),h(Y,g)]' = h[(X,f),(Y,g)]' + (\pi(X,f))(h)(Y,g),$$

which means that condition 1) of Definition 7.1 holds. We also have

$$\pi([(X,f),(Y,g)]') = [X,Y] - (\pi\circ(\Lambda,E)^\#)((d\omega,\omega)((X,f),(Y,g),\cdot)),$$

which is 2) of Definition 7.1. Moreover,

$$\mathrm{d}'(0,1) = -(d\omega,\omega)((\Lambda,E)^\#(0,1),\cdot,\cdot) = (d\omega,\omega)((-E,0),\cdot,\cdot)$$

and so 4) is also satisfied. The skew-symmetry of the morphism $(\Lambda,E)^\#$, allows us to conclude that

$$\mathrm{d}'^{(0,1)}(d\omega,\omega) = (0,0),$$

which is condition 5) of Definition 7.1.

Let us now consider the sections (X_1,f_1), (X_2,f_2) and (X_3,f_3) of $TM\times\mathbb{R}$. Then,

$$\begin{aligned}
&[[(X_1,f_1),(X_2,f_2)]',(X_3,f_3)]' + c.p. = ([[(X_1,f_1),(X_2,f_2)],(X_3,f_3)]\\
&\quad -(\Lambda,E)^\#((d\omega,\omega)([(X_1,f_1),(X_2,f_2)],(X_3,f_3),\cdot))\\
&\quad -[(\Lambda,E)^\#((d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)),(X_3,f_3)]\\
&\quad +(\Lambda,E)^\#((d\omega,\omega)((\Lambda,E)^\#((d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)),(X_3,f_3),\cdot))\big) + c.p.
\end{aligned} \tag{66}$$

First, we remark that, since $[\cdot,\cdot]$ is a Lie bracket on $\Gamma(TM\times\mathbb{R})$,

$$[[(X_1,f_1),(X_2,f_2)],(X_3,f_3)] + c.p. = (0,0).$$

Let (α, g) be an arbitrary section of $T^*M \times \mathbb{R}$. Then,

$$\begin{array}{rl}
& \langle(\alpha,g), -(\Lambda,E)^{\#}((d\omega,\omega)([(X_1,f_1),(X_2,f_2)],(X_3,f_3),\cdot)) + c.p.\rangle \\
= & (\pi(X_1,f_1)).((d\omega,\omega)((X_2,f_2),(X_3,f_3),(\Lambda,E)^{\#}(\alpha,g))) + c.p. \\
& -(\pi(\Lambda,E)^{\#}(\alpha,g)).((d\omega,\omega)((X_1,f_1),(X_2,f_2),(X_3,f_3))) \\
& -(d\omega,\omega)([(X_1,f_1),(\Lambda,E)^{\#}(\alpha,g)],(X_2,f_2),(X_3,f_3)) - c.p. \\
& +(0,1)\wedge(d\omega,\omega)((X_1,f_1),(X_2,f_2),(X_3,f_3),(\Lambda,E)^{\#}(\alpha,g))
\end{array} \tag{67}$$

and

$$\begin{array}{rl}
& \langle(\alpha,g), -[(\Lambda,E)^{\#}((d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)),(X_3,f_3)] + c.p.\rangle \\
= & -\left(i_{(d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)}(d_*^{\omega})^{(-E,0)}(X_3,f_3)\right)(\alpha,g) - c.p. \\
& -(\pi(X_1,f_1)).((d\omega,\omega)((X_2,f_2),(X_3,f_3),(\Lambda,E)^{\#}(\alpha,g))) - c.p. \\
& +(d\omega,\omega)([(X_1,f_1),(\Lambda,E)^{\#}(\alpha,g)],(X_2,f_2),(X_3,f_3)) + c.p. \\
& -f_1(d\omega,\omega)((X_2,f_2),(X_3,f_3),(\Lambda,E)^{\#}(\alpha,g)) - c.p. \\
& -(d\omega,\omega)((X_1,f_1),(X_2,f_2),((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(X_3,f_3)(\alpha,g)) - c.p.
\end{array} \tag{68}$$

On the other hand,

$$\begin{array}{rl}
& \langle(\alpha,g), (\Lambda,E)^{\#}((d\omega,\omega)((\Lambda,E)^{\#}((d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)),(X_3,f_3),\cdot)) \\
& \quad +c.p.\rangle \\
= & (d\omega,\omega)((X_1,f_1),(X_2,f_2),((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(X_3,f_3)(\alpha,g)) + c.p.
\end{array} \tag{69}$$

If we add up the terms of (67), (68) and (69), we obtain

$$\begin{array}{l}
-(\pi(\Lambda,E)^{\#}(\alpha,g)).((d\omega,\omega)((X_1,f_1),(X_2,f_2),(X_3,f_3))) \\
\quad +(0,d\omega)((X_1,f_1),(X_2,f_2),(X_3,f_3),(\Lambda,E)^{\#}(\alpha,g)) \\
\quad -\left(i_{(d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)}(d_*^{\omega})^{(-E,0)}(X_3,f_3)\right)(\alpha,g) - c.p. \\
\quad -f_1(d\omega,\omega)((X_2,f_2),(X_3,f_3),(\Lambda,E)^{\#}(\alpha,g)) - c.p. \\
\quad = -(d_*^{\omega})^{(-E,0)}((d\omega,\omega)((X_1,f_1),(X_2,f_2),(X_3,f_3)))((\alpha,g)) \\
\quad -\left(i_{(d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)}(d_*^{\omega})^{(-E,0)}(X_3,f_3)\right)(\alpha,g) - c.p.
\end{array}$$

and we conclude that

$$\begin{aligned}
&[[(X_1,f_1),(X_2,f_2)]',(X_3,f_3)]' + c.p. = \\
&\quad -(d_*^\omega)^{(-E,0)}((d\omega,\omega)((X_1,f_1),(X_2,f_2),(X_3,f_3))) \\
&\quad - \left(i_{(d\omega,\omega)((X_1,f_1),(X_2,f_2),\cdot)}(d_*^\omega)^{(-E,0)}(X_3,f_3) + c.p.\right),
\end{aligned}$$

which is condition 3) of Definition 7.1.

Finally, we must show that, for any $(P,P_0) \in \Gamma(\bigwedge^p(TM\times\mathbb{R}))$ and $(Q,Q_0) \in \Gamma(\bigwedge(TM\times\mathbb{R}))$,

$$\begin{aligned}
&(d_*^\omega)^{(-E,0)}[(P,P_0),(Q,Q_0)]'^{(0,1)} = \\
&= [(d_*^\omega)^{(-E,0)}(P,P_0),(Q,Q_0)]'^{(0,1)} + (-1)^{p+1}[(P,P_0),(d_*^\omega)^{(-E,0)}(Q,Q_0)]'^{(0,1)}.
\end{aligned} \tag{70}$$

As in the case of a Jacobi algebroid [3], it is enough to prove (70) in the cases where: i) (P,P_0) and (Q,Q_0) are both functions of M; ii) (P,P_0) is a section of $TM\times\mathbb{R}$ and (Q,Q_0) is a function of M; iii) (P,P_0) and (Q,Q_0) are both sections of $TM\times\mathbb{R}$.

We remark that, for any $f \in C^\infty(M,\mathbb{R})$ and $(P,P_0) \in \Gamma(\bigwedge^p(TM\times\mathbb{R}))$,

$$[(P,P_0),f]'^{(0,1)} = [(P,P_0),f]^{(0,1)}.$$

When $(P,P_0) = (f,0) \equiv f$ and $(Q,Q_0) = (g,0) \equiv g$, with $f,g \in C^\infty(M,\mathbb{R})$, equation (70) gives

$$[(d_*^\omega)^{(-E,0)}f,g]^{(0,1)} - [f,(d_*^\omega)^{(-E,0)}g]^{(0,1)} = (0,0),$$

or, equivalently,

$$[[(\Lambda,E),f]^{(0,1)},g]^{(0,1)} - [f,[(\Lambda,E),g]^{(0,1)}]^{(0,1)} = (0,0). \tag{71}$$

The graded Jacobi identity for the bracket $[\cdot,\cdot]^{(0,1)}$ on $\Gamma(TM\times\mathbb{R})$, ensures the validity of (71).

Let us now take $(P,P_0) = (X,f) \in \Gamma(TM \times \mathbb{R})$ and $(Q,Q_0) = g \in C^\infty(M,\mathbb{R})$. Then,

$$
\begin{aligned}
&(d_*^\omega)^{(-E,0)}[(X,f),g]'^{(0,1)} = [(\Lambda,E),[(X,f),g]^{(0,1)}]^{(0,1)}\\
&= \ [(X,f),[(\Lambda,E),g]^{(0,1)}]^{(0,1)} + [[(\Lambda,E),(X,f)]^{(0,1)},g]^{(0,1)}\\
&= \ [(X,f),(d_*^\omega)^{(-E,0)}g]^{(0,1)} + [(d_*^\omega)^{(-E,0)}(X,f),g]^{(0,1)}\\
&\quad -[((\Lambda,E)^\# \otimes 1)(d\omega,\omega)(X,f),g]^{(0,1)}\\
&= \ [(X,f),(d_*^\omega)^{(-E,0)}g]'^{(0,1)} + (\Lambda,E)^\# \Big((d\omega,\omega)((X,f),(d_*^\omega)^{(-E,0)}g,\cdot)\Big)\\
&\quad +[(d_*^\omega)^{(-E,0)}(X,f),g]'^{(0,1)} - [((\Lambda,E)^\# \otimes 1)(d\omega,\omega)(X,f),g]^{(0,1)}\\
&= \ [(X,f),(d_*^\omega)^{(-E,0)}g]'^{(0,1)} + [(d_*^\omega)^{(-E,0)}(X,f),g]'^{(0,1)},
\end{aligned}
\tag{72}
$$

which proves (70) in this case.

When $(P,P_0) = (X,f)$ and $(Q,Q_0) = (Y,g)$ are two sections of $TM \times \mathbb{R}$, equation (70) is given by

$$
\begin{aligned}
(d_*^\omega)^{(-E,0)}[(X,f),(Y,g)]'^{(0,1)} \ &= \ [(d_*^\omega)^{(-E,0)}(X,f),(Y,g)]'^{(0,1)}\\
&\quad +[(X,f),(d_*^\omega)^{(-E,0)}(Y,g)]'^{(0,1)}.
\end{aligned}
\tag{73}
$$

We compute,

$$
\begin{aligned}
&(d_*^\omega)^{(-E,0)}[(X,f),(Y,g)]'^{(0,1)} = [(\Lambda,E),[(X,f),(Y,g)]^{(0,1)}]^{(0,1)}\\
&\ + \ ((\Lambda,E)^\# \otimes 1)(d\omega,\omega)([(X,f),(Y,g)]^{(0,1)})\\
&\ - \ [(\Lambda,E),(\Lambda,E)^\# \left((d\omega,\omega)((X,f),(Y,g),\cdot)\right)]^{(0,1)}\\
&\ - \ ((\Lambda,E)^\# \otimes 1)(d\omega,\omega)((\Lambda,E)^\#((d\omega,\omega)((X,f),(Y,g),\cdot)))\\
&\ = \ [(X,f),[(\Lambda,E),(Y,g)]^{(0,1)}]^{(0,1)} + [[(\Lambda,E),(X,f)]^{(0,1)},(Y,g)]^{(0,1)}\\
&\ + \ ((\Lambda,E)^\# \otimes 1)(d\omega,\omega)([(X,f),(Y,g)]^{(0,1)})\\
&\ - \ ((\Lambda,E)^\# \otimes 1)(d\omega,\omega)((\Lambda,E)^\#((d\omega,\omega)((X,f),(Y,g),\cdot)))\\
&\ + \ (\Lambda,E)^\# \Big(\mathrm{d}^{(0,1)}((d\omega,\omega)((X,f),(Y,g),\cdot))\Big)\\
&\ + \ (\Lambda,E)^\#(d\omega,\omega)\left((d\omega,\omega)(X,f),(Y,g),\cdot\right)
\end{aligned}
\tag{74}
$$

where, in the last equality, we used (20), the graded Jacobi identity for the bracket $[\cdot,\cdot]^{(0,1)}$ and also the following formula, that holds for any section

(α, f) of $T^*M \times \mathbb{R}$:

$$[(\Lambda,E)^{\#}(\alpha,f),(\Lambda,E)] = (\Lambda,E)^{\#}(\mathrm{d}^{(0,1)}(\alpha,f)) + \frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}((\alpha,f)).$$

On the other hand,

$$\begin{aligned}
&[(d_*^{\omega})^{(-E,0)}(X,f),(Y,g)]'^{(0,1)} + [(X,f),(d_*^{\omega})^{(-E,0)}(Y,g)]'^{(0,1)} = \\
&= \ [[(\Lambda,E),(X,f)]^{(0,1)},(Y,g)]^{(0,1)} \\
&- \ \Big(\Big((\Lambda,E)^{\#}\otimes([(\Lambda,E),(X,f)]^{(0,1)})^{\#} \\
&+ \ ([(\Lambda,E),(X,f)]^{(0,1)})^{\#}\otimes(\Lambda,E)^{\#}\Big)\otimes 1\Big)(d\omega,\omega)(Y,g) \\
&+ \ [((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(X,f),(Y,g)]^{(0,1)} \\
&- \ \Big(\Big((\Lambda,E)^{\#}\otimes\big(((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(X,f)\big)^{\#} \\
&\quad + \big(((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(X,f)\big)^{\#}\otimes(\Lambda,E)^{\#}\Big)\otimes 1\Big)(d\omega,\omega)(Y,g) \\
&+ \ [(X,f),[(\Lambda,E),(Y,g)]^{(0,1)}]^{(0,1)} \\
&+ \ \Big(\Big((\Lambda,E)^{\#}\otimes([(\Lambda,E),(Y,g)]^{(0,1)})^{\#} \\
&+ \ ([(\Lambda,E),(Y,g)]^{(0,1)})^{\#}\otimes(\Lambda,E)^{\#}\Big)\otimes 1\Big)(d\omega,\omega)(X,f) \\
&+ \ [(X,f),((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(Y,g)]^{(0,1)} \\
&+ \ \Big(\Big((\Lambda,E)^{\#}\otimes\big(((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(Y,g)\big)^{\#} \\
&\quad + \big(((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(Y,g)\big)^{\#}\otimes(\Lambda,E)^{\#}\Big)\otimes 1\Big)(d\omega,\omega)(X,f).
\end{aligned} \tag{75}$$

Comparing the terms of (74) and (75), we conclude, after some computations, that (73) holds if and only if, for all $(\alpha,h),(\beta,l)\in\Gamma(T^*M\times\mathbb{R})$,

$$\begin{aligned}
&\mathrm{d}^{(0,1)}\left((d\omega,\omega)((X,f),(Y,g),\cdot)\right)\left((\Lambda,E)^{\#}(\alpha,h),(\Lambda,E)^{\#}(\beta,l)\right)\\
&\quad -(d\omega,\omega)([(X,f),(Y,g)]^{(0,1)},(\Lambda,E)^{\#}(\alpha,h),(\Lambda,E)^{\#}(\beta,l))\\
&\quad +(d\omega,\omega)((Y,g),(\Lambda,E)^{\#}(\alpha,h),([(\Lambda,E),(X,f)]^{(0,1)})^{\#}(\beta,l))\\
&\quad -(d\omega,\omega)((Y,g),(\Lambda,E)^{\#}(\beta,l),([(\Lambda,E),(X,f)]^{(0,1)})^{\#}(\alpha,h))\\
&\quad +[(Y,g),((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(X,f)]^{(0,1)}((\alpha,h),(\beta,l))\\
&\quad -(d\omega,\omega)((X,f),(\Lambda,E)^{\#}(\alpha,h),([(\Lambda,E),(Y,g)]^{(0,1)})^{\#}(\beta,l))\\
&\quad +(d\omega,\omega)((X,f),(\Lambda,E)^{\#}(\beta,l),([(\Lambda,E),(Y,g)]^{(0,1)})^{\#}(\alpha,h))\\
&\quad -[(X,f),((\Lambda,E)^{\#}\otimes 1)(d\omega,\omega)(Y,g)]^{(0,1)}((\alpha,h),(\beta,l))=0. \qquad (76)
\end{aligned}$$

After a long computation, we get that (76) is equivalent to

$$\left(\mathrm{d}^{(0,1)}(d\omega,\omega)\right)((X,f),(Y,g),(\Lambda,E)^{\#}(\alpha,h),(\Lambda,E)^{\#}(\beta,l))=0,$$

which holds since $\mathrm{d}^{(0,1)}(d\omega,\omega)=0$.

■

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