Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 07–43

A VARIATION ON THE TABLEAU SWITCHING AND A PAK-VALLEJO'S CONJECTURE

OLGA AZENHAS

ABSTRACT: Pak and Vallejo have defined fundamental symmetry map as any Young tableau bijection for the commutativity of the Littlewood-Richardson coefficients $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$. They have exhibited four fundamental symmetry maps and conjectured that they are all identical (2004). The three first ones are based on standard operations in Young tableau theory and, in this case, the conjecture was proved by Danilov and Koshevoy (2005). The fourth fundamental symmetry, given by the author in (1999;2000) and reformulated by Pak and Vallejo, is defined by nonstandard operations in Young tableau theory and is shown, in this paper, to be identical to the first one defined by the involution property of the Benkart-Sottile-Stroomer tableau switching. The proof of this equivalence exhibits *switching* as an operation satisfying the interlacing property between normal shapes of the pairs of tableaux and pairs of subtableaux. That property leads to a *jeu de taquin-like* on Littlewood-Richardson tableaux which explains the mentioned nonstandard operations and provides a variation of the *tableau switching* on Littlewood-Richardson tableau pairs.

KEYWORDS: commutativity of Littlewood-Richardson coefficients; equivalence of Young tableau bijections; fundamental symmetry; Gelfand-Tsetlin patterns; interlacing property; tableau switching.
AMS SUBJECT CLASSIFICATION (2000): 05E05, 05E10.

1. Introduction

Recently, with different approaches, several bijections exhibiting symmetries of Littlewood-Richardson coefficients have been constructed [PV2, KTW, HK, DK]. Also the relationship between different combinatorial objects has been studied [PV1]. In [KTW, HK] hives and octahedron recurrence are the main tools while in [PV2] the bijections are within Young tableaux. The fundamental symmetry map is defined in [PV2] as any bijection between sets of Littlewood-Richardson tableaux of shape λ/μ with weight ν, and those of shape λ/ν with weight μ. Namely in [PV2] four fundamental symmetry maps ρ_1, ρ_2, ρ_2^{-1} and ρ_3 are provided and it is conjectured that they are

Received December 31, 2007.
Supported by CMUC/FCT and SFRH/BSAB/515/2005, grant by the Portuguese Foundation of Science and Technology, FCT, at the Combinatorics Group, Fakultät für Mathematik, Universität Wien, Austria.

equivalent in the sense that in all of them the outcome is the same. The three first are based on standard algorithms in Young tableau theory, *jeu de taquin*, Schützenberger involution and tableau switching, while ρ_3 uses nonstandard operations in Young tableau theory and exhibits a Gelfand-Tsetlin pattern. The fundamental symmetry map ρ_3 is a *jeu de taquin-like algorithm* which rectifies a Littlewood-Richardson tableau of shape λ/μ such that the contracting slides decomposes the inner shape μ into a sequence of interlacing partitions defining a Gelfand-Tsetlin pattern of type $[\nu, \mu, \lambda]$ with ν the partition shape of the rectified Littlewood-Richardson tableau.

In [DK] it is shown that the Henriques-Kamnitzer commuter coincides with the Pak-Vallejo fundamental symmetries ρ_1, ρ_2, ρ_2^{-1}, and $\rho_2 = \rho_2^{-1}$. However the fundamental symmetry ρ_3 in [PV2] is left out. Fundamental symmetry map ρ_3 mentioned in [PV1] and slightly reformulated in [PV2], has appeared earlier in [AZ1, AZ2] and uses nonstandard operations in Young tableau theory which exhibits an interlacing property. Here we show that ρ_3 is equivalent to ρ_1 defined by the involution switching tableau property. The switching procedure [BSS] can be performed in almost any order without affecting the outcome. We impose an order on the operations of the tableau switching such that in the moving process if $S \cup T$ is the initial pair with S of normal shape and $R \cup X$ is an intermediate perforated pair, the shape of S always interlaces with the normal shape of the tableau obtained by full switching of any perforated subtableau consisting of the first i rows of R. When the left tableau is a Yamanouchi tableau the sequence of interlacing partitions encodes the Yamanouchi word of the right tableau of the outcome pair and this leads to a *jeu taquin-like* algorithm to exhibit the commutative property of Littlewood-Richardson coefficients.

The paper is divided into six sections. In the next section we give the basic definitions and terminology for what follows. In the third section we exhibit the tableau switching with the interlacing property of normal shapes. In the fourth section we show that the bijection ρ_3 is identical to the tableau switching by exhibiting it as a *jeu de taquin-like* algorithm on Littlewood-Richardson tableaux. This algorithm gives a special Littlewood-Richardson tableau switching. In the fifth section we translate the *jeu de taquin-like* operations to Littlewood-Richardson triangles [PV1] and get a Littlewood-Richardson triangle switching. In the last section we relate the interlacing property of the tableau switching with other occurrences of the interlacing phenomenon as the interlacing of invariant factors of matrices over principal ideal domains and of eigenvalues of Hermitian matrices [FP, EMSa, TH, QSSA]. Along the paper several examples are given.

2. Preliminaries

We think of $\mathbb{Z}\times\mathbb{Z}$ as consisting of boxes or dots $\bullet$ and number the rows and columns according the matrix style. Consider x and x' boxes in $\mathbb{Z}\times\mathbb{Z}$. We say that x is to the north of x' if the row containing x is above or equal the row containing x'; and x is to the west of x if the column containing x is to the left or equal to the column containing x'. The other compass directions are defined analogously. When x and x' are distinct adjacent boxes they are said *neighbours*. For instance, the neighbour to the north of x is the one directly above x. Often we label boxes or dots with integers (or with letters in a finite totally ordered alphabet) and in this case we identify these objects with the corresponding letters.

A *partition* (or normal shape) $\lambda = (\lambda_1, \cdots, \lambda_n)$ is a finite sequence (or infinite sequence with finite support) of nonnegative integers by weakly decreasing order. We ignore the distinction between two partitions that differ only at a string of zeros at the end. The diagram of λ consists of λ_1 boxes (or dots $\bullet$) in the first row, λ_2 boxes in the second row, etc, justified on the left. We look at partitions and diagrams indistinctly. If λ and μ are two partitions with $\lambda_i \geq \mu_i$ for all i, we write $\mu \subseteq \lambda$. The skew-diagram of shape λ/μ is the difference set of λ and μ. Whenever $\mu \subseteq \lambda$ we say λ/μ *extends* μ and the outer border of μ is the inner border of λ/μ. The *tableau* T of shape λ/μ, written $shT = \lambda/\mu$, is a filling (or labeling) of the skew-diagram λ/μ using letters of a finite totally ordered alphabet such that the entries increase weakly along rows and strictly down columns. The *weight* of a tableau is $\nu = (\nu_1, \cdots, \nu_n)$ where ν_i is the multiplicity of the letter i in the filling of the tableau. A *tableau* of (normal) shape λ is a tableau of shape $\lambda/0$. We say the tableau T extends the tableau S if the shape of T extends the shape of S.

A word is a sequence of letters over a finite totally ordered alphabet. We define the word of a tableau by reading the entries along the rows from left to right and bottom to top. The *Yamanouchi tableau* of shape λ, denoted $Y(\lambda)$, is the tableau whose shape and weight is λ, that is, the tableau obtained by filling the first row of λ with λ_1 1's, the second with λ_2 2's etc. A Yamanouchi word of weight λ is a word Knuth equivalent with $Y(\lambda)$. A Littlewood-Richardson (LR for short) tableau of type $[\mu, \nu, \lambda]$ is a tableau of shape λ/μ

and weight ν whose word is Yamanouchi. Denote by $LR[\mu,\nu,\lambda]$ the set of all LR tableaux of type $[\mu,\nu,\lambda]$. The cardinal of this set is the *Littlewood-Richardson coefficient* $c^{\lambda}_{\mu,\nu}$ [F, LLT, Sa, S].

Definition 2.1. [PV2] The fundamental symmetry is a bijection

$$\rho : LR[\mu,\nu,\lambda] \longrightarrow LR[\nu,\mu,\lambda].$$

In [PV2] the version ρ_1 of the fundamental symmetry is based on the involution property of the tableau switching [BSS]. In the last section we shall present the version ρ_3 of the fundamental symmetry [AZ1, AZ2, PV2] in terms of a *jeu de taquin-like* and this allows us to conclude that ρ_1 and ρ_3 have the same outcome.

We recall now the rules of *jeu de taquin slides.* Let us consider a black dot $\bullet$ with the two possible south-east neighbours $\begin{matrix}\bullet & \mathbf{b}\\ \mathbf{a} & \end{matrix}$. A *contracting slide* into the black dot $\bullet$ is performed according to the following rules (a) if it has only one neighbour, swap with that neighbour; (b) if it has two different neighbours, swap with the smaller one; (c) if it has equal neighbours, swap with the one to the south. In the case of two possible north-west neighbours $\begin{matrix} & \mathbf{d}\\ \mathbf{c} & \bullet\end{matrix}$, an *expanding slide* into the black dot $\bullet$ is performed analogously. Two tableaux are said *Knuth equivalent* if one of them can be transformed by contracting and expanding slides into the another one. The full contraction of a tableau is called *rectification.* Thus two tableaux are Knuth equivalent if they have the same rectification. The shape of the rectified tableau is called the *normal shape* of the tableau.

Thus another perspective for Litlewood-Richardson coefficients is that $c^{\lambda}_{\mu,\nu}$ counts the number of LR tableaux of type $[\mu,\nu,\lambda]$ that are Knuth equivalent to $Y(\nu)$ [F, LLT, Sa, S]. This point of view will be explored here.

Gelfand-Tsetlin (GT for short) patterns are related with LR tableaux as follows [GZ].

Definition 2.2. A Gelfand-Tsetlin pattern of size n is a map $G : \{(i,j) : 1 \leq j \leq i \leq n\} \to \mathbb{Z}$ such that $G(i,j) \geq G(i-1,j) \geq G(i,j+1)$ for all i and j.

The *base* of a Gelfand-Tsetlin pattern is the sequence of integers that appears on the bottom row and the *weight* of a GT pattern is the sequence of differences of row sums from top to bottom.

There exists a GT pattern of size n, base ν and weight $\lambda-\mu$ if and only if there exists a sequence of partitions $\nu^{(s)}=(\nu_1^{(s)},\ldots,\nu_s^{(s)})$, $s=1,...,n$, with $\nu^{(n)}=\nu$, satisfying the interlacing inequalities

$$\nu_j^{(i)} \geq \nu_j^{(i-1)} \geq \nu_{j+1}^{(i)},\ 1\leq j\leq i\leq n, \tag{2.1}$$

schematically

$$\begin{array}{ccccccccccc}
 & & & & & \nu_1^{(1)} & & & & & \\
 & & & & \nu_1^{(2)} & & \nu_2^{(2)} & & & & \\
 & & & \nu_1^{(3)} & & \nu_2^{(3)} & & \nu_3^{(3)} & & & \\
 & & & \cdots & & \cdots & & \cdots & & & \\
 & \nu_1^{(n-1)} & & & \nu_2^{(n-1)} & & \nu_3^{(n-1)} & \cdots & \nu_{n-1}^{(n-1)} & & \\
\nu_1^{(n)} & & & \nu_2^{(n)} & & \nu_3^{(n)} & \cdots & \nu_{n-1}^{(n)} & & & \nu_n^{(n)},
\end{array}$$

and the system of linear inequalities

$$\mu_{i-1}+\sum_{j=1}^{r-1}(\nu_j^{(i-1)}-\nu_j^{(i-2)}) \geq \mu_i+\sum_{j=1}^{r}(\nu_j^{(i)}-\nu_j^{(i-1)}),\ r=1,\ldots,i-1,\ i=2,\ldots,n, \tag{2.2}$$

$$\mu_i+\sum_{j=1}^{i}(\nu_j^{(i)}-\nu_j^{(i-1)})=\lambda_i,\ i=1,\ldots,n. \tag{2.3}$$

This GT pattern is said of type $[\mu,\nu,\lambda]$ and sometimes μ is called a *boundary* of the GT pattern. There is a standard bijection between Littlewood-Richardson tableaux of type $[\mu,\nu,\lambda]$ and the GT patterns of type $[\mu,\nu,\lambda]$ [GZ]. This bijection sends an LR tableau T to the GT pattern whose value at (i,j) is the number of $j's$ in the first i rows of T. An alternative way to look at this bijection is to put the i-th row of the GT pattern as the shape of the Yamanouchi tableau of the word in the first i rows of T. Equivalently the shape of the rectified LR subtableau defined by the first i rows of T. This point of view will be useful in the next section.

Here is an LR tableau of type $[\mu=(6,4,3,1);\nu=(5,4,2,1);\lambda=(9,7,6,4)]$ and the corresponding GT pattern with base ν, weight $\lambda-\mu=(3,3,3,3)$ and boundary μ,

$$\begin{array}{ccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 & 1 \\ \bullet & \bullet & \bullet & \bullet & 1 & 2 & 2 & & \\ \bullet & \bullet & \bullet & 2 & 3 & 3 & & & \\ \bullet & 1 & 2 & 4 & & & & & \end{array} \quad \longleftrightarrow \quad \begin{array}{ccccccc} & & & 3 & & & \\ & & 4 & & 2 & & \\ & 4 & & 3 & & 2 & \\ 5 & & 4 & & 2 & & 1 \end{array} .$$

We recall now the *switching procedure* and some related terminology [BSS]. A *perforated tableau* T of shape λ/μ is a labeling of some of the boxes satisfying some restrictions: whenever x and x' are letters in T and x is north-west of x', $x \leq x'$; within each column of T the letters are distinct. If S and T are perforated tableaux of some given shape λ and together they completely label λ such that no box is labeled twice, then $S \cup T$ (as union of subsets of $\mathbb{Z} \times \mathbb{Z}$) is called a perforated pair of shape λ. In particular, given two tableaux S and T of shapes μ and λ/μ respectively, $S \cup T$ is a perforated pair of shape λ.

For convenience, when considering pairs of perforated tableaux $S \cup T$, the letters in S and T belong to different alphabets either $\bar{\mathbf{0}} < \bar{\mathbf{1}} < \cdots < \bar{\mathbf{n}}$ or $\mathbf{0} < \mathbf{1} < \cdots < \mathbf{n}$. Let $S \cup T$ be a perforated pair and assume that $\bar{\mathbf{s}}$ and $\mathbf{t}$ are two adjacent letters $\bar{\mathbf{s}} \ \mathbf{t}$ or $\begin{array}{c} \bar{\mathbf{s}} \\ \mathbf{t} \end{array}$ from S and T respectively. Swapping $\bar{\mathbf{s}}$ and $\mathbf{t}$ is called a *switch* whenever we have simultaneously a contracting slide in T and a an expanding slide in S. The *switching procedure* starts with two tableaux S and T such that T extends S and, by switching letters from S with letters from T, transforms $S \cup T$ into a pair of tableaux $P \cup Q$ such that Q extends P, S is Knuth equivalent to Q and T is Knuth equivalent to P. We say that $P \cup Q$ is the switching of S and T. The switching transformation is an involution. In this paper the switching procedure always start with a pair $S \cup T$ where S has normal shape.

3. Tableau switching and shape interlacing property

We have seen that the i-th row of a GT-pattern of base μ encodes the normal shape of the rectified LR subtableau defined by the first i rows of an LR tableau of weight μ.

If $S \cup T$ is a pair of tableaux of shapes μ and λ/μ which switching procedure transforms into a pair $P \cup Q$ where P is a tableau of normal shape ν, then the sequence of normal shapes of the subtableaux defined by the first i rows of T is a GT pattern of type $[\mu, \nu, \lambda]$, and the sequence of normal shapes of the subtableaux defined by the first i rows of Q is a GT pattern of type $[\nu, \mu, \lambda]$. This follows from Haiman results on dual equivalence [H].

Theorem 3.1. [H, BSS] *Let U and V tableaux of the same shape. If W is any tableau that extends U and extends V, then switching transforms $U \cup W$ into $P \cup Q$ and $V \cup W$ into $P \cup R$ where Q and R are dual equivalent.*

Let $P^{(i)} \cup Q^{(i)}$ be the subpair defined by the first i rows of $P \cup Q$. If switching transforms $P^{(i)} \cup Q^{(i)}$ into $\hat{S}^{(i)} \cup \hat{T}^{(i)}$ for all i, and in each $\hat{S}^{(i)} \cup \hat{T}^{(i)}$ we substitute $\hat{S}^{(i)}$ with $Y(sh\,\hat{S}^{(i)})$, then switching will transform $Y(sh\,\hat{S}^{(i)}) \cup \hat{T}^{(i)}$ into $P^{(i)} \cup \tilde{Q}^{(i)}$ where $\tilde{Q}^{(i)}$ is an LR tableau of type $[sh\,P^{(i)}, sh\,\hat{S}^{(i)}, \lambda]$. Thus $\hat{S} = (sh\,\hat{S}^{(1)}, \cdots, sh\,\hat{S}^{(n)})$ define a GT pattern of type $[\nu, \mu, \lambda]$.

On the other hand, by symmetry, when by switching we pass from $P \cup Q$ to $S \cup T$ we get another GT pattern $\hat{P} = (sh\,\hat{P}^{(1)}, \cdots, sh\,\hat{P}^{(n)})$ of type $[\mu, \nu, \lambda]$ where $\hat{P}^{(i)}$ is the rectification of the subtableau defined by the first i rows of T. Thus switching transforms $S \cup T$ into $P \cup Q$ and at the same time transforms the GT pattern $\hat{P}$ of base ν and boundary μ into the GT pattern $\hat{S}$ of base μ and boundary ν.

Now we give an algorithmic approach to these results. We define a certain choice of order in the switching procedure which exhibits a pair of Gelfand-Tsetlin patterns. Let $S' \cup T'$ be an intermediate perforated pair of this procedure, $S'^{(i)} \cup T'^{(i)}$ the subpair defined by the first i rows, and $\hat{S}'^{(i)} \cup \hat{T}'^{(i)}$ the pair after full contraction of $S'^{(i)}$ and full extension of $T'^{(i)}$. Start with the GT pattern (μ_1), (μ_1, μ_2), $\ldots$, $\mu = (\mu_1, \ldots, \mu_n)$ associated with the tableau S of normal shape μ. Along the procedure, this pattern is transformed successively into patterns of base μ whose ith row is the shape of $\hat{S}'^{(i)}$. On the other hand, along the procedure, the GT pattern of base ν associated with T is transformed successively into patterns of base ν whose ith row is the shape of $\hat{T}'^{(i)}$ and eventually into the GT pattern (ν_1), (ν_1, ν_2), $\ldots$, $\nu = (\nu_1, \ldots, \nu_n)$.

Lemma 3.2. *Suppose S and T are two-row tableaux where S has shape $\mu = (\mu_1, \mu_2)$ and T extends S. Then we may switch S with T using at most $\mu_1 - \mu_2$ vertical switches. In particular, if switching transforms $S \cup T$ into $P \cup Q$ then the length of the first row of Q and μ define a a GT pattern of base μ.*

Proof: 1st *Step*: Switch horizontally the letters of the first row of T with the letters of the first row of S such that the letters of T get the leftmost possible positions.

2nd *Step*: Then switch horizontally the letters of the second row of S with the letters of the second row of T such that the letters of S get the rightmost possible positions. In this transformation possibly some of the letters of the second row of T will go to the left and we might have to switch again some of the letters of the first row of T with some letters of the first row of S to their left. At this point the letters of the first row of T are in the left most possible positions, and the letters of the second row of S are in the rightmost possible positions.

3th *Step*: We perform the vertical switches $\begin{matrix}\bar{\mathbf{s}}\\ \mathbf{t}\end{matrix}$.

4th *Step*: Now we may slide in the second and first rows of the perforated pair the letters of S totally to the right.

The number of vertical switches $\begin{matrix}\bar{\mathbf{s}}\\ \mathbf{t}\end{matrix}$ performed in 3th step is at most $\mu_1-\mu_2$ times, otherwise, after step 4, we would get a pair of tableaux where the skew tableau Knuth equivalent with S would have a row word of length at least μ_1+1. This is absurd since the shape of S is $\mu=(\mu_1,\mu_2)$ and therefore the longest row of any word Knuth equivalent with S has length μ_1. ■

Moreover, note that if $R\cup Q$ is an intermediate perforated pair with S the full contraction of R and T the full expansion of Q then the length of the first row of R interlaces with μ.

Example 3.1. Let $\mu=(3,1)$ and $\bar{\mathbf{d}}>\bar{\mathbf{b}}$. Then

$$S\cup T=\begin{array}{cccccc}\bar{\mathbf{a}}&\bar{\mathbf{b}}&\bar{\mathbf{c}}&2&3&4\\ \bar{\mathbf{d}}&2&3&4&&\end{array}\rightarrow\begin{array}{cccccc}\bar{\mathbf{a}}&\bar{\mathbf{b}}&2&3&4&\bar{\mathbf{c}}\\ \bar{\mathbf{d}}&2&3&4&&\end{array}\rightarrow\begin{array}{cccccc}\bar{\mathbf{a}}&\bar{\mathbf{b}}&2&3&4&\bar{\mathbf{c}}\\ 2&3&4&\bar{\mathbf{d}}&&\end{array}\rightarrow$$

$$\rightarrow\begin{array}{cccccc}\bar{\mathbf{a}}&2&3&4&\bar{\mathbf{b}}&\bar{\mathbf{c}}\\ 2&3&4&\bar{\mathbf{d}}&&\end{array}\rightarrow\begin{array}{cccccc}2&2&3&4&\bar{\mathbf{b}}&\bar{\mathbf{c}}\\ \bar{\mathbf{a}}&3&4&\bar{\mathbf{d}}&&\end{array}\rightarrow P\cup Q=\begin{array}{cccccc}2&2&3&4&\bar{\mathbf{b}}&\bar{\mathbf{c}}\\ 3&4&\bar{\mathbf{a}}&\bar{\mathbf{d}}&&\end{array}.$$

The GT pattern $\begin{matrix}&3&\\3&&1\end{matrix}$ of base μ, is transformed into the GT pattern $\begin{matrix}&2&\\3&&1\end{matrix}$ of base μ.

Proposition 3.3. *Let S and T be a three-row pair of tableaux, where S has shape μ and T extends S. Suppose switching transforms $S\cup T$ into $P\cup Q$ and let $P^{(i)}\cup Q^{(i)}$ be the sequence of pairs of tableaux defined by the first i rows of $P\cup Q=P^{(3)}\cup Q^{(3)}$, $1\le i\le 3$. If switching transforms $P^{(i)}\cup Q^{(i)}$*

into $\hat{S}^{(i)} \cup \hat{T}^{(i)}$ *for all* i, *then* $sh\,\hat{S}^{(1)}, sh\,\hat{S}^{(2)}, sh\,\hat{S}^{(3)}$ *define a GT pattern of base* μ.

Proof: We start with the GT pattern of base μ, defined by $sh\,S^{(1)}$, $sh\,S^{(2)}$, $sh\,S^{(3)}$, where $S^{(i)}$ is the tableau defined by the i first rows of S, for all i,

$$\begin{array}{ccccc} & & \mu_1 & & \\ & \mu_1 & & \mu_2 & \\ \mu_1 & & \mu_2 & & \mu_3 \end{array} . \tag{3.1}$$

Apply the 1st Step of switching, as in the Lemma, to the pair of tableaux defined by last two rows of $S \cup T$. Then either switching procedure involves only these two rows or some letters of the 1st row of S will slide down to the 2nd row of $S \cup T$. The last case occurs as follows. Suppose there is in the 2nd row a letter x of T with a west neighbour $\bar{\mathbf{a}}$ and north neighbour $\bar{\mathbf{b}}$ in S such that $\bar{\mathbf{a}} \leq \bar{\mathbf{b}}$. Moreover there is no smaller letters of T in the 1st row to the east of x. Then we have to switch vertically x with $\bar{\mathbf{b}}$,

$$\begin{array}{cccccc} & \dots & \bar{\mathbf{b}} & \dots & z \\ \bar{\mathbf{x}} & \bar{\mathbf{a}} & x & & \end{array} \rightarrow \begin{array}{cccccc} & \dots & x & \dots & z \\ \bar{\mathbf{x}} & \bar{\mathbf{a}} & \bar{\mathbf{b}} & & \end{array},\ x \leq z. \tag{3.2}$$

Otherwise either no letters in the 1st row of S are involved, or there is a smaller letter z of T, in the 1st row to the east of x which switches horizontally with $\bar{\mathbf{b}}$. Clearly, this operation does not slide down any letter of S to the second row of the perforated pair. Next apply to the outcome perforated pair the 2nd Step of the Lemma followed by the 3th Step. The number of vertical switches in the last two rows is at most equal to the number of those letters slid from 1st row to the 2nd row of S as shown in (3.2) plus $\mu_2 - \mu_3$. Denote the outcome perforated pair by $S' \cup T'$ and let $S'^{(i)} \cup T'^{(i)}$ be the perforated subpair defined by the first i rows of $S' \cup T'$. Let $\hat{S}'^{(i)} \cup \hat{T}'^{(i)}$ be the pair after full contraction of $S'^{(i)}$ and full extension of $T'^{(i)}$. Therefore, the second row of $S' \cup T'$ will have at least μ_3 letters of S. In the case the letters of the first row of S were not involved the GT pattern (3.1) is transformed into the GT pattern

$$\begin{array}{ccccc} & & \mu_1 & & \\ & \mu_1 & & \mu_2' & \\ \mu_1 & & \mu_2 & & \mu_3 \end{array} \tag{3.3}$$

with $\mu_2 \geq \mu_2' \geq \mu_3$. Otherwise the GT pattern (3.1) is first transformed into the GT (3.3), then to GT pattern

$$\begin{array}{ccccc} & & \mu_1'' & & \\ & \mu_1 & & \mu_2' & \\ \mu_1 & & \mu_2 & & \mu_3 \end{array} \tag{3.4}$$

with $\mu_1 \geq \mu_1'' \geq \mu_2'$, and then to the GT pattern defined by the interlacing shapes $shS'^{(1)} = (\mu_1'')$, $shS'^{(2)} = (\mu_1', \mu_2'')$ and $shS'^{(3)} = shS = \mu$,

$$\mu_1' \geq \mu_1'' \geq \mu_2'',$$

$$\mu_1 \geq \mu_1' \geq \mu_2 \geq \mu_2'' \geq \mu_3.$$

Schematically $shS'^{(1)} = (\mu_1'')$; $shS'^{(2)} = (\mu_1', \mu_2'')$ and $shS = \mu$ define a Gelfand-Tsetlin pattern of base μ

$$\begin{array}{ccccc} & & \mu_1'' & & \\ & \mu_1' & & \mu_2'' & \\ \mu_1 & & \mu_2 & & \mu_3 \end{array} \quad . \tag{3.5}$$

Now apply again the 1st Step of the Lemma to the first two rows of $S' \cup T'$ followed by the 2nd Step. Then apply Step 3 of Lemma and denote the outcome perforated pair by $\tilde{P} \cup \tilde{Q}$. After this the GT pattern of base μ (3.5) is transformed into another GT pattern of the same base as follows

$$\begin{array}{ccccc} & & \tilde{\mu}_1 & & \\ & \mu_1' & & \mu_2'' & \\ \mu_1 & & \mu_2 & & \mu_3 \end{array} \quad . \tag{3.6}$$

To conclude the process of switching S and T, the only vertical switches that we might to do are in the last two rows in consequence of the vertical switches performed in the two first rows in the last step. That is, some letters of T can be brought to the second row. Consider the two last rows of $\tilde{P} \cup \tilde{Q}$ and suppose that they are of the form $\begin{array}{cccccc} \cdots & \cdots & \cdots & \bar{\mathbf{e}} & \cdots & \cdots \\ \bar{\mathbf{a}} & \bar{\mathbf{b}} & \bar{\mathbf{c}} & x & y & z, \end{array}$ where $\bar{\mathbf{e}}$ is a letter that was not in the 2nd row of $S' \cup T'$. This means that, apart the situation described in (3.2), this letter $\bar{\mathbf{e}}$ slid from the first row. Now we have two kind of possible switches:

(1) if $\bar{\mathbf{c}} \leq \bar{\mathbf{e}}$, we have $\begin{array}{cccccc} \cdots & \cdots & \cdots & x & \cdots & \cdots \\ \bar{\mathbf{a}} & \bar{\mathbf{b}} & \bar{\mathbf{c}} & \bar{\mathbf{e}} & y & z, \end{array}$.

(2) if $\bar{\mathbf{c}} > \bar{\mathbf{e}}$, either we have $\begin{matrix} g & \dots & \dots & \bar{\mathbf{e}} & \dots & \dots \\ x & \bar{\mathbf{a}} & \bar{\mathbf{b}} & \bar{\mathbf{c}} & y & z, \end{matrix}$ or

$$\begin{matrix} \dots & \bar{\mathbf{g}} & \dots & \bar{\mathbf{e}} & \dots & \dots \\ \bar{\mathbf{a}} & x & \bar{\mathbf{b}} & \bar{\mathbf{c}} & y & z, \end{matrix} \quad \rightarrow \quad \begin{matrix} \dots & x & \dots & \bar{\mathbf{e}} & \dots & \dots \\ \bar{\mathbf{a}} & \bar{\mathbf{g}} & \bar{\mathbf{b}} & \bar{\mathbf{c}} & y & z. \end{matrix}$$

Clearly the number of vertical switches performed now is at most the number of vertical of switches performed from the first row of $\tilde{P} \cup \tilde{Q}$ to the second row. Therefore $sh\hat{S}'^{(2)}$ interlaces with the third and first rows of the previous GT pattern (3.6). The final GT is therefore $\begin{matrix} & & \tilde{\mu}_1 & & \\ & \hat{\mu}_1 & & \hat{\mu}_2 & \\ \mu_1 & & \mu_2 & & \mu_3 \end{matrix}$. This concludes the proof. ∎

The general case follows easily by successive application of previous proposition.

Theorem 3.4. *Let S and T be tableaux of shape μ and λ/μ respectively. Suppose switching transforms $S \cup T$ into $P \cup Q$ and let $P^{(i)} \cup Q^{(i)}$ be the sequence of pairs of tableaux defined by the first i rows of $P \cup Q = P^{(n)} \cup Q^{(n)}$, $1 \leq i \leq n$. If switching transforms $P^{(i)} \cup Q^{(i)}$ into $\hat{S}^{(i)} \cup \hat{T}^{(i)}$ for all i, then $sh\,\hat{S}^{(1)}, \cdots, sh\,\hat{S}^{(n)}$ define a GT pattern of base μ.*

Example 3.2. (1) We start to apply switching to the last two rows of SUT; then to the two first rows of $S' \cup T'$; and finally to the last two rows of $\tilde{P} \cup \tilde{Q}$.

$$S \cup T = \begin{matrix} \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{3}} & \bar{\mathbf{5}} & \bar{\mathbf{5}} \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & 1 & 4 \\ \bar{\mathbf{6}} & 2 & 3 & 4 & 5 \end{matrix} \rightarrow \begin{matrix} \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{3}} & 1 & \bar{\mathbf{5}} \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & 4 \\ 2 & 3 & 4 & \bar{\mathbf{6}} & 5 \end{matrix} \rightarrow \begin{matrix} \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{3}} & 1 & \bar{\mathbf{5}} \\ 2 & 3 & 4 & \bar{\mathbf{5}} & 4 \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & 5 \end{matrix} = S' \cup T' \rightarrow$$

$$\rightarrow \begin{matrix} 1 & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{3}} & \bar{\mathbf{5}} \\ 2 & 3 & 4 & \bar{\mathbf{5}} & 4 \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & 5 \end{matrix} \rightarrow \tilde{P} \cup \tilde{Q} = \begin{matrix} 1 & 3 & 4 & 4 & \bar{\mathbf{3}} \\ 2 & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{5}} & \bar{\mathbf{5}} \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & 5 \end{matrix} \rightarrow \begin{matrix} 1 & 3 & 4 & 4 & \bar{\mathbf{3}} \\ 2 & \bar{\mathbf{0}} & \bar{\mathbf{1}} & 5 & \bar{\mathbf{5}} \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \bar{\mathbf{6}} \end{matrix} \rightarrow$$

$$\rightarrow \begin{matrix} 1 & 3 & 4 & 4 & \bar{\mathbf{3}} \\ 2 & 5 & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{5}} \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \bar{\mathbf{6}} \end{matrix} .$$

Along the switching procedure we have the following sequence of GT patterns of base $\mu=(5,3,1)$ the shape of S,

$$\begin{array}{ccccc} & & 5 & & \\ & 5 & & 3 & \\ 5 & & 3 & & 1 \end{array} \rightarrow \begin{array}{ccccc} & & 4 & & \\ & 5 & & 3 & \\ 5 & & 3 & & 1 \end{array} \rightarrow \begin{array}{ccccc} & & 4 & & \\ & 4 & & 1 & \\ 5 & & 3 & & 1 \end{array} \rightarrow \begin{array}{ccccc} & & 1 & & \\ & 4 & & 1 & \\ 5 & & 3 & & 1 \end{array} \rightarrow$$

$$\rightarrow \begin{array}{ccccc} & & 1 & & \\ & 3 & & 1 & \\ 5 & & 3 & & 1 \end{array}.$$

(2) We do as in the previous example

$$S\cup T=\begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{5}} & 1 & \\ \bar{\mathbf{3}} & \bar{\mathbf{5}} & 2 & & & \end{array} \rightarrow \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & 1 & \bar{\mathbf{5}} & \\ \bar{\mathbf{3}} & \bar{\mathbf{5}} & 2 & & & \end{array} \rightarrow \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \\ \bar{\mathbf{3}} & \bar{\mathbf{5}} & 2 & & & \end{array} \rightarrow$$

$$\rightarrow \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & 2 & \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \\ \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{5}} & & & \end{array} = S'\cup T' \rightarrow \begin{array}{cccccc} \bar{\mathbf{1}} & 1 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & 2 & \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \\ \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{5}} & & & \end{array} \rightarrow$$

$$\rightarrow P\cup Q=\begin{array}{cccccc} 1 & 1 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} \\ 2 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \\ \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{5}} & & & \end{array}.$$

We have the the following GT pattern sequence of base $shS=(5,4,2)$

$$\begin{array}{ccccc} & & 5 & & \\ & 5 & & 4 & \\ 5 & & 4 & & 2 \end{array} \rightarrow \begin{array}{ccccc} & & 4 & & \\ & 5 & & 4 & \\ 5 & & 4 & & 2 \end{array} \rightarrow \begin{array}{ccccc} & & 4 & & \\ & 4 & & 4 & \\ 5 & & 4 & & 2 \end{array}.$$

(3) The same as in the previous examples with a perforated pair with four rows

$$(1) S\cup T=\begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & 1 & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & 2 & 4 & & \\ \bar{\mathbf{6}} & 3 & 3 & 5 & & \end{array} \rightarrow \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & 1 & \bar{\mathbf{5}} & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & 2 & 4 & & \\ \bar{\mathbf{6}} & 3 & 3 & 5 & & \end{array} \rightarrow (2) \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & 2 & 4 & & \\ \bar{\mathbf{6}} & 3 & 3 & 5 & & \end{array} \rightarrow$$

$$\rightarrow (3) \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{3}} & 2 & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & 4 & & \\ \bar{\mathbf{6}} & 3 & 3 & 5 & & \end{array} \rightarrow (4) \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{3}} & 2 & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \\ 3 & 3 & \bar{\mathbf{4}} & 4 & & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & 5 & & \end{array} \rightarrow \begin{array}{cccccc} \bar{\mathbf{1}} & 1 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & 1 \\ \bar{\mathbf{2}} & 2 & \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{5}} & \\ 3 & 3 & \bar{\mathbf{4}} & 4 & & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & 5 & & \end{array} \rightarrow$$

$$\rightarrow (5)\ \begin{array}{cccccc} 1 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & 1 \\ 2 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & 4 & \bar{\mathbf{5}} & \\ 3 & 3 & \bar{\mathbf{4}} & \bar{\mathbf{4}} & & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & 5 & & \end{array} \rightarrow (6)\ \begin{array}{cccccc} 1 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & 1 \\ 2 & 3 & \bar{\mathbf{3}} & 4 & \bar{\mathbf{5}} & \\ 3 & \bar{\mathbf{2}} & 5 & \bar{\mathbf{4}} & & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & & \end{array} \rightarrow \begin{array}{cccccc} 1 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} & 1 \\ 2 & 3 & 4 & \bar{\mathbf{3}} & \bar{\mathbf{5}} & \\ 3 & \bar{\mathbf{2}} & 5 & \bar{\mathbf{4}} & & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & & \end{array} \rightarrow$$

$$\rightarrow \begin{array}{cccccc} 1 & 1 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} \\ 2 & 3 & 4 & \bar{\mathbf{3}} & \bar{\mathbf{5}} & \\ 3 & \bar{\mathbf{2}} & 5 & \bar{\mathbf{4}} & & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & & \end{array} \rightarrow (7)\ \begin{array}{cccccc} 1 & 1 & 4 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} \\ 2 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{3}} & \bar{\mathbf{5}} & \\ 3 & \bar{\mathbf{2}} & 5 & \bar{\mathbf{4}} & & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & & \end{array} \rightarrow P \cup Q = \begin{array}{cccccc} 1 & 1 & 4 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{4}} \\ 2 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{3}} & \bar{\mathbf{5}} & \\ 3 & 5 & \bar{\mathbf{2}} & \bar{\mathbf{4}} & & \\ \bar{\mathbf{3}} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & \bar{\mathbf{6}} & & \end{array}.$$

The sequence (1); (2); (3); (4); (5); (6); (7) defines the GT pattern sequence of base $sh(S) = (5, 4, 2, 1)$

$$\begin{array}{ccccccc} &&&5&&& \\ &&5&&4&& \\ &5&&4&&2& \\ 5&&4&&2&&1 \end{array} \rightarrow \begin{array}{ccccccc} &&&4&&& \\ &&5&&4&& \\ &5&&4&&2& \\ 5&&4&&2&&1 \end{array} \rightarrow \begin{array}{ccccccc} &&&4&&& \\ &&4&&4&& \\ &5&&4&&2& \\ 5&&4&&2&&1 \end{array} \rightarrow$$

$$\begin{array}{ccccccc} &&&4&&& \\ &&4&&4&& \\ &4&&4&&1& \\ 5&&4&&2&&1 \end{array} \rightarrow \begin{array}{ccccccc} &&&4&&& \\ &&4&&3&& \\ &4&&3&&2& \\ 5&&4&&2&&1 \end{array} \rightarrow \begin{array}{ccccccc} &&&4&&& \\ &&4&&2&& \\ &4&&3&&1& \\ 5&&4&&2&&1 \end{array} \rightarrow$$

$$\rightarrow \begin{array}{ccccccc} &&&3&&& \\ &&4&&2&& \\ &4&&3&&1& \\ 5&&4&&2&&1 \end{array}.$$

If S is the Yamanouchi tableau $Y(\mu)$ then from previous Lemma and Proposition we conclude

Corollary 3.5. *Let T be a skew-tableau of shape λ/μ and suppose switching transforms $Y(\mu) \cup T$ into $U \cup V$. The following conditions hold*

1. *V is a Littlewood-Richardson tableau of type $[\nu, \mu, \lambda]$, where U of shape ν is Knuth equivalent to T.*
2. *Let $M_i = (m_1^{(i)}, \ldots, m_i^{(i)})$ with $m_k^{(i)}$ the number of letters slid down from the k-th row of $Y(\mu)$ to the i-th row of V (that is, the number of k's in the i-th row of V), $1 \leq k \leq i \leq n$. Then*
   (a) *$\mu^{(i)} = \mu^{(n)} - \sum_{j=i+1}^{n} M_j$, $1 \leq i \leq n$, is the GT pattern of type $[\nu, \mu, \lambda]$ defining V.*

(b) *if $U^{(i)} \cup V^{(i)}$ is the pair of tableaux defined by the first i rows of $U \cup V$, switching transforms $U^{(i)} \cup V^{(i)}$ into a pair $\hat{Y}^{(i)} \cup \hat{T}^{(i)}$ where $\hat{Y}^{(i)}$ is the Yamanouchi tableau of shape $\mu^{(i)}$, $1 \leq i \leq n$.*

Applying backwards Corollary 3.5, 2.(b) to $U \cup V$, with $U = Y(\nu)$, from top to bottom, we have for example

$$S \cup T = \begin{array}{ccccccccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 1 & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & 2 & & \\ \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & 1 & 2 & & & & & \\ \bar{\mathbf{4}} & 1 & 2 & 3 & 3 & & & & & & \end{array} \quad \rightarrow \begin{array}{ccccccccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 1 & 1 & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & 2 & 2 & & \\ \bar{\mathbf{3}} & \bar{\mathbf{3}} & 1 & 2 & 3 & 3 & & & & & \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & \end{array}$$

$$\rightarrow \begin{array}{ccccccccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 1 & 1 & 1 & 1 \\ \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & 2 & 2 & 2 & & \\ 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & & & & & \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & \end{array} \quad \rightarrow \begin{array}{ccccccccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \\ 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & & & & & \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & \end{array}$$

$$\rightarrow U \cup V = \begin{array}{ccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ 2 & 2 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \\ 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & & & & & \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & \end{array} .$$

Our aim is now to give operations realizing this information. Those operations will be described in the next section and are called *jeu de taquin-like slides*.

4. A *jeu de taquin-like* algorithm for Litllewood - Richardson tableaux

We shall now develop operations that shall make use of the information given in Corollary 3.5, 2.(b). The following technical statement defines the *jeu de taquin-like operations* and relate them with the switches in the switching transformation. This explains the nonstandard operations on the basis of the involution ρ_3 described with different flavours in [AZ1, AZ2] and [PV2].

Theorem 4.1. *Consider the following skew-tableau with inner shape μ*

$$T = \begin{array}{ccccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & z_0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \delta_1 & z_1 & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \gamma_2 & \theta_2 & \delta_2 & z_2 & & \\ \bullet & \alpha_3 & \beta_3 & \gamma_3 & \epsilon_3 & \theta_3 & \psi & \delta_3 & z_3 & & & \\ \lambda_4 & \alpha_4 & \gamma_4 & \theta_4 & \delta_4 & & & & & & & \end{array}$$

such that $z_i \geq \delta_{i+1}$*, for all* i*,*
$\delta_4 > \delta_3 > \delta_2 > \delta_1$*;* $\theta_4 > \theta_3 > \theta_2$*;* $\gamma_4 > \gamma_3 > \gamma_2$*;* $\alpha_4 > \alpha_3$*,*
and $\psi > \delta_2 \geq \theta_3$*;* $\epsilon_3 > \theta_2 > \delta_1 \geq \gamma_3$*;* $\beta_3 > \gamma_2 \geq \alpha_3 \geq \lambda_4$*. The following conditions hold*

(1) T *is Knuth equivalent to*

$$T' = \begin{array}{cccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \underline{\delta_1} & z_0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \underline{\gamma_2} & \underline{\theta_2} & \underline{\delta_2} & z_1 & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \underline{\alpha_3} & \underline{\gamma_3} & \underline{\theta_3} & \underline{\delta_3} & z_2 & & \\ \underline{\lambda_4} & \underline{\alpha_4} & \beta_3 & \underline{\gamma_4} & \epsilon_3 & \underline{\theta_4} & \psi & \underline{\delta_4} & z_3 & & & \end{array}, \qquad (4.1)$$

where T' *is obtained from* T *by sliding one row up the chains* $\delta_4 > \delta_3 > \delta_2 > \delta_1$*;* $\theta_4 > \theta_3 > \theta_2$*;* $\gamma_4 > \gamma_3 > \gamma_2$*;* $\alpha_4 > \alpha_3$ *and* λ_4*. The inner shape* μ' *of* T' *interlaces with the inner shape* μ *of* T*,*

$$\mu'_i \geq \mu_{i+1} \geq \mu'_{i+1},$$

such that $\mu_i - \mu'_i$ *is equal to the number of chains that have reached row* i *of* T*. (The underlines indicate the slid chains while the non underlined letters were kept fixed.) We call these sliding operations* jeu de taquin-like slides.

(2) *Suppose* $Y(\mu) \cup T$ *with* n *rows is by switching transformed into* $U \cup V$*. Then the last row of* $U \cup V$ *has* $(\mu_i - \mu'_i)$ i*'s, for all* i*, and* $U' \cup V'$*, defined by the first* $n-1$ *rows of* $U \cup V$*, can be transformed by switching into* $Y(\mu') \cup T'$*, with* T' *as in (4.1).*

Proof: Considering the inequalities above, by switches $Y(\mu) \cup T$ can be transformed into

$$\begin{array}{cccccccccccc} \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\ \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\ \bar{\mathbf{2}} & \bar{\mathbf{2}} & \gamma_2 & \bar{\mathbf{2}} & \theta_2 & \bar{\mathbf{2}} & \delta_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & & \\ \bar{\mathbf{3}} & \alpha_3 & \beta_3 & \gamma_3 & \epsilon_3 & \theta_3 & \psi & \delta_3 & z_3 & & & \\ \lambda_4 & \alpha_4 & \gamma_4 & \theta_4 & \delta_4 & & & & & & & \end{array}.$$

Then again by the following sequence of switches

$$\begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \theta_2 & \theta_3 & \delta_2 & \delta_3 & \bar{\mathbf{2}} & z_2 & & \\
\bar{\mathbf{2}} & \bar{\mathbf{2}} & \beta_3 & \bar{\mathbf{2}} & \epsilon_3 & \bar{\mathbf{2}} & \psi & \bar{\mathbf{2}} & z_3 & & & \\
\alpha_4 & \gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{3}} & & & & & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \theta_2 & \theta_3 & \delta_2 & \delta_3 & \bar{\mathbf{2}} & z_2 & & \\
\alpha_4 & \bar{\mathbf{2}} & \beta_3 & \bar{\mathbf{2}} & \epsilon_3 & \bar{\mathbf{2}} & \psi & \bar{\mathbf{2}} & z_3 & & & \\
\bar{\mathbf{2}} & \gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{3}} & & & & & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \theta_2 & \theta_3 & \delta_2 & \delta_3 & \bar{\mathbf{2}} & z_2 & & \\
\alpha_4 & \beta_3 & \epsilon_3 & \psi & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_3 & & & \\
\bar{\mathbf{2}} & \gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{3}} & & & & & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \theta_2 & \theta_3 & \delta_2 & \delta_3 & \bar{\mathbf{2}} & z_2 & & \\
\alpha_4 & \beta_3 & \epsilon_3 & \psi & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_3 & & & \\
\gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & & & & & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \theta_3 & \delta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\
\alpha_4 & \beta_3 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_3 & \bar{\mathbf{2}} & z_2 & & \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \epsilon_3 & \psi & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_3 & & & & \\
\gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & & & & & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \theta_3 & \delta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\
\alpha_4 & \beta_3 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_3 & \bar{\mathbf{2}} & z_2 & & \\
\gamma_4 & \bar{\mathbf{1}} & \epsilon_3 & \psi & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_3 & & & & \\
\bar{\mathbf{1}} & \theta_4 & \delta_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & & & & & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \theta_3 & \delta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\
\alpha_4 & \beta_3 & \theta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_3 & \bar{\mathbf{2}} & z_2 & & \\
\gamma_4 & \epsilon_3 & \bar{\mathbf{1}} & \psi & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_3 & & & \\
\bar{\mathbf{1}} & \theta_4 & \delta_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & & & & & & &
\end{array} \quad \rightarrow$$

$$\rightarrow \begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\alpha_4 & \beta_3 & \theta_2 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \theta_3 & \delta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\
\gamma_4 & \epsilon_3 & \bar{\mathbf{0}} & \psi & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_3 & \bar{\mathbf{2}} & z_2 & & \\
\theta_4 & \bar{\mathbf{0}} & \delta_4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_3 & & & \\
\bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & & & & & & &
\end{array} .$$

According to Corollary 3.5, in the last row of the previous perforated pair, the multiplicity of a letter $\overline{\mathbf{k-1}}$ is precisely the number of letters slid from the kth row of $Y(\mu)$ to the nth-row of $Y(\mu) \cup T$.

Now we show that the last perforated tableau pair with the last row suppressed

$$\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\alpha_4 & \beta_3 & \theta_2 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \theta_3 & \delta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 & \\
\gamma_4 & \epsilon_3 & \bar{\mathbf{0}} & \psi & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_3 & \bar{\mathbf{2}} & z_2 & & \\
\theta_4 & \bar{\mathbf{0}} & \delta_4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_3 & & &
\end{array}$$

can be transformed by switches into

$$T' = \begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \underline{\delta_1} & z_0 \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \underline{\gamma_2} & \underline{\theta_2} & \underline{\delta_2} & z_1 & \\
\bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \underline{\alpha_3} & \underline{\gamma_3} & \underline{\theta_3} & \underline{\delta_3} & z_2 & & \\
\lambda_4 & \underline{\alpha_4} & \beta_3 & \underline{\gamma_4} & \epsilon_3 & \underline{\theta_4} & \psi & \underline{\delta_4} & z_3 & & &
\end{array} . \tag{4.2}$$

Performing the following sequence of switches we get the desired result

$$\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\alpha_4 & \beta_3 & \theta_2 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \theta_3 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_2 & z_1 & \\
\gamma_4 & \epsilon_3 & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \psi & \bar{\mathbf{2}} & \delta_3 & z_2 & & \\
\theta_4 & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_4 & z_3 & & &
\end{array} \quad \rightarrow$$

$$\rightarrow \begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\alpha_4 & \beta_3 & \theta_2 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_2 & z_1 & \\
\gamma_4 & \epsilon_3 & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_3 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_3 & z_2 & & \\
\bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_4 & \psi & \delta_4 & z_3 & & &
\end{array} \quad \rightarrow$$

$$\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\alpha_4 & \beta_3 & \theta_2 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_2 & z_1 & \\
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \epsilon_3 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_3 & \delta_3 & z_2 & & \\
\gamma_4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_4 & \psi & \delta_4 & z_3 & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & z_0 \\
\alpha_4 & \beta_3 & \theta_2 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_2 & z_1 & \\
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_3 & \delta_3 & z_2 & & \\
\gamma_4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \epsilon_3 & \theta_4 & \psi & \delta_4 & z_3 & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \delta_1 & z_0 \\
\alpha_4 & \beta_3 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_2 & \delta_2 & z_1 & \\
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_3 & \delta_3 & z_2 & & \\
\bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \gamma_4 & \epsilon_3 & \theta_4 & \psi & \delta_4 & z_3 & & &
\end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \delta_1 & z_0 \\
\alpha_4 & \beta_3 & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_2 & \delta_2 & z_1 & \\
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \gamma_3 & \theta_3 & \delta_3 & z_2 & & \\
\bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \gamma_4 & \epsilon_3 & \theta_4 & \psi & \delta_4 & z_3 & & &
\end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \delta_1 & z_0 \\
\lambda_4 & \alpha_3 & \gamma_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_2 & \delta_2 & z_1 & \\
\alpha_4 & \beta_3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \gamma_3 & \theta_3 & \delta_3 & z_2 & & \\
\bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \gamma_4 & \epsilon_3 & \theta_4 & \psi & \delta_4 & z_3 & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \delta_1 & z_0 \\
\lambda_4 & \alpha_3 & \gamma_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_2 & \delta_2 & z_1 & \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \gamma_3 & \theta_3 & \delta_3 & z_2 & & \\
\bar{\mathbf{2}} & \alpha_4 & \beta_3 & \gamma_4 & \epsilon_3 & \theta_4 & \psi & \delta_4 & z_3 & & &
\end{array} \longrightarrow$$

$$\begin{array}{cccccccccccc}
\bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \bar{\mathbf{0}} & \delta_1 & z_0 \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \gamma_2 & \theta_2 & \delta_2 & z_1 & \\
\bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \alpha_3 & \gamma_3 & \theta_3 & \delta_3 & z_2 & & \\
\lambda_4 & \alpha_4 & \beta_3 & \gamma_4 & \epsilon_3 & \theta_4 & \psi & \delta_4 & z_3 & & &
\end{array} .$$

∎

Example 4.1. We may use the *jeu de taquin-like* slides defined in the previous theorem to conclude that

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & 4 \\
\bullet & \bullet & 2 & 4 & \\
3 & 5 & 6 & &
\end{array} \quad \text{is Knuth equivalent to} \quad \begin{array}{ccccc}
\bullet & \bullet & 2 & 4 & 4 \\
\bullet & 3 & 5 & 6 & \\
\bullet & \bullet & \bullet & &
\end{array} .$$

According to our previous study we now reformulate the bijection ρ_3 presented in [AZ1, AZ2] and [PV2], Section 6. In this context, since switching is an involution, it is easy to see that ρ_3 is also an involution. However in [AZ2] is defined another bijection ρ_3^{-1} which is shown, using a rather difficult argument, to coincide with ρ_3. This new bijection is not discussed here.

Algorithm 4.2. Start with an LR tableau T with n rows. Remove rows one by one from T, beginning with the bottom row, and replace each removed row by another one as follows. In each row to be removed, build a chain of integers in previous rows, starting with the last element and going to the first element. For each such element x, find the largest $y < x$ in the previous row, not used by the previous chains, starting from row containing x, then the largest element $z < y$ in the row above that of y not used by the previous chains, etc. This chain will finish either in a $\bullet$, in row $n-k$ of the inner shape of T whenever the length of the chain is k, or in the first row of T. This last situation occurs when the length of the chain is n. Now replace y with x, z with y, etc, until the top element of the chain remove a $\bullet$ in row $n-k$ of the inner shape of T unless the chain reaches the first row of T and, in this case, stay. The removed $\bullet$ is recorded as a letter $\overline{\mathbf{n-k}}$ in the row of T to be removed. Note that each entry of the last row of the inner shape of T forms a chain of length 0 which will be recorded in the same row as $\bar{\mathbf{n}}$.

Example 4.2. Consider the LR tableau T of type $[\mu = (8,7,4,1), \nu = (6,3,2); \lambda = (11,9,6,5)]$ and apply the *jeu de taquin-like*

$$T = \begin{array}{ccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 & 1 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 2 & & \\ \bullet & \bullet & \bullet & \bullet & 1 & 2 & & & & & \\ \bullet & 1 & 2 & 3 & 3 & & & & & & \end{array} \quad \rightarrow \quad \begin{array}{ccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 & 1 & 1 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 2 & 2 & & \\ \bullet & \bullet & 1 & 2 & 3 & 3 & & & & & \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & \end{array} \quad \rightarrow$$

$$\rightarrow \begin{array}{ccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 & 1 & 1 & 1 \\ \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 2 & 2 & 2 & & \\ 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & & & & & \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & \end{array} \quad \rightarrow \quad \begin{array}{ccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \\ 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & & & & & \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & \end{array} \quad \rightarrow$$

$$\rightarrow \begin{array}{ccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ 2 & 2 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \\ 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & & & & & \\ \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{4}}. & & & & & & \end{array}$$

The outcome is an LR tableau of type $[\nu, \mu, \lambda]$ with the LR tableau of type $[\mu, \nu, \lambda]$ rectified in the Yamanouchi tableau $Y(\nu)$. One has now operations realizing the last example in previous section.

5. Littlewood-Richardson triangle switching

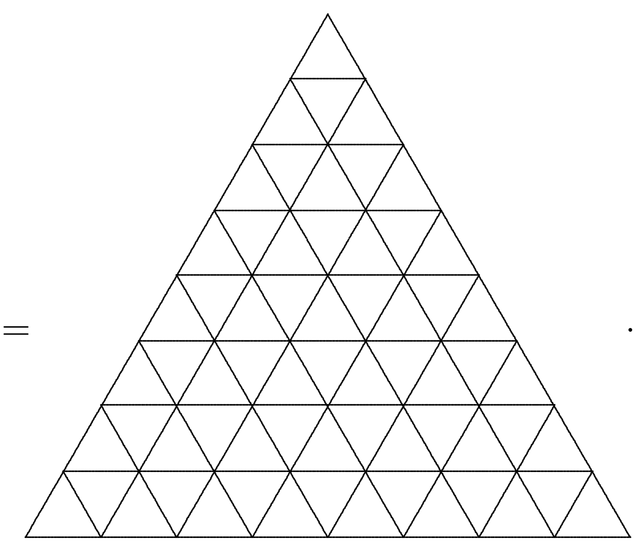
As we have seen in Section 3 the switching operation on a pair $S \cup T$ with S of normal shape μ and T of weight ν is accompanied by a pair of GT patterns, one with boundary ν and base μ, and the other one with boundary μ and base ν which switch between each other. This suggests that the *jeu de taquin-like* operations can be translated to Littlewood-Richardson triangles [PV1].

Let k be a positive integer and T_k the space of triangles of size k [KB] consisting of all sequences

$$A = (V^{(0)}, V^{(1)}, \dots, V^{(k)}),$$

where $V^{(j)} = (a_{jj}, \dots, a_{kj}) \in \mathbb{R}^{k-j+1}$, $0 \le j \le k$, and $a_{00} = 0$. As a vector space $T_k \simeq \mathbb{R}^{\frac{(k+1)(k+2)}{2}-1}$.

The hive graph Δ_k of size k is a graph in the plane with $\binom{k+2}{2}$ vertices arranged in a triangular grid, consisting of k^2 small equilateral triangles

$\Delta_8 =$  . (5.1)

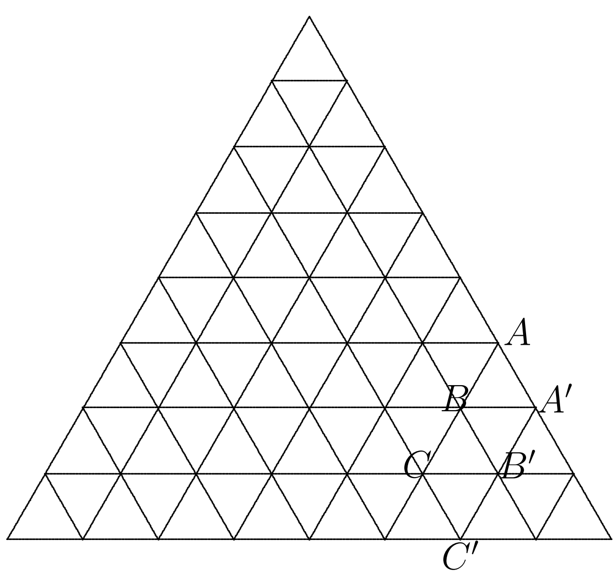
T_k is identified with the vector space of all labeling $A = (a_{ij})_{0 \le j \le i \le k}$ of Δ_k by real numbers such that $a_{00} = 0$. We write $A \in T_k$ as a triangular array of real numbers

$$A = \begin{matrix} & & & a_{00} & & & \\ & & a_{10} & & a_{11} & & \\ & a_{20} & & a_{21} & & a_{22} & \\ a_{30} & & a_{31} & & a_{32} & & a_{33} \end{matrix} \in T_3.$$

A Littlewood-Richardson (**LR**) triangle of size k [PV1] is an element $A = (a_{ij})_{0\leq j\leq i\leq k}$ of T_k that satisfies the following inequalities

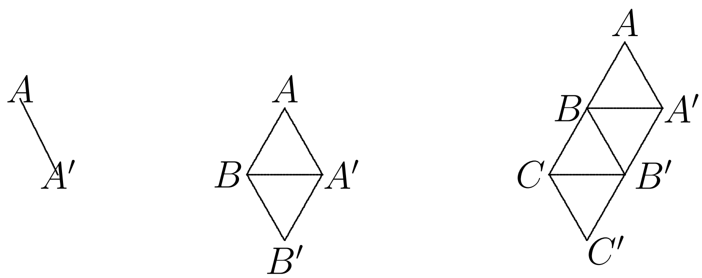
$$\begin{array}{lll} (P) & a_{ij} \geq 0, & 0 \leq i, j \leq k, \\ (I) & \sum_{q=j}^{i} a_{qj} \geq \sum_{q=j+1}^{i+1} a_{q,j+1}, & 1 \leq j \leq i < k, \\ (S) & \sum_{p=0}^{j-1} a_{ip} \geq \sum_{p=0}^{j} a_{i+1,p}, & 1 \leq j \leq i < k. \end{array}$$

For each $j = 1, \ldots, k-1$, we consider, in Δ_k (5.1),the labeled parallelogram $\mathbf{p}_j = [a_{jj}, \ldots, a_{k-1,j}, V^{(j+1)}]$. (We convention $\mathbf{p}_{k-1} = [a_{k-1,k-1}, a_{kk}]$ as a degenerated parallelogram.) The labels of parallelograms $\mathbf{p}_j$, $1 \leq j \leq k-1$, satisfy inequalities (I). For $k = 8$, we have the parallelogram $\mathbf{p}_5 = [ABC; A'B'C']$ in Δ_8



(5.2)

where the labels of

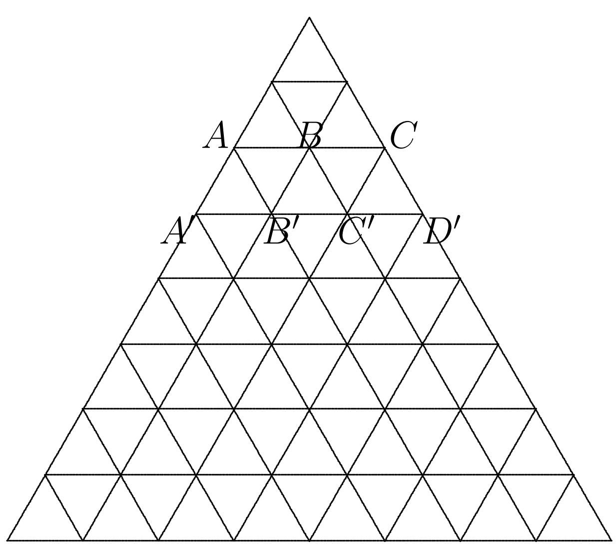


(5.3)

satisfy the inequalities

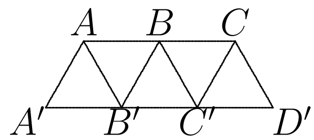
$$\begin{array}{l} A \geq A' \\ A+B \geq A'+B' \\ A+B+C \geq A'+B'+C'. \end{array}$$

For $i=1,\ldots,k-1$, we consider the labeled trapezoids $\mathbf{t}_i=[a_{i0},a_{i1},\ldots,a_{ii};a_{i+10},a_{i+11},\ldots,a_{i+1,i+1}]$ in Δ_k (5.4). The labels of $\mathbf{t}_j$, $1\leq j<k$, satisfy inequalities (S). For $k=8$, we have the trapezoid $\mathbf{t}_2=[ABC;A'B'C'D']$ in Δ_8



(5.4)

The labels of



(5.5)

satisfy
$A \geq A' + B'$,
$A + B \geq A' + B' + C'$,
$A + B + C \geq A' + B' + C' + D'$.

LR_k denotes the cone of all Littlewood-Richardson triangles in T_k, and is called the Littlewood-Richardson cone of order k.

To each triangle $A = (a_{ij})_{0 \leq j \leq i \leq k} \in T_k$ we associate the real vectors $\mu = (\mu_1, \dots, \mu_k)$, $\nu = (\nu_1, \dots, \nu_k)$ and $\lambda = (\lambda_1, \dots, \lambda_k)$, where

$$\begin{aligned} \mu_i &= a_{i0}, \quad 1 \leq i \leq k \\ \nu_j &= \Sigma_{q=j}^{k} a_{qj}, \quad 1 \leq j \leq k \\ \lambda_i &= \Sigma_{q=0}^{i} a_{iq}, \quad 1 \leq i \leq k. \end{aligned}$$

We call $[\mu, \nu, \lambda]$ the *type* of A, ν the *weight* of A, and μ the *boundary* of T_k. Note that μ is the label of the right edge of the hive graph Δ_k.

Let x be a real vector, and denote by $|x|$ the sum of its entries. If $A \in LR_k$, it follows from (P), (S) and (I) that the vectors μ, ν, and λ satisfy $\mu_1 \geq \cdots \geq \mu_k \geq 0$, $\nu_1 \geq \cdots \geq \nu_k \geq 0$, $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$, and $|\mu| + |\nu| = |\lambda|$, $\mu \leq \lambda$.

Let $LR_k(\mathbb{Z}) := LR_k \cap \mathbb{Z}^{\frac{(k+1)(k+2)}{2} - 1}$ be the set of all integral **LR** triangles of size k, that is, the set of integral points of LR_k. Since LR_k is a rational polyhedral cone, $LR_k(\mathbb{Z})$ is a finitely generated semigroup and the cone generated by $LR_k(\mathbb{Z})$ is LR_k.

Let P_k denote the set of all k-tuples $x = (x_1, \dots, x_k)$ of nonnegative integers such that $x_1 \geq \cdots \geq x_k \geq 0$. Let μ, ν, λ partitions in P_k such that $\mu \leq \lambda$ and $|\mu| + |\nu| = |\lambda|$. To each Littlewood-Richardson tableau T of type $[\mu, \nu, \lambda]$, we associate an integral Littlewood-Richardson triangle $A = (a_{ij})_{0 \leq j \leq i \leq k} \in T_k$

defined by

$$\begin{array}{l} a_{00}=0,\ \ a_{i0}=\mu_i,\ 1\le i\le k,\\ a_{ij}\ \text{ the number of } j\text{'s in row } i \text{ of } T,\ 0<j\le i\le k. \end{array}$$

For example,

$$T=\begin{array}{cccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 & 1\\ \bullet & \bullet & \bullet & 1 & 2 & 2 & & \\ \bullet & \bullet & 2 & 3 & 3 & & & \\ 1 & 2 & 4 & & & & & \end{array} \qquad \longleftrightarrow \quad A_T=\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 5 & & 3 & & & \\ & & 3 & & 1 & & 2 & & \\ & 2 & & 0 & & 1 & & 2 & \\ 0 & & 1 & & 1 & & 0 & & 1 \end{array}. \tag{5.6}$$

Proposition 5.1. [PV1] *Let μ, ν, λ partitions in P_k such that $|\lambda|=|\mu|+|\nu|$ and $\mu\le\lambda$. Then the correspondence $T\longleftrightarrow A_T$ is a bijection between the integral points of the set of LR triangles of type $[\mu,\nu,\lambda]$ and $LR[\mu,\nu,\lambda]$.*

Example 5.1. Let $\nu=(5,4,2,1)$ and $\mu=(5,3,2,0)$. We consider the two following triangles one of type $[\mu,\nu,\lambda]$ and the other one of type $[\nu,\mu,\lambda]$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & \mathbf{5} & & 3 & & & \\ & & \mathbf{3} & & 1 & & 2 & & \\ & \mathbf{2} & & 0 & & 1 & & 2 & \\ \mathbf{0} & & 1 & & 1 & & 0 & & 1 \end{array} \qquad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & \mathbf{5} & & 5 & & & \\ & & \mathbf{4} & & 0 & & 3 & & \\ & \mathbf{2} & & 0 & & 0 & & 2 & \\ \mathbf{1} & & 0 & & 0 & & 0 & & 0 \end{array}. \tag{5.7}$$

Now we attach the second triangle on the left edge of the first triangle, and translate the *jeu de taquin-like slides* on the LR tableau in (5.6) to the LR triangles (5.7)

$$\begin{array}{ccccccccccccc} \mathbf{1} & & \mathbf{2} & & \mathbf{4} & & \mathbf{5} & & 0 & & & & \\ & 0 & & 0 & & 0 & & \mathbf{5} & & 3 & & & \\ & & 0 & & 0 & & \mathbf{3} & & 1 & & 2 & & \\ & & & 0 & & \mathbf{2} & & 0 & & 1 & & 2 & \\ & & & & \mathbf{0} & & 1 & & 1 & & 0 & & 1 \end{array} \qquad \longrightarrow$$

$$\longrightarrow \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{5} & 0 \\ 0 & 0 & 0 & \mathbf{5} & 3 \\ 0 & 0 & \mathbf{3} & 1 & 2 \\ 1 & \mathbf{2-1} & 0 & 1+1 & 2 \\ \mathbf{0} & 1 & 1-1 & 0 & 1 \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{5} & 0 \\ 0 & 0 & 0 & \mathbf{5} & 3 \\ 0 & 0 & \mathbf{3} & 1 & 2 \\ 2 & \mathbf{1-1} & 0+1 & 2 & 2 \\ \mathbf{0} & 1-1 & 0 & 0 & 1 \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{5} & 0 \\ 0 & 1 & 0 & \mathbf{5-1} & 3+1 \\ 0 & 0 & \mathbf{3} & 1-1 & 2+1 \\ 2 & \mathbf{0} & 1 & 2-1 & 2 \\ \mathbf{0} & & & & 1 \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{5} & 0 \\ 0 & 1 & 0 & \mathbf{4} & 4 \\ 0 & 1 & \mathbf{3-1} & 0 & 3+1 \\ 2 & \mathbf{0} & 1 & 1-1 & 2 \\ \mathbf{0} & & & & 1 \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{5} & 0 \\ 0 & 1 & & \mathbf{4} & 4 \\ 0 & 1+1 & \mathbf{2-1} & 0+1 & 4 \\ 2 & \mathbf{0} & 1-1 & 0 & 2 \\ \mathbf{0} & & & & 1 \end{array} \longrightarrow$$

$$\longrightarrow \quad \begin{array}{ccccccccccccccccc}
\mathbf{1} & & \mathbf{2} & & \mathbf{4} & & \mathbf{5} & & & & 0 & & & & & & \\
 & 0 & & 1 & & 1 & & & \mathbf{4-1} & & & & 4+1 & & & & \\
 & & 0 & & 2 & & \mathbf{1} & & & & 1-1 & & & & 4 & & \\
 & & & 2 & & \mathbf{0} & & & & & & & & & & 2 & \\
 & & & & \mathbf{0} & & & & & & & & & & & & 1
\end{array} \quad \longrightarrow$$

$$\longrightarrow \quad \begin{array}{ccccccccccccc}
\mathbf{1} & & \mathbf{2} & & \mathbf{4} & & \mathbf{5} & & 0 & & & & \\
 & 0 & & 1 & & 1 & & \mathbf{3} & & 5 & & & \\
 & & 0 & & 2 & & \mathbf{1} & & 0 & & 4 & & \\
 & & & 2 & & \mathbf{0} & & 0 & & 0 & & 2 & \\
 & & & & \mathbf{0} & & 0 & & 0 & & 0 & & 1
\end{array} \quad .$$

The first triangle in (5.7) is thus transformed into the triangle with boundary $\mu = (5, 3, 2, 0)$ and weight $\nu = (5, 4, 2, 1)$

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & \mathbf{5} & & 5 & & & \\
 & & \mathbf{3} & & 0 & & 4 & & \\
 & \mathbf{2} & & 0 & & 0 & & 2 & \\
\mathbf{0} & & 0 & & 0 & & 0 & & 1
\end{array} \quad .$$

The LR tableau in (5.6) is therefore transformed into the LR tableau of type $[\nu, \mu, \lambda]$

$$\begin{array}{llllllll}
\bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 & 1 \\
\bullet & \bullet & \bullet & \bullet & 1 & 2 & & \\
\bullet & \bullet & 1 & 2 & 2 & & & \\
\bullet & 3 & 3 & & & & &
\end{array} \quad .$$

We observe that the *jeu de taquin-like* can be extended to any Litlewood-Richardson triangle in LR_k.

6. Final Remarks

To finish we may say that the interlacing property between the normal shape of a rectified tableau and the normal shape of any rectified subtableau gave rise to *jeu de taquin-like* operations. Moreover those operations give a variation of the tableau switching on Littlewood-Richardson tableau pairs and a Litlewood-Richardson triangle switching in the cone LR_k.

Interlacing inequalities occur in several contexts as matrix theory, module theory as well as in the combinatorics of Young tableaux. For instance in [QSSA] an explanation of analogies between interlacing of invariant factors of matrices over principal ideal domains and eigenvalues of Hermitian matrices is given. In [AZ1, AZ2], Introduction, and [AW] is also discussed a relationship between Littlewood-Richardson combinatorics and invariant factors of a product of matrices. In analogy with the interlacing property between the normal shape of a rectified tableau and the normal shape of any rectified subtableau, we recall two classical results in matrix theory:

[EMSa, TH] *Let R be a principal ideal domain and let $n \geq 2$ be a positive integer. Given elements $c_n | \dots | c_1$ and $a_{n-1} | \dots | a_1$ in R, there exists an $n \times n$ matrix over R with the c_i as invariant factors, containing an $(n-1) \times n$ submatrix with the a_i as invariant factors, if and only if $c_{i+1} | a_i | c_i$.*

[FP] *Given real numbers $\gamma_1 \geq \cdots \geq \gamma_n$ and $\alpha_1 \geq \cdots \geq \alpha_{n-1}$, there exists an $n \times n$ Hermitian matrix with the γ_i as eigenvalues if and only if $\gamma_{i+1} \leq \alpha_i \leq \gamma_i$.*

Acknowledgements: The author is grateful to Christian Krattenthaler, Igor Pak and Ernesto Vallejo for stimulating conversations and helpful remarks on bijection ρ_3. In particular, to draw her attention to the paper [BSS].

References

[AW] Glenn Appleby and Tamsen Whitehead. *Matrix pairs over discrete valuation rings determine Littlewood-Richardson fillings*, available at http://www.arXiv:math/0710.3013.

[AZ1] Olga Azenhas. *Littlewood-Richardson fillings and their symmetries*, Matrices and Groups Representations, Textos de Matemática, Série B 19 (1999), 81-82.

[AZ2] Olga Azenhas. *On an involution on the set of Littlewood-Richardson tableaux and the hidden commutativity*, Pré-Publicações Departamento de Matemática Universidade de Coimbra 00-27 (2000); available at http://www.mat.uc.pt/preprints/2000.html.

[BSS] Georgia Benkart, Frank Sottile and Jeffrey Stroomer. *Tableau switching:algorithms and applications*, J. Combin. Theory, Ser. A, vol. 76 (1996), 11-43.

[DK] Vladimir I. Danilov and Gleb A. Koshevoy. *Arrays and the octahedron recurrence*, available at http://www.arXiv:math/0504299.

[FP] Ky Fan and Gordon Pall. *Imbedding conditions for Hermitian and normal matrices*, Canad. J. Math. 9 (1957), 298-304.

[F] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, United Kingdom, 1997.

[H] Mark Haiman. *Dual equivalence with applications, including a conjecture of Proctor*, Discrete Math. 99 (1992), 79–113.

[HK] André Henriques, Joel Kamnitzer. *The octahedron recurrence and gl_n crystals*, Adv. Math. 206 (2006), no. 1, 211–249.

[KB] Anatol N. Kirillov, Arkady D. Berenstein. *Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux*, Algebra i Analiz 7, (1995), .

[KTW] Allen Knutson, Terence Tao and Christopher T. Woodward. *A positive proof of the Littlewood-Richardson rule using the octahedron recurrence*, Electron. J. Combin. 11 (2004), no. 1, Research Paper 61, 18 pp.

[LLT] Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon. *The plactic monoid* in M. Lothaire (ed.), Algebraic Combinatorics on Words, Vol. 90 of Enciclopedia of Mathematics and its Applications, pp. 164–196, Cambridge University Press, Cambridge, United Kingdom, 2002.

[PV1] Igor Pak and Ernesto Vallejo. *Combinatorics and geometry of Littlewood-Richardson cones*, Europ. J. Combinatorics, vol. 26 (2005), 995-1008.

[PV2] Igor Pak and Ernesto Vallejo. *Reductions of Young tableau bijections*, available at http://www.arXiv:math/0408171http://www.math.mit.edu/ pak/tab7.pdf.

[QSSA] João Filipe Queiró, Eduardo Marques de Sá, Ana Paula Santan, Olga Azenhas. *Interlacing of eigenvalues and invariant factors*, Mathematical Inequalities & Apllications, vol 3, number 2 (2000), 149-154.

[Sa] Bruce Sagan. *The symmetric group Reperesentations, Combinatorial Algorithms, and Symmetric functions*, Springer Verlag New York, 2001.

[EMSa] Eduardo Marques de Sá. *Imbedding conditions for λ-matrices*, Linear Algebra and its Applications, 24 (1979), 33-50.

[S] Richard Stanley. *Enumerative Combinatorics*, Vol 2, Cambridge University Press, Cambridge, United Kingdom, 2001.

[TH] Robert C. Thompson. *Interlacing inequalities for invariant factors*, Linear Algebra and its Applications, 24 (1979), 1-32.

[GZ] Israel Gelfand and Andrei Zelevinsky. *Multiplicities and proper bases of gl_n*, Group theoretical methods in physics, Vol. II (Yurmala, 1985), 147–159, VNU Sci. Press, Utrecht, 1986.

Olga Azenhas
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: oazenhas@mat.uc.pt