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AN OBSERVATION ON PREORDERS AND INTERNAL CATEGORIES

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Dedicated to René Guitart on the occasion of his sixty-fifth birthday

Abstract: We prove that in a regular category all reflexive and transitive relations are symmetric if and only if every internal category is an internal groupoid. In particular, these conditions hold when the category is n-permutable for some n.

Keywords: Mal'tsev, Goursat, n-permutable category; preorder; equivalence relation; internal category, groupoid.
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We take $\mathscr{C}$ to be a regular category. It is well known that any internal preorder, being a reflexive and transitive relation (R, r_1, r_2) on an object X of $\mathscr{C}$, may be considered as an internal category in $\mathscr{C}$. In fact, a preorder is the same thing as a skeletal category, an internal category of which the domain and codomain morphisms r_1, $r_2\colon R \to X$ are jointly monic. This internal category will be a groupoid precisely when the given reflexive and transitive relation R is symmetric, so that *if in $\mathscr{C}$ every internal category is an internal groupoid, then all of its internal preorders are equivalence relations.*

The converse implication is interesting due to its close relation with the following question: "What conditions does a regular category need to satisfy for all internal categories in it to be internal groupoids?" One of the main results of [1] gives a sufficient condition: the Mal'tsev property, that is, 2-permutability $RS = SR$ of internal equivalence relations or, equivalently, congruences R, S. But when $\mathscr{C}$ is a variety, already the strictly weaker n-permutability condition ($RSRS\cdots = SRSR\cdots$ with n factors R or S on each side) is sufficient [6]. Furthermore—here we follow a remark in [5]—a variety is n-permutable if and only if [2] all of its internal preorders are equivalence relations (= congruences). Altogether:

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Proposition. *If $\mathscr{V}$ is a variety of universal algebras, then the following conditions are equivalent:*

(i) *all preorders in $\mathscr{V}$ are congruences;*
(ii) *all internal categories in $\mathscr{V}$ are internal groupoids;*
(iii) *$\mathscr{V}$ is n-permutable for some n.* ■

This result is no longer true for regular categories. The number n in the third condition is obtained through a construction on a free algebra, and it cannot be replaced by a purely categorical argument—see [4] for a counterexample. On the other hand, the equivalence between the upper two conditions makes sense in general and, given any n-permutable category, we may ask whether they hold or not. As it turns out, the situation is as good as it could possibly be:

Theorem. *If $\mathscr{C}$ is a regular category, then the following conditions are equivalent:*

(i) *all preorders in $\mathscr{C}$ are equivalence relations;*
(ii) *all internal categories in $\mathscr{C}$ are internal groupoids.*

Furthermore, these conditions hold if $\mathscr{C}$ is n-permutable for some n.

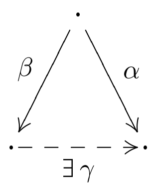
Proof: We already recalled that the second condition is stronger than the first. As for the final statement, the article [4] gives new equivalent conditions for n-permutable categories, based on the varietal case [3], which easily imply (i). So we are left with proving (i) $\Rightarrow$ (ii), for which it suffices to observe that the argument given by Carboni, Pedicchio and Pirovano in the Mal'tsev context [1, Theorem 2.2] is still valid. For the sake of completeness, let us briefly sketch how it goes.

Consider an internal category

$$M * M \xrightarrow{m} M \underset{c}{\overset{d}{\rightrightarrows}} O \quad (i\colon O \to M)$$

and the induced relation S on M defined by

$$\beta S \alpha \qquad \Leftrightarrow \qquad \exists_{\gamma \in M} \quad \gamma\beta = \alpha$$



as on page 103 of [1]. Then the relation S is not just reflexive as mentioned there, but it is clearly also transitive. As a consequence, condition (i) tells us that S is an equivalence relation on M. Now given any $\alpha \in M$, we have that $1_{d(\alpha)} S \alpha$. Hence $\alpha S 1_{d(\alpha)}$ yields an element ${}^\bullet\alpha$ of M such that ${}^\bullet\alpha\alpha = 1_{d(\alpha)}$. Via a similar argument we obtain $\alpha^\bullet \in M$ satisfying $\alpha\alpha^\bullet = 1_{c(\alpha)}$. We have

$$ {}^\bullet\alpha = {}^\bullet\alpha(\alpha\alpha^\bullet) = ({}^\bullet\alpha\alpha)\alpha^\bullet = \alpha^\bullet, $$

so ${}^\bullet\alpha = \alpha^\bullet$ is a two-sided inverse for α. Finally, given any other such inverse α^{-1},

$$ \alpha^{-1} = \alpha^{-1}(\alpha\alpha^\bullet) = (\alpha^{-1}\alpha)\alpha^\bullet = \alpha^\bullet, $$

which proves its uniqueness. ■

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