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SINGLETON FREE SET PARTITIONS AVOIDING A 3-ELEMENT SET

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ABSTRACT: The definition and study of pattern avoidance for set partitions, which is an analogue of pattern avoidance for permutations, begun with Klazar. Sagan continued his work by considering set partitions which avoids a single partition of three elements, and Goyt generalized these results by considering partitions which avoids any family of partitions of a 3-element set. In this paper we enumerate and describe set partitions, even set partitions and odd set partitions without singletons which avoids any family of partitions of a 3-element set. The characterizations of these families allows us to conclude that the corresponding sequences are P-recursive. We also construct Gray codes for the sets of singletons free partitions that avoids a single partition of three elements.

KEYWORDS: Set partitions; Pattern avoidance; P-recursion; Gray code.
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1. Introduction

Enumeration of pattern-avoiding objects such as permutations, words or compositions, is a very active area of research, with connections to several areas of mathematics. In 1996, Klazar [4] extended the notion of pattern avoidance for permutations, words and compositions to set partitions by analyzing set partitions that avoids the patterns *abab* and *aabb*. Sagan [9] continued this work by considering set partitions which avoids a single partition of a 3-element set. Since then this notion has been studied by many authors (see [6] and the references therein for a comprehensive survey). In particular, Goyt [2] generalized Sagan's results by considering partitions, even partitions and odd partitions that avoids any family of partitions of a 3-element set. In this paper, we will focus on set partitions without singletons, that is set partitions whose blocks have at least two elements, that avoids any family of partitions of a 3-element set. To this end, we need some definitions.

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For integers $m \leq n$ define the interval $[m, n] = \{m, m+1, \ldots, n\}$ with special case $[1, n] = [n]$. A *partition* π of a set $S \subseteq [n]$, $n \geq 1$, is a collection of nonempty disjoint subsets $B_1, \ldots, B_t$ of S, called *blocks*, whose union is S. We will write $\pi \vdash S$ and $b(\pi) = t$ to denote the number of blocks of π. A block with only one element is said to be a *singleton*. A partition is said to be in *standard form* if it is written as $\pi = B_1/B_2/\cdots/B_t$, where the blocks are listed in ascending order according to their smallest element. Generally, we will not use braces and commas in the blocks unless they are needed for clarity. For example, if $\pi = 13/245/6/7$ then $\pi \vdash [7]$ with $b(\pi) = 4$.

The set of all set partitions of $[n]$, $n \geq 1$, will be denoted by

$$\Pi_n = \{\pi : \pi \vdash [n]\}.$$

If S is a subset of the integers with cardinality $\#S = n$, then the standardization map corresponding to S is the unique order-preserving bijection $st_S : S \to [n]$. When S is clear from the context we drop the subscript. For example, if $S = \{2, 5, 7\}$ then $st(2) = 1, st(5) = 2$ and $st(7) = 3$. Thus, if $\pi = 27/5$ its standardization is $st(\pi) = 13/2$.

A set *subpartition* of a set partition $\pi = B_1/B_2/\cdots/B_t$ of S is a set partition π' of $S' \subseteq S$ such that each block of π' is contained in a different block of π. For example, $27/5$ is a subpartition of $1356/27/4$ but not of $1357/26/4$. Let $\pi \in \Pi_k$ be a given set partition called the *pattern*. We say that a partition $\sigma \in \Pi_n$ *contains the pattern* π if there exists a set subpartition σ' of σ such that $st(\sigma') = \pi$. In this case, σ' is called an occurrence of the pattern π in σ. If σ as no occurrences of π, then we say that σ *avoids* the pattern π. For example, $\sigma = 16/23/45$ avoids the pattern 123 but contains the pattern $13/2$ since the standardization of the subpartition $\sigma' = 16/2$ is $13/2$. In this context, for $R \subseteq \Pi_k$ we use the notation

$$\Pi_n(R) = \{\sigma \in \Pi_n : \sigma \text{ avoids every pattern } \pi \in R\}.$$

The set $\Pi(R)$, with $R \subseteq \Pi_3$, was studied by Sagan [9] when $\#R = 1$ and by Goyt [2] for $\#R \geq 2$. Denote by Π'_n the set of all singleton free partitions of $[n]$, and given $R \subseteq \Pi_k$ a subset of patterns, let

$$\Pi'_n(R) = \{\sigma \in \Pi'_n : \sigma \text{ avoids every pattern } \pi \in R\}$$

be the set of all singleton free partitions of $[n]$ that avoids all partitions of R. When $R = \{\pi\}$, we simplify the notation and write $\Pi'_n(\pi)$.

In the next section we characterize the set $\Pi'_n(\pi)$, and give exact formulas and generating functions for its cardinal, for various patterns π, including all $\pi \vdash [3]$. We then use these results to characterize and enumerate $\Pi'_n(R)$, for any $R \subset \Pi_3$. In section 3 we present the notion of sign of a partition, defined in [2], and enumerate the set of singleton free signed partitions of $[n]$ which avoids any family of patterns of Π_3. The study of P-recursiveness associated with permutation patterns begun with Gessel [1] and Noonan-Zeilberger [8], and was applied to set partitions by Sagan [9]. In section 4 we show that although Π'_n is not P-recursive, the sets of singleton free partitions and singleton free sign partitions that avoids any pattern $\pi \vdash [3]$ are P-recursive. Finally, in the last section we construct Gray codes for the sets $\Pi'_n(\pi)$, for all $\pi \vdash [3]$ for which the set is not trivial, where each partition in the list is obtained from its immediate predecessor by changing the block of at most two elements.

2. Singleton free set partitions

We start by considering the case $\Pi'_n(\pi)$, with π a pattern in Π_3, namely $123, 1/23, 12/3, 1/2/3$ and $13/2$. Following the notation of [9] for exponential generating functions, we let

$$F_I(x) = \sum_{i \in I} \frac{x^i}{i!}, \tag{2.1}$$

for I a set of nonnegative integers. In particular, when $I = [0, m]$, we write

$$\exp_m(x) = \sum_{i=0}^{m} \frac{x^i}{i!}.$$

Let $a^I_{n,\ell}$ denote the number of partitions of $[n]$ with ℓ blocks with cardinalities in the set $I \subseteq \mathbb{N}$. As $F_I(x)$ is the exponential generating function for the number of ways an n-set can form a block with size in the set I, it follows that (see, for example, [7] or [13])

$$\sum_{n \geq 0} a^I_{n,\ell} \frac{x^n}{n!} = \frac{F_I(x)^\ell}{\ell!} \tag{2.2}$$

is the exponential generating function for the number of partitions of $[n]$ with ℓ blocks, each of them having sizes in the set I. Finally, given a pattern π, we write

$$F_\pi(x) = \sum_{n \geq 0} \#\Pi'_n(\pi) \frac{x^n}{n!}. \tag{2.3}$$

The distinction between (2.1) and (2.3) will be clear, since we denote patterns by Greek letters and sets of integers by capital Latin letters.

For example, with $I = \mathbb{N} \setminus \{1\}$, it follows that $\#\Pi'_n$ is the sum over all $\ell \geq 0$ of the numbers $a^I_{n,\ell}$, and thus the exponential generating function for the number of singleton free set partitions of $[n]$ is

$$\begin{aligned} F(x) &= \sum_{n\geq 0} \#\Pi'_n \frac{x^n}{n!} = \sum_{n\geq 0}\sum_{\ell\geq 0} a^I_{n,\ell}\frac{x^n}{n!} = \sum_{\ell\geq 0}\sum_{n\geq 0} a^I_{n,\ell}\frac{x^n}{n!} \\ &= \sum_{\ell\geq 0} \frac{(e^x-1-x)^\ell}{\ell!} = \exp(e^x-1-x). \end{aligned} \tag{2.4}$$

A partition $\sigma \vdash [n]$ is *layered* if it is of the form $[1,i]/[i+1,j]/[j+1,k]/\cdots/[\ell+i,n]$. A partition σ is said to be a *matching* if $\#B \leq 2$, for all block B of σ. When the cardinality of each block is exactly 2 the partition is called a *perfect matching*. The characterization of the set partitions in $\Pi_n(\pi)$, for $\pi \in \Pi_3$, obtained by Sagan [9], will be used repeatedly, so we state it below.

Theorem 2.1 (Sagan)**.** *For $n \geq 1$,*

$$\begin{aligned} \Pi_n(1/2/3) &= \{\sigma \in \Pi_n : b(\sigma) \leq 2\}, \\ \Pi_n(123) &= \{\sigma \in \Pi_n : \sigma \textit{ is a matching}\}, \\ \Pi_n(13/2) &= \{\sigma \in \Pi_n : \sigma \textit{ is layered}\}. \end{aligned}$$

Given positive integers $i < m$, let π^i_m be the layered pattern

$$1/2/\cdots/i-1/i(i+1)/i+2/\cdots/m$$

in Π_m, where all blocks are singletons with the exception of $B_i = \{i, i+1\}$.

Theorem 2.2. *For $n \geq 2$,*

$$\begin{aligned} \Pi'_n(\pi^i_m) &= \{\sigma \in \Pi'_n : b(\sigma) \leq m-2\}, \\ F_{\pi^i_m}(x) &= \exp_{m-2}(\exp(x)-1-x). \end{aligned}$$

Proof: Since π^i_m has $m-1$ blocks, it is clear that if $b(\sigma) \leq m-2$ then σ avoids the pattern π^i_m. Reciprocally, let $\sigma \in \Pi'_n(\pi^i_m)$ and assume that $b(\sigma) \geq m-1$. Let $B_1, \ldots, B_{m-1}$ be $m-1$ blocks of σ, each of them with at least two elements, ordered by their least element: $B_j = \{a_j, \ldots\}$, with

$$a_1 < a_2 < \cdots < a_{m-1}.$$

Next, let $B'_i, \ldots, B'_{m-1}$ be the blocks $B_i, \ldots, B_{m-1}$ ordered by their largest element: $B'_j = \{\ldots, b_j\}$, with

$$b_i < b_{i+1} < \cdots < b_{m-1}.$$

Then,

$$a_1 < a_2 < \cdots < a_i < b_i < b_{i+1} < \cdots < b_{m-1}$$

and $a_1/a_2/\cdots/a_i\, b_i/b_{i+1}/\cdots/b_{m-1}$ is a copy of π_m^i in σ, a contradiction.

It follows that the number of partitions in $\Pi'_n(\pi_m^i)$ is the sum over all $0 \le \ell \le m-2$ of the number $a_{n,\ell}^I$ of partitions of $[n]$ with ℓ blocks, each of them with at most two elements:

$$\#\Pi'_n(\pi_m^i) = \sum_{\ell=0}^{m-2} a_{n,\ell}^I,$$

with $I = \{2, 3, \ldots\}$. Thus, we can use (2.2) to write

$$\begin{aligned} F_{\pi_m^i}(x) &= \sum_{n\ge 0} \#\Pi'_n(\pi_m^i)\frac{x^n}{n!} = \sum_{n\ge 0}\sum_{\ell=0}^{m-2} a_{n,\ell}^I \frac{x^n}{n!} = \sum_{\ell=0}^{m-2}\left(\sum_{n\ge 0} a_{n,\ell}^I \frac{x^n}{n!}\right) \\ &= \sum_{\ell=0}^{m-2} \frac{F_I(x)^\ell}{\ell!} = \exp_{m-2}(\exp(x) - 1 - x). \end{aligned}$$

■

Two patterns σ and π are said to be *Wilf-equivalent* [6], denoted by $\sigma \sim \pi$, if the number of elements of the sets $\Pi'_n(R)$ and $\Pi'_n(T)$ are the same for all $n \ge 1$. The last result shows that $\pi_m^i \sim \pi_m^j$, for $i, j < m$.

Corollary 2.3. *The patterns π_m^i, for $1 \le i \le m-1$, are Wilf-equivalent.*

Corollary 2.4. *For $n \ge 2$,*

$$\begin{aligned} &\Pi'_n(12/3) = \Pi'_n(1/23) = \{12\cdots n\}, \\ &F_{1/23}(x) = F_{12/3}(x) = e^x - x. \end{aligned}$$

Proof: It follows from the last results since $12\cdots n$ is the only partition in Π'_n with a single block. ■

Theorem 2.5. *For $n \ge 2$,*

$$\begin{aligned} &\Pi'_n(12\cdots m) = \{\sigma \in \Pi_n : 2 \le \#B \le m-1, \textit{ for all block } B \in \sigma\}, \\ &F_{12\cdots m}(x) = \exp(\exp_{m-1}(x) - 1 - x). \end{aligned}$$

Proof: The characterization of the elements of $\Pi'_n(12\cdots m)$ is clear, since a partition contains a copy of $12\cdots m$ if and only if it has a block with at least m elements. It follows that $\#\Pi'_n(12\cdots m)$ is the sum of the numbers $a^I_{n,\ell}$ of partitions of $[n]$ with $\ell \geq 0$ blocks with cardinalities in the set $I = [2, m-1]$. Again, we use (2.2) to write:

$$\begin{aligned} F_{12\cdots m}(x) &= \sum_{n\geq 0} \#\Pi'_n(12\cdots m)\frac{x^n}{n!} = \sum_{n\geq 0}\sum_{\ell\geq 0} a^I_{n,\ell}\frac{x^n}{n!} = \sum_{\ell\geq 0}\left(\sum_{n\geq 0} a^I_{n,\ell}\frac{x^n}{n!}\right) \\ &= \sum_{\ell\geq 0}\frac{F_I(x)^\ell}{\ell!} = \exp(\exp_{m-1}(x) - 1 - x). \end{aligned}$$

■

The *double factorial* of an odd positive integer $2i-1$ is defined as the product of all positive odd integers up to $2i-1$:

$$(2i-1)!! = (2i-1)(2i-3)\cdots 5\cdot 3\cdot 1.$$

Corollary 2.6. *For $n \geq 2$,*

$$\Pi'_n(123) = \{\sigma \in \Pi_n : \sigma \textit{ is a perfect matching}\},$$

$$\#\Pi'_n(123) = \begin{cases} (2k-1)!! & \textit{if } n = 2k \\ 0 & \textit{otherwise} \end{cases}.$$

Proof: The characterization of $\Pi'_n(123)$ is a consequence of the previous theorem, and it can also be deduced from Theorem 2.1. Moreover, we can write

$$F_{123}(x) = \sum_{n\geq 0}\frac{\left(\frac{x^2}{2!}\right)^n}{n!} = \sum_{n\geq 0}\frac{(2n)!}{2^n n!}\frac{x^{2n}}{(2n)!},$$

and since $\dfrac{(2n)!}{2^n n!} = (2n-1)!!$ the result follows. ■

Theorem 2.7. *For $n \geq 2$,*

$$\Pi'_n(1/2/\cdots/m) = \{\sigma \in \Pi'_n : b(\sigma) \leq m-1\},$$

$$F_{1/2/\cdots/m}(x) = \exp_{m-1}(\exp(x) - 1 - x).$$

Proof: The characterization of the partitions of $[n]$ that avoids the pattern $1/2/\cdots/m$ is clear from the definitions, and the generating function follows from (2.1), since:

$$F_{1/2/\cdots/m}(x) = \sum_{n\geq 0} \#\Pi'_n(1/2/\cdots/m)\frac{x^n}{n!} = \sum_{n\geq 0}\sum_{\ell=0}^{m-1} a^I_{n,\ell}\frac{x^n}{n!} = \sum_{\ell=0}^{m-1}\left(\sum_{n\geq 0} a^I_{n,\ell}\frac{x^n}{n!}\right)$$
$$= \sum_{\ell=0}^{m-1}\frac{F_I(x)^\ell}{\ell!} = \exp_{m-1}(\exp(x)-1-x),$$

where $a^I_{n,\ell}$ is the number of partitions of $[n]$ with $\ell \leq m-1$ blocks, and I is the set of all integers greater than, or equal to 2. ∎

Corollary 2.8. *We have*

$$\Pi'_n(1/2/3) = \{\sigma \in \Pi'_n : b(\sigma) \leq 2\},$$
$$\#\Pi'_n(1/2/3) = 2^{n-1} - n, \text{ for } n \geq 3,$$

with $\#\Pi'_0(1/2/3) = \#\Pi'_2(1/2/3) = 1$ *and* $\#\Pi'_1(1/2/3) = 0$.

Proof: From the generating function given in the last result, we have

$$F_{1/2/3}(x) = 1 + (e^x - 1 - x) + \frac{(e^x-1-x)^2}{2} = \frac{1}{2} + \frac{x^2}{2} + \frac{e^{2x}}{2} - xe^x$$
$$= 1 + \frac{x^2}{2} + \sum_{n\geq 1}(2^{n-1}-n)\frac{x^n}{n!},$$

and the result follows. ∎

The *Eulerian number* $e(n,m)$ is the number of permutations $p_1p_2\cdots p_n$ of $[n]$ with exactly m descents, that is, m places in which $p_j > p_{j+1}$, for $1 \leq j \leq n-1$. Let $E(n,m)$ be the set of all permutations of $[n]$ with exactly m descents.

Theorem 2.9. *There is a bijection between* $\Pi'_n(1/2/3)$ *and* $E(n-1,1)$, *for* $n \geq 1$.

Proof: Using the description of $\Pi'_n(1/2/3)$ as the partitions of Π'_n having one or two blocks, its cardinality $2^{n-1} - n$ for $n \geq 3$ can be obtain directly as follows. If $\sigma \in \Pi'_n$ has only one block then $\sigma = 12\cdots n$. Otherwise, $\sigma = B_1/B_2$, with

$$B_1 = \{1\} \cup S,$$

where $S \subset [2,n]$ has i elements, for some $1 \le i \le n-3$. Thus,

$$\#\Pi'_n(1/2/3) = 1 + \sum_{i=1}^{n-3} \binom{n-1}{i} = \sum_{i=0}^{n-1} \binom{n-1}{i} - n = 2^{n-1} - n.$$

On the other hand, a permutation $p = p_1 p_2 \cdots p_{n-1}$ of $[n-1]$ with exactly one descent must satisfy

$$p_1 < \cdots < p_k, \quad p_k > p_{k+1}, \quad p_{k+1} < \cdots < p_{n-1},$$

for some $1 \le k \le n-2$. Thus, to give such a permutation is to give a set $S = \{p_1, \ldots, p_k\}$ with k elements of $[n-1]$ such that $p_1 < \cdots < p_k$ and $p_{k+1} < \cdots < p_n$. There will be a descent at position k if and only if $S \neq \{1, \ldots, k\}$. We identify permutations in $E(n-1,1)$ with sets $S \subset [n-1]$ such that $S \neq [k]$. Therefore,

$$e(n-1,1) = \sum_{k=1}^{n-1} \left(\binom{n-1}{k} - 1 \right) = \left(\sum_{k=0}^{n-1} \binom{n-1}{k} \right) - n = 2^{n-1} - n.$$

We can now give an explicit bijection $\psi : E(n-1,1) \to \Pi'_n(1/2/3)$, for $n \ge 3$. Note that for $n = 1$ or 2 the result is trivial.

Let $S = \{p_1, \ldots, p_k\} \subset [n-1]$, $S \neq [k]$, with $p_1 < \cdots < p_k$. If $\#S \neq n-2$, we set

$$\psi(S) = \{1, p_1+1, \ldots, p_k+1\}/B,$$

where B is the complement of $\{1, p_1+1, \ldots, p_k+1\}$ in $[n]$, having $\#B \ge 2$. If $\#S = n-2$, then we must have $S = \{1, \ldots, \hat{i}, \ldots, n-1\}$, for some $i \in [n-2]$, where $\hat{i}$ means that the integer i is not in S. In this case, we put

$$\psi(S) = \begin{cases} \{1,2,\ldots,i\}/\{i+1,\ldots,n\}, & \text{if } i \neq 1 \\ \{1,2,\ldots,n\}, & \text{if } i = 1 \end{cases}.$$

From its construction, the partition $\psi(S)$ has one or two blocks, each with at least 2 elements. Moreover, note that the partition $\{1,2,\ldots,i\}/\{i+1,\ldots,n\}$ must be obtained via the map ψ from a uniquely determined set $S \subset [n-1]$ with $\#S = n-2$, for otherwise we would have $S = \{1, \ldots, i-1\}$, a contradiction. Henceforth, we can easily conclude that ψ is a bijection. ■

Denote by F_n the n-th Fibonacci number which is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \ge 2,$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$.

Theorem 2.10. *For $n \geq 1$,*

$$\Pi'_n(13/2) = \{\sigma \in \Pi'_n : \sigma \text{ is layered}\},$$
$$\#\Pi'_n(13/2) = F_{n-1}.$$

Proof: It is clear that if σ is layered then σ avoids the pattern 13/2. Reciprocally, let B_1 be the block of $\sigma \in \Pi'_n(13/2)$ having the integer 1, and let $i > 1$ be the largest integer of B_1. Note that if there is an integer $1 < j < i$ such that j is not in B_1, then $st(1i/j) = 13/2$. Thus, we must have $B_1 = [1, i]$. Iterating this process we find that σ is layered.

For the enumeration part, note that $\#\Pi'_1(13/2) = F_0 = 0$ and $\#\Pi'_2(13/2) = F_1 = 1$. We claim that the number of elements of $\Pi'_n(13/2)$ is equal to the sums of the cardinals of $\Pi'_{n-2}(13/2)$ and $\Pi'_{n-1}(13/2)$, for $n \geq 3$. Consider the map

$$\phi : \Pi'_{n-2}(13/2) \cup \Pi'_{n-1}(13/2) \longrightarrow \Pi'_n(13/2),$$

where the image of the singleton free layered partition σ of, respectively, $[n-2]$ or $[n-1]$ is obtained by adding, respectively, the block $\{n-1, n\}$ to σ, or by adding the integer n to the block containing the letter $n-1$. The map ϕ is a bijection, since if τ is a layered partition of $[n]$, then the block B containing n must also contain the integer $n-1$. Therefore, if $B = \{n-1, n\}$, then τ is the image of the layered partition of $[n-2]$ obtained by removing B from τ, and if $\#B \geq 3$, then it is the image of the layered partition of $[n-1]$ obtained from τ by removing the letter n. Thus, we find that $\#\Pi'_n(13/2) = \#\Pi'_{n-2}(13/2) + \#\Pi'_{n-1}(13/2)$ and the result follows. ■

π	$\Pi'_n(\pi)$	$\#\Pi'_n(\pi)$
12/3	$12\cdots n$	1
1/23	$12\cdots n$	1
1/2/3	partitions with at most 2 blocks	$2^{n-1} - n$
13/2	layered partitions	F_{n-1}
123	perfect matchings	$(2k-1)!!$ if $n = 2k$ 0 otherwise

TABLE 2.1. Singleton free partitions avoiding a 3-letter pattern

Corollary 2.11. *The number of layered set partitions of $[n]$ with at least one singleton is given by $2^{n-1} - F_{n-1}$.*

Proof: It follows from the previous result and the number 2^{n-1} of layered partitions of $[n]$ obtained by Sagan [9]. ■

We consider now the classification and enumeration of the set of singleton free partitions that avoid a set R of patterns of Π_3, with $\#R \geq 2$. Note that since $12/3 \sim 1/23$, if both patterns $12/3$ and $1/23$ are in R, then $\Pi'_n(R) = \Pi'_n(R \setminus \{1/23\})$. Therefore, without loss of generality we may consider only the patterns $12/3, 1/2/3, 13/2$ and 123. The following proposition is a consequence of Corollaries 2.4, 2.6, 2.8 and Theorem 2.10.

Proposition 2.12. *Let $R = \{12/3, \pi\} \subset \Pi_3$. Then, for $n \geq 3$*

$$\Pi'_n(R) = \begin{cases} \emptyset, & \text{if } \pi = 123 \\ \{12\cdots n\}, & \text{otherwise} \end{cases}.$$

It follows that $\Pi'_n(\Pi_3) = \emptyset$. The results for $\Pi'_n(R)$, with $\#R = 2$ or 3, are easy to prove, so we omit the proofs. Table 2.2 describes these sets and gives their enumeration for $n \geq 3$.

R	$\Pi'_n(R)$	$\#\Pi'_n(R)$
$\{12/3, \pi\}$	$\emptyset$ if $\pi = 123$ $\{12\cdots n\}$ if $\pi \neq 123$	0 if $\pi = 123$ 1 if $\pi \neq 123$
$\{123, 13/2\}$	$\{12/34/\cdots/(n-1)n\}$ if n even $\emptyset$ if n odd	1 if n even 0 if n odd
$\{123, 1/2/3\}$	$\emptyset$ if $n \neq 4$ $\{12/34, 13/24, 14/23\}$ if $n = 4$	0 if $n \neq 4$ 3 if $n = 4$
$\{13/2, 1/2/3\}$	$\{1\cdots i/(i+1)\cdots n : i \in [2, n-2]\} \cup \{12\cdots n\}$	$n-2$
$\{12/3, 13/2, 1/2/3\}$	$\{12\cdots n\}$	1
$\{12/3, 123, \pi\}$	$\emptyset$ for $\pi = 1/2/3$ or $\pi = 13/2$	0
$\{13/2, 123, 1/2/3\}$	$\{12/34\}$ if $n = 4$ $\emptyset$ if $n \neq 4$	1 if $n = 4$ 0 if $n \neq 4$

TABLE 2.2. Singleton free partitions with more than one restriction

3. Even and Odd Singleton Free Set Partitions

In this section we consider the number of even and odd singleton free set partitions that avoids a set R of patterns of Π_3. A partition $\sigma \vdash [n]$ with

$b(\sigma) = k$ has *sign*

$$sgn(\sigma) = (-1)^{n-k}.$$

Definition 1. A set partition σ of $[n]$ is *even* if $sgn(n) = 1$, and is *odd* if $sgn(n) = -1$ [2]. We denote by $E\Pi'_n$ (resp. $O\Pi'_n$) the set of all singleton free even (resp. odd) set partitions of $[n]$. Given $R \subset \Pi_3$, let $E\Pi'_n(R)$ (resp. $O\Pi'_n(R)$) be the set of all singleton free even (resp. odd) set partitions of $[n]$ that avoids the patterns in R.

The *complement* σ^c of a set partition $\sigma = B_1/B_2/\cdots/B_k \vdash [n]$, is the set partition $\sigma = B_1^c/B_2^c/\cdots/B_k^c$ where

$$B_i^c = \{n - a + 1 : a \in B_i\}.$$

As mentioned in [2], the sign of σ is the same as the sign of σ^c. Therefore, since $12/3 \sim 1/23$, we obtain the following lemma.

Lemma 3.1. *For* $n \geq 1$,

$$\#E\Pi'_n(12/3) = \#E\Pi'_n(1/23),$$
$$\#O\Pi'_n(12/3) = \#O\Pi'_n(1/23).$$

We start by considering single restrictions.

Theorem 3.2. *For* $n \geq 1$,

$$E\Pi'_n(12/3) = \begin{cases} \emptyset, & \textit{if } n \textit{ is even} \\ \{12\cdots n\}, & \textit{if } n \textit{ is odd} \end{cases},$$

and

$$O\Pi'_n(12/3) = \begin{cases} \emptyset, & \textit{if } n \textit{ is odd} \\ \{12\cdots n\}, & \textit{if } n \textit{ is even} \end{cases}.$$

Proof: By Corollary 2.4, the set $\Pi'_n(12/3)$ has only the one block partition $12\cdots n$, which will be even if n is odd, and will be odd otherwise. ■

Theorem 3.3. *For* $n \geq 1$,

$$E\Pi'_n(1/2/3) = \begin{cases} \{\sigma \in \Pi'_n : b(\sigma) = 2\}, & \textit{if } n \textit{ is even} \\ \{12\cdots n\}, & \textit{if } n \textit{ is odd} \end{cases},$$

$$\#E\Pi'_n(1/2/3) = \begin{cases} 2^{n-1} - n - 1, & \textit{if } n \textit{ is even} \\ 1, & \textit{if } n \textit{ is odd} \end{cases};$$

and

$$O\Pi'_n(1/2/3) = \begin{cases} \{\sigma \in \Pi'_n : b(\sigma) = 2\}, & \text{if } n \text{ is odd} \\ \{12\cdots n\}, & \text{if } n \text{ is even} \end{cases},$$

$$\#O\Pi'_n(1/2/3) = \begin{cases} 2^{n-1} - n - 1, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}.$$

Proof: By Corollary 2.8, the $2^{n-1} - n$ partitions of $[n]$ that avoids the pattern $1/2/3$ are the ones having one or two blocks. As in the previous result, the only partition $12\cdots n$ with one block is even if n is odd, and is odd otherwise. On the other hand, if σ is one of the $2^{n-1} - n - 1$ partitions of $[n]$ with two blocks, then it will have the same parity as n. Thus, the result holds. ■

Theorem 3.4. *If n is an odd integer then $E\Pi'_n(123) = O\Pi'_n(123) = \emptyset$.*
If $n = 2k \geq 1$, then

$$E\Pi'_n(123) = \Pi'_n(123) \text{ and } O\Pi'_n(123) = \emptyset, \text{ if } k \text{ is even}$$

and

$$O\Pi'_n(123) = \Pi'_n(123) \text{ and } E\Pi'_n(123) = \emptyset, \text{ if } k \text{ is odd.}$$

Proof: It follows from Corollary 2.6, since when $n = 2k$, all perfect matchings of $[n]$ have k blocks, and thus its parity is the same of that of k. ■

Theorem 3.5. *For $n \geq 1$,*

$$E\Pi'_n(13/2) = \{\sigma \in \Pi'_n : \sigma \text{ is layered and } b(\sigma) \text{ has the parity of } n\},$$

$$\#E\Pi'_n(13/2) = \frac{1}{2}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) - \frac{1}{2}\left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right),$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad \gamma = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \delta = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

are the roots of the equation $x^4 + 2x^3 + x^2 - 1 = 0$.

Proof: The description of the set $E\Pi'_n(13/2)$ follows from theorem 2.10 and the definitions. For the enumeration part, we start by noticing that

$$\#E\Pi'_n(13/2) = \#O\Pi'_{n-2}(13/2) + \#O\Pi'_{n-1}(13/2)$$

since, as in the proof of Theorem 2.10, any partition $\sigma \in \#E\Pi'_n(13/2)$ is uniquely obtained from a partition in $\Pi'_{n-2}(13/2)$, with parity different from

n, by adding the block $\{n-1, n\}$, or from a partition from $\Pi'_{n-1}(13/2)$, with parity different from n, by adding the integer n to the block having the letter $n-1$. Therefore, using Theorem 2.10 we can write

$$\begin{aligned}\#E\Pi'_n(13/2) &= \#O\Pi'_{n-2}(13/2) + \#O\Pi'_{n-1}(13/2)\\ &= \#\Pi'_{n-2}(13/2) - \#E\Pi'_{n-2}(13/2) + \#\Pi'_{n-1}(13/2) - \#E\Pi'_{n-1}(13/2)\\ &= F_{n-3} + F_{n-2} - \#E\Pi'_{n-2}(13/2) - \#E\Pi'_{n-1}(13/2).\end{aligned}$$

Thus, the sequence formed by the cardinalities $a_n := \#E\Pi'_n(13/2)$, for $n \geq 0$, satisfies the recurrence relation

$$a_n = F_{n-3} + F_{n-2} - a_{n-2} - a_{n-1}, \text{ for } n \geq 3 \tag{3.1}$$

with the initial conditions $a_0 = a_1 = a_2 = 0$.

Recalling that $F(x) = \dfrac{x}{1-x-x^2}$ is the generating functions for the Fibonacci numbers (see [5]), and setting $G(x) = \sum_{n\geq 0} a_n x^n$, from the recurrence (3.1) we obtain

$$\begin{aligned}G(x) &= \sum_{n\geq 3}\left(F_{n-3} + F_{n-2} - a_{n-2} - a_{n-1}\right) x^n\\ &= x^3 \sum_{n\geq 0} F_n x^n + x^2 \sum_{n\geq 1} F_n x^n - x^2 G(x) - xG(x)\\ &= x^2(x+1)F(x) - (x^2+x)G(x),\end{aligned}$$

that is, the generating function for the number of partitions in $E\Pi'_n(13/2)$ is

$$G(x) = \frac{x^2(x+1)}{(1-x-x^2)(1+x+x^2)}.$$

Let $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}, \gamma = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \delta = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ be the roots of the equation $(1-x-x^2)(1+x+x^2) = 0$. By the Binet formula [5], we have

$$\frac{x}{1-x-x^2} = \sum_{n\geq 0} \frac{\alpha^n - \beta^n}{\alpha - \beta} x^n.$$

In a similar way, we can write $1 + x + x^2 = (1-\gamma x)(1-\delta x)$, and thus

$$\frac{x}{1+x+x^2} = \frac{x}{(1-\gamma x)(1-\delta x)} = \frac{1}{\gamma - \delta}\left(\frac{1}{1-\gamma x} - \frac{1}{1-\delta x}\right) = \sum_{n\geq 0} \frac{\gamma^n - \delta^n}{\gamma - \delta} x^n.$$

Finally, noticing that

$$G(x) = \frac{1}{2}\left(\frac{x}{1-x-x^2}\right) - \frac{1}{2}\left(\frac{x}{1+x+x^2}\right),$$

we get the desired result. ■

Since the set $\Pi'_n(13/2)$ is the union of the disjoint sets $E\Pi'_n(13/2)$ and $O\Pi'_n(13/2)$, from the last theorem we get the analogous result for singleton free odd set partitions that avoids the pattern $13/2$.

Corollary 3.6. *For $n \geq 1$,*

$$O\Pi'_n(13/2) = \{\sigma \in \Pi'_n : \sigma \textit{ is layered and } b(\sigma) \textit{ has not the parity of } n\},$$

$$\#O\Pi'_n(13/2) = \frac{1}{2}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + \frac{1}{2}\left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right),$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}, \quad \gamma = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \delta = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

are the roots of the equation $x^4 + 2x^3 + x^2 - 1 = 0$.

Proof: If $H(x)$ is the generating function for the numbers $\#O\Pi'_n(13/2)$, then by the previous theorem,

$$H(x) = F(x) - G(x) = \frac{1}{2}\sum_{n\geq 0}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\gamma^n - \delta^n}{\gamma - \delta}\right)x^n,$$

and the result follows. ■

We consider next the description and enumeration of the sets $E\Pi'_n(R)$ and $O\Pi'_n(R)$ where $\#R \geq 2$ and $n \geq 2$. As before, by Lemma 3.1, we have $E\Pi'_n(R) = E\Pi'_n(R \setminus \{12/3\})$ and $O\Pi'_n(R) = O\Pi'_n(R \setminus \{12/3\})$, so we need to consider only the patterns $12/3, 1/2/3, 123$ and $13/2$. Tables 3.1 and 3.2 give the results for $E\Pi'_n(R)$ and $O\Pi'_n(R)$, $n \geq 2$. The proofs are direct consequences of the theorems above.

R	$E\Pi'_n(R)$	$\#E\Pi'_n(R)$
$\{12/3, 1/2/3\}$	$\emptyset$ if n is even $\{12\cdots n\}$ if n is odd	0 1
$\{12/3,123\}$	$\emptyset$	0
$\{12/3,13/2\}$	$\emptyset$ if n is even $\{12\cdots n\}$ if n is odd	0 1
$\{1/2/3,123\}$	$\emptyset$ if $n \neq 4$ $\{12/34, 13/24, 14/23\}$ if $n = 4$	0 3
$\{1/2/3,13/2\}$	$\{1\cdots i/(i+1)\cdots n : 2 \leq i \leq n-2\}$, n even $\{12\cdots n\}$ if n is odd	$n-3$ 1
$\{123,13/2\}$	$\{12/34/\cdots/(n-1)n\}$ if $n = 2k$ with k even $\emptyset$ otherwise	1 0
$\{12/3, 123, \pi\}$	$\emptyset$ for $\pi = 1/2/3$ or $\pi = 13/2$	0
$\{12/3, 1/2/3, 13/2\}$	$\emptyset$ if n is even $\{12\cdots n\}$ if n is odd	0 1
$\{123,1/2/3,13/2\}$	$\{12/34\}$ if $n = 4$ $\emptyset$ if $n \neq 4$	1 0
$\#R \geq 4$	$\emptyset$	0

Table 3.1. Singleton free even partitions with more than one restriction

4. P-recursion

A sequence $(a_n)_{n\geq 0}$ is said to be P-recursive (short for *polynomial recursive*) if there exist polynomials $p_0(x), p_1(x) \ldots, p_d(x)$ with $p_d(x) \neq 0$, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0,$$

for all $n \geq 0$. That is, $(a_n)_{n\geq 0}$ satisfies a homogeneous linear recurrence of finite degree with polynomial coefficients [11]. The above relation defines a_{n+d} in terms of the values of $a_n, a_{n+1}, \ldots, a_{n+d-1}$, provided $p_d(n) \neq 0$, and can be used to compute the sequence of values a_{n+d} with relatively low computational cost, for n large enough. Our objective in this section is to identify the sequences $\#\Pi'_n(\pi)$, $\#E\Pi'_n(\pi)$ and $\#O\Pi'_n(\pi)$, $n \geq 1$, for $\pi \vdash [3]$, which are P-recursive.

Closed related with P-recursive sequences is the notion of D-finite (short for *differentiably finite*) formal power series [10]. A power series $f(x)$ is D-finite if there exist finitely many polynomials $p_0(x), p_1(x), \ldots, p_m(x)$ with

R	$O\Pi'_n(R)$	$\#O\Pi'_n(R)$
$\{12/3, 1/2/3\}$	$\emptyset$ if n is odd $\{12\cdots n\}$ if n is even	0 1
{12/3,123}	$\emptyset$	0
{12/3,13/2}	$\emptyset$ if n is odd $\{12\cdots n\}$ if n is even	0 1
{1/2/3,123}	$\emptyset$	0
{1/2/3,13/2}	$\{1\cdots i/(i+1)\cdots n : 2 \leq i \leq n-2\}$, n odd $\{12\cdots n\}$ if n is even	$n-3$ 1
{123,13/2}	$\{12/34/\cdots/(n-1)n\}$ if $n=2k$ with k odd $\emptyset$ otherwise	1 0
T={12/3,1/2/3,13/2}	$\emptyset$ if n is odd $\{12\cdots n\}$ if n is even	0 1
$\#R \geq 3,\ R \neq T$	$\emptyset$	0

TABLE 3.2. Singleton free odd partitions with more than one restriction

$p_m(x) \neq 0$ such that

$$p_0(x)f(x) + p_1(x)f^{(1)}(x) + \cdots + p_m(x)f^{(m)}(x) = 0, \tag{4.1}$$

where $f^{(i)}(x) = d^i f/dx^i$.

An example of a D-finite function is $f(x) = e^x$, since $f(x) - f'(x) = 0$. Similarly, any linear combination of series of the form $x^m e^{ax}$ $(m \in \mathbb{N}, a \in \mathbb{R})$ is D-finite, since such series satisfy a linear homogeneous differential equation with constant coefficients.

The following result, proved by Stanley in [10], was also mentioned in Jungen [3].

Theorem 4.1. *A sequence $(a_n)_{n\geq 0}$ is P-recursive if and only if its ordinary generating function $f(x) = \sum_{n\geq 0} a_n x^n$ is D-finite.*

Sagan [9] proved the following analogous result for exponential generating functions.

Theorem 4.2. *A sequence $(a_n)_{n\geq 0}$ is P-recursive if and only if its exponential generating function $f(x) = \sum_{n\geq 0} a_n x^n/n!$ is D-finite.*

A formal power series is said to be *algebraic* if there exist polynomials $p_0(x), \ldots, p_d(x)$, not all zero, such that

$$p_0(x) + p_1(x)f(x) + \cdots + p_d(x)f(x)^d = 0. \tag{4.2}$$

The smallest positive integer d for which (4.2) hold is called the *degree* of $f(x)$. It is simple to see that an algebraic power series $f(x)$ has degree 1 if and only if $f(x)$ is rational. The following result asserts that all algebraic power series are D-finite (see [11]).

Theorem 4.3. *If $f(x)$ is an algebraic power series then $f(x)$ is D-finite*

The converse of this result is false, since, for instance, the power series $f(x) = e^x$ is D-finite but not algebraic.

We will also need the following result of Stanley [11].

Theorem 4.4. *If $f(x)$ and $g(x)$ are D-finite, then any linear combination $af(x) + bg(x)$ is also D-finite.*

If $f(x)$ is D-finite and $g(x)$ is algebraic with $g(0) = 0$, then the composition $f(g(x))$ is D-finite.

We start our analysis by showing that $\#\Pi'_n$, $n \geq 1$, do not form a P-recursive sequence.

Proposition 4.5. *The sequence $\#\Pi'_n$, $n \geq 1$, is not P-recursive.*

Proof: The proof follows essentially the same argument used by Sagan in [9] to show that $\#\Pi_n$ is not P-recursive. By contradiction, assume that the sequence $\#\Pi'_n$ is P-recursive. Then, its generating function

$$F(x) = e^{e^x - 1 - x},$$

determined in (2.4), must be D-finite by Theorem 4.2, and so it must satisfy equation (4.1) for some polynomials $p_0(x), p_1(x), \ldots, p_d(x)$. A simple induction shows that the i-th derivative of $F(x)$ can be written as

$$\frac{d^i}{dx^i} F(x) = F(x) \left(a_0^i + a_1^i e^x + a_2^i e^{2x} + \cdots + a_{i-1}^i e^{(i-1)x} + e^{ix} \right),$$

for some constants a_j^i, $j = 0, 1, \ldots, i-1$. Thus, taking the derivatives in equation (4.1) and dividing by $F(x)$, which is never zero, we get

$$q_0(x) + q_1(x) e^x + \cdots + q_d(x) e^{dx} = 0,$$

where

$$q_i(x) = p_i(x) + \sum_{k=i+1}^{d} a_i^k p_k(x).$$

Moreover, since the $p_i(x)$ are not all zero, the same is true for the $q_i(x)$. But this imply that e^x is algebraic, a contradiction. ■

Theorem 4.6. *For any $m \geq 1$, the following sequences are P-recursive, for $n \geq 1$:*

$$\#\Pi'_n(12\cdots m), \quad \#\Pi'_n(\pi^i_m), \quad \#\Pi'_n(1/2/\cdots/m).$$

Furthermore, for any $\pi \vdash [3]$, the sequences $\#\Pi'_n(\pi)$, $\#E\Pi'_n(\pi)$ and $\#O\Pi'_n(\pi)$, $n \geq 1$, are P-recursive.

Proof: The exponential generating function for the numbers $\#\Pi'_n(12\cdots m)$, $n \geq 1$, is given by $F_{12\cdots m}(x) = \exp(\exp_{m-1}(x) - 1 - x)$. We have already seen that $f(x) = e^x$ is D-finite, and $g(x) = \exp_{m-1}(x) - 1 - x$ is algebraic since it is a polynomial. Thus, by Theorem 4.4 the composition $f(g(x)) = F_{12\cdots m}(x)$ is D-finite.

The exponential generating functions $\exp_{m-2}(e^x - 1 - x)$ and $\exp_{m-1}(e^x - 1 - x)$, respectively, for the numbers $\#\Pi'_n(\pi^i_m)$ and $\#\Pi'_n(1/2/\cdots/m)$, $n \geq 1$, are D-finite since this functions are linear combinations of series of the form $x^m e^{ax}$, with $m \in \mathbb{N}$ and $a \in \mathbb{R}$, and thus satisfy a linear homogeneous differential equation with constant coefficients.

Finally, note that by the results of sections 2 and 3, the generating functions for $\#\Pi'_n(\pi)$, $\#E\Pi'_n(\pi)$ and $\#O\Pi'_n(\pi)$, for each $\pi \vdash [3]$, are either specifications of the functions above, or rational functions, and thus are D-finite.

Since the generating functions of all sequences considered are D-finite, we can use Theorems 4.1 and 4.2 to conclude that all these sequences are P-recursive. ■

5. Gray Codes

A Gray code for a class of combinatorial objects is a list of these objects so that the transition from one object in the list to its successor takes only a "small change" (see [12] for a comprehensive survey). The definition of "small change" depends on the particular class of objects. In our case, we define the distance $d(\pi, \omega)$ between two partitions π, ω of $[n]$ as the minimum number of letters that must be moved between blocks of π, possibly creating a new block, so that the resulting partition is ω.

If the maximum distance between any two consecutive elements of a Gray code is k, then we say that the Gray code has distance k.

In this section, we describe Gray codes with distance 2 for the sets $\Pi'_n(\pi)$, for $\pi = 1/2/3, 123, 13/2$. The remaining cases $\pi = 12/3$ and $1/23$ are trivial. We point out that 2 is the minimum possible distance for a Gray code for these sets. Except for $\pi = 123$, the partition $12\cdots n$ belongs to $\Pi'_n(\pi)$, and therefore, the distance between $12\cdots n$ and any other partition must be at least equal to 2. The set $\Pi'_{2n}(123)$ is formed by perfect matchings, and again in this case, 2 is the minimum distance between two elements of this set.

We start with the case $\Pi'_n(13/2)$, for which we need the following definitions.

Definition 2. Given a singleton free partition $\sigma = B_1/\cdots/B_t$ of $[n-j]$, $j = 1, 2$, define the partition σ^n of $[n]$ as

$$\sigma^n = \begin{cases} B_1/\cdots/B_t \cup \{n\}, & \text{if } j = 1 \\ B_1/\cdots/B_t/\{n-1, n\}, & \text{if } j = 2 \end{cases}.$$

Definition 3. Let $\sigma = B_1/\cdots/B_{t-1}/B_t$ and π be layered singleton free partitions of $[n]$. We say that σ and π forms a *good pair* if whenever $\#B_{t-1} \geq 3$ and $B_t = \{n-1, n\}$, then $B_{t-1} \cup \{n-1, n\}$ is not a block of π.

Lemma 5.1. *If σ, π is a good pair of $\Pi'_{n-j}(13/2)$ and $d(\sigma, \pi) \leq 2$ then σ^n, π^n is also good pair of $\Pi'_n(13/2)$ and $d(\sigma^n, \pi^n) \leq 2$, for $j = 1, 2$.*

Proof: If σ, π is a good pair, it follows from the definitions of good pair and σ^n that σ^n, π^n is also a good pair. Assume that $d(\sigma, \pi) \leq 2$. This means that one or two integers moved between blocks of σ to get ω, and the same is true for the partitions σ^n and π^n. Since σ^n and π^n are obtained from σ and π by inserting n is the last block, or by inserting the block $\{n-1, n\}$, the only non trivial situation to analyze is when $j = 1$ and the last block, say $B_t = \{n-2, n-1\}$, of σ vanishes in π. That is, $\sigma = B_1/\cdots/B_{t-1}/B_t$ and $\tau = B_1/\cdots/B_{t-1} \cup B_t$. In this case, we have $\sigma^n = B_1/\cdots/B_{t-1}/B_t \cup \{n\}$ and $\tau^n = B_1/\cdots/B_{t-1} \cup B_t \cup \{n\}$. But since σ and π are good pairs, we must have $B_{t-1} = \{n-4, n-3\}$, and therefore π^n is obtained from σ^n by moving the integers $n-4$ and $n-3$ to the last block. It follows that $d(\sigma^n, \pi^n) = 2$. ■

Note that if we drop the good pair condition in the last lemma, we may have layered singleton free partitions σ and π of $[n-1]$ with distance 2 such that the distance of σ^n and π^n is greater than 2. For instance, $d(123/45, 12345) = 2$ but $d(123/456, 123456) = 3$.

Theorem 5.2. *For each $n \geq 4$ there is a Gray code sequence with distance 2,*

$$\pi_1, \pi_2 \ldots, \pi_s,$$

for $\Pi'_n(13/2)$ such that any two consecutive elements are good pairs, $\pi_1 = 12\cdots n$ and $\pi_s = 12\cdots(n-2)/(n-1)n$.

Proof: The list $1234, 12/34$ is a good pair and forms a Gray code with distance 2 for $\Pi'_4(13/2)$. Assume the result for integers less than n, with $n > 4$, and let

$$\alpha_1, \ldots, \alpha_s \text{ and } \beta_1, \ldots, \beta_t$$

be Gray codes with distance 2 for $\Pi'_{n-2}(13/2)$ and $\Pi'_{n-1}(13/2)$, respectively, in the conditions of the theorem. Then

$$\begin{aligned}
\beta_1^n &= 12\cdots(n-1)n,\\
\beta_t^n &= 12\cdots(n-3)/(n-2)(n-1)n,\\
\alpha_1^n &= 12\cdots(n-2)/(n-1)n, \text{ and}\\
\alpha_s^n &= 12\cdots(n-4)/(n-3)(n-2)/(n-1)n.
\end{aligned}$$

Thus, β_t^n and α_s^n is a good pair with $d(\beta_t^n, \alpha_s^n) = 2$ and we may use Lemma 5.1 to conclude that any other two consecutive partitions of the sequence

$$\beta_1^n, \ldots, \beta_t^n, \alpha_s^n, \ldots, \alpha_1^n. \tag{5.1}$$

forms a good pair and have distance less than, or equal to 2. Moreover, from the construction used in the proof of Theorem 2.10, we find that this sequence is an exhaustive list of the elements of $\Pi'_n(13/2)$. This means that the list (5.1) is a Gray code with distance 2 for $\Pi'_n(13/2)$ in the conditions of the theorem. ■

$\Pi'_2(13/2)$	12
$\Pi'_3(13/2)$	123
$\Pi'_4(13/2)$	1234, 12/34
$\Pi'_5(13/2)$	12345, 12/345, 123/45
$\Pi'_6(13/2)$	123456, 12/3456, 123/456, 12/34/56, 1234/56
$\Pi'_7(13/2)$	1234567, 12/34567, 123/4567, 12/34/567, 1234/567, 123/45/67, 12/345/67, 12345/67
$\Pi'_8(13/2)$	12345678, 12/345678, 123/45678, 12/34/5678, 1234/5678, 123/45/678, 12/345/678, 12345/678, 1234/56/78, 12/34/56/78, 123/456/78, 12/3456/78, 123456/78

TABLE 5.1. Gray codes for $\Pi'_n(13/2)$, $n = 2, \ldots, 8$

Theorem 5.3. *For each $n \geq 4$ there is a Gray code sequence with distance 2 for $\Pi'_n(1/2/3)$ which starts with $12\cdots n$ and is followed by $1n/2\cdots(n-1)$.*

Proof: For $n = 4$, the list 1234, 14/23, 13/24, 12/34 is a Gray code with distance 2. By induction, assume that

$$\alpha_0, \alpha_1, \ldots, \alpha_t$$

is a Gray code sequence with distance 2 for $\Pi'_{n-1}(1/2/3)$, for some $n-1 \geq 4$, with $\alpha_0 = 12\cdots(n-1)$ and $\alpha_1 = 1(n-1)/23\cdots(n-2)$. Recalling that each partition in $\Pi'_{n-1}(1/2/3)$ has one or two blocks, given $\alpha = B_1/B_2 \in \Pi'_{n-1}(1/2/3)$ define

$$^{n}\alpha = B_1 \cup \{n\}/B_2 \quad \text{and} \quad \alpha^n = B_1/B_2 \cup \{n\}.$$

For each $i = 1, \ldots, n-1$, let $\beta_i = i\,n/1\cdots\widehat{i}\cdots(n-1)$, where $\widehat{i}$ means that the integer i is not in the block, and let L be the sequence of partitions in $\Pi'_n(1/2/3)$ defined by:

$$L = 12\cdots n,\ \beta_1,\ \beta_2,\ \ldots, \beta_{n-1}, \alpha_1^n,\ \alpha_2^n, \ldots, \alpha_t^n,\ {}^{n}\alpha_t, \ldots, {}^{n}\alpha_2,\ {}^{n}\alpha_1.$$

It is clear from the definitions that each consecutive partitions in L has distance 2. Moreover, note that by Corollary 2.8, the number of elements in L is

$$\begin{aligned} \#L &= 2\,(\#\Pi'_{n-1}(1/2/3) - 1) + n \\ &= 2\left(2^{n-2} - (n-1) - 1\right) + n \\ &= 2^{n-1} - n. \end{aligned}$$

That is, L is an exhaustive list of the elements in $\Pi'_n(1/2/3)$, and therefore is a Gray code sequence with distance 2 for $\Pi'_n(1/2/3)$. ■

$\Pi'_2(1/2/3)$	12
$\Pi'_3(1/2/3)$	123
$\Pi'_4(1/2/3)$	1234, 14/23, 24/13, 12/34
$\Pi'_5(1/2/3)$	12345, 15/234, 25/134, 35/124, 45/123, 14/235, 24/135, 12/345, 125/34, 245/13, 145/23
$\Pi'_6(1/2/3)$	123456, 16/2345, 26/1345, 36/1245, 46/1235, 56/1234, 15/2346, 25/1346, 35/1246, 45/1236, 14/2356, 24/1356, 12/3456, 125/346, 245/136, 145/236, 1456/23, 2456/13, 1256/34, 126/345, 246/135, 146/235, 456/123, 356/124, 256/134, 156/234

TABLE 5.2. Gray codes for $\Pi'_n(1/2/3)$, $n = 2, 3, 4, 5, 6$

In the next theorem we construct a Gray code with distance 2 for the perfect matchings of $[2k]$, that is, for the set $\Pi'_{2k}(123)$, $k \geq 2$. The next lemma, whose proof is clear from the definitions, characterize perfect matchings with distance 2.

Lemma 5.4. *Two perfect matchings of $[2k]$ have distance 2 if and only if all but two of their blocks are equal.*

Let $\alpha = B_1/\cdots/B_{k-1}$ be a perfect matching of $[n]$ with $n = 2(k-1)$, written in standard form. For each $j = 1, \ldots, k-1$ let $B_j = \{a, b\}$ with $a < b$, and define

$$\begin{aligned} \alpha^0 &= B_1/\cdots/B_j/\cdots/B_{k-1}/\{n-1, n\}, \\ \alpha_i^{j1} &= B_1/\cdots/B_{j-1}/\{a, n\}/B_{j+1}/\cdots/B_{k-1}/\{b, n-1\}, \text{ and} \\ \alpha_i^{j2} &= B_1/\cdots/B_{j-1}/\{b, n\}/B_{j+1}/\cdots/B_{k-1}/\{a, n-1\}. \end{aligned}$$

Lemma 5.5. *Let α and α_1 be two perfect matchings of $[2(k-1)]$ with distance 2, and $j \in [k-1]$. Then,*

(1) $d(\alpha^0, \alpha_1^0) = 2$;
(2) $d(\alpha^0, \alpha^{j\ell}) = 2$, *for* $\ell = 1, 2$;
(3) $d(\alpha^{j1}, \alpha^{j2}) = 2$;
(4) $d(\alpha^{j\ell}, \alpha_1^{j\ell}) = 2$ *for* $\ell = 1, 2$.

Proof: The first three conditions are clear since all but two of the blocks of each of the pairs of partitions α^0, α_1^0, $\alpha^0, \alpha^{j\ell}$ and α^{j1}, α^{j2} are equal.

Let $\alpha = B_1/\cdots/B_{k-1}$ and $\alpha_1 = B'_1/\cdots/B'_{k-1}$ be perfect matchings of $[2(k-1)]$, written in standard form and such that $d(\alpha, \alpha_1) = 2$. Let $n = 2(k-1)$, $j \in [k-1]$ and assume that $B_j = \{a, b\}$, with $a < b$, so that

$$\alpha^{j1} = B/\{a, n\}/\{c, d\}/\{b, n-1\} \quad \text{and} \quad \alpha^{j2} = B/\{b, n\}/\{c, d\}/\{a, n-1\}, \tag{5.2}$$

where $B = B_1/\cdots/\widehat{B}_j/\cdots/\widehat{B}_q/\cdots/B_{k-1}$. Since the distance between α and α_1 is 2, there must be a block $B_q = \{c, d\}$ of α, with $c < d$, and two integers $j', q' \in [k-1]$ such that $B'_\ell = B_\ell$ for $\ell \neq j', q'$, and either

$$B'_{j'} = \{a, c\} \text{ and } B'_{q'} = \{b, d\} \quad \text{or} \quad B'_{j'} = \{a, d\} \text{ and } B'_{q'} = \{b, c\}.$$

Now, if $B_j = B'_j$, then it is clear that $d(\alpha^{j\ell}, \alpha_1^{j\ell}) = 2$ since all but two of the blocks of these partitions are equal, for $\ell = 1, 2$. So, assume that $B_j \neq B'_j$. We have two cases to consider: $a < c$ or $c < a$. We consider only the case

$a < c$, the other case is analogous. Then, we have $j < q$ and $j' < q'$, and this implies that

$$B'_j = B'_{j'} = \{a, c\} \text{ and } B'_{q'} = \{b, d\} \quad \text{or} \quad B'_j = B'_{j'} = \{a, d\} \text{ and } B'_{q'} = \{b, c\}.$$

In the first case we have

$$\alpha_1^{j1} = B/\{a, n\}/\{b, d\}/\{c, n-1\} \text{ and } \alpha_1^{j2} = B/\{c, n\}/\{b, d\}/\{a, n-1\},$$

and in the second

$$\alpha_1^{j1} = B/\{a, n\}/\{b, c\}/\{d, n-1\} \text{ and } \alpha_1^{j2} = B/\{d, n\}/\{b, c\}/\{a, n-1\},$$

In both cases, comparing the expressions of $\alpha^{j\ell}$ given in (5.2) with that of $\alpha_1^{j\ell}$, for $\ell = 1, 2$, we conclude that their distance is 2. ■

Theorem 5.6. *For each integer $k \geq 1$, there is a Gray code sequence for $\Pi'_{2k}(123)$ with distance 2.*

Proof: The proof is by induction on $k \geq 1$. For $k = 1$ and $k = 2$, the lists 12 and 12/34, 13/24, 14/23 are Gray codes with distance 2. Assume the result for $k - 1 \geq 2$, and let

$$L_{k-1} = \alpha_1, \alpha_2, \ldots, \alpha_s,$$

be a Gray code sequence for $\Pi'_{2(k-1)}(123)$ with distance 2, where $s = (2k-3)!!$ by Corollary 2.6.

For each $i = 1, \ldots, k-1$, let R_i be the list off all $2s$ partitions $\alpha_j^{i\ell}$, $j = 1, \ldots, s$ and $\ell = 1, 2$, starting with α_i^{i1} and ending in α_{i+1}^{i1}, defined by:

$$R_i = \alpha_i^{i1},\ \alpha_{i-1}^{i1}, \ldots,\ \alpha_1^{i1},\ \alpha_1^{i2},\ \alpha_2^{i2}, \ldots,\ \alpha_s^{i2},\ \alpha_s^{i1},\ \alpha_{s-1}^{i1}, \ldots, \alpha_{i+1}^{i1}.$$

Finally, let

$$L_k = \alpha_1^0,\ R_1,\ \alpha_2^0\ R_2, \ldots, \alpha_{k-1}^0,\ R_{k-1},\ \alpha_k^0,\ \alpha_{k+1}^0, \ldots, \alpha_s^0.$$

By its construction, all partitions in L_k are perfect matchings and, by Lemma 5.5, any two consecutive partitions in L_k are distinct and have distance 2. Moreover, the list L_k exhaust all elements of $\pi_{2k}(123)$, since its cardinal is given by

$$\begin{aligned}
\#L_k &= s + (k-1)2s \\
&= (2k-3)!! + (k-1)2\,((2k-3)!!) \\
&= (1 + 2k - 2)((2k-3)!!) \\
&= (2k-1)!!
\end{aligned}$$

Therefore, L_k is a Gray code with distance 2 for $\Pi'_{2k}(123)$. ■

$\Pi'_2(123)$	12
$\Pi'_4(123)$	12/34, 13/24, 14/23
$\Pi'_6(123)$	12/34/56, 16/34/25, 26/34/15, 36/24/15, 46/23/15, 16/23/45, 16/24/35, 13/24/56, 13/26/45, 12/36/45, 12/46/35, 13/46/25, 14/36/25, 14/26/35, 14/23/45

TABLE 5.3. Gray codes for $\Pi'_n(123)$, $n = 2, 4, 6$

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