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LIKELIHOOD RATIO COMPARISONS AMONG SPACINGS RELATED TO BOTH ONE OR TWO SAMPLES

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ABSTRACT: In this paper, we investigate some stochastic comparisons in terms of likelihood ratio ordering between spacings from independent random variables exponentially distributed with different scale parameters. We partially solve some open problems in Wen et al. [16] for a one-sample problem and in Hu et al. [5] for a two-sample problem. Specifically, we prove that the second spacing is always smaller than the third spacing in terms of the likelihood ratio order and we provide the ordering among all spacings in the case $n = 4$. In the two-sample case, we establish comparisons between the second spacings related to each sample under certain conditions.

KEYWORDS: stochastic comparisons, heterogeneous random variables, second spacing, multiple-outlier models.
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1. Introduction

Given a set of independent random variables, $X_1, X_2, \ldots, X_n$, let the order statistics of these variables be $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. Then, the random variables

$$D_{i:n} = X_{i:n} - X_{i-1:n},$$

for $i = 1, \ldots, n$, with $X_{0:n} \equiv 0$, are called spacings.

Spacings are of great interest in many areas. In particular, in auction theory, the second and the last spacings represent reverse auction in the second-price business auction and auction rent's in buyer's auction, respectively (see Xu and Li [18]). In the reliability context, they correspond to times elapsed between successive failures of components in a system. In addition, there

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are many goodness-of-fit test based on spacings (see, e.g., Balakrishnan and Rao [1, 2]) very useful in the context of life testing.

Many researchers have investigated stochastic relations in terms of stochastic orderings between spacings of a random sample from independent and identically distributed random variables. Some early references for this case are Barlow and Proschan [3], Pledger and Proschan [12] and Kochar and Kirmani [7]. The corresponding problem in single-outlier exponential model has been studied by Khaledi and Kochar [6]. This topic has also been studied by Wen et al. [16] and Xu et al. [17] in the multiple-outlier exponential model. When observations are heterogeneous, there are few references due to the complicated distribution form of the spacings. From Kochar and Korwar [8], we know that the survival function of $D_{2:n}$ is Schur convex in $(\lambda_1, \ldots, \lambda_n)$ and that the hazard rate of $D_{2:2}$ is Schur concave in (λ_1, λ_2). Note that for $n \geq 3$, the hazard rate of $D_{2:n}$ is not Schur concave. Wen et al. [16] conjectured that $D_{i:n} \leq_{lr} D_{i+1:n}$ for $i = 1, \ldots, n-1$. Hu et al. [4] proved that $D_{1:n} \leq_{lr} D_{2:n}$ and $D_{2:3} \leq_{lr} D_{3:3}$ for all λ_i's; and if $\lambda_{n+1} \geq \lambda_i$, $i = 1, \ldots, n$, then $D_{2:n+1} \leq_{lr} D_{2:n}$.

For two samples, Kochar and Rojo [9] and Kochar an Xu [10] established condition for different stochastic orderings among spacings when one of those samples is from heterogeneous exponential random variables and the other one is from homogeneous exponential random variables. The case in which both samples are from heterogeneous exponential random variables is investigated in Torrado and Lillo [15].

In this work, we investigate stochastic order relations among successive spacings from a sample and also between spacings from two samples, in both cases we consider that the observations are independent but not identically distributed. In particular, for the one sample problem, we partially solve open problems in Wen et al. [16]. When spacings are from two heterogeneous samples, we stochastically compare second spacings in the sense of the likelihood ratio order by solving open problems in Hu et al. [5].

The rest of this article is organized as follows. We recall the definition of likelihood ratio order, as well as, of the density function of spacings in Section 2. Also in this section, we give some useful lemmas used in the sequel. In Section 3, we present some advances on the conjecture in [16]. In particular, we prove this conjecture for $n = 4$ and show that the second and third spacings for any n are ordered according to the likelihood ratio order.

In addition, we solve in Section 4 an open problem in [5] for the second spacing. Finally, Section 5 makes some concluding remarks.

2. Preliminaries

First, let us introduce some notations and definitions. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $n > 1$ a vector with positive components, then

$$S(\mathbf{x}) = \sum_{i=1}^{n} x_i\,, \tag{2.1}$$

is the first elementary symmetrical function of the positive $x_1, x_2, \ldots, x_n$.

For two random variables X and Y with densities f_X and f_Y respectively, X is said to be smaller than Y in the likelihood ratio order, denoted by $X \leq_{lr} Y$, if $f_Y(t)/f_X(t)$ is increasing in t. It is known that the likelihood ratio order implies both the hazard rate and the reversed hazard rate orders (see, e.g., [13]).

For heterogeneous but independent exponential random variables, Kochar and Korwar [8] proved that, for $i \in \{2, \ldots, n\}$, the distribution of D_i is a mixture of independent exponential random variables. Following Torrado et al. [14] the density function of D_i can be written as

$$f_i(t) = \sum_{j=1}^{M_i} \Delta(\beta^i_{m_j}, n)\beta^{(i)}_{m_j} e^{-t\beta^{(i)}_{m_j}}, \tag{2.2}$$

with $M_i = \binom{n}{n-i+1}$,

$$\beta^{(i)}_{m_j} = S(\lambda_{m_j}), \tag{2.3}$$

where $S(\cdot)$ is defined as in (2.1) and m_j indicates a group of indices of size $n - i + 1$, and

$$\Delta(\beta^{(i)}_{m_j}, n) \;=\; \sum_{\mathbf{r}_{i-1,m_j}} \left(\prod_{k \in H_{m_j}} \lambda_k \right) \left[\prod_{\ell=1}^{i-1} \left\{ \sum_{\substack{u=\ell \\ r_u \in H_{m_j}}}^{i-1} \lambda_{r_u} + \beta^{(i)}_{m_j} \right\} \right]^{-1}, \tag{2.4}$$

where $H_{m_j} = \{1, \ldots, n\} - m_j$ and the outer summation is being taken over all permutations of the elements of H_{m_j}.

Before proceeding to our main results, we recall four lemmas, which will be used repeatedly in the following sections.

Lemma 2.1 (Lemma 3.1, in Kochar and Korwar [8]). *Let $\Delta(\beta_{m_j}^{(i)}, n)$ be as defined in (2.4). Suppose that m_1 and m_2 are two subsets of $\{1, \ldots, n\}$ of size $n-i+1$ $(1 < i \leq n)$ and having all but one element in common. Denote the uncommon element in m_1 by a_1 and that in m_2 by a_2. Then,*

$$\lambda_{a_1}\Delta(\beta_{m_1}^{(i)}, n) \geq \lambda_{a_2}\Delta(\beta_{m_2}^{(i)}, n), \qquad \text{if} \quad \lambda_{a_2} \geq \lambda_{a_1}.$$

Lemma 2.2 (Chebyshev's sum inequality, Theorem 1, in Mitrinovic [11]). *Let $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$ be two decreasing sequences of real numbers. Then*

$$n \sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

Lemma 2.3 (Lemma A.1, in Torrado et al. [14]). *Let $\Delta(\beta_{m_j}^{(i)}, n)$ be as in (2.4) and $\beta_{m_j}^{(i)}$ as in (2.3). Then,*

$$\Delta(\beta_{(3,4)}^{(3)}, 4)\Delta(\beta_h^{(4)}, 4) \geq \Delta(\beta_{(h,4)}^{(3)}, 4)\Delta(\beta_3^{(4)}, 4) \geq \Delta(\beta_{(h,3)}^{(3)}, 4)\Delta(\beta_4^{(4)}, 4),$$

for $h = 1, 2$.

Lemma 2.4 (Lemma A.2, in Torrado et al. [14]). *Under the same assumptions as those in Lemma 2.3*

(a) $\Delta(\beta_{(2,h)}^{(3)}, 4)\Delta(\beta_1^{(4)}, 4) \geq \Delta(\beta_{(1,2)}^{(3)}, 4)\Delta(\beta_h^{(4)}, 4)$,
(b) $\Delta(\beta_{(1,h)}^{(3)}, 4)\Delta(\beta_2^{(4)}, 4) \geq \Delta(\beta_{(1,2)}^{(3)}, 4)\Delta(\beta_h^{(4)}, 4)$,

for $h = 3, 4$.

Remark that in Lemmas 2.3 and 2.4, from (2.1), we get that

$$\beta_j^{(4)} = S(\lambda_j) = \lambda_j \quad \text{and} \quad \beta_{(j,\ell)}^{(3)} = S(\lambda_j, \lambda_\ell) = \lambda_j + \lambda_\ell.$$

3. Comparisons between spacings related to a sample

Observing equation (2.2), note that $D_{i:n} \leq_{lr} D_{i+1:n}$ if and only if

$$\frac{f_{i+1}(t)}{f_i(t)} = \frac{\sum_{j=1}^{M_{i+1}} \Delta(\beta_{m_j}^{(i+1)}, n)\beta_{m_j}^{(i+1)} e^{-t\beta_{m_j}^{(i+1)}}}{\sum_{j=1}^{M_i} \Delta(\beta_{m_j}^{(i)}, n)\beta_{m_j}^{(i)} e^{-t\beta_{m_j}^{(i)}}},$$

is increasing in t. Differentiating this equation with respect to t we have to prove

$$\sum_{j=1}^{M_{i+1}}\sum_{k=1}^{M_i}\Delta(\beta_{m_k}^{(i)},n)\Delta(\beta_{m_j}^{(i+1)},n)\beta_{m_k}^{(i)}\beta_{m_j}^{(i+1)}e^{-t\left(\beta_{m_k}^{(i)}+\beta_{m_j}^{(i+1)}\right)}\left(\beta_{m_k}^{(i)}-\beta_{m_j}^{(i+1)}\right)\geq 0. \tag{3.1}$$

Note that if $\left(\beta_{m_k}^{(i)}-\beta_{m_j}^{(i+1)}\right)$ is positive for all m_k and m_j, then the Eq.(3.1) holds. Throughout this paper we suppose without loss of generality that the λ_i's are in increasing order.

Wen et al [16] conjectured that successive spacings from heterogeneous exponential random variables are increasing in likelihood ratio ordering. Next, we show that the second and the third simple spacings are ordered according to likelihood ratio ordering for any n, but first we need to prove the following result.

Lemma 3.1. *Let $\beta_{-j}^{(2)}=S(\boldsymbol{\lambda})-\lambda_j$ and $\beta_{(-j,-k)}^{(3)}=S(\boldsymbol{\lambda})-\lambda_j-\lambda_k$ be for $1\leq j<k<\ell\leq n$, where $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$. Then,*

$$\beta_{-j}^{(2)}\beta_{(-k,-\ell)}^{(3)}+\beta_{-k}^{(2)}\beta_{(-j,-\ell)}^{(3)}\geq\beta_{-\ell}^{(2)}\beta_{(-j,-k)}^{(3)}.$$

Proof: Note that,

$$\begin{aligned}
\beta_{-j}^{(2)}\beta_{(-k,-\ell)}^{(3)} &= S^2(\boldsymbol{\lambda})-S(\boldsymbol{\lambda})\left(\lambda_j+\lambda_k+\lambda_\ell\right)+\lambda_j(\lambda_k+\lambda_\ell),\\
\beta_{-k}^{(2)}\beta_{(-j,-\ell)}^{(3)} &= S^2(\boldsymbol{\lambda})-S(\boldsymbol{\lambda})\left(\lambda_j+\lambda_k+\lambda_\ell\right)+\lambda_k(\lambda_j+\lambda_\ell),\\
\beta_{-\ell}^{(2)}\beta_{(-j,-k)}^{(3)} &= S^2(\boldsymbol{\lambda})-S(\boldsymbol{\lambda})\left(\lambda_j+\lambda_k+\lambda_\ell\right)+\lambda_\ell(\lambda_k+\lambda_j),
\end{aligned}$$

where $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$. Then

$$\beta_{-j}^{(2)}\beta_{(-k,-\ell)}^{(3)}+\beta_{-k}^{(2)}\beta_{(-j,-\ell)}^{(3)}-\beta_{-\ell}^{(2)}\beta_{(-j,-k)}^{(3)}=S(\boldsymbol{\lambda})^2-S(\boldsymbol{\lambda})\left(\lambda_j+\lambda_k+\lambda_\ell\right)+2\lambda_j\lambda_k\geq 0,$$

since $S(\boldsymbol{\lambda})-\lambda_j-\lambda_\ell-\lambda_k\geq 0$. ■

Theorem 3.2. *Let $X_1,\ldots,X_n$ be independent exponential random variables such that X_i has hazard rate λ_i for $i=1,\ldots,n$, then*

$$D_{2:n}\leq_{lr}D_{3:n}.$$

Proof: We have to show that (3.1) holds. Note that, for $j<k<\ell$,

$$\beta_{-k}^{(2)}-\beta_{(-j,-\ell)}^{(3)}=(\lambda_j+\lambda_\ell)-\lambda_k,$$

since $\beta^{(2)}_{-k} = S(\boldsymbol{\lambda})-\lambda_k$ and $\beta^{(3)}_{(-j,-\ell)} = S(\boldsymbol{\lambda})-\lambda_j-\lambda_\ell$. Let us denote $a_{u,1} = \beta^{(2)}_{-j} - \beta^{(3)}_{(-k,-\ell)}$, $a_{u,2} = \beta^{(2)}_{-k} - \beta^{(3)}_{(-j,-\ell)}$ and $a_{u,3} = \beta^{(2)}_{-\ell} - \beta^{(3)}_{(-j,-k)}$, for $u = 1, \ldots, \binom{n}{3}$. It is easy to see that $a_{u,1}, a_{u,2} \geq 0$ and $a_{u,3}$ can be positive or negative. Moreover, these elements are ordered as follows:

$$a_{u,1} = (\lambda_k + \lambda_\ell) - \lambda_j \geq a_{u,2} = (\lambda_j + \lambda_\ell) - \lambda_k \geq a_{u,3} = (\lambda_j + \lambda_k) - \lambda_\ell.$$

Since

$$\beta^{(2)}_{-j} + \beta^{(3)}_{(-k,-\ell)} = \beta^{(2)}_{-k} + \beta^{(3)}_{(-j,-\ell)} = \beta^{(2)}_{-\ell} + \beta^{(3)}_{(-j,-k)} = 2S(\boldsymbol{\lambda}) - \lambda_j - \lambda_k - \lambda_\ell,$$

then $e^{-t\left(\beta^{(2)}_{-j}+\beta^{(3)}_{(-k,-\ell)}\right)} = e^{-t\left(\beta^{(2)}_{-k}+\beta^{(3)}_{(-j,-\ell)}\right)} = e^{-t\left(\beta^{(2)}_{-\ell}+\beta^{(3)}_{(-j,-k)}\right)}$. Hence, we have to prove

$$B_u = a_{u,1}\, b_{u,1}\beta^{(2)}_{-j}\beta^{(3)}_{(-k,-\ell)} + a_{u,2}\, b_{u,2}\beta^{(2)}_{-k}\beta^{(3)}_{(-j,-\ell)} + a_{u,3}\, b_{u,3}\beta^{(2)}_{-\ell}\beta^{(3)}_{(-j,-k)} \geq 0,$$

where

$$\begin{aligned} b_{u,1} &= \Delta(\beta^{(2)}_{-j}, n)\Delta(\beta^{(3)}_{(-k,-\ell)}, n), \\ b_{u,2} &= \Delta(\beta^{(2)}_{-k}, n)\Delta(\beta^{(3)}_{(-j,-\ell)}, n), \\ b_{u,3} &= \Delta(\beta^{(2)}_{-\ell}, n)\Delta(\beta^{(3)}_{(-j,-k)}, n), \end{aligned}$$

for $u = 1, \ldots, \binom{n}{3}$. By Lemma 2.1, we get $b_{u,1} \geq b_{u,2} \geq b_{u,3}$, then,

$$B_u \geq a_{u,2}\, b_{u,2}\left(\beta^{(2)}_{-j}\beta^{(3)}_{(-k,-\ell)} + \beta^{(2)}_{-k}\beta^{(3)}_{(-j,-\ell)}\right) + a_{u,3}\, b_{u,3}\beta^{(2)}_{-\ell}\beta^{(3)}_{(-j,-k)} = \sum_{h=2}^{3} a_{u,h}c_{u,h}\ ,$$

where

$$\begin{aligned} c_{u,2} &= b_{u,2}\left(\beta^{(2)}_{-j}\beta^{(3)}_{(-k,-\ell)} + \beta^{(2)}_{-k}\beta^{(3)}_{(-j,-\ell)}\right) \text{ and} \\ c_{u,3} &= b_{u,3}\beta^{(2)}_{-\ell}\beta^{(3)}_{(-j,-k)}. \end{aligned}$$

It follows by Lemmas 2.2 and 3.1 that

$$B_u \geq \sum_{h=2}^{3} a_{u,h}c_{u,h} \geq \frac{1}{2}\left(\sum_{h=2}^{3} a_{u,h}\right)\left(\sum_{h=2}^{3} c_{u,h}\right) \geq 0,$$

for $u = 1, \ldots, \binom{n}{3}$. From all these inequalities, the required result follows immediately. ∎

Hu et al [4] proved for $n = 3$ the conjecture in [16], that is

$$D_{1:3} \leq_{lr} D_{2:3} \leq_{lr} D_{3:3}.$$

In the following result, we extend the result in [4] to $n = 4$. Hu et al [4] also proved that $D_{1:n} \leq_{lr} D_{2:n}$ for any n, and by Theorem 3.2 we know that $D_{2:n} \leq_{lr} D_{3:n}$ for any n, so we have to show that $D_{3:4} \leq_{lr} D_{4:4}$.

Theorem 3.3. *Let $X_1, \ldots, X_4$ be independent exponential random variables such that X_i has hazard rate λ_i for $i = 1, \ldots, 4$, then*

$$D_{3:4} \leq_{lr} D_{4:4}.$$

Proof: We have to show that 3.1 holds. Here, the matrix of $\beta^{(3)}_{m_k} - \beta^{(4)}_{m_j}$ is

$$\begin{pmatrix} \lambda_3+\lambda_4-\lambda_1 & \lambda_3+\lambda_4-\lambda_2 & \lambda_4 & \lambda_3 \\ \lambda_2+\lambda_4-\lambda_1 & \lambda_4 & \lambda_2+\lambda_4-\lambda_3 & \lambda_2 \\ \lambda_2+\lambda_3-\lambda_1 & \lambda_3 & \lambda_2 & \lambda_2+\lambda_3-\lambda_4 \\ \lambda_4 & \lambda_1+\lambda_4-\lambda_2 & \lambda_1+\lambda_4-\lambda_3 & \lambda_1 \\ \lambda_3 & \lambda_1+\lambda_3-\lambda_2 & \lambda_1 & \lambda_1+\lambda_3-\lambda_4 \\ \lambda_2 & \lambda_1 & \lambda_1+\lambda_2-\lambda_3 & \lambda_1+\lambda_2-\lambda_4 \end{pmatrix} \tag{3.2}$$

To simplify the notation, we define $\beta^{(3)}_{(j,k)} = \lambda_j + \lambda_k$ and $\beta^{(4)}_j = \lambda_j$. It is easy to check that there are only four negative coefficients $a_{u,3} = \lambda_j + \lambda_k - \lambda_\ell$ for $j < k < \ell$ and $u \notin \{j, k, \ell\}$. We can consider the terms $a_{u,1} = \lambda_k + \lambda_\ell - \lambda_j \geq a_{u,2} = \lambda_j + \lambda_\ell - \lambda_k \geq 0$ for $u = 1, \ldots, 4$. Notice that $\exp\left\{-t(\beta^{(3)}_{(k,\ell)} + \beta^{(4)}_j)\right\} = \exp\left\{-t(\beta^{(3)}_{(j,\ell)} + \beta^{(4)}_k)\right\} = \exp\left\{-t(\beta^{(3)}_{(j,k)} + \beta^{(4)}_\ell)\right\}$. Hence, we have to prove

$$\begin{aligned} B_u &= \Delta(\beta^{(3)}_{(k,\ell)}, 4)\Delta(\beta^{(4)}_j, 4)\beta^{(3)}_{(k,\ell)}\beta^{(4)}_j (\lambda_k + \lambda_\ell - \lambda_j) \\ &\quad + \Delta(\beta^{(3)}_{(j,\ell)}, 4)\Delta(\beta^{(4)}_k, 4)\beta^{(3)}_{(j,\ell)}\beta^{(4)}_k (\lambda_j + \lambda_\ell - \lambda_k) \\ &\quad + \Delta(\beta^{(3)}_{(j,k)}, 4)\Delta(\beta^{(4)}_\ell, 4)\beta^{(3)}_{(j,k)}\beta^{(4)}_\ell (\lambda_j + \lambda_k - \lambda_\ell) \geq 0, \end{aligned} \tag{3.3}$$

where $u \notin \{j, k, \ell\}$.

Now, if $u = 1$ or 2, using Lemma 2.3, we find that

$$\Delta(\beta^{(3)}_{(3,4)}, 4)\Delta(\beta^{(4)}_{3-u}, 4) \geq \Delta(\beta^{(3)}_{(3-u,4)}, 4)\Delta(\beta^{(4)}_3, 4) \geq \Delta(\beta^{(3)}_{(3-u,3)}, 4)\Delta(\beta^{(4)}_4, 4).$$

And, if $u = 3$ or 4, by Lemma 2.4, we have that

$$\Delta(\beta^{(3)}_{(2,u)}, 4)\Delta(\beta^{(4)}_1, 4) \geq \Delta(\beta^{(3)}_{(1,2)}, 4)\Delta(\beta^{(4)}_u, 4),$$

and

$$\Delta(\beta^{(3)}_{(1,u)}, 4)\Delta(\beta^{(4)}_2, 4) \geq \Delta(\beta^{(3)}_{(1,2)}, 4)\Delta(\beta^{(4)}_u, 4).$$

From this, we conclude that

$$B_u \geq a_{u,2} \, \min\{b_{u,1}, b_{u,2}\} \left(\beta^{(3)}_{(k,\ell)}\beta^{(4)}_j + \beta^{(3)}_{(j,\ell)}\beta^{(4)}_k\right) + a_{u,3}\, b_{u,3}\beta^{(3)}_{(j,k)}\beta^{(4)}_\ell,$$

where

$$\begin{aligned} b_{u,1} &= \Delta(\beta^{(3)}_{(k,\ell)}, 4)\Delta(\beta^{(4)}_j, 4), \\ b_{u,2} &= \Delta(\beta^{(3)}_{(j,\ell)}, 4)\Delta(\beta^{(4)}_k, 4), \\ b_{u,3} &= \Delta(\beta^{(3)}_{(j,k)}, 4)\Delta(\beta^{(4)}_\ell, 4), \end{aligned}$$

for $u = 1, \ldots, 4$. Note that

$$\beta^{(3)}_{(k,\ell)}\beta^{(4)}_j + \beta^{(3)}_{(j,\ell)}\beta^{(4)}_k \geq \beta^{(3)}_{(j,k)}\beta^{(4)}_\ell,$$

for $j < k < \ell$, since

$$\beta^{(3)}_{(k,\ell)}\beta^{(4)}_j + \beta^{(3)}_{(j,\ell)}\beta^{(4)}_k - \beta^{(3)}_{(j,k)}\beta^{(4)}_\ell = 2\lambda_j\lambda_k \geq 0.$$

Hence, by Lemma 2.2

$$B_u \geq \sum_{h=2}^{3} a_{u,h}c_{u,h} \geq \frac{1}{2}\left(\sum_{h=2}^{3} a_{u,h}\right)\left(\sum_{h=2}^{3} c_{u,h}\right) \geq 0,$$

where

$$\begin{aligned} c_{u,2} &= min\{b_{u,1}, b_{u,2}\} \left(\beta^{(3)}_{(k,\ell)}\beta^{(4)}_j + \beta^{(3)}_{(j,\ell)}\beta^{(4)}_k\right), \\ c_{u,3} &= b_{u,3}\beta^{(3)}_{(j,k)}\beta^{(4)}_\ell, \end{aligned}$$

for $u = 1, \ldots, 4$. This proves the required result. ∎

4. Comparisons between spacings related to two samples

In the two samples problem, when $X_1, \ldots, X_n$ are independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$ and $Y_1, \ldots, Y_n$ are another random sample with Y_i having hazard rate θ_i, $i = 1, \ldots, n$, Kochar and Rojo [9] proved that simple spacings are ordered according to likelihood ratio order when the parameters are ordered in the majorization order for $n = 2$, i.e., if $(\theta_1, \theta_2) \leq^m (\lambda_1, \lambda_2)$, then $C_{2:2} \leq_{lr} D_{2:2}$, where $D_{2:2}$ and $C_{2:2}$ are the second simple spacings from X_i's and Y_i's, respectively. As pointed out Kochar and Korwar [8], the hazard rate of $D_{2:n}$ is not Schur-concave for $n \geq 3$. Therefore the result in [9] can not be extended to $n \geq 3$. This topic has been studied for multiple-outlier exponential models in [5, 15].

Let $X_1, \ldots, X_n$ be independent exponential distributions such that X_i has hazard rate λ_1, for $i = 1, \ldots, p$ and X_j has hazard rate λ_* for $j = p+1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be another set of independent exponential distributions such that Y_i has hazard rate λ_2, for $i = 1, \ldots, p$ and Y_j has hazard rate λ_* for $j = p+1, \ldots, n$. In particular, Hu et al. [5] showed that $C_{i:n} \leq_{lr} D_{i:n}$, for $i = 1, \ldots, n$ if $\lambda_1 \leq \lambda_* \leq \lambda_2$, where $D_{2:n}$ and $C_{2:n}$ are the second simple spacing from X_i's and Y_i's, respectively. Note that this is a particular case of Theorem 3.3 in Torrado and Lillo [15].

For $\lambda_1 \leq \lambda_2 \leq \lambda_*$, Hu et al. [5] showed that spacings are increasing in the likelihood ratio order for $n = 2, 3$. In the following result, we prove this open problem for the second spacing for arbitrary n.

Theorem 4.1. *Let $X_1, \ldots, X_n$ be independent exponential distributions such that X_i has hazard rate λ_1, for $i = 1, \ldots, p$ and X_j has hazard rate λ_* for $j = p+1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be another set of independent exponential distributions such that Y_i has hazard rate λ_2, for $i = 1, \ldots, p$ and Y_j has hazard rate λ_* for $j = p+1, \ldots, n$. If $\lambda_1 \leq \lambda_2 \leq \lambda_*$, then*

$$C_{2:n} \leq_{lr} D_{2:n},$$

where $D_{2:n}$ and $C_{2:n}$ are the second simple spacing from X_i's and Y_i's, respectively.

Proof: From (2.2), we know that the density function of $D_{2:n}$ is

$$f_{2:n}(t) = \frac{1}{S(\boldsymbol{\lambda}_1)} \left(p\lambda_1 \left(S(\boldsymbol{\lambda}_1) - \lambda_1\right) e^{-t\left(S(\boldsymbol{\lambda}_1) - \lambda_1\right)} + q\lambda_* \left(S(\boldsymbol{\lambda}_1) - \lambda_*\right) e^{-t\left(S(\boldsymbol{\lambda}_1) - \lambda_*\right)} \right).$$

where $\boldsymbol{\lambda}_1 = (\lambda_1, \ldots, \lambda_1, \lambda_*, \ldots, \lambda_*)$. Analogously, we get the density function of $C_{2:n}$ by interchanging λ_1 and λ_2. On differentiating $f_{2:n}(t)$ with respect to

t, we get

$$f'_{2:n}(t) = \frac{-1}{S(\boldsymbol{\lambda}_1)}\left(p\lambda_1\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)^2 e^{-t\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)} + q\lambda_*\left(S(\boldsymbol{\lambda}_1)-\lambda_2\right)^2 e^{-t\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)}\right),$$

Thus, we have to prove that

$$\frac{p\lambda_2\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right)^2 e^{\lambda_2 t} + q\lambda_*\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right)^2 e^{\lambda_* t}}{p\lambda_2\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right) e^{\lambda_2 t} + q\lambda_*\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right) e^{\lambda_* t}} \geq \frac{p\lambda_1\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)^2 e^{\lambda_1 t} + q\lambda_*\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)^2 e^{\lambda_* t}}{p\lambda_1\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right) e^{\lambda_1 t} + q\lambda_*\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right) e^{\lambda_* t}}.$$

After some computations we get that the above expression is equivalent to

$$\begin{aligned}
&p^2\lambda_1\lambda_2\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right)e^{(\lambda_1+\lambda_2)t}\Big(\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right)-\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)\Big)+\\
&pq\lambda_1\lambda_*\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right)e^{(\lambda_1+\lambda_*)t}\Big(\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right)-\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)\Big)+\\
&pq\lambda_2\lambda_*\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right)e^{(\lambda_2+\lambda_*)t}\Big(\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right)-\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)\Big)+\\
&pq\lambda_*^2\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right)e^{2\lambda_* t}\Big(\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right)-\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)\Big)\geq 0.
\end{aligned}$$

Note that

$$(S(\boldsymbol{\lambda}_2)-\lambda_2)-(S(\boldsymbol{\lambda}_1)-\lambda_1) = p\lambda_2+q\lambda_*-p\lambda_1-q\lambda_*+\lambda_1-\lambda_2 = (p-1)(\lambda_2-\lambda_1)\geq 0, \tag{4.1}$$

and

$$\begin{aligned}
(S(\boldsymbol{\lambda}_2)-\lambda_*)-(S(\boldsymbol{\lambda}_1)-\lambda_*) = S(\boldsymbol{\lambda}_2)-S(\boldsymbol{\lambda}_1) &= p\lambda_2+q\lambda_*-p\lambda_1-q\lambda_*\\
&= p(\lambda_2-\lambda_1)\geq 0,
\end{aligned}$$

since $\lambda_1 \leq \lambda_2$ and $p \geq 1$. Therefore, what remains to be proved is

$$\begin{aligned}
&pq\lambda_* e^{\lambda_* t}\Big(\lambda_1 e^{\lambda_1 t}\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right)\big(\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right)-\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)\big)\\
&+\lambda_2 e^{\lambda_2 t}\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right)\big(\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right)-\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)\big)\Big)\geq 0.
\end{aligned}$$

Note that $(S(\boldsymbol{\lambda}_2)-\lambda_2)-(S(\boldsymbol{\lambda}_1)-\lambda_*) = (S(\boldsymbol{\lambda}_2)-S(\boldsymbol{\lambda}_1))+(\lambda_*-\lambda_2)\geq 0$ since $\lambda_1\leq\lambda_2\leq\lambda_*$ and $e^{\lambda_2 t}\geq e^{\lambda_1 t}$, then the above expression is equivalent to

$$\begin{aligned}
&\lambda_1\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right)\big(\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right)-\left(S(\boldsymbol{\lambda}_1)-\lambda_1\right)\big)\\
+\;&\lambda_2\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right)\big(\left(S(\boldsymbol{\lambda}_2)-\lambda_2\right)-\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)\big)\geq 0.
\end{aligned}$$

Hence, the required result follows from (4.1),

$$\begin{aligned}(S(\boldsymbol{\lambda}_2)-\lambda_2)-(S(\boldsymbol{\lambda}_1)-\lambda_*) &\geq (S(\boldsymbol{\lambda}_1)-\lambda_1)-(S(\boldsymbol{\lambda}_2)-\lambda_*)\\ &\Leftrightarrow (p-1)\lambda_2-p\lambda_1+\lambda_*\geq(p-1)\lambda_1-p\lambda_1+\lambda_*\\ &\Leftrightarrow \lambda_1\leq\lambda_2,\end{aligned}$$

and

$$\begin{aligned}\lambda_2\left(S(\boldsymbol{\lambda}_1)-\lambda_*\right)\geq\lambda_1\left(S(\boldsymbol{\lambda}_2)-\lambda_*\right) &\Leftrightarrow p\lambda_1\lambda_2+(q-1)\lambda_2\lambda_*\geq p\lambda_1\lambda_2+(q-1)\lambda_1\lambda_*\\ &\Leftrightarrow \lambda_1\leq\lambda_2.\end{aligned}$$

■

Recall that normalized spacings are defined as $D^*_{i:n}=(n-i+1)D_{i:n}$, for $i=1,\ldots,n$. Note that, from Theorem 3.1 in Torrado and Lillo [15] and Theorem 4.1, we have that

$$\lambda_1\leq\lambda_2\leq\lambda_*\Rightarrow C^*_{2:2}\leq_{lr}D^*_{2:2},$$

where $D^*_{2:2}$ and $C^*_{2:2}$ are the second normalized spacings from two multiple-outlier exponential models as before.

Theorem 4.2. *Let $X_1,\ldots,X_n$ be independent exponential distributions such that X_i has hazard rate λ_i, for $i=1,\ldots,n-1$ and X_n has hazard rate λ_*. Let $Y_1,\ldots,Y_n$ be another set of independent exponential distributions such that Y_i has hazard rate λ_i, for $i=1,\ldots,n-1$ and Y_n has hazard rate θ_*. If $\lambda_*\leq\lambda_i\leq\theta_*$ for $i=1,\ldots,n-1$. Then,*

$$C_{2:n}\leq_{lr}D_{2:n},$$

where $D_{2:n}$ and $C_{2:n}$ are the second simple spacing from X_i's and Y_i's, respectively.

Proof: Again, from (2.2), we know that the density function of $D_{2:n}$ is

$$f_{2:n}(t)=\frac{1}{S(\boldsymbol{\lambda})}\left(\lambda_*\left(S(\boldsymbol{\lambda})-\lambda_*\right)e^{-t\left(S(\boldsymbol{\lambda})-\lambda_*\right)}+\sum_{i=1}^{n-1}\lambda_i\left(S(\boldsymbol{\lambda})-\lambda_i\right)e^{-t\left(S(\boldsymbol{\lambda})-\lambda_i\right)}\right).$$

where $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_{n-1},\lambda_*)$. Analogously, we get the density function of $C_{2:n}$ by interchanging λ_* and θ_*. On differentiating $f_{2:n}(t)$ with respect to t, we get

$$f'_{2:n}(t)=\frac{-1}{S(\boldsymbol{\lambda})}\left(\lambda_*\left(S(\boldsymbol{\lambda})-\lambda_*\right)^2e^{-t\left(S(\boldsymbol{\lambda})-\lambda_*\right)}+\sum_{i=1}^{n-1}\lambda_i\left(S(\boldsymbol{\lambda})-\lambda_i\right)^2e^{-t\left(S(\boldsymbol{\lambda})-\lambda_i\right)}\right),$$

Thus, we have to prove that

$$\frac{\theta_*\left(S(\boldsymbol{\theta})-\theta_*\right)^2 e^{\theta_* t}+\sum_{i=1}^{n-1}\lambda_i\left(S(\boldsymbol{\theta})-\lambda_i\right)^2 e^{\lambda_i t}}{\theta_*\left(S(\boldsymbol{\theta})-\theta_*\right) e^{\theta_* t}+\sum_{i=1}^{n-1}\lambda_i\left(S(\boldsymbol{\theta})-\lambda_i\right) e^{\lambda_i t}}$$
$$\geq \frac{\lambda_*\left(S(\boldsymbol{\lambda})-\lambda_*\right)^2 e^{\lambda_* t}+\sum_{i=1}^{n-1}\lambda_i\left(S(\boldsymbol{\lambda})-\lambda_i\right)^2 e^{\lambda_i t}}{\lambda_*\left(S(\boldsymbol{\lambda})-\lambda_*\right) e^{\lambda_* t}+\sum_{i=1}^{n-1}\lambda_i\left(S(\boldsymbol{\lambda})-\lambda_i\right) e^{\lambda_i t}}.$$

After some computations, we get that the above expression is equivalent to

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\lambda_i\lambda_j\left(S(\boldsymbol{\theta})-\lambda_i\right)\left(S(\boldsymbol{\lambda})-\lambda_j\right)e^{(\lambda_i+\lambda_j)t}\Big(\left(S(\boldsymbol{\theta})-\lambda_i\right)-\left(S(\boldsymbol{\lambda})-\lambda_j\right)\Big)+$$
$$\lambda_*\left(S(\boldsymbol{\lambda})-\lambda_*\right)e^{\lambda_* t}\sum_{i=1}^{n-1}\lambda_i\left(S(\boldsymbol{\theta})-\lambda_i\right)e^{\lambda_i t}\Big(\left(S(\boldsymbol{\theta})-\lambda_i\right)-\left(S(\boldsymbol{\lambda})-\lambda_*\right)\Big)+$$
$$\theta_*\left(S(\boldsymbol{\theta})-\theta_*\right)e^{\theta_* t}\sum_{i=1}^{n-1}\lambda_i\left(S(\boldsymbol{\lambda})-\lambda_i\right)e^{\lambda_i t}\Big(\left(S(\boldsymbol{\theta})-\theta_*\right)-\left(S(\boldsymbol{\lambda})-\lambda_i\right)\Big)+$$
$$\lambda_*\left(S(\boldsymbol{\lambda})-\lambda_*\right)e^{\lambda_* t}\theta_*\left(S(\boldsymbol{\theta})-\theta_*\right)e^{\theta_* t}\Big(\left(S(\boldsymbol{\theta})-\theta_*\right)-\left(S(\boldsymbol{\lambda})-\lambda_*\right)\Big)\geq 0.$$

Since $(S(\boldsymbol{\theta})-\theta_*)-(S(\boldsymbol{\lambda})-\lambda_*)=0$, the last term of the above expression is equal to zero. Note hat

$$\begin{aligned}\left(S(\boldsymbol{\theta})-\lambda_i\right)-\left(S(\boldsymbol{\lambda})-\lambda_*\right) &= \theta_*-\lambda_i\geq 0,\\ \left(S(\boldsymbol{\theta})-\theta_*\right)-\left(S(\boldsymbol{\lambda})-\lambda_i\right) &= \lambda_i-\lambda_*\geq 0,\end{aligned}$$

and

$$\left(S(\boldsymbol{\theta})-\lambda_i\right)-\left(S(\boldsymbol{\lambda})-\lambda_j\right)\geq\left(S(\boldsymbol{\theta})-\theta_*\right)-\left(S(\boldsymbol{\lambda})-\lambda_j\right)=\lambda_i-\lambda_*\geq 0,$$

since $\lambda_*\leq\lambda_i\leq\theta_*$ for $i=1,\ldots,n-1$. Hence, the required result follows immediately. ■

Again, from Theorem 3.1 in Torrado and Lillo [15], we have a similar result as Theorem 4.2 for normalized spacing, that is:

$$\lambda_*\leq\lambda_i\leq\theta_*,\text{ for } i=1,\ldots,n-1\Rightarrow C^*_{2:2}\leq_{lr} D^*_{2:2},$$

where $D^*_{2:2}$ and $C^*_{2:2}$ are the second normalized spacings from two multiple-outlier exponential models as before.

5. Discussion and Concluding Remarks

Let $X_1, \ldots, X_n$ be independent exponential random variables such that X_i has hazard rate λ_i for $i = 1, \ldots, n$, we have established the conjecture by Wen et al. [16] for $n = 4$, that is, $D_{1:4} \leq_{lr} D_{2:4} \leq_{lr} D_{3:4} \leq_{lr} D_{4:4}$. In addition, we have proved that the second spacing is smaller than the third spacing in the likelihood ratio order and for all n. These results have extended the known results for spacings for the exponential case, but the general case still remains an open problem.

For the two random samples problem, we have established that

$$\lambda_1 \leq \lambda_2 \leq \lambda_* \Rightarrow C_{2:n} \leq_{lr} D_{2:n},$$

where $D_{2:n}$ and $C_{2:n}$ are the second simple spacings from two multiple-outlier exponential models with hazard rate $\boldsymbol{\lambda}_1 = (\lambda_1, \ldots, \lambda_1, \lambda_*, \ldots, \lambda_*)$ and $\boldsymbol{\lambda}_2 = (\lambda_2, \ldots, \lambda_2, \lambda_*, \ldots, \lambda_*)$, respectively. This result solves an open problem in the literature (see Hu et al. [5]) for the second spacing from multiple-outlier exponential model. We have also proved that

$$\lambda_* \leq \lambda_i \leq \theta_* \text{ for } i = 1, \ldots, n-1 \Rightarrow C_{2:n} \leq_{lr} D_{2:n},$$

where $D_{2:n}$ and $C_{2:n}$ are the second simple spacings from two samples of exponential random variables with hazard rate $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_*)$ and $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \theta_*)$, respectively.

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