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CONSTRUCTION OF ALGEBRAIC COVERS

COVER HOMOMORPHISMS

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ABSTRACT: Given an algebraic variety Y and a locally free $\mathcal{O}_Y$-module of rank 2, $\mathcal{E}$, Miranda defined the notion of triple cover homomorphism as a map $S^2\mathcal{E} \to \mathcal{E}$ that determines a triple cover of Y. In this paper we generalize the definition of cover homomorphism and present a method to compute them.

The main theorem shows how to use cover homomorphisms to describe the section ring of polarized varieties $(X, \mathcal{L})$ when $\mathcal{L}$ induces a covering map. Furthermore, we study in detail the case of Gorenstein covering maps for which the direct image of $\mathcal{O}_X$ admits an orthogonal decomposition.

Finally we apply the results to determine Gorenstein covers of degree 6 satisfying some mild conditions, obtaining the structure of a codimension 4 Gorenstein ideal, and study the ideals that determine a S_3-Galois branched cover.

1. Introduction

Let $\varphi\colon X \to Y$ be a covering map of degree d, i.e. a finite and flat morphism between irreducible schemes over an algebraically closed field k. It is known that $\varphi_*\mathcal{O}_X$ can be decomposed as $\mathcal{O}_Y \oplus \mathcal{E}$, where $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of rank $d-1$, called the trace free or Tschirnhausen module of φ. For $d = 3$, Miranda defines in [Mir85] a *triple cover homomorphism* as a map in $\mathrm{Hom}(S^2\mathcal{E}, \mathcal{E})$ that is locally of the form

$$\begin{aligned} \phi(z_1^2) &= c_1z_1 + c_0z_2 \\ \phi(z_1z_2) &= -c_2z_1 - c_1z_2 \\ \phi(z_2^2) &= c_3z_1 + c_2z_2, \end{aligned} \tag{1.1}$$

where $\{z_1, z_2\}$ is a local basis for $\mathcal{E}$ and $c_i \in \mathcal{O}_Y$. It is proven then that a multiplication rule in $\mathcal{O}_X$ defines and is defined by a triple cover homomorphism. In this paper we start by extending the definition of covering homomorphism and show a computational method to determine them.

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We extend the definition not by looking for a multiplication rule on $\varphi_*\mathcal{O}_X$ as an associative $\mathcal{O}_Y$-linear map $S^2\mathcal{O}_X \to \varphi_*\mathcal{O}_X$ but by defining this multiplication via the study of an ideal sheaf in an ambient $\mathcal{O}_Y$-algebra containing $\varphi_*\mathcal{O}_X$. Specifically, consider $\mathcal{F}$ a direct summand of $\varphi_*\mathcal{O}_X$ such that $\varphi_*\mathcal{O}_X$ is itself a direct summand of $\mathcal{R}(\mathcal{F}) := \bigoplus_{n=0}^{\infty} \operatorname{Sym}^n \mathcal{F}$. One can define a multiplication on $\mathcal{O}_X$ by describing $\mathcal{I}_X$, an $\mathcal{R}(\mathcal{F})$ ideal such that the following sequence is exact

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{R}(\mathcal{F}) \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Notice that for a cover of degree $d = 3$, taking $\mathcal{F} = \mathcal{E}$ is exactly the problem studied by Miranda. Theorem 2.14 states that for a general d, a degree d cover defines and is defined by a morphism $S^2\mathcal{E} \to \mathcal{E}$ satisfying some local conditions. These conditions are obtained using local generators for the initial ideal $\operatorname{in}(\mathcal{I}_X)$. Applying deformation–theoretic methods as presented by Reid [Rei90], we study when the free resolution of $\operatorname{in}(\mathcal{I}_X)$ lifts to one for $\mathcal{I}_X$. This is done by considering $\operatorname{in}(\mathcal{I}_X)$ as an hyperplane section in a larger ambient variety. We then define the set of *cover homomorphisms* as the set of all extensions of $\operatorname{in}(\mathcal{I}_X)$ in $\mathcal{R}(\mathcal{F})$ and denote it by $\operatorname{CHom}(\operatorname{in}(\mathcal{I}_X), \varphi_*\mathcal{O}_X)$.

Our aim is the description of graded rings for varieties polarized by an ample line bundle $\mathcal{L}$ on X, i.e. the ring

$$R(X, \mathcal{L}) = \bigoplus_{n=0}^{\infty} H^0\left(X, \mathcal{L}^{\otimes n}\right).$$

In the case that $\varphi\colon X \to Y$ is generated by $\mathcal{L} = \varphi^*\mathcal{M}$, where $\mathcal{M}$ is a very ample line bundle on Y, by the projection formula [Har77, Ch. III] we have that

$$H^0\left(X, \mathcal{L}^{\otimes n}\right) \cong H^0\left(Y, \varphi_*\mathcal{O}_X \otimes \mathcal{M}^{\otimes n}\right).$$

As we are considering $\varphi_*\mathcal{O}_X$ to be generated by a locally free $\mathcal{O}_Y$-module $\mathcal{F}$, there is an epimorphism

$$\bigoplus_{n=0}^{\infty} \operatorname{Sym}^n \left(\bigoplus_{k=0}^{\infty} H^0\left(Y, \mathcal{F} \otimes \mathcal{M}^{\otimes k}\right) \right) \to \bigoplus_{n=0}^{\infty} H^0\left(Y, \varphi_*\mathcal{O}_X \otimes \mathcal{M}^{\otimes n}\right).$$

Such map is determined by a cover homomorphism and corresponds to an embedding

$$X \hookrightarrow \mathbb{A}_Y\left(\mathcal{F}^\vee\right) \twoheadrightarrow Y,$$

where $\mathbb{A}_Y(\mathcal{F}^\vee)$ is the relative affine scheme of sections of $\operatorname{Sym}\mathcal{F}^\vee$, i.e.

$$\operatorname{Spec}_Y\left(\bigoplus_{n=0}^{\infty}\operatorname{Sym}^n\mathcal{F}^\vee\right).$$

An application of this will be Proposition 3.3, where we show that if $\mathcal{L} = \varphi^*\mathcal{M}$ induces a covering map $\varphi\colon X \to Y$, and the canonical bundles of Y and X are $\mathcal{M}^{\otimes k_Y}$ and $\mathcal{L}^{\otimes k_X}$, then $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{F} \oplus \mathcal{M}^{\otimes k_Y - k_X}$ and $\mathcal{M}^{\otimes k_Y - k_X}$ is a direct summand of $S^2\mathcal{F}$. Furthermore, there is an embedding $X \hookrightarrow \mathbb{A}(\mathcal{F}^\vee)$ and φ is determined by a cover homomorphism $\mathcal{Q} \to \mathcal{F}$, where $\mathcal{Q}$ is the kernel of the map $S^2\mathcal{F} \to \mathcal{M}^{\otimes k_Y - k_X}$.

Such covering maps are called Gorenstein covering maps (all its fibres are Gorenstein), and were first studied by Casnati and Ekedahl in [CE96]. A structure theorem found by them is transcribed in Section 3 so that we can show the relation with our results. In simple cases, a Gorenstein ring corresponds to a hypersurface, a codimension 2 complete intersection, or a codimension 3 Pfaffian. For higher codimensions there is no structure theorem.

In section 3.1 we study the case of a Gorenstein covering map such that

$$\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus M \oplus M \oplus \wedge^2 M,$$

where M is a simple $\mathcal{O}_Y$-sheaf of rank 2. Although its fibre is given by a codimension 4 Gorenstein ideal, the conditions imposed on $\varphi_*\mathcal{O}_X$ enable us to determine $\operatorname{in}(\mathcal{I}_X)$ and implement our method via an explicit SAGE algorithm presented in the Appendix.

This algorithm results in Theorem 3.5 where we prove that such covering maps are locally parametrized by the spinor embedding of the affine orthogonal Grassmannian $\operatorname{aOGr}(5,10)$ in $\mathbb{P}^{15}$. Finally, in Section 3.2 we find that a linear section of $\operatorname{aOGr}(5,10)$ corresponds to an S_3-Galois branched covering map.

Throughout this paper, k will denote an algebraically closed field, $\operatorname{ch}(k) = 0$.

2. Algebraic covers

Definition 2.1. Let X, Y be schemes over a field k and assume Y is irreducible and integral. A covering map of degree d is a flat and finite morphism $\varphi\colon X \to Y$, such that the locally free $\mathcal{O}_Y$-module $\varphi_*\mathcal{O}_X$ has rank d.

Lemma 2.2. [HM99, 2.2] *Let $\varphi\colon X \to Y$ be a covering map. Then $\varphi_*\mathcal{O}_X$ splits as $\mathcal{O}_Y \oplus \mathcal{E}$, where $\mathcal{E}$ is the sub-module of trace-zero elements.*

Proof: For an open set $\mathcal{U} \subset Y$, the multiplication by $x \in \mathcal{O}_X\left(\varphi^{-1}(\mathcal{U})\right)$ defines a morphism

$$\varphi_*\mathcal{O}_X(\mathcal{U}) \xrightarrow{x} \varphi_*\mathcal{O}_X(\mathcal{U}).$$

After a choice of basis for $\varphi_*\mathcal{O}_X(\mathcal{U})$, this morphism is represented by a matrix, M_x, with entries in $\mathcal{O}_Y(\mathcal{U})$. If $\operatorname{char} k \nmid d$, we can define the map $\frac{1}{d}\operatorname{tr}\colon \varphi_*\mathcal{O}_X(\mathcal{U}) \to \mathcal{O}_Y(\mathcal{U})$ by $x \mapsto \frac{1}{d}\operatorname{tr}(M_x)$. This map is well defined as the trace of a matrix is unchanged by a change of basis and so we get the following short exact sequence

$$0 \to \mathcal{E} \to \varphi_*\mathcal{O}_X \xrightarrow{\frac{1}{d}tr} \mathcal{O}_Y \to 0.$$

This sequence splits as for $y \in \mathcal{O}_Y(\mathcal{U})$ the matrix is $\operatorname{diag}(y, y, \ldots, y)$ and so $\frac{1}{d}\operatorname{tr}(y) = y$ and $\frac{1}{d}\operatorname{tr}\circ\varphi^* = \operatorname{id}$. ■

2.1. Local analysis. Assume in this section that X and Y are local schemes over k. Let $\varphi\colon X \to Y$ be a covering map of degree d, and let $\mathcal{F}$ be an $\mathcal{O}_Y$-module with rank r, such that $\varphi_*\mathcal{O}_X$ is a direct summand of $\mathcal{R}(\mathcal{F}) := \bigoplus_{n\geq 0}\operatorname{Sym}^n\mathcal{F}$. Detecting the algebra structure on $\varphi_*\mathcal{O}_X$ is the same as determining the ideal $\mathcal{I}_X$ in the following exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{R}(\mathcal{F}) \xrightarrow{p} \mathcal{O}_X \to 0. \tag{2.1}$$

Let $(z_1, \ldots, z_r)$ be a choice of basis for $\mathcal{F}$ so that $\mathcal{R}(\mathcal{F}) \cong \mathcal{O}_Y[z_i]$. Given an element $f \in \mathcal{R}(\mathcal{F})$ we use this isomorphism to define $\operatorname{in}(f)$ as the sum of the terms with maximal degree in the variables z_i of f, and the initial ideal of $\mathcal{I}_X$ as

$$\operatorname{in}(\mathcal{I}_X) = \{\operatorname{in}(f)\colon f \in \mathcal{I}_X\}.$$

Lemma 2.3. *Let $\mathcal{O}_Y$ be a local k-algebra which is an integral domain and $\mathcal{O}_X$ a finite flat $\mathcal{O}_Y$-algebra such that $\mathcal{O}_X \cong \mathcal{R}(\mathcal{F})/\mathcal{I}_X$, where $\mathcal{F}$ is a free $\mathcal{O}_Y$-module with rank r. Then*

1. *the ideal sheaf $\mathcal{I}_X$ is Cohen-Macaulay,*
2. *the ideal $\operatorname{in}(\mathcal{I}_X)$ is independent of the choice of basis for $\mathcal{F}$,*
3. *the free resolution of the ideal $\operatorname{in}(\mathcal{I}_X)$ has the same format as the one for $\mathcal{I}_X$, in particular $\operatorname{in}(\mathcal{I}_X)$ is also CM.*

Proof: 1: As $\mathcal{O}_X$ is a finite flat $\mathcal{O}_Y$-algebra

$$\dim \mathcal{O}_X = \dim \mathcal{O}_Y \Rightarrow \operatorname{codim}(\mathcal{I}_X) = r.$$

Take $n \in \mathbb{N}$ big enough such that $\operatorname{Sym}^n \mathcal{F} \cap \varphi_* \mathcal{O}_X = \emptyset$. Then the sequence $(z_j^n - \overline{p(z_j^n)})_{1 \leq j \leq r}$ is a $\mathcal{R}(\mathcal{F})$-sequence in the ideal $\operatorname{in}(\mathcal{I}_X)$, where $\overline{p(z_j^n)}$ is the image of z_j^n in $\varphi_* \mathcal{O}_X$ considered as element of $\mathcal{R}(\mathcal{F})$. Then we have that

$$r \leq \operatorname{depth}(\mathcal{I}_X) \leq \operatorname{codim}(\mathcal{I}_X) = r,$$

which asserts that $\mathcal{I}_X$ is CM.

2: Denote by $\operatorname{in}(\mathcal{I}_X)_{(z_i)}$ the initial ideal when defined with respect to the basis (z_i) for $\mathcal{F}$. A change of basis for $\mathcal{R}(\mathcal{F})$ is given by a $\mathcal{O}_Y$-linear map, $\Psi \colon \mathcal{F} \to \mathcal{O}_Y \oplus \mathcal{F}$. Decomposing Ψ as $\Psi_1 \oplus \Psi_2$, where $\Psi_1 \colon \mathcal{F} \to \mathcal{O}_Y$ and $\Psi_2 \colon \mathcal{F} \to \mathcal{F}$, we have that Ψ_2 is an isomorphism which sends $\operatorname{in}(\mathcal{I}_X)_{(z_i)}$ into $\operatorname{in}(\mathcal{I}_X)_{\Psi(z_i)}$ so we are done.

3: Use a change of variables $z_i \mapsto z_i/z_0$ to homogenize $\mathcal{I}_X$ and take the quotient by (z_0). As z_0 is a non-zero divisor, taking the tensor of a minimal free resolution of $\mathcal{I}_X$ with $\mathcal{O}_Y[z_i]/(z_0)$ gives a resolution of the ideal $\operatorname{in}(\mathcal{I}_X)$, so we have the right depth. As codimension of $\operatorname{in}(\mathcal{I}_X)$ is smaller or equal to r we are done. ∎

For a general covering map of degree d such that $\varphi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$, taking $\mathcal{F} = \mathcal{E}$ gives that $\operatorname{in}(\mathcal{I}_X) = (z_i z_j)$, for all $1 \leq i \leq j \leq d-1$. As we will see in Section 3, if we assume X and Y to be Gorenstein, we can determine a decomposition of $\varphi_* \mathcal{O}_X$ with $\mathcal{F}$ as a direct summand and, under certain assumptions, describe $\operatorname{in}(\mathcal{I}_X)$.

In such cases denote by q_i the generators of the ideal $\operatorname{in}(\mathcal{I}_X)$ for a fixed basis $(z_1, \ldots, z_r)$ of $\mathcal{F}$. We want to determine the relations between coefficients $c_{ij} \in \mathcal{O}_Y$, $(i, j) \in \mathbb{N} \times \mathbb{N}^r$, such that the ideal generated by the polynomials

$$f_i := q_i - \sum_j c_{ij} \overline{z}^j$$

where $\overline{z}^j = z_1^{j_1} \cdots z_r^{j_r}$, has the same resolution as $\operatorname{in}(\mathcal{I}_X)$. By definition q_i is the sum of the monomials with maximal degree in f_i so $c_{ij} = 0$ if $\sum_i j_i \geq \deg(q_i)$. We have then a finite number of non-zero coefficients c_{ij}.

Definition 2.4. Let $\mathcal{O}_Y$ be a local k-algebra which is an integral domain, $\mathcal{O}_X$ a flat $\mathcal{O}_Y$-algebra and $\mathcal{F}$ a free $\mathcal{O}_Y$-module of rank r such that $\mathcal{O}_X$ is a direct summand of $\mathcal{R}(\mathcal{F})$. Given a minimal set of generators $q = (q_1, \ldots, q_m)$ for the ideal $\mathrm{in}(\mathcal{I}_X) \subset \mathcal{R}(\mathcal{F})$ we denote by

$$\mathcal{I}_q \subset k[c_{ij}],$$

where $(i,j) \in \mathbb{N} \times \mathbb{N}^r$ and $\sum_i j_i < \deg(q_i)$, the *ideal of q-relations*, which is generated by those relations between the c_{ij} for which the polynomials $q_i - \sum_j c_{ij}\overline{z}^j$ generate an ideal with the same resolution as $\mathrm{in}(\mathcal{I}_X)$.

We will now present a computational algorithm to compute the ideal $\mathcal{I}_q$ for a given q. Notice that by the Nakayama lemma, the image of the generators of $\mathcal{I}_X$ in $\mathcal{I}_X \otimes k$ are the generators of $\mathcal{I}_X \otimes k$. As Y is integral, the relations between the c_{ij} are also kept, so one can study these relations in the case $\mathcal{O}_Y = k$.

2.2. Ideal of q-relations. In this section we follow the paper [Rei90] by Miles Reid, where the following extension problem is tackled. *Given a Noetherian graded ring $\overline{R}$ and a_0 a non negative integer, determine the set of all pairs (R, z_0), where R is a graded ring and $z_0 \in R$ is a homogeneous, nonzero divisor, of degree a_0 such that $R/(z_0) = \overline{R}$.*

The hyperplane section principle (see [Rei90, §1.2]) says that the generators of R, their relations and syzygies reduce modulo z_0 to those of $\overline{R}$ and occur in the same degrees. In particular, if $\overline{R} = \overline{S}/\overline{I}$, where $\overline{S} = k[z_1, \ldots, z_r]$, then we have an exact sequence

$$\overline{R} \leftarrow \overline{S} \xleftarrow{f} \bigoplus \overline{S}(-a_i) \xleftarrow{l} \bigoplus \overline{S}(-b_j), \tag{2.2}$$

with a_i, b_j positive integers, f a vector with the generators of $\overline{I}$ as entries, l its syzygy matrix.

Proposition 2.5. *Given a ring $\overline{R} = \overline{S}/\overline{I}$ with a presentation as in (2.2), if every relation $\sigma_j := (\sum_i l_i f_i \equiv 0)$ lifts to a relation $\Sigma_j := (\sum_i L_i F_i \equiv 0)$, then the resolution lifts to a resolution of R and $\overline{R} = R/(z_0)$.*

Proof: [Rei90, §1.2 - The hyperplane section principle] ■

A ring $\overline{R}^{(k)}$ together with a homogeneous element $z_0 \in \overline{R}^{(k)}$ of degree a_0 such that $\overline{R} = \overline{R}^{(k)}/(z_0)$ is a k^{th} order *infinitesimal extension* of $\overline{R}$ if $z_0^{k+1} = 0$ and $\overline{R}^{(k)}$ is flat over the subring $k[z_0]/(z_0^{k+1})$ generated by z_0.

Definition 2.6. The Hilbert scheme of k^{th} order infinitesimal extensions of $\overline{R}$ by a variable of degree a_0 is the set

$$\mathbb{H}^{(k)}\left(\overline{R}, a_0\right) = \left\{(R, z_0) \mid \deg(z_0) = a_0,\ R/\left(z_0^{k+1}\right) \cong \overline{R}\right\}.$$

Notice that if $k \geq \max_i\{\deg(\sigma_i)\}$, then $\mathbb{H}^{(k)}\left(\overline{R}, a_0\right)$ is the solution to the extension problem, denoted by $\mathbb{H}\left(\overline{R}, a_0\right)$. It is constructed as a tower of schemes

$$\mathbb{H}\left(\overline{R}, a_0\right) \cdots \to \mathbb{H}^{(k)}\left(\overline{R}, a_0\right) \to \mathbb{H}^{(k-1)}\left(\overline{R}, a_0\right) \to \cdots \to \mathbb{H}^{(0)}\left(\overline{R}, a_0\right) = \mathrm{pt},$$

where each morphism is induced by the forgetful map

$$k[z_0]/\left(z_0^{k+1}\right) \to k[z_0]/\left(z_0^{k}\right).$$

Notice that $\mathbb{H}^{(k)}\left(\overline{R}, a_0\right)$ is independent of a choice of variables so we introduce the following scheme for the sake of computations.

Definition 2.7. We call big Hilbert scheme of k^{th} order infinitesimal extensions of $\overline{R}$ by a variable z_0 of degree a_0 the following set

$$\left\{ \begin{array}{c} F_i = f_i + z_0 f_i' + \cdots + z_0^k f_i^{(k)}, \\ \Sigma_i : L_{ij} = l_{ij} + z_0 l_{ij}' + \cdots + z_0^k l_{ij}^{(k)} \end{array} \;\middle|\; \sum_i L_{ij} F_i \mod z_0^{k+1} \equiv 0 \right\}$$

where F_i, L_{ij} are homogeneous polynomials and denote it by $\mathrm{BH}^{(k)}\left(\overline{R}, a_0\right)$. This set has the structure of an affine scheme with coordinates given by the coefficients of $f_i^{(m)}$ and $l_{ij}^{(m)}$.

The Hilbert scheme is the quotient of the big Hilbert scheme by linear changes of variables. As we will see in the next Theorem, all the constructions are independent of choices and we can see the scheme $\mathrm{BH}\left(\overline{R}, a_0\right)$ as a choice of basis for the scheme $\mathbb{H}\left(\overline{R}, a_0\right)$.

Theorem 2.8.

- $\mathbb{H}^{(1)}\left(\overline{R}, a_0\right) = \mathrm{Hom}_{\overline{R}}\left(\overline{I}/\overline{I}^2, \overline{R}\right)_{-a_0}$.
- $\psi\colon\ \mathrm{BH}^{(k-1)} \to \bigoplus \overline{R}(s_j)_{-ka_0}$ *factors through a morphism of schemes*

$$\Psi\colon \mathbb{H}^{(k-1)}\left(\overline{R}, a_0\right) \to \bigoplus \overline{R}(s_j)_{-ka_0}.$$

- $\Psi\colon \mathbb{H}^{(k-1)}\left(\overline{R}, a_0\right) \to \ker \delta_1 \subset \bigoplus \overline{R}(s_j)_{-ka_0}$.

Therefore, the middle square in the diagram

$$
\begin{tikzcd}
 & \mathbb{H}^{(k)}\left(\overline{R}, a_0\right) \arrow[r, "\varphi_k"] \arrow[d] & \mathbb{H}^{(k-1)}\left(\overline{R}, a_0\right) \arrow[d] \arrow[dr, "\Psi"] & \\
0 \arrow[r] & \operatorname{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0} \arrow[r] & \bigoplus \overline{R}(d_i)_{-ka_0} \arrow[r, "\delta_0"] & (\ker \delta_1)_{-ka_0} \\
 & & & \xrightarrow{\pi} \operatorname{Ext}^1_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0} \to 0
\end{tikzcd}
\tag{2.3}
$$

is Cartesian.

In other words, the morphism $\varphi_k \colon \mathbb{H}^{(k)}\left(\overline{R}, a_0\right) \to \mathbb{H}^{(k-1)}\left(\overline{R}, a_0\right)$ *has the following structure: its image* $\operatorname{im}\varphi_k$ *is the scheme-theoretic fibre over* 0 *of the morphism* $\pi \circ \Psi \colon \mathbb{H}^{(k-1)}\left(\overline{R}, a_0\right) \to \operatorname{Ext}^1_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0}$*; so* $\pi \circ \Psi(R^{(k-1)})$ *is the obstruction to extending* $R^{(k-1)}$ *to* k*-th order and* $\varphi_k \colon \mathbb{H}^{(k)}\left(\overline{R}, a_0\right) \to \operatorname{im}\varphi_k \subset \mathbb{H}^{(k-1)}\left(\overline{R}, a_0\right)$ *is a fibre bundle (in the Zariski topology) with fibre an affine space over* $\operatorname{Hom}_{\overline{R}}(\overline{I}/\overline{I}^2, \overline{R})_{-ka_0}$.

Proof: [Rei90, Theorem 1.10 and Theorem 1.15] ■

We now use these tools to describe I_q. Let $q = (q_1, \dots, q_m)$ be a minimal set of homogeneous generators of an ideal with codimension $r \geq 2$ in $k[z_1, \dots, z_r]$. Furthermore, consider that for all i, $\deg(q_i) = 2$, and the first syzygy matrix of the ideal $(q_1, \dots, q_m)$ is linear. Our goal is to determine the relations between the c_{ij} and d_i such that the ideal generated by the polynomials

$$f_i := q_i - \sum_{j=1}^{r} c_{ij} z_j - d_i,$$

is an extension of the ideal $(q_1, \dots, q_r)$.

Homogenizing each f_i with a non-zero divisor z_0 of degree 1, we get the ideal

$$I = \left(F_i := q_i - \left(\sum_{k=1}^{r} c_{ij} z_j \right) z_0 - d_i z_0^2 \right).$$

Let $S = k[z_0, z_1, \dots, z_{d-1}]$ and $R = S/I$. Then we have the following exact sequence,

$$R \leftarrow S \xleftarrow{(F_i)} S(-2)^{\oplus m} \xleftarrow{L} S(-3)^{\oplus n_2} \leftarrow \cdots \leftarrow S(-d)^{\oplus n_r} \leftarrow 0. \tag{2.4}$$

As z_0 is a non-zero divisor, the resolution of $\overline{R} = R/(z_0)$ as a $\overline{S}$–module, $\overline{S} := S/(z_0)$, has the same format. Hence, the relations between the c_{ij}, d_i

we want to determine are given by $\mathbb{H}\left(\overline{R},1\right)$, i.e. the set of all pairs (R,z_0), $\deg z_0=1$, such that $\overline{R}=R/(z_0)$.

To implement the algorithm to construct $\mathrm{BH}(\overline{R})$, take the following matrices

- $q:=(q_1,q_2,\dots,q_m)\in \mathrm{Mat}_{1\times m}(\mathcal{A})$,
- $\overline{z}:=(z_1,\dots,z_r)\in \mathrm{Mat}_{(1\times r)}(\mathcal{A})$,
- $C:=[c_{ij}]\in \mathrm{Mat}_{r\times m}(\mathcal{A})$,
- $D:=[d_i]\in \mathrm{Mat}_{1\times m}(\mathcal{A})$,
- $l\in \mathrm{Mat}_{m\times n_2}(\mathcal{A})$, the first syzygy matrix of $\overline{I}$,
- $N:=[n_{ij}]\in \mathrm{Mat}_{m\times n_2}(\mathcal{A})$.

where $\mathcal{A}=k[z_i,c_{ij},d_i,n_{ij}]$. Notice that $\mathrm{BH}\left(\overline{R},1\right)$ is the set of relations between the variables c_{ij},d_i and n_{ij} that make the following equality true

$$\left(q-\overline{z}Cz_0-Dz_0^2\right)(l+Nz_0)=0.$$

Decomposing the equality in powers of z_0 we will get a tower of schemes

$$\mathrm{BH}\left(\overline{R},1\right)=\mathrm{BH}^{(3)}\left(\overline{R},1\right)\to \mathrm{BH}^{(2)}\left(\overline{R},1\right)$$
$$\to \mathrm{BH}^{(1)}\left(\overline{R},1\right)\to \mathrm{BH}^{(0)}\left(\overline{R},1\right)=\mathrm{pt},$$

where each $\mathrm{BH}^{(k)}\left(\overline{R},1\right)$ contains the relations given by the coefficients of z_0^k.

(1) $ql=0$,
which is true by definition of syzygy.

(2) $z_0\left(qN-(\overline{z}C)\,l\right)=0$.
This equality allows us to write all n_{ij} as linear combination of the c_{ij} and get linear equations between the c_{ij}.

(3) $z_0^2\left(\overline{z}CN+Dl\right)=0$.
As we can write N with the entries of C, this equation gives us identities $d_i=h_i(c_{ij})$, where the h_i are quadratic polynomials, and quadratic equations between the c_{ij}.

(4) $z_0^3\,(DN)=0$.
This last equation should give us cubic equations in the c_{ij} but we shall see that they are already contained in the ideal generated by the equations found before.

Remark 2.9. By step (3) the d_i are determined by the c_{ij}, so we consider I_q to be the ideal defined by the relations between the c_{ij} as they completely parametrize $\mathrm{BH}(\overline{R},1)$.

Proposition 2.10. *Let $q = (q_1, \dots, q_m)$ be a minimal set of homogeneous generators of an ideal with codimension $r \geq 2$ in $k[z_1, \dots, z_r]$ such that for all i, $\deg(q_i) = 2$, and the first syzygy matrix of the ideal $(q_1, \dots, q_m)$ is linear. Then the ideal of q-relations I_q is generated by quadratic polynomials. Furthermore, if q' is obtained from q by a linear change of variables, then $I_q \cong I_{q'}$.*

Proof: As q is generated by quadratic polynomials, $\mathrm{Hom}_{\overline{R}}(q/q^2, \overline{R})_{-3} = 0$. Therefore, we just have to prove that $\varphi_3 \colon \mathbb{H}(\overline{R}, 1)^{(3)} \to \mathbb{H}(\overline{R}, 1)^{(2)}$ is surjective, which by Theorem 2.8, is equivalent to $\mathrm{Ext}^1_{\overline{R}}(q/q^2, \overline{R})_{-3} = 0$.

The ideal q is CM, hence $\mathrm{Ext}^i_{\overline{R}}(q/q^2, \overline{R}) \cong \mathrm{Ext}^i_{\overline{R}}(q, \overline{R}) \cong \mathrm{Ext}^i_{\overline{S}}(q, \overline{R}) \cong \mathrm{Ext}^i_{\overline{S}}(q, \overline{S}) \otimes \overline{R} \neq 0$ if and only if $i = 0$ or $i = r$. Therefore, the result is true for $r \geq 3$. For $r = 2$, as q is generated by quadratic polynomials and the syzygy matrix is linear, $q = (z_1^2, z_1 z_2, z_2^2)$. This is a triple cover which is a determinantal variety determined by the 2×2 minors of its syzygy matrix, whose entries only satisfy linear relations. Therefore I_q only contains the quadratic equations defining d_i (see Example (2.11)).

The second statement is direct as a linear change of variables on the z_i induces one in the c_{ij}. ■

Example 2.11. [Triple Covers] Let $\overline{z} = (z_1, z_2)$, $q = (z_1^2, z_1 z_2, z_2^2)$ and

$$l = \begin{pmatrix} 0 & z_2 \\ -z_2 & -z_1 \\ z_1 & 0 \end{pmatrix}.$$

We want to determine the matrices $C_{2\times 3}$, $N_{3\times 2}$ and $D_{1\times 3}$. Applying a change of variables so that each of the z_i is trace free and using $(qN - \overline{z}Cl) = 0$ we get that

$$(q - \overline{z}C) = \begin{pmatrix} z_1^2 - c_1 z_1 - c_0 z_2 & z_1 z_2 + c_2 z_1 + c_1 z_2 & z_2^2 - c_3 z_1 - c_2 z_2 \end{pmatrix},$$

$$(l + N) = \begin{pmatrix} c_3 & z_2 + c_2 \\ -z_2 + 2c_2 & -z_1 + 2c_1 \\ z_1 + c_1 & c_0 \end{pmatrix}.$$

The left kernel of the matrix $(l + N)$ over S is generated by its 2×2 minors so we get

$$D = \begin{pmatrix} 2(c_0 c_2 - c_1^2) & -(c_0 c_3 - c_1 c_2) & 2(c_1 c_3 - c_2^2) \end{pmatrix}.$$

Computing D using the equation $\overline{z}CN + Dl = 0$ would give us the same result, so we find that $\mathbb{H}(\overline{R}, 1) = \mathbb{H}^{(2)}(\overline{R}, 1)$. This was expected from the work of Miranda, a triple cover is determined by an element of $\mathrm{Hom}(S^3E, \bigwedge^2 E)$, which is naturally isomorphic to $\mathrm{TCHom}(S^2E, E)$ (see [Mir85, Prop. 3.3]).

Notice that for $q = (z_i z_j)_{1 \le i \le j \le d-1}$ the ideal I_q gives us the local conditions under which a map $S^2\mathcal{E} \to \mathcal{E}$ induces an associative map $S^2\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{E}$, where $\mathcal{E}$ is a locally free rank $d-1$ $\mathcal{O}_Y$-module. If we add the linear conditions on the c_{ij} that make the variables z_i trace free, we get the local structure of a general covering map of degree d.

The case of general covering maps with degree 4 is worked out in [HM99] where it is proven that I_q + (trace free conditions) are the equations of an affine cone over $\mathrm{Gr}(2, 6)$, the Grassmannian of two dimensional subspaces of a six dimensional space, under its natural Plücker embedding in $\mathbb{P}^{14}$.

Another case of interest is when q consists of $\binom{d-2}{2} - 1$ quadratic polynomials in the ring $k[z_1, \ldots, z_{d-2}]$ that generate a Gorenstein ideal. This case fits the conditions of Proposition 2.10.

2.3. Global analysis. In this section we generalize the notion of cover homomorphism introduced by Miranda.

Definition 2.12. Given an integral scheme Y and $\mathcal{F}$ a locally free $\mathcal{O}_Y$-module of rank r, let $\mathcal{Q}$ be a direct summand of the algebra $\mathcal{R}(\mathcal{F})$ such that $\mathcal{R}(\mathcal{F})/(\mathcal{R}(\mathcal{F})\mathcal{Q})$ is a finitely generated $\mathcal{O}_Y$-module and $\mathcal{Q} \cap \mathcal{O}_Y \oplus \mathcal{F} = 0$. Then $\Phi \in \mathrm{Hom}\left(\mathcal{Q}, \mathcal{R}(\mathcal{F})/(\mathcal{R}(\mathcal{F})\mathcal{Q})\right)$ is called a **cover homomorphism** if for any open $\mathcal{U} \subset Y$ the ideal defined by the polynomials

$$q_i - \sum_{\sum_i j_i < \deg(q_i)} \left(\Phi^* \overline{z}^j\right)(q_i) \overline{z}^j \in \mathrm{Sym}^n \mathcal{F}^\vee|_{\mathcal{U}},$$

is an extension of the ideal defined by the polynomials in q, where $q = (q_i)_{1 \le i \le m}$ is a local basis of $\mathcal{Q}^\vee$ written using a local basis $(z_i)_{1 \le i \le r}$ of $\mathcal{F}^\vee$.

If q satisfies the conditions of Proposition 2.10, i.e. $\mathrm{rk}(\mathcal{F}) \ge 2$, $\mathcal{Q} \subset \mathrm{Sym}^2 \mathcal{F}$ and the first sygyzy matrix of the ideal defined by q is linear, then a cover homomorphism is locally determined by a morphism in $\mathrm{Hom}(\mathcal{Q}, \mathcal{F})$. In these cases we denote the set of cover homomorphisms by $\mathrm{CHom}\left(\mathcal{Q}, \mathcal{F}\right)$.

Lemma 2.13. *Let $\Phi \in \mathrm{Hom}(\mathcal{Q}, \mathcal{R}(\mathcal{F})/(\mathcal{R}(\mathcal{F})\mathcal{Q}))$ and $y \in Y$. Then Φ is a cover homomorphism if and only if $\Phi|_y \in \mathrm{Hom}(\mathcal{Q}_y, \mathcal{R}(\mathcal{F}_y)/(\mathcal{R}(\mathcal{F}_y)\mathcal{Q}_y)) = \mathrm{Hom}(\mathcal{Q}, \mathcal{R}(\mathcal{F})/(\mathcal{R}(\mathcal{F})\mathcal{Q})) \otimes \mathcal{O}_{Y,y}$ is a cover homomorphism.*

Proof: Let $(z_1, \ldots, z_r)$ be a basis for $\mathcal{F}^\vee$ and $(q_1, \ldots, q_m)$ a set of generators for $\mathcal{Q}^\vee \otimes \mathcal{O}_{Y,y}$. As Φ is a cover homomorphism, the $c_{ij} \equiv \Phi^*(\overline{z}^j)(q_i) \in \mathcal{O}_{Y,y}$ satisfy the relations in I_q.

Let $\mathcal{U} \subset Y$ an open set for which all $c_{ij} \in \mathcal{O}_Y(\mathcal{U})$, then for any $y' \in \mathcal{U}$, the image of the c_{ij} in $\mathcal{O}_{Y,y'}$ also satisfy the relations in I_q, as Y is integral by assumption. Furthermore, as Y is irreducible and the transition morphisms are given by $\mathcal{O}_Y$-linear automorphisms, for a set of generators $f'_i \equiv q'_i - \sum c'_{ij}\overline{z}^j$ of $\mathcal{I}_X|_{\mathcal{U}'}$, for an open set $\mathcal{U}' \subset Y$, the polynomials q'_i are linear combinations of the q_i, hence $I_q \cong I_{q'}$ and the c'_{ij} satisfy the relations in $I_{q'}$. ■

Summing up all the results we get to the following construction theorem of algebraic covers.

Theorem 2.14. *Let Y be an integral scheme, $\mathcal{L}$ a very ample line bundle on Y and $\mathcal{F}$ a locally free $\mathcal{O}_Y$-module of finite rank such that $H^0(Y, \mathcal{F}) = 0$ and $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n > 0$. If $\mathcal{Q}$ is a direct summand of $\mathcal{R}(\mathcal{F})$ such that $\mathcal{Q} \cap \mathcal{F} = 0$ and* $\operatorname{rank}(\mathcal{R}(\mathcal{F})/(\mathcal{R}(\mathcal{F})\mathcal{Q})) = d$*, then a morphism $\Phi|_y \in \operatorname{CHom}(\mathcal{Q}_y, \mathcal{R}(\mathcal{F}_y)/(\mathcal{R}(\mathcal{F}_y)\mathcal{Q}_y))$, where $y \in Y$, determines a covering map $\varphi \colon X \to Y$ of degree d such that*

1. *$\varphi^*\mathcal{L}$ is an ample line bundle on X,*
2. *$\varphi_*\mathcal{O}_X = \mathcal{R}(\mathcal{F})/(\mathcal{R}(\mathcal{F})\mathcal{Q})$,*
3. *φ factorizes as $p \circ i$ where $i \colon X \to \mathbb{A}(\mathcal{F}^\vee)$ is an embedding and $p \colon \mathbb{A}(\mathcal{F}^\vee) \to Y$ is the relative affine space*
$$\operatorname{Spec}_Y\left(\bigoplus_{i=0} \operatorname{Sym}^i H^0(Y, \mathcal{F} \otimes \mathcal{L}^n)\right)$$
over Y.

Furthermore, if $\mathcal{Q}$ is locally generated by quadratic polynomials and the ideal defined by them has a linear first syzygy matrix, then i is defined by a morphism in $\operatorname{CHom}(\mathcal{Q}, \mathcal{F}) \subset \operatorname{Hom}(\mathcal{Q}, \mathcal{F})$.

Theorem 2.14 is not a global theorem as the theorems presented in [Mir85, HM99] for triple and quadruple covers are. Nonetheless, this theorem will allow to explicitly construct the section ring of algebraic varieties polarized by ample line bundles $\varphi^*\mathcal{L}$ that induce covering morphisms over Y, a scheme for which $\mathcal{L}$ is a very ample line bundle. An important case we analyze in the next section is the case of Gorenstein covers for which we have a concrete decomposition of $\varphi_*\mathcal{O}_X$.

3. Gorenstein Covers

A covering map $\varphi\colon X \to Y$ is called a Gorenstein covering map if all its fibres are Gorenstein. This was the definition used by Casnati and Ekedahl for which they proved the following result.

Theorem 3.1. [CE96, Theorem 2.1] *Let X and Y be schemes, Y integral and let $\varphi\colon X \to Y$ be a Gorenstein cover of degree $d \geq 3$. There exists a unique $\mathbb{P}^{d-2}$-bundle $\pi\colon \mathbb{P} \to Y$ and an embedding $i\colon X \hookrightarrow \mathbb{P}$ such that $\varphi = \pi \circ i$ and $X_y := \varphi^{-1}(y) \subseteq \mathbb{P}_y := \pi^{-1}(y) \cong \mathbb{P}^{d-2}$ is a non–degenerate arithmetically Gorenstein subscheme for each $y \in Y$. Moreover the following hold.*

(1) *$\mathbb{P} \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E}^\vee \cong \operatorname{coker} \varphi^\#$, $\varphi^\#\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$.*
(2) *The composition $\varphi\colon \varphi^*\mathcal{E} \to \varphi^*\varphi_*\omega_{X|Y} \to \omega_{X|Y}$ is surjective and the ramification divisor R satisfies $\mathcal{O}_X(R) \cong \omega_{X|Y} \cong \mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.*
(3) *There exists an exact sequence $\mathcal{N}_*$ of locally free $\mathcal{O}_\mathbb{P}$-sheaves*

$$0 \leftarrow \mathcal{O}_X \leftarrow \mathcal{O}_\mathbb{P} \xleftarrow{\alpha_1} \mathcal{N}_1(-2) \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{d-3}} \mathcal{N}_{d-3}(-d+2) \xleftarrow{\alpha_{d-2}} \mathcal{N}_{d-2}(-d) \leftarrow 0$$

unique up to isomorphisms and whose restriction to the fibre $\mathbb{P}_y := \pi^{-1}(y)$ over y is a minimal free resolution of the structure sheaf of $X_y := \varphi^{-1}(y)$, in particular $\mathcal{N}_i$ is fibrewise trivial. $\mathcal{N}_{d-2}$ is invertible and, for $i = 1, \dots, d-3$, one has

$$\operatorname{rk} \mathcal{N}_i = \beta_i = \frac{i(d-2-i)}{d-1}\binom{d}{i+1},$$

hence $X_y \subset \mathbb{P}_y$ is an arithmetically Gorenstein subscheme. Moreover $\pi^*\pi_*\mathcal{N}_* \cong \mathcal{N}_*$ and $\mathcal{H}om\,(\mathcal{N}_*, \mathcal{N}_{d-2}(-d)) \cong \mathcal{N}_*$.
(4) *If $\mathbb{P} \cong \mathbb{P}(\mathcal{E}')$, then $\mathcal{E}' \cong \mathcal{E}$ if and only if $\mathcal{N}_{d-2} \cong \pi^* \det \mathcal{E}'$ in the resolution (3) computed with respect to the polarization $\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$.*

The result above is a structure theorem for Gorenstein covers. The use of $\mathcal{E}$, where $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}^\vee$, instead of $\mathcal{F}$, a generator for $\varphi_*\mathcal{O}_X$, is just a choice of notation that can be obtained by homogenizing the local equations, so the Theorem is still valid for our constructions. An important remark is that triple covers are fibrewise Gorenstein, although the global structure is only Cohen–Macaulay, so for the method we want to use we need a stronger assumption.

Definition 3.2. A covering map $\varphi\colon X \to Y$ is called a Gorenstein covering map if it is induced by an ample line bundle $\varphi^*\mathcal{L}$, where $\mathcal{L}$ is a very ample line bundle on Y such that $\omega_Y \cong \mathcal{L}^{\otimes k_Y}$ and $\omega_X \cong \varphi^*\mathcal{L}^{\otimes k_X}$, for $k_X, k_Y \in \mathbb{Z}$.

The usefulness of this definition is that for such coverings, the structure of a fibre is directly given by the structure of $\varphi_*\mathcal{O}_X$.

Proposition 3.3. *Let X and Y be schemes, Y integral, $\varphi\colon X \to Y$ a Gorenstein covering map induced by $\varphi^*\mathcal{L}$. Then, in the notation of Definition 3.2*

$$\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{F} \oplus \mathcal{L}^{\otimes k_Y - k_X}.$$

Furthermore, $\varphi_*\mathcal{O}_X$ is contained in $\mathcal{R}(\mathcal{F})$ and there is an embedding $X \hookrightarrow \mathbb{A}(\mathcal{F}^\vee)$.

Proof: By Lemma 2.2, $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$ for a locally free $\mathcal{O}_Y$-module $\mathcal{E}$, from where we get

$$\varphi_*\omega_X \cong \varphi_*\left(\varphi^*\mathcal{L}^{\otimes k_X}\right) = \mathcal{L}^{\otimes k_X} \oplus \left(\mathcal{E} \otimes \mathcal{L}^{\otimes k_X}\right).$$

At the same time, by adjunction formula, we have

$$\varphi_*\omega_X = \mathcal{H}om(\varphi_*\mathcal{O}_X, \omega_Y) = (\varphi_*\mathcal{O}_X)^\vee \otimes \mathcal{L}^{\otimes k_Y} = \mathcal{L}^{\otimes k_Y} \oplus \left(\mathcal{E}^\vee \otimes \mathcal{L}^{\otimes k_Y}\right),$$

which implies that $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{F} \oplus \mathcal{L}^{\otimes(k_Y - k_X)}$ and $\mathcal{F}^\vee \otimes \mathcal{L}^{\otimes(k_Y-k_X)} \cong \mathcal{F}$. Hence, there is an isomorphism between $\omega_X = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y)$ and $\mathcal{O}_X$.

The duality of $\varphi_*\mathcal{O}_X$ goes down to the stalks at points $y \in Y$, hence $\mathcal{O}_{X,y}$ is a Gorenstein local ring and the socle [Eis95, Ch. 21.1] of $\mathcal{O}_{X,y} \otimes k(y)$ is the term $\mathcal{L}^{\otimes(k_Y-k_X)} \otimes k(y)$. Therefore, there is a surjective map $S^2\mathcal{F}_y \otimes k(y) \mapsto \mathcal{L}^{\otimes(k_Y-k_X)} \otimes k(y)$. By Nakayama's lemma we can drop the $- \otimes k(y)$ and we get a surjective morphism on the stalks

$$S^2\mathcal{F}_y \to \mathcal{L}^{\otimes(k_Y-k_X)} \otimes \mathcal{O}_{Y,y}.$$

As the morphism is defined locally and the changes of variables are given by $\mathcal{O}_Y$ automorphisms, there is a surjective schemes morphism $S^2\mathcal{F} \to \mathcal{L}^{\otimes(k_Y-k_X)}$. ■

Using Proposition 3.3 and Theorem 2.14 we get the following theorem that gives us a method to explicitly construct Gorenstein covering maps.

Theorem 3.4. *Let X and Y be schemes, Y integral and $\varphi\colon X \to Y$ a Gorenstein covering map induced by $\varphi^*\mathcal{L}$. Then φ is determined by a cover homomorphism $\Phi \in \mathrm{CHom}(\mathcal{Q}, \mathcal{F})$, where $\mathcal{F}$ is the 'middle' component of $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{F} \oplus \mathcal{L}^{\otimes k_X - k_Y}$ and $\mathcal{Q}$ is the kernel of the map $S^2\mathcal{F} \to \mathcal{L}^{\otimes k_X - k_Y}$.*

Although $\mathcal{F}$ is determined for a Gorenstein cover, $\mathcal{Q}$ may not be. As example take the surfaces S_d such that $K(S_d) = d, p_g(S_d) = 3, q(S_d) = 0,$

$d \geq 2$. In [Cas96], Casnati proves that the canonical map of S_d induces a Gorenstein cover of degree d, $\varphi\colon X_d \to \mathbb{P}^2$, such that

$$\varphi_*\mathcal{O}_{X_d} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{d-2} \oplus \mathcal{O}_{\mathbb{P}^2}(-4).$$

As $S^2\left(\mathcal{O}_{\mathbb{P}^2}(-2)^{d-2}\right)$ is generated by $\binom{d-2}{2}$ copies of $\mathcal{O}_{\mathbb{P}^2}(-4)$, there are different possible choices for $\mathcal{Q}$. As the fibres of φ are Gorenstein ideals of codimension $d-2$, one can only completely describe the surfaces S_d for $d \leq 5$, i.e. up to codimension 3. In the next section we will construct a model for Gorenstein covers of degree 6 when we have certain extra assumptions on $\mathcal{F}$.

3.1. Codimension 4 Gorenstein Ideal.

Theorem 3.5. *Let $\varphi\colon X \to Y$ be a Gorenstein covering map such that*

$$\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus M \oplus M \oplus \wedge^2 M,$$

where M is a simple $\mathcal{O}_Y$-module with rank 2. Then

(1) *φ determines and is determined by a morphism in*

$$\mathrm{CHom}\left((S^2M)^{\oplus 3}, M^{\oplus 2}\right);$$

(2) *if $\{z_1, z_2, w_1, w_2\}$ is a local basis for $M \oplus M$ (where w_i is the image of z_i by an isomorphism from M to M), we get that $\mathrm{in}(\mathcal{I}_X)$ is locally generated by the polynomials*

$$q = \begin{pmatrix} z_1^2 & z_1z_2 & z_2^2 & z_1w_1 & \frac{1}{2}(z_1w_2 + z_2w_1) & z_2w_2 & w_1^2 & w_1w_2 & w_2^2 \end{pmatrix}; \qquad (3.1)$$

(3) *the ideal I_q determining the local structure of a fibre of φ is given by the polynomials defining the spinor embedding of the affine orthogonal Grassmannian $\mathrm{aOGr}(5,10)$ in $\mathbb{P}^{15}$;*

(4) *φ is a deformation of an S_3-Galois branched cover (which corresponds to a linear section of the embedding of $\mathrm{aOGr}(5,10)$ in $\mathbb{P}^{15}$).*

Proof: By Theorem 3.4, φ is determined by an element of $\mathrm{CHom}(\mathcal{Q}, M \oplus M)$, where $\mathcal{Q}$ is a direct summand of $S^2(M \oplus M)$. As M is simple

$$\begin{aligned} \mathrm{Hom}(M,M) = k &\Leftrightarrow \mathrm{Hom}(M \otimes M^\vee, \mathcal{O}_Y) = k \\ &\Leftrightarrow \mathrm{Hom}(M \otimes M^\vee \otimes \wedge^2 M, \wedge^2 M) = k \\ &\Leftrightarrow \mathrm{Hom}(M \otimes M, \wedge^2 M) = k, \end{aligned}$$

hence $(S^2M)^{\oplus 3}$ is the kernel of the morphism $S^2(M \oplus M) \to \wedge^2 M$, proving (1). (2) is a consequence of (1) as q is a basis for $(S^2M)^{\oplus 3}$.

Using q, we run the algorithm described in Section 2.2 (see Appendix) and get

$$C^t = \begin{pmatrix} c_{32}+2c_{43} & -c_{33} & -c_{13} & c_{03} \\ c_{53} & c_{32} & -c_{23} & c_{13} \\ -c_{52} & 2c_{42}+c_{53} & c_{22} & c_{23} \\ c_{73} & -c_{63} & c_{32} & c_{33} \\ -\frac{1}{2}c_{72}+\frac{1}{2}c_{83} & \frac{1}{2}c_{62}-\frac{1}{2}c_{73} & c_{42} & c_{43} \\ -c_{82} & c_{72} & c_{52} & c_{53} \\ -c_{71} & c_{61} & c_{62} & c_{63} \\ -c_{81} & c_{71} & c_{72} & c_{73} \\ c_{80} & c_{81} & c_{82} & c_{83} \end{pmatrix}.$$

With a change of variables

$$\begin{pmatrix} z_0 \\ z_1 \\ w_0 \\ w_1 \end{pmatrix} \mapsto \begin{pmatrix} z_0-(c_{32}+c_{43}) \\ z_1-(c_{42}+c_{53}) \\ w_0-(c_{62}+c_{73}) \\ w_1-(c_{72}+c_{83}) \end{pmatrix}$$

we get the matrix

$$C^t = \begin{pmatrix} c_{43} & -c_{33} & -c_{13} & c_{03} \\ c_{53} & -c_{43} & -c_{23} & c_{13} \\ -c_{52} & -c_{53} & c_{22} & c_{23} \\ c_{73} & -c_{63} & -c_{43} & c_{33} \\ c_{83} & -c_{73} & -c_{53} & c_{43} \\ -c_{82} & -c_{83} & c_{52} & c_{53} \\ -c_{71} & c_{61} & -c_{73} & c_{63} \\ -c_{81} & c_{71} & -c_{83} & c_{73} \\ c_{80} & c_{81} & c_{82} & c_{83} \end{pmatrix} = \begin{pmatrix} c'_{11} & c'_{10} & c'_{01} & c'_{00} \\ -c'_{12} & -c'_{11} & -c'_{02} & -c'_{01} \\ c'_{13} & c'_{12} & c'_{03} & c'_{02} \\ -c'_{21} & -c'_{20} & -c'_{11} & -c'_{10} \\ c'_{22} & c'_{21} & c'_{12} & c'_{11} \\ -c'_{23} & -c'_{22} & -c'_{13} & -c'_{12} \\ c'_{31} & c'_{30} & c'_{21} & c'_{20} \\ -c'_{32} & -c'_{31} & -c'_{22} & -c'_{21} \\ c'_{33} & c'_{32} & c'_{23} & c'_{22} \end{pmatrix}.$$

In the last matrix we renamed the variables to have a better look at the structure of C and we see that C has the same decomposition as a triple cover homomorphism where each of the entries is a triple cover homomorphism,

$$C^t = \begin{pmatrix} C_1 & C_0 \\ -C_2 & -C_1 \\ C_3 & C_2 \end{pmatrix}, C_i = \begin{pmatrix} c_{i1} & c_{i0} \\ -c_{i2} & -c_{i1} \\ c_{i3} & c_{i2} \end{pmatrix}. \tag{3.2}$$

With the second step we get the vector D

$$D^t = \begin{pmatrix} -2c_{11}^2 + 2c_{10}c_{12} + 2c_{01}c_{21} - c_{02}c_{20} - c_{00}c_{22} \\ -c_{10}c_{13} + c_{11}c_{12} - 2c_{02}c_{21} + c_{03}c_{20} + c_{01}c_{22} \\ 2c_{11}c_{13} - 2c_{12}^2 - c_{03}c_{21} - c_{01}c_{23} + 2c_{02}c_{22} \\ -c_{01}c_{31} + c_{00}c_{32} + c_{11}c_{21} + c_{12}c_{20} - 2c_{10}c_{22} \\ \frac{1}{2}(-c_{00}c_{33} + c_{01}c_{32} - 5c_{12}c_{21} + c_{13}c_{20} + 4c_{11}c_{22}) \\ c_{01}c_{33} - c_{02}c_{32} + c_{13}c_{21} - 2c_{11}c_{23} + c_{12}c_{22} \\ 2c_{11}c_{31} - c_{12}c_{30} - c_{10}c_{32} - 2c_{21}^2 + 2c_{20}c_{22} \\ c_{12}c_{31} + c_{10}c_{33} - 2c_{11}c_{32} - c_{20}c_{23} + c_{21}c_{22} \\ -c_{13}c_{31} - c_{11}c_{33} + 2c_{12}c_{32} + 2c_{21}c_{23} - 2c_{22}^2 \end{pmatrix}$$

and all the quadratic relations that the c_{ij} need to satisfy

$$I_q = \begin{pmatrix} c_{00}c_{13} - 3c_{01}c_{12} + 3c_{02}c_{11} - c_{03}c_{10} \\ c_{00}c_{23} - 3c_{01}c_{22} + 3c_{02}c_{21} - c_{03}c_{20} \\ c_{10}c_{33} - 3c_{11}c_{32} + 3c_{12}c_{31} - c_{13}c_{30} \\ c_{20}c_{33} - 3c_{21}c_{32} + 3c_{22}c_{31} - c_{23}c_{30} \\ c_{00}c_{31} - c_{01}c_{30} - 3c_{10}c_{21} + 3c_{11}c_{20} \\ c_{00}c_{32} - c_{02}c_{30} - 3c_{10}c_{22} + 3c_{12}c_{20} \\ c_{00}c_{33} - c_{03}c_{30} - 9c_{11}c_{22} + 9c_{12}c_{21} \\ c_{01}c_{33} - c_{03}c_{31} - 3c_{11}c_{23} + 3c_{13}c_{21} \\ c_{02}c_{33} - c_{03}c_{32} - 3c_{12}c_{23} + 3c_{13}c_{22} \\ c_{01}c_{32} - c_{02}c_{31} - c_{10}c_{23} + c_{13}c_{20} \end{pmatrix}.$$

Let us recall the definition of $\mathrm{OGr}(5,10)$ as presented in [CR02]. Consider the vector space $V = \mathbb{C}^{10}$ with p a nondegenerate quadratic form. Using a change of basis we can put p in the form

$$p = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \text{ that is, } V = U \oplus U^\vee.$$

A vector space $F \subset V$ is isotropic if p is identically zero on F, for example U is an isotropic 5-space. The orthogonal Grassmannian variety $\mathrm{OGr}(5,10)$ is defined as

$$\mathrm{OGr}(5,10) = \left\{ F \in \mathrm{Gr}(5,V) \;\middle|\; \begin{array}{c} \text{F is isotropic for } p \\ \text{and } \dim F \cap U \text{ is odd} \end{array} \right\}.$$

The equations of the affine spinor embedding of $\mathrm{aOGr}(5,10)$ on $\mathbb{P}^{15}$ can be written as the 10 equations centered at x

$$xv - \operatorname{Pfaff} M = 0 \text{ and } Mv = 0,$$

where

$$M = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ & x_{23} & x_{24} & x_{25} \\ & & x_{34} & x_{35} \\ -\,\mathrm{sym} & & & x_{45} \end{pmatrix} \text{ and } v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

Applying to I_1, consider the matrices

$$M = \begin{pmatrix} 3c_{21} & c_{30} & -c_{33} & -c_{31} \\ & -c_{00} & c_{03} & c_{01} \\ & & 3c_{12} & -c_{10} \\ \text{-sym} & & & c_{13} \end{pmatrix}, v = \begin{pmatrix} c_{02} \\ c_{32} \\ c_{23} \\ c_{20} \\ 3c_{22} \end{pmatrix}$$

then I_q is given by the equations

$$\begin{cases} 3c_{11}v - \mathrm{Pfaff}(M) = 0 \\ Mv = 0 \end{cases},$$

that up to rescaling are the equations of $\mathrm{aOGr}(5, 10)$ centered at $3c_{11}$.

Taking each $C_i = k_i\widetilde{C}$, where $k_i \in k$ and $\widetilde{C}$ is a triple cover block

$$\widetilde{C} = \begin{pmatrix} c_1 & c_0 \\ -c_2 & -c_1 \\ c_3 & c_2 \end{pmatrix},$$

all the relations in I_q are satisfied (notice the index homogeneity of each polynomial defining I_q). This is a linear component of $\mathrm{aOGr}(5, 10)$. In Theorem 3.7 we prove it is an S_3-Galois branched cover. As $\mathrm{aOGr}(5, 10)$ is connected, statement (4) holds. ■

3.2. Linear component - an S_3-Galois branched cover. In this section we study a component of the degree 6 covering maps described above, i.e. the component of $\mathrm{aOGr}(5, 10)$ whose local equations are given by $C_i = k_i\widetilde{C}$, for $k_i \in k$ and $\widetilde{C}$ a 'triple cover block'

$$\widetilde{C} = \begin{pmatrix} c_1 & c_0 \\ -c_2 & -c_1 \\ c_3 & c_2 \end{pmatrix}.$$

One can write such covering homomorphism as

$$\begin{array}{rcl}\Phi\left(S^2(Z)\right) & = & k_1\widetilde{C}Z + k_0\widetilde{C}W \\ \Phi\left(S^2(Z,W)\right) & = & -k_2\widetilde{C}Z - k_1\widetilde{C}W \\ \Phi\left(S^2(W)\right) & = & k_3\widetilde{C}Z + k_2\widetilde{C}W,\end{array} \tag{3.3}$$

where, by abuse of notation, $Z = \begin{pmatrix} z_1 & z_2\end{pmatrix}^t, W = \begin{pmatrix} w_1 & w_2\end{pmatrix}^t$ and $S^2(Z)$, $S^2(Z,W)$ and $S^2(W)$ are vectors with the second symmetric powers of z_i and w_i as entries.

This allows one to think of Z and W as a basis for $\mathbb{A}^2_Y$ and the cover homomorphism above as a cover homomorphism defining three points over a field. In particular, for $g \in \mathrm{GL}(2,k)$, one has the following action on a local basis

$$\begin{pmatrix} z_i & w_i \end{pmatrix} \mapsto g \begin{pmatrix} z_i & w_i \end{pmatrix}, \;\; i \in \{1,2\}.$$

In the next proposition we study such actions.

Proposition 3.6. *Let $\{q_1, q_2, q_3\} \subset \mathbb{A}^2 = \mathrm{Spec}\,(k[z,w])$ be three points defined by the vanishing of the following polynomials*

$$\begin{pmatrix} z^2 - k_1 z - k_0 w + 2(k_0k_2 - k_1^2) \\ zw + k_2 z + k_1 w - (k_0k_3 - k_1k_2) \\ w^2 - k_3 z - k_2 w + 2(k_1k_3 - k_2^2)\end{pmatrix}.$$

Then the following hold

(1) *the q_i are distinct if and only if*

$$\Delta_{tc}(k_0,k_1,k_2,k_3) := k_0^2k_3^2 + 4k_0k_2^3 - 3k_1^2k_2^2 + 4k_1^3k_3 - 6k_0k_1k_2k_3 \neq 0;$$

The branch locus of a triple cover is defined by $\Delta_{tc} = 0$.

(2) $\sum_i z(q_i) = \sum_i w(q_i) = 0$, *i.e. the origin is the barycenter of the points;*

(3) *via a linear change of variables on the basis $\{z,w\}$ we can change any quadruple $(k_i)_{0\le i\le 3}$ defining three distinct points into any other for which $\Delta_{tc}(k_i') \neq 0$;*

(4) *each quadruple (k_0,k_1,k_2,k_3) not in the branch locus is fixed by a representation of S_3, the symmetric group of order 3, in* $\mathrm{GL}(2,k)$.

Statement (1) was already proved in [Mir85, Lemma 4.5]. We will show a different proof.

Proof: (1): Introducing a variable t to homogenize the three equations and eliminating the variable t from the ideal we get the polynomial

$$\begin{aligned}(k_1k_2k_3 - \tfrac{1}{3}k_0k_3^2 - \tfrac{2}{3}k_2^3)z^3 + (2k_1^2k_3 - k_1k_2^2 - k_0k_2k_3)z^2w \\ +(k_1^2k_2 + k_0k_1k_3 - 2k_0k_2^2)zw^2 + (\tfrac{2}{3}k_1^3 - k_0k_1k_2 + \tfrac{1}{3}k_0^2k_3)w^3.\end{aligned} \tag{3.4}$$

Recall that the discriminant of a cubic polynomial $p(x) = ax^3+bx^2+cx+d$ is

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

(see [Muk03, Lemma 1.19, Example 1.22]), hence we get the result by direct computation.

(2): Assume that the points are not collinear. Applying a linear change of variables on (z, w), we can assume that q_1, q_2 are vertex points, i.e. $z(q_1) = w(q_2) = 1$ and $z(q_2) = w(q_1) = 0$.

Evaluating the polynomial $z^2 - k_1z - k_0w + 2(k_0k_2 - k_1^2)$ on the three points

$$\begin{cases} 0 &= 1 - k_1 + 2(k_0k_2 - k_1^2) \\ 0 &= -k_0 + 2(k_0k_2 - k_1^2) \\ 0 &= z^2(q_3) - k_1z(q_3) - k_0w(q_3) + 2(k_0k_2 - k_1^2) \end{cases}$$

one concludes that $k_1 = 1 + \frac{z^2(q_3)-z(q_3)}{z(q_3)+w(q_3)-1}$. Notice that $z(q_3) + w(q_3) \neq 1$ as equality would imply collinearity of the points. From the polynomial $zw + k_2z + k_1w - (k_0k_3 - k_1k_2)$ we get the following equations

$$\begin{cases} 0 &= k_2 - (k_0k_3 - k_1k_2) \\ 0 &= k_1 - (k_0k_3 - k_1k_2) \\ 0 &= zw(q_3) + k_2z(q_3) + k_1w(q_3) - (k_0k_3 - k_1k_2), \end{cases}$$

which implies that $k_1 = -\frac{zw(q_3)}{z(q_3)+w(q_3)-1}$ and therefore, $z(q_3) = -1$. By the same argument $w(q_3) = -1$ which implies that $\sum_i z(q_i) = \sum_i w(q_i) = 0$. This still holds after any change of variables. We are left with the case of three collinear points.

If q_1, q_2, q_3 are distinct but lie in a line l, by a linear change of variables, one can assume that $l = \{z = 0\}$. Then, as $z(q_i) = 0$ for all i, the equation $w^2 - k_2w + 2(k_1k_3 - k_2^2) = 0$ has only two solutions which is a contradiction. We conclude that the points are collinear if and only if they are in the ramification locus.

As the points in the ramification locus can be obtained as the image of a linear map of three distinct points in $\mathbb{A}^2$, and a $\mathrm{GL}(2, k)$ map on (z, w) gives the same linear transformation of the pair $(\sum_i z(q_i), \sum_i w(q_i))$, this pair is always $(0,0)$.

(3): We proved above that for each quadruple $(k_i) \notin \Delta_{tc}$, the polynomials vanish in three non-collinear points with the origin as their barycenter, i.e. these points are defined by any two of them. With a linear change of basis one can send them to any points at will, which proves the result as three points correspond to a single quadruple (k_i).

(4): Using (3), take $(k_0, k_1, k_2, k_3) = (1,0,0,1)$. Then $q_1 = (1,1)$, $q_2 = (\epsilon, \epsilon^2)$ and $q_3 = (\epsilon^2, \epsilon)$, where ϵ is a cubic root of unity. The action of S_3 on these three points is generated by a rotation r sending q_i into q_{i+1} and a reflection ι changing q_2 with q_3. The representation of S_3 on k^2 is given by

$$r \mapsto \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}, \ \iota \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

■

Theorem 3.7. *Let $\varphi\colon X \to Y$ be a Gorenstein covering map of degree 6 such that $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus M \oplus M \oplus \wedge^2 M$. Assume that φ is defined by a general linear morphism $\Phi \in \mathrm{CHom}\left((S^2M)^{\oplus 3}, M^{\oplus 2}\right)$, i.e. for a local choice of basis $\{z_i, w_i\}_{i=1,2}$ it can be written as in (3.3) with $(k_i) \in k^4 \backslash \Delta(k_i)_{tc}$. Then X is an S_3-Galois branched cover. Furthermore*

(1) *with a linear change of coordinates I_X can locally be written in the form*

$$\begin{pmatrix} S^2(Z) - \widetilde{C}W \\ S^2(Z,W) - D \\ S^2(W) - \widetilde{C}Z \end{pmatrix},$$

with

$$\widetilde{C} = \begin{pmatrix} c_1 & c_0 \\ -c_2 & -c_1 \\ c_3 & c_2 \end{pmatrix}, \widetilde{D} = \begin{pmatrix} 2(-c_0c_2 + c_1^2) \\ c_0c_3 - c_1c_2 \\ 2(-c_1c_3 + c_2^2) \end{pmatrix}.$$

(2) *$X/\mathbb{Z}_2$ is a triple cover $\varphi_2\colon X/\mathbb{Z}_2 \to Y$ such that $\varphi_{2*}\mathcal{O}_{X/\mathbb{Z}_2} = \mathcal{O}_Y \oplus M$;*

(3) *$X/\mathbb{Z}_3$ is a double cover $\varphi_3\colon X/\mathbb{Z}_3 \to Y$ such that $\varphi_{3*}\mathcal{O}_{X/\mathbb{Z}_3} = \mathcal{O}_Y \oplus \wedge^2 M$.*

Proof: By Proposition 3.6 - (4), the S_3 action is just the action fixing the quadruple (k_i). Using Proposition 3.6 - 3, we can set (k_i) to be $(1,0,0,1)$ proving (1).

To prove (2) and (3), just notice that $(1, 0, 0, 1)$ is fixed by the actions of r and ι presented in the proof of Proposition 3.6 - (4). Locally, the submodule of $\varphi_*\mathcal{O}_X$ left invariant by the action of ι is generated by $\{1, z_1 + w_1, z_2 + w_2\}$. Hence the ideal defining $\mathcal{O}_{X/\mathbb{Z}_2}$ is locally defined by the vanishing of the polynomials

$$S^2(Z) + 2S^2(Z, W) + S^2(W) - \widetilde{C}(Z + W) - D.$$

The submodule left invariant by the action of r is generated by $\{1, z_1w_2 - z_2w_1\}$. The ideal of $\mathcal{O}_{X/\mathbb{Z}_3}$ is locally generated by the vanishing of the polynomial

$$\left(\frac{z_1w_2 - z_2w_1}{2}\right)^2 - \left((c_0c_3 - c_1c_2)^2 - 4(-c_1c_3 + c_2^2)(-c_0c_2 + c_1^2)\right)$$

$$\Leftrightarrow \left(\frac{z_1w_2 - z_2w_1}{2}\right)^2 - (c_0^2c_3^2 + 4c_0c_2^3 - 3c_1^2c_2^2 + 4c_1^3c_3 - 6c_0c_1c_2c_3).$$

Notice that $(z_1w_2 - z_2w_1)$ is a local generator of $\wedge^2 M$. Furthermore, as all these actions are independent of the choice of basis, we conclude

$$(\varphi_*\mathcal{O}_X)^{\mathbb{Z}_2} \cong \mathcal{O}_Y \oplus M \text{ and } (\varphi_*\mathcal{O}_X)^{\mathbb{Z}_3} \cong \mathcal{O}_Y \oplus \wedge^2 M.$$

∎

4. Appendix

```
S.<z0, z1, w0, w1, nij, di, cij>=QQ[]
from sage.libs.singular.function_factory import
                                  singular_function
minbase = singular_function('minbase')
I = S.ideal([z0**2, z0*z1, z1**2, z0*w0,
       (1/2)*(z0*w1 + z1*w0), z1*w1, w0**2, w1*w0, w1**2])
M = I.syzygy_module()
F = matrix(S, 9,1, lambda i,j: I.gens()[i]);
v=matrix(S,4,1,[z0,z1,w0,w1])
N = matrix(S, 16, 9, lambda i, j: 'n'+ str(i) + str(j));
C = matrix(S, 9, 4, lambda i, j: 'c'+ str(i) + str(j));
D = matrix(S, 9, 1, lambda i, j: 'd'+ str(i) + str(j));
V = [z0**2, z0*z1, z1**2,w0**2, w1*w0,
                  w1**2, z0*w0, z1*w1, z0*w1, z1*w0]
Z2 = M*C*v+N*F
```

```
a = [c32 + c43,c42 + c53, c72 + c83,  c62 + c73]
for i in range(Z2.nrows()):
for j in range(len(V)):
a.append(Z2[i][0].coefficient(V[j]))
J = S.ideal(a)
R=S.quotient_ring(J)
R.inject_variables()
N = matrix(R, 16, 9, lambda i, j: 'n'+ str(i) + str(j));
C = matrix(R, 9, 4, lambda i, j: 'c'+ str(i) + str(j));
Clift = matrix(S, C.nrows(), C.ncols(),
                           lambda i, j: C[i][j].lift())
Nlift = matrix(S, N.nrows(), N.ncols(),
                           lambda i, j: N[i][j].lift())

Z1 = Nlift*Clift*v + M*D
b=[]
for i in range(Z1.nrows()):
for j in range(v.nrows()):
b.append(Z1[i][0].coefficient(v[j][0]))
JJ = S.ideal(b)
```

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