Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–09

LINEAR PRESERVERS FOR THE q-PERMANENT, CYCLE q-PERMANENT EXPANSIONS, AND POSITIVE CROSSINGS IN DIGRAPHS

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Abstract: The q-permanent linear preservers are described. We give several expansion formulas for the q-permanent of a square matrix, based on the cycle factorization of permutations. Some of these formulas are valid for all matrices, but others are not; for each such formula Φ we determine all digraphs D such that Φ holds for all matrices with digraph D. Proof techniques are based on combinatorial results, relating the length (number of inversions) of a permutation, the lengths of its cycles, and a delicate counting of crossings, jumps, and arc-under-arc relations in digraphs. We get new algebraic characterizations of noncrossing [acyclic] graphs.

Keywords: q-permanent, determinant, polynomial identities, digraphs, permutations.
Math. Subject Classification (2010): 15A15, 15A86, 05C20, 05C50.

1. Introduction

This paper is a continuation of [19], and gives a substantial improvement to the main results of a previous preprint [18]. The protagonist is the q-permanent of an n-square matrix $A = (a_{ij})$, a polynomial given by

$$\operatorname{per}_q A = \textstyle\sum_{\sigma\in\mathscr{S}_n} q^{\ell(\sigma)} \prod_{i=1}^{n} a_{i\sigma_i}\,.$$

Here, $\mathscr{S}_n$ is the symmetric group of order n, and $\ell(\sigma)$ denotes the *length* of σ, defined as the number of inversions of the permutation σ. In [13, 5, 21, 24] the reader will find the genesis and uses of this function in the areas of mathematical physics, and quantum groups and algebras. Further developments may be found in [1, 10, 2, 11].

Section 3 describes the q-permanent linear preservers. In [22, 23] the q-permanent is generalized to multivariable quantum parameters and, in this context, some expansions are obtained for the q-permanent which are reminiscent of the archetypal expansions of Laplace along a set of rows or columns.

Received March 20, 2018.
This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

The expansions considered below (in sections 4, 7, and 8) are of a different nature, in that we collect the q-permanent terms according to the cycle structure of the digraph of the matrix A, as has been done for the determinant in [15, 17, 16]. In section 5, we relate the length of a permutation, the lengths of its cycles, and the number of positive crossings in the corresponding digraphs. In section 6 we show how to express the number of positive crossings, using jumps of arcs over vertices, and arc-under-arc relations. This paves the way to sections 7 and 8, where the main q-permanent expansion formula is modified in several ways. Each modified formula Φ is *combinatorially solved*, *i.e.*, all digraphs D are found such that Φ holds for the generic matrices with digraph D.

In the referred combinatorial solutions to our tentative expansions of the q-permanent, new algebraic characterizations emerge for interesting classes of graphs, like noncrossing graphs, and noncrossing acyclic graphs.

2. Preliminaries

On digraphs, graphs and matrices we follow the traditional concepts as may be seen in, *e.g.*, [6, 7], with minor changes. The set $V(D)$ of the vertices of a [di]graph D is a subset of $[n] = \{1, \dots, n\}$. Notations like $(i, j) \in E \subseteq D$ mean that (i, j) is an arc of E, and E is a subdigraph of D; an arc is also denoted $i \to j$. We write $[r, s]$, $]r, s]$, etc., to refer *integer intervals*. By *disjoint digraphs* we mean *vertex* disjoint digraphs, unless otherwise specified.

On the concept of *(oriented) cycle*, as a digraph and as a permutation, we follow the conventions of [19], except that a loop is considered here as a cycle. Thus a k-cycle, often denoted in short notation, $c = (v_1 v_2 \cdots v_k)$, is a digraph with vertex set $V(c) = \{v_1, \dots, v_k\}$, and arcs $v_i \to v_{i+1}$, with i read modulo k. The set of all cycles through a given vertex v is denoted by $\mathscr{C}_v$, or $\mathscr{C}_v(n)$ if needed; we may identify $\mathscr{C}_v$ with a set of cyclic permutations of $\mathscr{S}_n$. The sole 1-cycle of $\mathscr{C}_v$ is the loop (v).

The set of n-square matrices over a field $\mathbb{F}$ is denoted by $\mathfrak{M}_n$. For $A \in \mathfrak{M}_n$ and $S \subseteq [n]$, $A(S)$ or A_S denote the principal submatrix obtained by eliminating the rows and columns of A indexed by the elements of S; the notations $A_{\{i\}}, A_{\{i,j\}}, A_{V(E)}$, where E is a digraph, will be simplified to A_i, A_{ij}, A_E. The matrix $A_S^{\checkmark}$ is obtained by zeroing out the rows and columns of A corresponding to the elements of S, except the diagonal elements a_{ii}, that are replaced with 1's, for $i \in S$; the abbreviations $A_i^{\checkmark}, A_{ij}^{\checkmark}, A_E^{\checkmark}$ will be used with the obvious meanings.

The digraph of A is denoted $D(A)$. A matrix A is said to be *generic* if its nonzero entries are independent commuting variables over the base field. For a digraph E and a permutation σ, the *weight of E in A* and the *total weight of σ in A* are defined by

$$\mathrm{wt}_E(A) = \prod\nolimits_{(i,j)\in E} a_{ij}, \quad \text{and} \quad \mathrm{twt}_\sigma(A) = \prod\nolimits_{i\in[n]} a_{i\sigma_i}.$$

We let P_σ be the permutation matrix with ij-entry $\delta_{\sigma(i),j}$. So $\mathrm{per}_q\, P_\sigma = q^{\ell(\sigma)}$. The symbols A^T and A^R denote, respectively, the transpose and the *reverse* of A, the latter being obtained by reversing the order of the rows, and of the columns of A. Hence $A^R = P_{w_\circ} A P_{w_\circ}$, where $w_\circ$ is the so-called *reversal permutation*, given by $w_\circ(i) = n + 1 - i$. We also use the upper-scripts E^T and E^R, for a digraph E. Thus (i, j) is an arc of E^T [E^R] iff (j, i) [resp., $(w_\circ(i), w_\circ(j))$] is an arc of E. Clearly then $D(A)^T = D(A^T)$, and $D(A)^R = D(A^R)$.

3. q-Permanent linear preservers

This section is a contribution to the theory of linear preservers (see, *e.g.*, [14, 4, 12, 8]). A linear mapping $\mathscr{L} : \mathfrak{M}_n \to \mathfrak{M}_n$ is said to *preserve the q-permanent*, if $\mathrm{per}_q \mathscr{L}(X) = \mathrm{per}_q X$, for all $X \in \mathfrak{M}_n$.

Theorem 3.1. *The set of linear operators $\mathscr{L}$ on $\mathfrak{M}_n$ that preserve the q-permanent is the group generated by the transformations of the following three kinds:* (a) $X \rightsquigarrow \Gamma X \Delta$, *where Γ and Δ are diagonal matrices such that* $\mathrm{per}(\Gamma\Delta) = 1$*;* ($b$) $X \rightsquigarrow X^T$*; and* (c) $X \rightsquigarrow X^R$.

Proof: By [14, 4], the set of the permanent linear preservers is the group generated by (a), (b), and the transformations $X \rightsquigarrow PXQ$, where P, Q are permutation matrices. Clearly, (a) preserves the q-permanent, because $\mathrm{twt}_\sigma(\Gamma X \Delta) = \mathrm{twt}_\sigma(X)$, for all σ. Moreover, (b) preserves the q-permanent, because $\mathrm{twt}_\sigma(X^T) = \mathrm{twt}_{\sigma^{-1}}(X)$, and $\ell(\sigma^{-1}) = \ell(\sigma)$.

To handle the case of a permutational equivalence, $X \rightsquigarrow PXQ$, let α and β be the permutations such that $P_\alpha = Q$ and $P_\beta = P$. Suppose that $\mathrm{per}_q(PXQ) = \mathrm{per}_q X$, for all X. For an arbitrary $\sigma \in \mathscr{S}_n$, we replace X with P_σ, to obtain $\mathrm{per}_q(P_\beta P_\sigma P_\alpha) = \mathrm{per}_q P_\sigma$. As $P_\beta P_\sigma P_\alpha = P_{\alpha\sigma\beta}$, we have $q^{\ell(\alpha\sigma\beta)} = q^{\ell(\sigma)}$. Therefore $\ell(\alpha\sigma\beta) = \ell(\sigma)$.

Conversely, suppose that $\ell(\alpha\sigma\beta) = \ell(\sigma)$, for all σ. We first compute the ij-entry of PXQ, and relate total weights in X and PXQ:

$$\begin{aligned} \big[PXQ\big]_{ij} &= \big[P_\beta X P_\alpha\big]_{ij} = x_{\beta(i),\alpha^{-1}(j)} \\ \operatorname{twt}_{\alpha\sigma\beta}(PXQ) &= \textstyle\prod_{i=1}^n \big[PXQ\big]_{i,\alpha\sigma\beta(i)} = \prod_{i=1}^n x_{\beta(i),\sigma\beta(i)} = \operatorname{twt}_\sigma(X). \end{aligned}$$

From this we get

$$\operatorname{per}_q(PXQ) = \textstyle\sum_{\sigma\in\mathscr{S}_n} q^{\ell(\alpha\sigma\beta)} \operatorname{twt}_{\alpha\sigma\beta}(PXQ) = \operatorname{per}_q X.$$

We thus proved that $X \rightsquigarrow PXQ$ preserves the q-permanent, if and only if $\ell(\alpha\sigma\beta) = \ell(\sigma)$ for all σ. Plugging $\sigma = \mathrm{id}$ in this equation, we get $\ell(\alpha\beta) = \ell(\mathrm{id})$. So $\beta = \alpha^{-1}$, because id is the only permutation of length 0. In this way, we are left with the problem of finding the inner automorphisms of $\mathscr{S}_n$,

$$\sigma \rightsquigarrow \alpha\sigma\alpha^{-1}$$

that preserve the length. It is well-known [3, p. 38 *ff*], that the only inner automorphisms preserving length, are the identity, and the reversal permutation. An elementary proof of this for the Coxeter group $\mathscr{S}_n$, is the following. Write $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ (in complete notation). Put $\sigma = (i\ i+1)$; then we have $\ell((\alpha_i\ \alpha_{i+1})) = 1$. This means that α_i, α_{i+1} are consecutive integers, for all $i < n$. As the α_i are distinct indices, we either have $\alpha = 12\cdots n$, or $\alpha = n\cdots 21$. That is, $\alpha \in \{\mathrm{id}, w_\circ\}$.

Therefore, only two linear mappings of the kind $X \rightsquigarrow PXQ$ preserve the q-permanent, namely the identity, and the reversion. The remaining proof details are left to the reader. ■

4. Factors and cycle expansions

The Laplace expansions of $\det A$, along a set of rows (or columns) of A, follow a well-known strategy. Namely, each building block is the determinant of a square submatrix, times the determinant of the respective complementary submatrix; then $\det A$ is expressed as a linear combination of such building blocks. For the permanent, the same method goes nicely. However, for the q-permanent, the complementary minor strategy does not apply *stricto sensu*. For example, suppose we wish to expand $\operatorname{per}_q A$ along row v, aiming at a combination of the form

$$\operatorname{per}_q A = \textstyle\sum_{i=1}^n \psi_j(q) a_{vj} \operatorname{per}_q A(v|j), \tag{1}$$

with polynomial coefficients $\psi_j \in \mathbb{F}[q]$, where $A(v|j)$ is the complementary submatrix of a_{vj}. For $v = 1$ and $v = n$, (1) holds for appropriate values of the $\psi_j(q)$ [22, 24, 2]. However, for $1 < v < n$, the equation (1) fails (as a polynomial identity in the variables q and the entries of A), for all choices of $\psi_1, \ldots, \psi_n$ (hint: in (1), replace A with the identity matrix, and then with the permutation matrix $P_{(1n)}$). We may overcome this difficulty, by replacing the submatrix $A(v|j)$ with the matrix, denoted $A^{\checkmark}_{(v|j)}$, resulting from A by making $a_{vj} = 1$, and zeroing out all other entries of A in row v, and in column j. We trivially have

$$\operatorname{per}_q A = \textstyle\sum_{i=1}^{n} a_{vj} \operatorname{per}_q A^{\checkmark}_{(v|j)}. \tag{2}$$

This is essentially done in [22, 23], in the more general setting of multivariable quantum parameters. Moreover, [22, 23] show that, in (2), we may replace $A^{\checkmark}_{(v|j)}$ with a matrix obtained by multiplying by q some entries of $A(v|j)$. In such multiparameter setting, [22, 23] also give expansions of $\operatorname{per}_q A$, in Laplace style, along r rows, say $v_1, \ldots, v_r$, in case $\{v_1, \ldots, v_r\}$ is a leading, or trailing subinterval of $[n]$. To my knowledge, the quest for this sort of Laplace expansions is still unfinished.

A different expansion strategy for $\det A$ has been followed in [15, 17, 16]. For each digraph $\mathfrak{f}$, that is a union of disjoint cycles, we take the product of the weight $\mathrm{wt}_{\mathfrak{f}}(A)$, by the determinant of $A_{\mathfrak{f}}$. Then we try to express $\det A$ as a linear combination of the possible products $\mathrm{wt}_{\mathfrak{f}}(A) \det A_{\mathfrak{f}}$. There are many ways to achieving this goal, that we may call *cycle expansions* for $\det A$. We now generalize those procedures to get cycle expansions for $\operatorname{per}_q A$.

Given $\sigma \in \mathscr{S}_n$, the *permutation digraph*, Δ_σ, has $i \to j$ as arc, iff $\sigma(i) = j$. A *factor* is a digraph, $\mathfrak{f}$, that consists of a union of (vertex) disjoint cycles [16]. A *spanning factor* is a factor having $[n]$ as set of vertices. Thus, 'spanning factor' and 'permutation digraph' are synonymous. If $\mathfrak{f}$ and $\mathfrak{g}$ are disjoint factors, the union of $\mathfrak{f}$ and $\mathfrak{g}$ is still a factor, that we represent in juxtaposed form, $\mathfrak{f}\mathfrak{g}$. Thus, if $\mathfrak{c}_1, \ldots, \mathfrak{c}_m$ are the cycles of $\mathfrak{f}$, we may write $\mathfrak{f} = \mathfrak{c}_1 \cdots \mathfrak{c}_m$. A factor $\mathfrak{f}$ determines a permutation $p(\mathfrak{f})$, which is the product of the cycles of $\mathfrak{f}$ viewed as cyclic permutations. The length of $p(\mathfrak{f})$, is also called *length of* $\mathfrak{f}$, and denoted by $\ell(\mathfrak{f})$. If $\mathfrak{f}$ and $\mathfrak{g}$ are disjoint factors, we have $p(\mathfrak{f}\mathfrak{g}) = p(\mathfrak{f})p(\mathfrak{g})$. With due care, we sometimes identify $\mathfrak{f}$ with $p(\mathfrak{f})$; in such cases, we may use the notations $\mathfrak{f}(i)$, or $\mathfrak{f}_i$ with the traditional meaning in $\mathscr{S}_n$. The *trace of* *a factor* $\mathfrak{f}$ *on a set* $K \subseteq [n]$, is the factor whose cycles are those of $\mathfrak{f}$ that intersect K.

To illustrate these concepts, and others to come, our digraphs are depicted in 'linear style' [20]. Thus, all vertices are uniformly and orderly disposed on

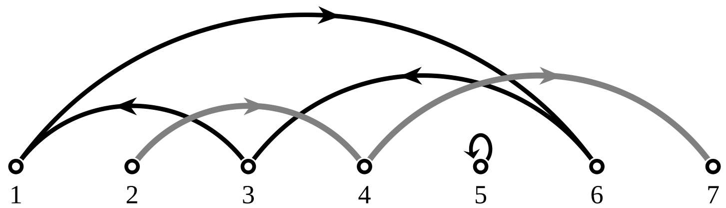

FIGURE 1. A digraph depicted in linear style.

a line, and non-loop arcs are represented by similar circular arrows. In Figure 1, we represent, for $n = 7$, the cycle (163), the path $2{\to}4{\to}7$, and the loop (5). Clearly, (163) and (163)(5) are distinct factors, with the same corresponding permutation, namely (163). The permutation digraph $\Delta_{(163)}$ is the spanning factor (163)(2)(4)(5)(7). The trace of $\Delta_{(163)}$ on the set $\{2, 3, 7\}$ is (163)(2)(7).

Fix a set $K \subseteq [n]$. Let $\mathfrak{F}_K$ be the set of all factors $\mathfrak{f}$ satisfying the conditions: each cycle of $\mathfrak{f}$ intersects K, and $V(\mathfrak{f})$ contains K. For $\mathfrak{f} \in \mathfrak{F}_K$ let $\mathcal{E}_\mathfrak{f}$ be the set of all $\sigma \in \mathscr{S}_n$ such that $\mathfrak{f} \subseteq \Delta_\sigma$. Note that, for any $\sigma \in \mathscr{S}_n$, the trace of Δ_σ on K is the *unique* member of $\mathfrak{F}_K$ that is a sub-factor of Δ_σ. Therefore $\{\mathcal{E}_\mathfrak{f}\}_{\mathfrak{f}\in\mathfrak{F}_K}$ is a partition of $\mathscr{S}_n$. We may think of $\mathcal{E}_\mathfrak{f}$ as the set of spanning factors having $\mathfrak{f}$ as trace.

Given a matrix $A \in \mathfrak{M}_n$, and a factor $\mathfrak{f}$, we let $\mathcal{Z}(A, \mathfrak{f})$ be the matrix obtained from A by zeroing out all rows and all columns indexed by the vertices of $\mathfrak{f}$, except the entries in the positions $(i, j) \in \mathfrak{f}$, which are replaced by 1's. In (3) we show a pattern of an example of $\mathcal{Z}(A, \mathfrak{f})$, for $n = 11$, and where $\mathfrak{f}$ is the factor (186)(11). We convention that absent entries are 0, and the a's indicate the surviving entries of A in their original places.

$$\mathcal{Z}(A,(186)(11)) = \begin{bmatrix} &&&&&&&1&&& \\ &a&a&a&a&&a&&a&a& \\ &a&a&a&a&&a&&a&a& \\ &a&a&a&a&&a&&a&a& \\ &a&a&a&a&&a&&a&a& \\ 1&&&&&&&&&& \\ &a&a&a&a&&a&&a&a& \\ &&&&&1&&&&& \\ &a&a&a&a&&a&&a&a& \\ &a&a&a&a&&a&&a&a& \\ &&&&&&&&&&1 \end{bmatrix}. \tag{3}$$

For comparison, we also show the pattern of the matrix $A^{\checkmark}_{\{1,6,8,11\}}$:

$$A^{\checkmark}_{\{1,6,8,11\}} = \begin{bmatrix} 1 & & & & & & & & & & \\ & a & a & a & a & & a & & a & a & \\ & a & a & a & a & & a & & a & a & \\ & a & a & a & a & & a & & a & a & \\ & a & a & a & a & & a & & a & a & \\ & & & & & 1 & & & & & \\ & a & a & a & a & & a & & a & a & \\ & & & & & & & 1 & & & \\ & a & a & a & a & & a & & a & a & \\ & a & a & a & a & & a & & a & a & \\ & & & & & & & & & & 1 \end{bmatrix}.$$

The 1's in $A^{\checkmark}_{\mathfrak{f}}$ are located in main diagonal positions, while in $\mathcal{Z}(A, \mathfrak{f})$, the 1's zigzag according to the factor $\mathfrak{f}$.

Theorem 4.1. *For any set $K \subseteq [n]$ and matrix A, we have*

$$\operatorname{per}_q A = \textstyle\sum_{\mathfrak{f} \in \mathfrak{F}_K} \operatorname{wt}_{\mathfrak{f}}(A) \operatorname{per}_q \mathcal{Z}(A, \mathfrak{f}). \tag{4}$$

Proof: We assume A is a generic matrix without zero entries. Fix any $\mathfrak{f} \in \mathfrak{F}_K$. For $\sigma \in \mathcal{E}_{\mathfrak{f}}$, we clearly have

$$\operatorname{twt}_\sigma(A) = \operatorname{wt}_{\mathfrak{f}}(A) \operatorname{twt}_\sigma(\mathcal{Z}(A, \mathfrak{f})).$$

Now let $\sigma \notin \mathcal{E}_{\mathfrak{f}}$. This means that, viewing $\mathfrak{f}$ as a permutation, $\sigma(u) \neq \mathfrak{f}(u)$, for some $u \in K$. In row u of $\mathcal{Z}(A, \mathfrak{f})$ the only nonzero entry lies in position $(u, \mathfrak{f}(u))$; therefore $\operatorname{twt}_\sigma(\mathcal{Z}(A, \mathfrak{f})) = 0$. So we get

$$\begin{aligned} \operatorname{per}_q A &= \sum_{\mathfrak{f} \in \mathfrak{F}_K} \sum_{\sigma \in \mathcal{E}_{\mathfrak{f}}} q^{\ell(\sigma)} \operatorname{twt}_\sigma(A) = \sum_{\mathfrak{f} \in \mathfrak{F}_K} \operatorname{wt}_{\mathfrak{f}}(A) \sum_{\sigma \in \mathscr{S}_n} q^{\ell(\sigma)} \operatorname{twt}_\sigma(\mathcal{Z}(A, \mathfrak{f})) \\ &= \textstyle\sum_{\mathfrak{f} \in \mathfrak{F}_K} \operatorname{wt}_{\mathfrak{f}}(A) \operatorname{per}_q \mathcal{Z}(A, \mathfrak{f}), \end{aligned}$$

and the theorem is proved. ∎

Suppose that $\mathfrak{f} \in \mathfrak{F}_K$ has r vertices. Let ω be the permutation $\omega = \omega_1\omega_2 \cdots \omega_n$, where the ω_i are defined by the conditions

$$\begin{aligned} V(\mathfrak{f}) &= \{\omega_1, \ldots, \omega_r\}, \quad \omega_1 < \cdots < \omega_r \\ [n] \smallsetminus V(\mathfrak{f}) &= \{\omega_{r+1}, \ldots, \omega_n\}, \quad \omega_{r+1} < \cdots < \omega_n. \end{aligned}$$

Obviously, ω depends on $\mathfrak{f}$. As a permutation of $[n]$, $\mathfrak{f}$ fixes all points outside $V(\mathfrak{f})$. Therefore, $\omega^{-1}\mathfrak{f}\omega$ fixes all points of $]r, n]$. Denote by $\mathfrak{f}'$ the restriction of $\omega^{-1}\mathfrak{f}\omega$ to $[r]$. Note that $\mathfrak{f}'$ is a member of $\mathscr{S}_r$; accordingly, $P_{\mathfrak{f}'}$ is an $r \times r$ permutation matrix. With these conventions,

$$\mathcal{Z}(A, \mathfrak{f}) = P_\omega^T \big(P_{\mathfrak{f}'} \oplus A_{\mathfrak{f}}\big) P_\omega. \tag{5}$$

We point out that $\mathcal{Z}(A,\mathfrak{f}) \rightsquigarrow P_\omega \mathcal{Z}(A,\mathfrak{f}) P_\omega^T$ is the permutation similarity that drags, to the leading positions, the rows and columns of $\mathcal{Z}(A,\mathfrak{f})$ corresponding to the vertices of $\mathfrak{f}$. Moreover, that similarity preserves the original order of the rows/columns $\omega_1,\ldots,\omega_r$, and of the rows/columns $\omega_{r+1},\ldots,\omega_n$. To illustrate this procedure, let us pick the above example, with pattern described in (3). The permutation ω is $168\overline{11}234579\overline{10}$. The transformed matrix has the pattern

$$P_\omega \mathcal{Z}(A,\mathfrak{f}) P_\omega^T = \begin{bmatrix} & & 1 & & & & & & & & \\ 1 & & & & & & & & & & \\ & 1 & & & & & & & & & \\ & & & 1 & & & & & & & \\ & & & & a & a & a & a & a & a & a \\ & & & & a & a & a & a & a & a & a \\ & & & & a & a & a & a & a & a & a \\ & & & & a & a & a & a & a & a & a \\ & & & & a & a & a & a & a & a & a \\ & & & & a & a & a & a & a & a & a \\ & & & & a & a & a & a & a & a & a \end{bmatrix},$$

where the 7×7 block of a's represents the submatrix $A_{\{1,6,8,\overline{11}\}}$.

Clearly, for the traditional permanent and the determinant, we get from (5)

$$\operatorname{per}_q \mathcal{Z}(A,\mathfrak{f}) = q^{\ell(\mathfrak{f})} \operatorname{per}_q A_{\mathfrak{f}} \qquad \text{(in case } q=\pm 1\text{).} \tag{6}$$

Thus we get from Theorem 4.1, the following expansion, which essentially is Theorem 3 of [16]:

$$\operatorname{per}_q A = \textstyle\sum_{\mathfrak{f}\in\mathfrak{F}_K} q^{\ell(\mathfrak{f})} \operatorname{wt}_{\mathfrak{f}}(A) \operatorname{per}_q A_{\mathfrak{f}}. \qquad \text{(in case } q=\pm 1\text{)} \tag{7}$$

Following the trail of [17, 16], let $\mathcal{M}_K$ be the set of the *minimal factors* $\mathfrak{f}\in\mathfrak{F}_K$, *i.e.*, those satisfying $V(\mathfrak{f}) = K$. If in (7) we collect the terms corresponding to $\mathfrak{f}\in\mathcal{M}_K$, we get easily $\operatorname{per}_q A_K \operatorname{per}_q A_L$, where L denotes the complement $[n]\smallsetminus K$. So we get, in case $q=\pm 1$, (*cf.* [17, 16]):

$$\operatorname{per}_q A = \operatorname{per}_q A_K \operatorname{per}_q A_L + \textstyle\sum_{\mathfrak{f}\in\mathfrak{F}_K\smallsetminus\mathcal{M}_K} q^{\ell(\mathfrak{f})} \operatorname{wt}_{\mathfrak{f}}(A) \operatorname{per}_q A_{\mathfrak{f}}. \tag{8}$$

This methodology does not work for a variable q. The main reason is found in Theorem 3.1, according to which we cannot, in general, get rid of the similarity by P_ω, as we have done in (6). Nevertheless, the case when K *does not cross its complement*, $L := [n]\smallsetminus K$, deserves some attention, because we may then retrieve some features of (8). Note that K and L are noncrossing sets, iff one of K, L is an interval.

Theorem 4.2. *In case K and $L=[n]\smallsetminus K$ do not cross each other, we have*

$$\operatorname{per}_q A = \operatorname{per}_q A_K^{\checkmark} \operatorname{per}_q A_L^{\checkmark} + \textstyle\sum_{\mathfrak{f}\in\mathfrak{F}_K\smallsetminus\mathcal{M}_K} \operatorname{wt}_{\mathfrak{f}}(A) \operatorname{per}_q \mathcal{Z}(A,\mathfrak{f}). \tag{9}$$

Proof: For any $\mathfrak{f} \in \mathcal{M}_K$, the matrix $\mathcal{Z} := \mathcal{Z}(A, \mathfrak{f})$ is of one of the following forms,

$$\underbrace{\mathcal{Z} = \begin{bmatrix} * & 0 & * \\ 0 & \mathcal{Z}[K] & 0 \\ * & 0 & * \end{bmatrix},}_{\text{when } K \text{ is an interval}} \quad \text{or} \quad \underbrace{\mathcal{Z} = \begin{bmatrix} * & 0 & * \\ 0 & \mathcal{Z}[L] & 0 \\ * & 0 & * \end{bmatrix},}_{\text{when } L \text{ is an interval}} \tag{10}$$

where the stars denote unspecified blocks. In both cases, $\mathcal{Z}[K]$ is the permutation matrix $P_{\mathfrak{f}'}$ of (5), and $\mathcal{Z}[L]$ is the submatrix $A_{\mathfrak{f}}$. We identify $\mathcal{M}_K$ $[\mathcal{M}_L]$ with the set of members of $\mathscr{S}_n$ that fix each index of L [resp., K]. Then, any $\sigma \in \mathscr{S}_n$ such that Δ_σ is a subdigraph of $D(\mathcal{Z}(A, \mathfrak{f}))$, factorizes as $\sigma = \mathfrak{f}\mathfrak{h}$, with $\mathfrak{h} \in \mathcal{M}_L$. Moreover, as K does not cross L, no orbit of $\mathfrak{f}$ crosses an orbit of $\mathfrak{h}$. By [19, Theorem 4.2], we have $\ell(\sigma) = \ell(\mathfrak{f}) + \ell(\mathfrak{h})$. Thus, for $\mathfrak{f} \in \mathcal{M}_K$, we get

$$\operatorname{per}_q \mathcal{Z}(A, \mathfrak{f}) = q^{\ell(\mathfrak{f})} \textstyle\sum_{\mathfrak{h} \in \mathcal{M}_L} q^{\ell(\mathfrak{h})} \operatorname{wt}_{\mathfrak{h}}(A) = q^{\ell(\mathfrak{f})} \operatorname{per}_q A_K^{\checkmark}.$$

Therefore

$$\begin{aligned} \textstyle\sum_{\mathfrak{f} \in \mathcal{M}_K} \operatorname{wt}_{\mathfrak{f}}(A) \operatorname{per}_q \mathcal{Z}(A, \mathfrak{f}) &= \operatorname{per}_q A_K^{\checkmark} \textstyle\sum_{\mathfrak{f} \in \mathcal{M}_K} q^{\ell(\mathfrak{f})} \operatorname{wt}_{\mathfrak{f}}(A) \\ &= \operatorname{per}_q A_K^{\checkmark} \operatorname{per}_q A_L^{\checkmark}. \end{aligned}$$

We then apply Theorem 4.1 to finish the proof. ∎

Let us check what happens in case K is a singleton, say $K = \{v\}$. Then we have $\mathfrak{F}_K = \mathscr{C}_v$, the set of all cycles through v. Therefore, we get the following extension of [16, Theorem 2] to arbitrary q:

Corollary 4.3. *Let A be any matrix. For any vertex $v \in [n]$, we have*

$$\operatorname{per}_q A = \textstyle\sum_{c \in \mathscr{C}_v} \operatorname{wt}_c(A) \operatorname{per}_q \mathcal{Z}(A, c). \qquad ∎ \tag{11}$$

Let us briefly consider the case $\sharp K = n - 1$. Then L, the complement of K, is a singleton, say $L = \{s\}$. We clearly have

$$\operatorname{per}_q A_K^{\checkmark} \operatorname{per}_q A_L^{\checkmark} = a_{ss} \operatorname{per}_q A_s^{\checkmark}.$$

In (9), the expression just displayed collects the monomials $q^{\ell(\sigma)} \operatorname{twt}_\sigma(A)$, corresponding to the factors in $\mathcal{M}_K$, *i.e.*, the $(n-1)!$ permutations σ that fix s. On the other hand, each factor $\mathfrak{f} \in \mathfrak{F}_K \smallsetminus \mathcal{M}_K$ spans $[n]$, and has no loop at s; therefore, $\mathcal{Z}(A, \mathfrak{f})$ is a permutation matrix, with 0 at the position (s, s). Hence, in (9), $\operatorname{per}_q \mathcal{Z}(A, \mathfrak{f})$ is simply $q^{\ell(\mathfrak{f})}$. This is alright, but not so interesting.

The equation (7) is false in general, for a variable q. The quest for combinatorial solutions to (7) (for variable q), and to some of its modifications, is an interesting challenge that motivates the combinatorial theorems of sections 5 and 6. The modifications we have in mind consist in replacing $A_{\mathfrak{f}}$ with various principal submatrices of the 'augmented submatrix' $A_{\mathfrak{f}}^{\checkmark}$. For each such trial, we give combinatorial solutions to the resulting expansion. Expansions using $A_{\mathfrak{f}}^{\checkmark}$ have proved to be useful (as in [19], for the q-derivative of $\mathrm{per}_q\, A$), due to the fact that the matrices considered are positive definite, whenever A is positive definite.

5. Lengths of permutations, and positive crossings

The *orientation* of an arc $i{\to}j$ is the sign of $j-i$. We say that $i{\to}j$ and $r{\to}s$ have a *positive [negative] crossing*, whenever $\{i,j\}$ and $\{r,s\}$ are crossing sets, and the arcs have the same [resp., opposite] orientation. For digraphs D and E, we define $\mathsf{X}^+(D,E)$ $[\mathsf{X}^-(D,E)]$ as the number of pairs (δ,ϵ), such that δ is an arc of D, ϵ is an arc of E, and δ has a positive [resp., negative] crossing with ϵ. For permutations σ and τ, the notations $\mathsf{X}^{\pm}(\Delta_\sigma,\Delta_\tau)$, will be simplified to $\mathsf{X}^{\pm}(\sigma,\tau)$. Note that, in the following obvious properties, we may reverse the roles of positive and negative crossings:

$$\mathsf{X}^+(D,E)=\mathsf{X}^+(E,D)=\mathsf{X}^-(D^T,E)=\mathsf{X}^+(D^R,E^R) \tag{12}$$

$$\mathsf{X}^+(\sigma,\tau)=\mathsf{X}^+(\tau,\sigma)=\mathsf{X}^-(\sigma^{-1},\tau)=\mathsf{X}^+(w_\circ\sigma w_\circ,w_\circ\tau w_\circ) \tag{13}$$

$$\mathsf{X}^+(D,E_1\cup\cdots\cup E_u)=\textstyle\sum_{i=1}^u \mathsf{X}^+(D,E_i), \tag{14}$$

where the $E_1,\ldots,E_u$ are pairwise edge-disjoint digraphs.

Examples. In Figure 1, the arc $2{\to}4$ crosses twice, negatively, the cycle $c=(163)$. Moreover, $\mathsf{X}^+(2{\to}4,c)=0$. We have $\mathsf{X}^+(2{\to}4{\to}7,c)=1$, and $\mathsf{X}^-(2{\to}4{\to}7,c)=3$. Clearly, loops do not generate crossings.

Theorem 5.1. *For disjoint permutations, σ and τ, the numbers $\mathsf{X}^+(\sigma,\tau)$ and $\mathsf{X}^-(\sigma,\tau)$ are both even. Moreover, if one is positive, the other is positive as well.*

Proof: We argue by induction on the cardinality of $\mathrm{Mov}(\tau)$, the set of the points moved by τ (*i.e.*, the non-fixed points of τ). The theorem is trivial if $\mathrm{Mov}(\tau)$ is empty. For a nonempty $\mathrm{Mov}(\tau)$, let m be the maximum of $\mathrm{Mov}(\tau)$, and define a and w by $a=\tau(m)$ and $\tau(w)=m$. So $w{\to}m{\to}a$ is a path in Δ_τ. Let τ' be the permutation $\tau\cdot(w\,m)$. Clearly, m is a fixed point of τ', and $\mathrm{Mov}(\tau')=\mathrm{Mov}(\tau)\smallsetminus\{m\}$. In digraphic language, the transformation

$\tau \rightsquigarrow \tau'$, corresponds to replacing the path $w \to m \to a$ of Δ_τ, with the shortcut $m \to a$. The situation is depicted in figure 2, where we assume that $a \leqslant w$. This assumption is made with no loss of generality, because the roles of a and w may be reversed, by inverting σ and τ, and applying the elementary properties (13). In figure 2, the symbols $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ denote the intervals

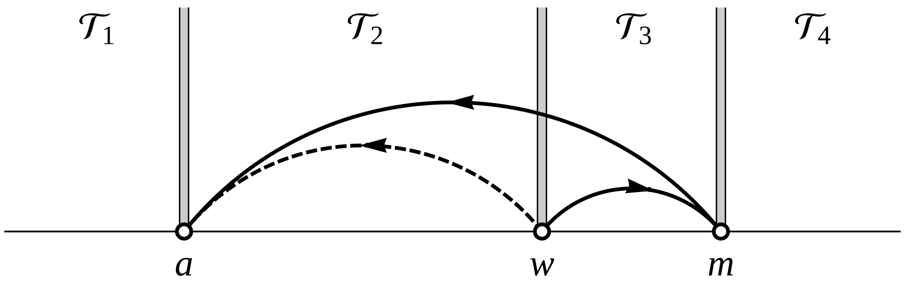

FIGURE 2. Replacing $w \to m \to a$ with the shortcut $w \to a$.

$[1, a[,]a, w[,]w, m[,]m, n]$, respectively. We let $T_i = \sharp \mathcal{T}_i$, and let t_{ij} be the number of indices of $\mathcal{T}_i$ that are mapped by σ into $\mathcal{T}_j$. For example, in Figure 2, $t_{i,i+1}$ is the number of arcs of Δ_σ that cross, from left to right, the vertical grey bar separating $\mathcal{T}_i$ from $\mathcal{T}_{i+1}$. An essential fact is that T_k *is the sum of row (column)* k *of the matrix* (t_{ij}). Clearly, we have

$$\mathsf{X}^+(\sigma,\tau) - \mathsf{X}^+(\sigma,\tau') = \mathsf{X}^+(\sigma, w \to m) + \mathsf{X}^+(\sigma, m \to a) - \mathsf{X}^+(\sigma, w \to a).$$

Each term on the right hand side is easy to determine using the t_{ij} numbers:

$$\begin{aligned} \mathsf{X}^+(\sigma,\tau) - \mathsf{X}^+(\sigma,\tau') &= \\ &= (t_{13} + t_{23} + t_{34}) + (t_{21} + t_{31} + t_{42} + t_{43}) - (t_{21} + t_{32} + t_{42}) \\ &= T_3 - t_{33} - t_{32} + t_{34} + t_{31} = 2t_{31} + 2t_{34} \end{aligned} \tag{15}$$

If $\mathrm{Mov}(\tau)$ is empty, $\mathsf{X}^+(\sigma,\tau) = 0$. By induction, $\mathsf{X}^+(\sigma,\tau')$ is even. Therefore, (15) tells that $\mathsf{X}^+(\sigma,\tau)$ is even as well. As $\mathsf{X}^-(\sigma,\tau) = \mathsf{X}^+(\sigma^{-1},\tau)$, then $\mathsf{X}^-(\sigma,\tau)$ is also even.

We now assume that $\mathsf{X}^+(\sigma,\tau)$ is positive, and show that $\mathsf{X}^-(\sigma,\tau)$ is positive as well. By (14), we may assume that τ is a cycle. Applying the algorithm $\tau \rightsquigarrow \tau'$ and replacing σ with σ^{-1}, the matrix (t_{ij}) transforms into its transpose. Therefore,

$$\mathsf{X}^-(\sigma,\tau) - \mathsf{X}^-(\sigma,\tau') = 2t_{13} + 2t_{43}.$$

So, when we pass from τ to τ', the number of positive [negative] crossings does not increase. We may adopt a similar procedure, say $\tau \rightsquigarrow \tau''$, by taking the minimum of $\mathrm{Mov}(\tau)$, instead of the maximum m (this, in fact, amounts to the former procedure applied to the conjugate $w_\circ \tau w_\circ$, $w_\circ \sigma w_\circ$; the result may be seen as the original procedure, reflected with respect to a vertical mirror on Figure 2).

The procedure $\tau \rightsquigarrow \tau'$ [$\tau \rightsquigarrow \tau''$] does not eliminate the arcs of τ, except those incident with the maximum [resp., minimum] vertex of $\mathrm{Mov}(\tau)$; and this vertex is transformed into a fixed point of τ' [resp., τ''].

There exist positively crossing arcs, $r{\to}s$ and $x{\to}y$, belonging to Δ_σ and Δ_τ, respectively. We may perform the above procedures, repeatedly, until the cycle τ is reduced to a cycle ρ, such that $x{\to}y$ is an arc of Δ_ρ, and x, y are the extreme values of $\mathrm{Mov}(\rho)$. Let us write the cycle ρ, in short notation, as $\rho = (y_0 y_1 \ldots y_k)$, where $y_0 = y$, and $y_k = x$. Without loss of generality, we assume $x < y$. As $r{\to}s$ and $x{\to}y$ cross positively, we either have $x < r < y < s$, or $r < x < s < y$. Suppose $x < r < y < s$ (the other case is similarly treated). The path $y_0{\to}y_1{\to}\ldots{\to}y_k$ goes from y to x, and all its points lie in $[x, y]$. As $x < r < y$, one of the arcs $y_u{\to}y_{u+1}$ has origin $y_u > r$, and terminal $y_{u+1} < r$ (*e.g.*, take u as the largest such that all $y_0, y_1, \ldots, y_u$ are $> r$). Therefore, $y_u{\to}y_{u+1}$ crosses negatively $r{\to}s$. We have just proven that $\mathsf{X}^-(\sigma, \rho)$ is positive. Then the last part of the theorem follows from $\mathsf{X}^-(\sigma, \tau) \geqslant \mathsf{X}^-(\sigma, \rho)$. ∎

Theorem 5.2. *For disjoint permutations, σ and τ, we have*

$$\ell(\sigma\tau) = \ell(\sigma) + \ell(\tau) - \mathsf{X}^+(\sigma, \tau). \tag{16}$$

Proof: We follow the strategy and the notation of the proof of Theorem 5.1. We go by induction on the cardinality of $\mathrm{Mov}(\tau)$. If $\mathrm{Mov}(\tau)$ is empty, then (16) is trivial.

Recall, from the proof of 5.1, the definition of m, w, a, and $\tau' = \tau \cdot (wm)$. To simplify subsequent formulas, we denote the products $\sigma\tau$ and $\sigma\tau'$, respectively by $\overline{\pi}$ and π. We shall use the one line notation for some of these permutations, namely, we let $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$, $\pi = \pi_1\pi_2 \cdots \pi_n$, and $\tau' = \tau'_1\tau'_2 \cdots \tau'_n$. For a permutation $\theta = \theta_1\theta_2 \cdots \theta_n$, let K_θ be the integer

$$K_\theta = \sharp\{i \in]w, m[: \theta_i \in]a, m[\}. \tag{17}$$

Note the right hand side action of the transposition (wm):

$$\pi = \overline{\pi} \cdot (wm), \quad \text{and} \quad \tau' = \tau \cdot (wm).$$

As $\overline{\pi}$ and τ both have an inversion at (w, m), we get from [19, Lemma 4.1]

$$\ell(\overline{\pi}) - \ell(\pi) = 2K_\pi + 1, \quad \text{and} \quad \ell(\tau) - \ell(\tau') = 2K_{\tau'} + 1. \tag{18}$$

In the current notation, the target formula, and the induction hypothesis are, respectively

$$\ell(\overline{\pi}) = \ell(\sigma) + \ell(\tau) - \mathsf{X}^+(\sigma, \tau),$$
$$\ell(\pi) = \ell(\sigma) + \ell(\tau') - \mathsf{X}^+(\sigma, \tau').$$

Subtracting term by term these equations, we get

$$\ell(\overline{\pi}) - \ell(\pi) = \ell(\tau) - \ell(\tau') + \mathsf{X}^+(\sigma, \tau') - \mathsf{X}^+(\sigma, \tau).$$

If, in this equation, we take (15) and (18) into account, we get

$$K_\pi - K_{\tau'} + t_{31} + t_{34} = 0. \tag{19}$$

This equation is all we need to prove.

Let M_σ be the set of those $i \in]w, m[$ that are moved by σ. For $i \notin M_\sigma$, we have $\pi_i = \tau'_i$; for $i \in M_\sigma$, we have $\tau'_i = i$. Therefore

$$\begin{aligned} K_\pi - K_{\tau'} &= \sharp\{i \in]w, m[: \pi_i \in]a, m[\} - \sharp\{i \in]w, m[: \tau'_i \in]a, m[\} \\ &= \sharp\{i \in M_\sigma : \sigma_i \in]a, m[\} - \sharp\{i \in M_\sigma : c_i \in]a, m[\} \\ &= \sharp\{i \in M_\sigma : \sigma_i \in]a, m[\} - \sharp M_\sigma \\ &= -\sharp\{i \in M_\sigma : \sigma_i \notin]a, m[\} = -\sharp\{i \in]w, m[: \sigma_i \notin]a, m[\} \\ &= -x_{31} - x_{34}. \end{aligned}$$

This proves the desired formula (19). ∎

Corollary 5.3. *For pairwise disjoint permutations, $\omega_1, \ldots, \omega_u$, we have*

$$\ell(\omega_1 \cdots \omega_u) = \ell(\omega_1) + \cdots + \ell(\omega_u) - \textstyle\sum_{1 \leqslant i < j \leqslant u} \mathsf{X}^+(\omega_i, \omega_j).$$

∎

Note that [19, Theorem 4.2] and [19, Corollary 4.3], follow easily from the results in this section.

6. Jumps, crossings, and arc-under-arc relations

We have seen that a factor $\mathfrak{f}$ behaves much like its associated permutation, $p(\mathfrak{f})$. For disjoint factors, $\mathfrak{f}$ and $\mathfrak{g}$, we have $p(\mathfrak{f}\mathfrak{g}) = p(\mathfrak{f})p(\mathfrak{g})$. With the natural definition of length of a factor, namely $\ell(\mathfrak{f}) = \ell(p(\mathfrak{f}))$, we may rephrase Theorems 5.1 and 5.2 in terms of factors.

Corollary 6.1. *Let $\mathfrak{f}$ and $\mathfrak{g}$ be disjoint factors. The numbers $\mathsf{X}^+(\mathfrak{f},\mathfrak{g})$ and $\mathsf{X}^-(\mathfrak{f},\mathfrak{g})$ are both even, and, if one is positive, the other is positive as well. Moreover, we have*

$$\ell(\mathfrak{f}\mathfrak{g}) = \ell(\mathfrak{f}) + \ell(\mathfrak{g}) - \mathsf{X}^+(\mathfrak{f},\mathfrak{g}). \qquad \blacksquare \tag{20}$$

By these results, for disjoint factors, it makes sense to say that *they cross*, or *do not cross*, with no reference to positive/negative crossings.

We may naturally wonder what happens if one of $\mathfrak{f},\mathfrak{g}$ is not a factor. Then, equation (20) does not make sense, because the length is not defined for non-factors. On the other hand, trivial examples have been given, just below (14), where $\mathsf{X}^+(\mathfrak{f},D)$ and $\mathsf{X}^-(\mathfrak{f},D)$ are both odd, and examples where one of these numbers is positive and the other is zero. Nevertheless, we may say something more about crossings.

Corollary 6.2. *If a factor $\mathfrak{f}$ and a digraph D are disjoint, then the numbers $\mathsf{X}^+(\mathfrak{f},D)$ and $\mathsf{X}^-(\mathfrak{f},D)$ have the same parity.*

Proof: Let $a{\to}b$ be an arc, disjoint from $\mathfrak{f}$. As $a{\to}b{\to}a$ and $\mathfrak{f}$ are disjoint factors, the number $\mathsf{X}^+(\mathfrak{f},a{\to}b{\to}a)$ is even. We clearly have

$$\mathsf{X}^+(\mathfrak{f},a{\to}b{\to}a) = \mathsf{X}^+(\mathfrak{f},a{\to}b) + \mathsf{X}^-(\mathfrak{f},a{\to}b).$$

So, the corollary holds when D is an edge. Then it holds in general. $\blacksquare$

An arc $i{\to}j$ is said to *jump over* a vertex w, if w lies strictly in between i and j. The *number of jumps* of a digraph D over a set $W \subseteq [n]$, denoted $\mathcal{J}(D,W)$, or just $\mathcal{J}^D_W$, is the number of pairs (a,w), such that a is an arc of D that jumps over $w \in W$. We simplify $\mathcal{J}^D_{\{w\}}$ to $\mathcal{J}^D_w$. For any factors, $\mathfrak{s}$ and $\mathfrak{g}$, we abbreviate $\mathcal{J}^{\mathfrak{s}}_{V(\mathfrak{g})}$ to $\mathcal{J}^{\mathfrak{s}}_{\mathfrak{g}}$.

Given two arcs, $i{\to}j$ and $r{\to}s$, we say that $i{\to}j$ *lies under* $r{\to}s$ (and $r{\to}s$ *lies above* $i{\to}j$) if $r < i \leqslant j < s$ or $r > i \geqslant j > s$. So, in case $i = j$, the loop $i{\to}i$ lies under $r{\to}s$, iff $r{\to}s$ jumps over i. In case $i \neq j$, it is clear that $i{\to}j$ lies under $r{\to}s$, iff i,j lie strictly in between r,s, and the arcs $r{\to}s$, $i{\to}j$ have the same orientation. We now define the following arc-under-arc count, where δ and ϵ are arcs:

$$\mathcal{U}^\delta_\epsilon = \begin{cases} 1\,, & \text{if } \epsilon \text{ lies under } \delta, \text{ and } \epsilon \text{ is not a loop} \\ 1/2\,, & \text{if } \epsilon \text{ lies under } \delta, \text{ and } \epsilon \text{ is a loop} \\ 0\,, & \text{if } \epsilon \text{ does not lie under } \delta. \end{cases}$$

For digraphs D and E, we define $\mathcal{U}^D_E$ (also denoted $\mathcal{U}(D,E)$), as the sum of all $\mathcal{U}^\delta_\epsilon$, for δ an arc of D, and ϵ an arc of E.

Note that, if σ is a permutation, and w is a fixed point of σ, then $\mathcal{J}(\Delta_\sigma, w)$ is even. Therefore, we have

$$\mathcal{U}(\Delta_\sigma, (w)) = \tfrac{1}{2}\,\mathcal{J}(\Delta_\sigma, w),$$

and this number is an integer. Here are some elementary properties:

$$\begin{aligned} \mathcal{J}(D, W) &= \mathcal{J}(D^T, W) = \mathcal{J}(D^R, w_\circ(W)) \\ \mathcal{U}(D, E) &= \mathcal{U}(D^T, E^T) = \mathcal{U}(D^R, E^R). \end{aligned}$$

Examples. In Figure 1, the path $2{\to}4{\to}7$ has 3 jumps over the factor $(163)(5)$, but only 2 over the cycle (163). We have
$\mathcal{U}(2{\to}4{\to}7, (163)(5)) = 1/2$, $\mathcal{U}((163), 2{\to}4) = 1$, and $\mathcal{U}((163)^{-1}, 2{\to}4) = 0$.

Theorem 6.3. *For any disjoint factors,* $\mathfrak{f}$ *and* $\mathfrak{g}$*, we have*

$$\mathcal{J}^{\mathfrak{f}}_{\mathfrak{g}} = \mathsf{X}^+(\mathfrak{f}, \mathfrak{g}) + 2\,\mathcal{U}^{\mathfrak{f}}_{\mathfrak{g}}\,. \tag{21}$$

Proof: No term of (21) depends on the possible loops of $\mathfrak{f}$. The set Λ of loops of $\mathfrak{g}$ enters on the left, and on the right hand sides of (21) with the same count, namely $\mathcal{J}^{\mathfrak{f}}_{\Lambda} = 2\,\mathcal{U}^{\mathfrak{f}}_{\Lambda}$. So, we only need to prove the theorem in case $\mathfrak{g}$ is loopless. Let σ and τ be the permutations associated with $\mathfrak{f}$ and $\mathfrak{g}$, respectively.

The present proof follows the strategy, and most of the notation of Theorem 5.1. So we go by induction on the cardinality of $\mathrm{Mov}(\tau) = V(\mathfrak{g})$. If $\mathrm{Mov}(\tau)$ is empty, then $\mathfrak{g}$ is the empty factor, and (21) is trivial.

For $\mathrm{Mov}(\tau)$ nonempty, we let m be the maximum of $\mathrm{Mov}(\tau)$. Then, we define a, w, and τ' as in the proof of 5.1. Let $\mathfrak{g}'$ be the loopless factor having τ' as associated permutation. Note that $V(\mathfrak{g}') = V(\mathfrak{g}) \smallsetminus \{m\}$. The Figure 2 also applies in the present situation.

We first note that $\mathcal{J}^{\mathfrak{f}}_{\mathfrak{g}} - \mathcal{J}^{\mathfrak{f}}_{\mathfrak{g}'} = \mathcal{J}^{\mathfrak{f}}_{m}$. This leads us to count the number of arcs of $\mathfrak{f}$ that intersect the vertical grey bar over m, in Figure 2. We thus get

$$\mathcal{J}^{\mathfrak{f}}_{\mathfrak{g}} - \mathcal{J}^{\mathfrak{f}}_{\mathfrak{g}'} = t_{14} + t_{24} + t_{34} + t_{41} + t_{42} + t_{43} = 2(T_4 - t_{44}). \tag{22}$$

On the other hand, we clearly have

$$\mathcal{U}^{\mathfrak{f}}_{\mathfrak{g}} - \mathcal{U}^{\mathfrak{f}}_{\mathfrak{g}'} = \mathcal{U}^{\mathfrak{f}}_{w\to m} + \mathcal{U}^{\mathfrak{f}}_{m\to a} - \mathcal{U}^{\mathfrak{f}}_{w\to a}\,.$$

Counting the crossings of the arcs of $\mathfrak{f}$ with the appropriate vertical bars in Figure 2, we get

$$\mathcal{U}^{\mathfrak{f}}_{\mathfrak{g}} - \mathcal{U}^{\mathfrak{f}}_{\mathfrak{g}'} = (t_{14} + t_{24}) + t_{41} - (t_{31} + t_{41}) = t_{14} + t_{24} - t_{31}. \tag{23}$$

The difference $\mathsf{X}^+(\mathfrak{f},\mathfrak{g}) - \mathsf{X}^+(\mathfrak{f},\mathfrak{g}')$ equals $2t_{31} + 2t_{34}$, as determined in (15). Therefore, combining (15) with (22)-(23), we get

$$\mathcal{J}^{\mathfrak{f}}_{\mathfrak{g}} - \mathcal{J}^{\mathfrak{f}}_{\mathfrak{g}'} = \mathsf{X}^+(\mathfrak{f},\mathfrak{g}) - \mathsf{X}^+(\mathfrak{f},\mathfrak{g}') + 2(\mathcal{U}^{\mathfrak{f}}_{\mathfrak{g}} - \mathcal{U}^{\mathfrak{f}}_{\mathfrak{g}'}). \tag{24}$$

By the induction hypothesis, $\mathcal{J}^{\mathfrak{f}}_{\mathfrak{g}'} = \mathsf{X}^+(\mathfrak{f},\mathfrak{g}') + 2\mathcal{U}^{\mathfrak{f}}_{\mathfrak{g}'}$. Combining this with (24) proves the target formula (21). ■

7. Tentative q-permanent expansions

For a factor $\mathfrak{f}$, and a subset $\alpha \subseteq V(\mathfrak{f})$, we let $A(\mathfrak{f},\alpha)$ be the matrix obtained by eliminating the rows and columns of $A^{\checkmark}_{\mathfrak{f}}$ corresponding to the indices in the set α.

As in section 4, let K be any subset of $[n]$. We now assume that, for each factor $\mathfrak{f}$, a set $\alpha(\mathfrak{f}) \subseteq V(\mathfrak{f})$ has been fixed. Each function $\mathfrak{f} \rightsquigarrow \alpha(\mathfrak{f})$ determines the following tentative expansion of the q-permanent of a generic matrix A:

$$\operatorname{per}_q A = \textstyle\sum_{\mathfrak{f}\in\mathfrak{F}_K} q^{\ell(\mathfrak{f})} \operatorname{wt}_{\mathfrak{f}}(A) \operatorname{per}_q A(\mathfrak{f},\alpha(\mathfrak{f})). \tag{25}$$

A nice feature of these expansions, when compared with (4) and (11), is that the matrices $A(\mathfrak{f},\alpha(\mathfrak{f}))$ are symmetric/Hermitian, whenever A is symmetric/Hermitian (a property that $\mathcal{Z}(A,\mathfrak{f})$ does not satisfy, in case $\mathfrak{f}$ has a cycle of order > 2). Moreover, if A is positive definite, then all $A(\mathfrak{f},\alpha(\mathfrak{f}))$ are positive definite as well. So (25) is prone to induction, a fact we have used in the last section of [19]. The handicap of (25) is, of course, the fact that it does not hold for all matrices. To give a combinatorial solution to this equation, we need some notation.

For two factors, $\mathfrak{f}$ and $\mathfrak{g}$, the difference $\mathfrak{f} \setminus \mathfrak{g}$ is the factor composed by the cycles of $\mathfrak{f}$ that are not in $\mathfrak{g}$. We say that $\mathfrak{f}$ and $\mathfrak{g}$ are *complementary in a digraph D*, whenever they are disjoint, and the union, $\mathfrak{f}\mathfrak{g}$ is a spanning factor of D. Of course, a factor may have more than one complement in D, and may have none.

Theorem 7.1. *The expansion* (25) *holds for a generic matrix A with digraph D, if and only if, for any complementary factors of D, $\mathfrak{f}$ and $\mathfrak{g}$, such that $\mathfrak{f} \in \mathfrak{F}_K$, we have*

$$\mathsf{X}^+(\mathfrak{g},\mathfrak{f}) = \mathcal{J}(\mathfrak{g},\alpha(\mathfrak{f})). \tag{26}$$

Remark. By Theorem 6.3, the condition (26) is equivalent to

$$2\mathcal{U}(\mathfrak{g},\mathfrak{f}) = \mathcal{J}(\mathfrak{g}, V(\mathfrak{f}) \setminus \alpha(\mathfrak{f})).$$

In case $\alpha(\mathfrak{f}) = V(\mathfrak{f})$, this condition reduces to $\mathcal{U}(\mathfrak{g},\mathfrak{f}) = 0$.

Proof: Denote by RHS the right hand side of (25). For any spanning factor $\mathfrak{s}$, let $\mathrm{RHS}_{\mathfrak{s}}$ be the sum of all monomials of RHS that are multiple of $\mathrm{wt}_{\mathfrak{s}}(A)$. Then RHS is the sum of all $\mathrm{RHS}_{\mathfrak{s}}$, and each $\mathrm{RHS}_{\mathfrak{s}}$ has the form $\mathrm{RHS}_{\mathfrak{s}} = Q_{\mathfrak{s}}(q)\,\mathrm{wt}_{\mathfrak{s}}(A)$, where $Q_{\mathfrak{s}}(q)$ is a polynomial in q, to be determined below. Clearly, (25) is equivalent to the system of polynomial equations in $\mathbb{Z}[q]$

$$q^{\ell(\mathfrak{s})} = Q_{\mathfrak{s}}(q), \quad \text{for all spanning } \mathfrak{s} \text{ such that } \mathrm{wt}_{\mathfrak{s}}(A) \neq 0. \tag{27}$$

To find $\mathrm{RHS}_{\mathfrak{s}}$, we zero out in RHS all entries of A except $a_{1\mathfrak{s}_1}, a_{2\mathfrak{s}_2}, \ldots, a_{n\mathfrak{s}_n}$. The matrix so-obtained is the Hadamard product $A \circ P_{\mathfrak{s}}$. We get

$$\mathrm{RHS}_{\mathfrak{s}} = \textstyle\sum_{\mathfrak{f}\in\mathfrak{F}_K} q^{\ell(\mathfrak{f})}\,\mathrm{wt}_{\mathfrak{f}}(P_{\mathfrak{s}} \circ A)\ \mathrm{per}_q\big((P_{\mathfrak{s}} \circ A)(\mathfrak{f}, \alpha(\mathfrak{f}))\big). \tag{28}$$

Clearly $\mathrm{wt}_{\mathfrak{f}}(P_{\mathfrak{s}} \circ A) = \mathrm{wt}_{\mathfrak{f}}(P_{\mathfrak{s}})\,\mathrm{wt}_{\mathfrak{f}}(A)$. A simple application of the definitions yields

$$\mathrm{wt}_{\mathfrak{f}}(P_{\mathfrak{s}}) = \textstyle\prod_{i\in V(\mathfrak{f})} \delta_{\mathfrak{s}(i),\mathfrak{f}(i)},$$

where δ_{ij} is the Kronecker delta. Therefore, we have $\mathrm{wt}_{\mathfrak{f}}(P_{\mathfrak{s}}) = 1$, if $\mathfrak{f}$ is a sub-factor of $\mathfrak{s}$; and $\mathrm{wt}_{\mathfrak{f}}(P_{\mathfrak{s}}) = 0$, otherwise. So, in the sum of (28), the only relevant summand is the one corresponding to the trace of $\mathfrak{s}$ in K. Accordingly, we assume $\mathfrak{f}$ be the trace of $\mathfrak{s}$, and let $\mathfrak{g}$ be the factor whose cycles are those of $\mathfrak{s}$ that are not cycles of $\mathfrak{f}$. Thus, $\mathfrak{f}$ and $\mathfrak{g}$ are disjoint, and $\mathfrak{s} = \mathfrak{f}\mathfrak{g}$. Therefore

$$\mathrm{RHS}_{\mathfrak{s}} = q^{\ell(\mathfrak{f})}\,\mathrm{wt}_{\mathfrak{f}}(A)\,\mathrm{per}_q\big((P_{\mathfrak{s}} \circ A)(\mathfrak{f}, \alpha(\mathfrak{f}))\big). \tag{29}$$

On the other hand, as $\mathrm{wt}_{\mathfrak{s}}(A) = \mathrm{wt}_{\mathfrak{f}}(A)\,\mathrm{wt}_{\mathfrak{g}}(A)$, we have

$$\mathrm{per}_q\big((A \circ P_{\mathfrak{s}})(\mathfrak{f}, \alpha(\mathfrak{f}))\big) = \mathrm{wt}_{\mathfrak{g}}(A)\,\mathrm{per}_q\,P_{\mathfrak{s}}(\mathfrak{f}, \alpha(\mathfrak{f})).$$

Clearly, $P_{\mathfrak{s}}(\mathfrak{f}, \alpha(\mathfrak{f}))$ is a permutation matrix of order $n - \sharp\alpha(\mathfrak{f})$. In fact, the corresponding permutation is the restriction to the set $U := [n] \smallsetminus \alpha(\mathfrak{f})$, of the permutation associated with $\mathfrak{g}$. Therefore, we have

$$\mathrm{per}_q\,P_{\mathfrak{s}}(\mathfrak{f}, \alpha(\mathfrak{f})) = q^{\ell_U(\mathfrak{g})}.$$

Applying [19, Lemma 3.1], $\ell_U(\mathfrak{g}) = \ell(\mathfrak{g}) - \mathcal{J}(\mathfrak{g}, \alpha(\mathfrak{f}))$. From (29), we get

$$Q_{\mathfrak{s}}(q) = q^{\ell(\mathfrak{f})+\ell(\mathfrak{g})-\mathcal{J}(\mathfrak{g},\alpha(\mathfrak{f}))}.$$

Taking (27) into account, we see that (25) holds, if and only if

$$\ell(\mathfrak{s}) = \ell(\mathfrak{f}) + \ell(\mathfrak{g}) - \mathcal{J}(\mathfrak{g}, \alpha(\mathfrak{f}))$$

holds, for any spanning factor $\mathfrak{s} = \mathfrak{f}\mathfrak{g} \subseteq D$. By Theorem 6.1, the last displayed identity is equivalent to (26). ■

Truncated expansions. Let A be a generic matrix with digraph D. We say that a digraph E is *relevant* (in, or with respect to D), whenever E can be extended to a spanning factor of D. Of course, a factor E is relevant iff the expression $\mathrm{wt}_E(A)$ (effectively) occurs in $\mathrm{per}_q A$. For example, the arc $i{\to}j$ is relevant iff a_{ij} occurs in $\mathrm{per}_q A$. A permutation π is *relevant*, whenever $\Delta_\pi \subseteq D$. Note that Theorem 7.1 may be rephrased as follows:

> *Equation* (25) *holds if and only if, for any relevant permutation* π*, we have* $\mathsf{X}^+(\mathfrak{g},\mathfrak{f}) = \mathcal{J}(\mathfrak{g},\alpha(\mathfrak{f}))$*, where* $\mathfrak{f}$ *is the trace of* Δ_π *on* K*, and* $\mathfrak{g}$ *denotes the complement* $\Delta_\pi \setminus \mathfrak{f}$*.*

If A satisfies (25), then A still satisfies that equation, with the sum extended to only the relevant $\mathfrak{f} \in \mathfrak{F}_K$. Note that a union of disjoint relevant factors, may not be relevant.

> **Examples**. From the complete digraph $[n]^2$, remove the loop at v, and call E the so-obtained digraph. Partition $[n]$ into three parts, $\{v\}$, U, and W. Let $\mathfrak{u}$ and $\mathfrak{w}$ be factors with vertex sets U and W, respectively. Then $\mathfrak{u}$ and $\mathfrak{w}$ are disjoint, and relevant in E, but $\mathfrak{u}\mathfrak{w}$ is not relevant.

An obvious advantage of a digraph D with all loops, is that every factor of D is relevant in D.

Theorem 7.2. *Let* D *be a digraph with all loops, and let* $\mathfrak{F} \subseteq \mathfrak{F}_K$*. The equation*

$$\mathrm{per}_q A = \textstyle\sum_{\mathfrak{f}\in\mathfrak{F}} q^{\ell(\mathfrak{f})}\, \mathrm{wt}_{\mathfrak{f}}(A)\ \mathrm{per}_q A(\mathfrak{f},\alpha(\mathfrak{f})), \tag{30}$$

holds for a generic matrix A *with digraph* D*, if and only if* (*i*) $\mathfrak{F}$ *contains all members of* $\mathfrak{F}_K$ *that are sub-digraphs of* D*, and* (*ii*) *for any disjoint factors of* D*,* $\mathfrak{f}$ *and* ω*, such that* ω *is a cycle and* $\mathfrak{f} \in \mathfrak{F}_K$*, we have*

$$\mathsf{X}^+(\omega,\mathfrak{f}) = \mathcal{J}(\omega,\alpha(\mathfrak{f})). \tag{31}$$

Proof: *The only if part.* (*i*) Let $\mathfrak{h}$ be a factor of D, in $\mathfrak{F}_K$. Adding loops, extend $\mathfrak{h}$ to a spanning factor $\mathfrak{s}$ in D. Then $\mathrm{wt}_{\mathfrak{s}}(A)$ occurs in $\mathrm{per}_q A$, and therefore in the right hand side of (30). So, there exists $\mathfrak{f} \in \mathscr{F}$, that is a subfactor of $\mathfrak{s}$. The uniqueness of the trace implies $\mathfrak{f} = \mathfrak{h}$. So $\mathfrak{h} \in \mathscr{F}$.

(*ii*) Let $\mathfrak{f},\omega$ be as (*ii*) prescribes. By joining the adequate loops to ω, we get a factor $\mathfrak{g}$, such that $\mathfrak{f}$ and $\mathfrak{g}$ are complementary in D. From Theorem 7.1, we get (26). This implies (31), because the loops of $\mathfrak{g}$ are irrelevant in (26).

The if part. Let $\mathfrak{f},\mathfrak{g}$ be any complementary factors of D, such that $\mathfrak{f} \in \mathfrak{F}_K$. Factorize $\mathfrak{g}$ as $\mathfrak{g} = \omega_1 \cdots \omega_u$, where the ω_i are pairwise disjoint cycles. Then,

(ii) yields

$$\mathsf{X}^+(\omega_i, \mathfrak{f}) = \mathcal{J}(\omega_i, \alpha(\mathfrak{f})),$$

for $i = 1, \ldots, u$. Summing term-by-term these u identities, we get (26). Therefore, Theorem 7.1 implies the expansion (25). Finally, the hypothesis (i) yields (30). ■

Examples. Suppose D has all loops. Let $\mathfrak{t}$ be the trivial factor, consisting of a loop at each vertex of K. If $\mathfrak{t}$ is not in $\mathfrak{F}$, then Theorem 7.2 is vacuously true, in the sense that (30) and (i) are both false.

Let us check the case $\mathfrak{F} = \{\mathfrak{t}\}$. Then the expansion (30) is

$$\operatorname{per}_q A = \left(\prod_{i \in K} a_{ii}\right) \operatorname{per}_q A(\mathfrak{t}, \alpha(\mathfrak{t})). \tag{32}$$

Condition (i) of Theorem 7.2 says that no non-loop cycle of D goes through a vertex of K. In (31), we must have $\mathfrak{f} = \mathfrak{t}$; therefore, (31) says that no cycle of D jumps over a vertex of $\alpha(\mathfrak{t})$. In case $\alpha(\mathfrak{t})$ is empty, (ii) is trivially true; therefore, we have: (32) *holds, iff the cycles of D that intersect K are loops.*

A more interesting case occurs when $\mathfrak{F}$ is the set, denoted $\mathcal{O}_m$, of all factors in $\mathfrak{F}_K$, whose cycles have orders $\leqslant m$. The expansion (30), for $\mathfrak{F} = \mathcal{O}_m$, implies that all cycles of D through a vertex of K have orders $\leqslant m$. We have just seen the case $\mathcal{O}_1 = \{\mathfrak{t}\}$. Other cases will be briefly considered when K is a singleton.

8. The singleton case

Recall that, if $K = \{v\}$, then $\mathfrak{F}_K$ is the set $\mathscr{C}_v$ of all cycles through v. Therefore, Theorem 7.1 has the following translation:

Theorem 8.1. *Choose v in $[n]$. The equation*

$$\operatorname{per}_q A = \sum_{c \in \mathscr{C}_v} q^{\ell(c)} \operatorname{wt}_c(A) \operatorname{per}_q A(c, \alpha(c)) \tag{33}$$

holds for a generic matrix A with digraph D, if and only if, for any disjoint permutations, τ and c, such that c is a cycle through v, and $\Delta_{\tau c} \subseteq D$, we have

$$\mathsf{X}^+(\tau, c) = \mathcal{J}(\tau, \alpha(c)). \qquad ■ \tag{34}$$

Theorem 8.2. *Let A be generic matrix with digraph D, that has all loops.*

(I) *Let $\mathscr{C} \subseteq \mathscr{C}_v$. The equation* (33) *holds, with the sum restricted to $c \in \mathscr{C}$, if and only if:* (i) *$\mathscr{C}$ contains all cycles of D through v, and* (ii) *for all disjoint cycles ω, c of D, such that $v \in V(c)$, we have $\mathsf{X}^+(\omega, c) = \mathcal{J}(\omega, \alpha(c))$.*

(II) *The equations* (33) *hold for all $v = 1, \ldots, n$, if and only if, for all disjoint cycles ω, c of D, we have $\mathsf{X}^+(\omega, \tau) = \mathcal{J}(\omega, \alpha(c))$.* ■

Note that (34) does not, in general, imply $\mathsf{X}^+(\omega, c) = \mathcal{J}(\omega, \alpha(c))$, for ω a cycle of τ. Let us illustrate this with an example.

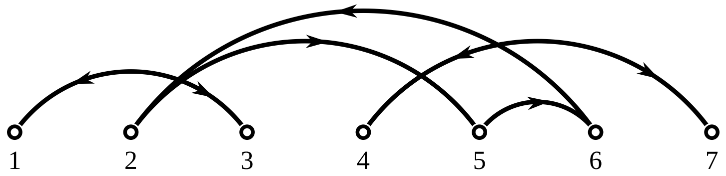

FIGURE 3. The permutation digraph of $\pi = (13)(47)(256)$.

Examples. The digraph D depicted in Figure 3 is the permutation digraph of $\pi = (13)(47)(256)$. We let $v = 2$, and $\alpha((256)) = \{5, 6\}$. The expansion (33), where A is the generic matrix with digraph D, has the following simple form

$$\operatorname{per}_q A = q^{\ell((256))}\,\mathrm{wt}_{(256)}(A)\,\operatorname{per}_q A((256), \{5, 6\}). \tag{35}$$

Let us show this is true, using simple algebra. On one hand, $\ell(\pi) = 10$, and so $\operatorname{per}_q A = q^{10}\,\mathrm{twt}_\pi$. On the other hand, we have $\ell((256)) = 6$, and

$$A((256), \{5, 6\}) = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & 1 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & a_{47} \\ a_{74} & 0 \end{bmatrix}.$$

Therefore, $\operatorname{per}_q A((256), \{5, 6\}) = q^4\,\mathrm{wt}_{(13)(47)}$. This obviously implies (35). This conclusion may be readily achieved via the cross-jump formula (34), which reads, in the current situation

$$\mathsf{X}^+((13)(47), (256)) = \mathcal{J}((13)(47), \{5, 6\}).$$

This holds, as a simple look at Figure 3 shows. Therefore, (35) holds as well. Note that the equation just displayed is equivalent to

$$\mathsf{X}^+((13), (256)) + \mathsf{X}^+((47), (256)) = \mathcal{J}((13), \{5, 6\}) + \mathcal{J}((47), \{5, 6\}).$$

However, if we compare the parts corresponding to (13) and (47), we get

$$\begin{aligned} \mathsf{X}^+((13), (256)) &\neq \mathcal{J}((13), \{5, 6\}), \\ \mathsf{X}^+((47), (256)) &\neq \mathcal{J}((47), \{5, 6\}). \end{aligned} \tag{36}$$

Now imagine adding to the current D a loop at each vertex. Clearly A now has nonzero diagonal entries, and (33) (for $v = 2$) has two summands, corresponding to the cycles (2) and (256). By Theorem 8.2, the inequalities (36) imply the failure of the expansion (33).

The case of the submatrices A_c. We now treat the tentative expansion obtained with the choice $\alpha(c) = V(c)$, for all c, namely

$$\operatorname{per}_q A = \textstyle\sum_{c \in \mathscr{C}_v} q^{\ell(c)}\,\mathrm{wt}_c(A)\,\operatorname{per}_q A_c. \tag{37}$$

We assume the reader is now acquainted with the tricks involving the all loops condition. So, the statements concerning a general digraph, and a digraph with all loops, have been compressed in one.

Theorem 8.3. *Let $v \in [n]$. The equation* (37) *holds for a generic matrix A with digraph D [D having all loops], if and only if, for any complementary factors* [resp., *for any disjoint cycles*] *of D, say $\mathfrak{g}$ and ω, such that ω is a cycle through v, no arc of ω lies under an arc of $\mathfrak{g}$.*

Proof: This follows from Theorem 7.1, taking into account that, for $\alpha(\omega) = V(\omega)$, (26) is equivalent to $\mathcal{U}(\mathfrak{g}, \omega) = 0$. ■

> **Remark**. In page 227 of [C. Fonseca, *Linear and Multilinear Algebra*, 53(2005), pp. 225-230], the equation (37) has been accepted, with no proof, as an identity valid for arbitrary q, A, and $v \in [n]$, with correspondingly flawed corollaries. Theorem 8.3 gives a systematic way to get counter-examples to (37). For example, let c and τ be disjoint cycles, say
>
> $$c = (v_1 v_2 \cdots v_k), \quad \tau = (w_1 w_2 \cdots w_s),$$
>
> where $v_1 = v$, and $w_1 < v_1 < v_2 < w_2$. Then, (37) fails when A is replaced with $P_{c\tau}$, because $v_1 \to v_2$ lies under $w_1 \to w_2$. ■

Our results may be partly translated to the language of graphs. Given a graph G, the *underlying digraph*, denoted D_G, has (i, j) as arc iff $i = j$, or $\{i, j\}$ is an edge of G. Of course, a 2-cycle of D_G is translated as an edge of G, and the oriented k-cycles of D_G, for $k \geqslant 3$, give rise to non-oriented cycles of G. We shall use expressions like *the edge $\{i, j\}$ jumps over the vertex v*, or *the edge lies under* [*above*] *another edge*, with obvious meanings.

The expression A *is a generic matrix with graph G* means that A is generic, and has digraph D_G. Clearly, each edge of G is *relevant*, in the sense that the corresponding 2-cycle of D_G is relevant in D_G.

Corollary 8.4. *Let A be a generic matrix with graph G. For a given $v \in [n]$, the matrix A satisfies* (37)*, if and only if the following two conditions hold:* (i) *no edge of G jumps over v; and* (ii) *if γ is a cycle of G through v, no edge of G disjoint from all edges of γ lies above an edge of γ.*

The equations (37) *hold for all $v = 1, \ldots, n$, if and only if G is a subgraph of the path* 1–2–3–…–n *(this means that A is a tridiagonal matrix).* ■

Proof: We show that $(i) \wedge (ii)$ is equivalent to the condition of Theorem 8.3,

(*) *for any disjoint cycles of D_G, say $\mathfrak{g}$ and ω, with ω a cycle through v, no arc of ω lies under an arc of $\mathfrak{g}$*,

The existence of an edge $\{r, s\}$ of G such that $r < v < s$, contradicts (*), with $\omega = (v)$, and $\mathfrak{g} = (rs)$; so (*) implies (i). Now, suppose (ii) is false. Let γ be a cycle of G through v, and $\{r, s\}$ an edge of G, disjoint from all edges of γ, which lies above an edge of γ. This situation contradicts (*), with $\mathfrak{g} = (rs)$, and ω obtained by choosing an orientation for γ. So (*) implies (ii).

Conversely, assume that (*) fails for some $\mathfrak{g}$ and ω. Let $\omega = (v_1 \ldots v_k)$, where $v_1 = v$. For some j, the arc $v_j \to v_{j+1}$ lies under an arc $u \to w$ of $\mathfrak{g}$. The symmetry of D_G allows us to assume $u < v_j \leqslant v_{j+1} < w$. If $k = 1, 2$ then $u < v < w$, contradicting (i). For $k \geqslant 3$, $\{v_j, v_{j+1}\}$ and $\{u, w\}$ are edges of G, and the former lies under the latter; this goes against (ii). Thus, $(i) \wedge (ii)$ implies (*).

The last part of the corollary is obvious. ■

Corollary 8.5. *Let $v \in \{1, n\}$. If G is a graph with no cycles through v (in particular, if G is acyclic), then the expansion* (37) *holds for a generic matrix with graph G.* ■

We now consider the expansion

$$\operatorname{per}_q A = a_{vv} \operatorname{per}_q A_v + \textstyle\sum_{i \neq v} q^{\ell((vi))} a_{vi} a_{iv} \operatorname{per}_q A_{vi}. \tag{38}$$

This is (37), with the sum restricted to the set of cycles through v, of orders $k \leqslant 2$. The following result is immediate from Theorem 7.2 and Corollary 8.4.

Corollary 8.6. *Let A be a generic matrix with graph G. The equation* (38) *holds, if and only if G is a graph with no cycles through v, and no edges jumping over v.* ■

The case of the augmented submatrices $A_c^{\checkmark}$. Now, we concentrate on the case $\alpha(c) = \varnothing$, for every cycle in $\mathscr{C}_v$. So we are considering the tentative expansion

$$\operatorname{per}_q A = \textstyle\sum_{c \in \mathscr{C}_v} q^{\ell(c)} \operatorname{wt}_c(A) \operatorname{per}_q A_c^{\checkmark}. \tag{39}$$

Applying Theorem 8.1 to the present situation, the cross-jump condition (34) reads $\mathsf{X}^+(\tau, c) = 0$. So we get

Corollary 8.7. *Let $v \in [n]$. The generic matrix A satisfies* (39) *if and only if, for any permutation digraph $\Delta_\pi \subseteq D(A)$, the orbit of π containing v does not cross any other orbit of π.* ■

Examples. Suppose that $D(A)$ has no relevant arc jumping over v, and that any relevant cycle through v has vertex set contained in $\{v-1, v, v+1\}$. For example, tridiagonal matrices satisfy these conditions. Then no relevant arc of $D(A)$ lies above, or crosses, an arc of a relevant cycle through v. Therefore, for such matrices, both expansions (37) and (39) hold. ■

Corollary 8.8. *A generic matrix A with graph G satisfies* (39)*, if and only if the following two conditions hold:* $(i)'$ *no edge of G with endpoint v is crossed by another edge of G, and* $(ii)'$ *if γ is a cycle of G through v, no edge of G disjoint from all edges of γ crosses an edge of γ.*

Proof: This is analogous to Corollary 8.4, with the 'crossing' relation replacing the 'lie above' relation of 8.4. The proof may follow the same pattern, with adequate changes that are left to the reader. ■

Finally, we consider the truncated expansion

$$\mathrm{per}_q\, A = a_{vv}\, \mathrm{per}_q\, A_v^{\checkmark} + \textstyle\sum_{i\neq v} q^{\ell((vi))} a_{vi} a_{iv}\ \mathrm{per}_q\, A_{vi}^{\checkmark}. \tag{40}$$

As the condition $(ii)'$ of Corollary 8.8 is trivially true, we have

Corollary 8.9. *Let A be a generic matrix with graph G. Equation* (40) *holds, if and only if G is a graph with no cycles through v, and no edge of G with endpoint v is crossed by another edge of G.* ■

The last result gives new characterizations of noncrossing [acyclic] graphs.

Corollary 8.10. *A graph G is noncrossing, if and only if a generic matrix A with graph G satisfies all equations* (39)*, for $v = 1, \ldots, n$. Moreover, G is a noncrossing acyclic graph, if and only if a generic matrix A with graph G satisfies all equations* (40)*, for $v = 1, \ldots, n$.* ■

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