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CYCLE CONVEXITY AND THE TUNNEL NUMBER OF LINKS

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Abstract: In this work, we introduce and study a relationship between two considerably distinct areas of mathematics: Knot Theory and Graph Convexity.

A well-known invariant of a knot or a link is its tunnel number. A collection of disjoint arcs $\mathcal{T} = \{t_1, \ldots, t_m\}$ properly embedded in the exterior of a knot K in S^3 is a *unknotting tunnel system for* K if the exterior $K \cup \mathcal{T}$ can be ambient isotoped into the exterior of a plane graph in S^3. The *tunnel number of* K, denoted by $t(K)$, is the minimum cardinality of an unknotting tunnel system of K. Given a plane graph $G = (V, E)$, the *face convexity* is the graph convexity obtained by iteratively applying the following interval function (the final set is called the *face convex hull of* S):

$$\mathrm{I}_{\mathrm{fc}}(S) = S \cup \{v \in V(G) \mid \text{there is a face } F \text{ of } G \text{ such that } F - v \subseteq S\}.$$

A subset S is a *face hull set of* G if its convex hull equals $V(G)$, and the *face hull number of* G is the size of a smallest face hull set of G; it is denoted by $\mathrm{hn}_{\mathrm{fc}}(G)$. A close relationship between $\mathrm{t}(K)$ and $\mathrm{hn}_{\mathrm{fc}}(D(K))$ is known, where $D(K)$ is a diagram of the knot K, i.e. a 4-regular planar graph obtained from K by its vertical projection into the plane.

Because of its dependency on the embedding of the graph, the face convexity is of hard approach. This is why we lose the constraint and introduce the *cycle convexity*. The interval function in this convexity is the following, and it can be defined for any graph G (the final set is called the *convex hull of* S):

$$\mathrm{I}_{\mathrm{cc}} = S \cup \{v \in V(G) \mid d_{G'}(v) \geq 2, \text{ for some component } G' \text{ of } G[S]\}.$$

A subset $S \subseteq V(G)$ is a *hull set of* G if its convex hull equals $V(G)$, and the *hull number of* G is the size of a smallest convex set of G; it is denoted by $\mathrm{hn}_{\mathrm{cc}}(G)$.

In this article, we prove that: 1. the hull number of a 4-regular planar graph is at most half of its vertices; 2. computing the hull number of a planar graph is an NP-complete problem; and 3. computing the hull humber of chordal graphs, P_4-sparse graphs and grids can be done in polynomial time. As an important open question, we ask whether $\mathrm{hn}_{\mathrm{cc}}(G)$ is an upper bound for $\mathrm{hn}_{\mathrm{fc}}(G)$; if the answer is positive, our result gives an upper bound for the tunnel number of knots based on the related diagram graph.

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1. Introduction

We tried to present most of the needed notions in this section, but if the reader finds any gap, we refer him or her to [1, 2, 3]. In this work, we link two distinct areas of knowledge: Graph Convexity and Knot Theory. Let us first shortly describe the notions we need from both areas, alongside the related works in the literature, before stating our results.

1.1. Basic Graph Theory. A *graph* G is a pair (V, E), where V is any finite set called the *vertex set of* G, and E is a multi-set of subsets of V of size at most 2; it is called the *edge set of* G. When V, E are not given, they are denoted by $V(G), E(G)$, respectively. If $\{u, v\} \in E$, we write simply uv. We say that uv has *multiplicity* k if uv occurs k times in E, and we write $\mu(uv) = k$. If $e = \{u\} \in E$, we call e a *loop*, and we say that G is *simple* if G has no loops and every edge of G has multiplicity one. Here, we consider only loopless graphs, which is not really a big restriction to our problem, as we will later see.

Given a graph G, the *neighborhood of a vertex* $u \in V(G)$ is the set $N(u) = \{v \in V(G) \mid uv \in E(G)\}$; the *neighborhood of a subset* $X \subseteq V(G)$ is the set $N(X) = \bigcup_{v\in X} N(v) \setminus X$; the *neighborhood of* u *in* $X \subseteq V(G)$ is the set $N_X(u) = N(u) \cap X$; and the *neighborhood of a subset* $X \subseteq V(G)$ *in* $X' \subseteq V(G)$ is the set $N_{X'}(X) = N(X) \cap X'$. The *degree of* u *in* G is denoted by $d(u)$ and equals $\sum_{v\in N(u)} \mu(uv)$. The *minimum degree of* G is the minimum value over $d(u)$, $u \in V(G)$, and is denoted by $\delta(G)$. We say that G is *k-regular* if $d(u) = k$ for every $k \in V(G)$; this is also sometimes called *k-valent*.

Given a subset $C \subseteq V(G)$, the *subgraph of* G *induced by* C is the graph $G[C] = (C, E_C)$, where $uv \in E_C$ if and only if $\{u, v\} \subseteq C$ and $uv \in E(G)$. If $S \subseteq V(G)$, then we define $G \setminus S = G[V(G) \setminus S]$. If $H \subseteq G$ is an induced subgraph of G and $u \in V(G)$, the *degree of* u *in* H, denoted by $d_H(u)$, is the number of edges incident to u in H, counting multiplicities.

A *path* in G (between v_1 and v_q) is a sequence of vertices $(v_1, \cdots, v_q)$ such that either $q = 1$ or $v_iv_{i+1} \in E(G)$ for every $i \in \{1, \cdots, q-1\}$. If $v_q = v_1$, the path is called a *cycle of length* $q-1$. A graph is said to be *connected* is there exists a path between u and v, for every pair of vertices $u, v \in V(G)$. Also, a *component* of G is a maximal connected induced subgraph of G. A graph G is a *tree* if it is connected and acyclic (alternatively, a tree is a minimal connected graph).

A vertex $u \in V(G)$ is a *cut-vertex* of G if $G - u$ has more components than G. A graph G is *2-connected* if G is connected and does not have any cut-vertex. A *block of* G is a maximal 2-connected subgraph of G. It is well-known that the block structure of a connected graph is that of a tree.

A graph G is *planar* if G can be embedded on the plane in a way that its edges intersect only at their endpoints. A graph thus embedded is called a *plane graph*. Given a plane graph G, each region of the plane minus G is called a *face of* G, and the unbounded face is called the *outer face of* G. We say that a face F has *degree* k if its bounded by k edges (edges that bound only F are counted twice); a face of degree k is also called a k*-face*.

In Section 4, we investigate some graph classes. A *chordal graph* is a graph with no induced cycles of length greater than 4. A P_4*-sparse graph* is a graph in which every subset of 5 vertices induces at most one path on 4 vertices. And for positive integers m and n, the $m \times n$ *grid*, denoted by $G_{m,n}$ is the graph on vertices $\{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ having as edge set:

$$\begin{array}{l}\{(i,j)(i,j+1) \mid i \in \{1,\ldots,m\}, j \in \{1,\ldots,n-1\}\} \cup \\ \{(i,j)(i+1,j) \mid i \in \{1,\ldots,m-1\}, j \in \{1,\ldots,n\}\}.\end{array}$$

1.2. Graph Convexities. A *convexity space* is an ordered pair $(V, \mathcal{C})$, where V is an arbitrary set and $\mathcal{C}$ is a family of subsets of V, called *convex sets*, that satisfies:

- **(C1)** $\emptyset, V \in \mathcal{C}$;
- **(C2)** For all $\mathcal{C}' \subseteq \mathcal{C}$, we have $\bigcap \mathcal{C}' \in \mathcal{C}$; and
- **(C3)** The union of any non-decreasing (w.r.t. inclusion) sequence of elements of $\mathcal{C}$ belongs to $\mathcal{C}$.

In Graph Convexity, convexity spaces are defined over the vertex set of a graph G. Thus, we always consider the set V to be finite and non-empty. Given a subset $S \subseteq V$, the *convex hull of* S (with respect to $(V, \mathcal{C})$) is the minimum $C \in \mathcal{C}$ containing S and we denoted it by $\mathrm{Hull}(S)$. If $\mathrm{Hull}(S) = V$, then we say that S is a *hull-set*. The *hull number* of $(V, \mathcal{C})$ is the minimum cardinality of a hull-set of $(V, \mathcal{C})$ and it is denoted by $\mathrm{hn}(V, \mathcal{C})$.

Let V be a set and $I : 2^V \to 2^V$ be a mapping such that $C \subseteq I(C)$, for all $C \subseteq V$. A subset $C \subseteq V$ is said to be *I-closed* if $I(C) = C$. The family $\mathcal{C}_I$ of I-closed subsets satisfies axioms (C1), (C2) and (C3). The convexity space $(V, \mathcal{C}_I)$ is said to be an *interval-convexity space*, and the function I is called *interval function*. Observe that in this case the convex hull of a subset $C \subseteq V$ equals the set $\bigcup_{k\in\mathbb{N}} I^k(C)$, where I^k denotes k applications of the interval function; for instance, $I^2(C) = I(I(C))$.

Given a graph G, the interval function $\mathrm{I_{cc}} : 2^{V(G)} \to 2^{V(G)}$ in the *cycle convexity* is defined as follows:

$$\mathrm{I_{cc}}(C) = C \cup \{v \in V(G) \mid \text{there exists a cycle in } G[C \cup \{v\}] \text{ containing } v\}.$$

For a plane graph G we also define the *face convexity* $\mathrm{I_{fc}} : 2^{V(G)} \to 2^{V(G)}$ as follows:

$$\mathrm{I_{fc}}(C) = C \cup \left\{ v \in V(G) \mid \begin{array}{l} \text{there is a face } F \text{ of } G \text{ containing } v \\ \text{such that } F - \{v\} \subseteq C \end{array} \right\}.$$

We denote the hull number of $(V(G), \mathcal{C}_{\mathrm{I_{cc}}})$ by $\mathrm{hn_{cc}}(G)$ and of $(V(G), \mathcal{C}_{\mathrm{I_{fc}}})$ by $\mathrm{hn_{fc}}(G)$. For shortness we refer to $\mathrm{hn_{cc}}(G)$ simply as the *hull number of* G and to $\mathrm{hn_{fc}}(G)$ as the *face hull number of* G. Clearly $\mathrm{hn_{cc}}(G) \leq \mathrm{hn_{fc}}(G)$. Similarly, by hull set, we mean the hull set on the cycle convexity, and by *face hull set*, a hull set in the face convexity.

As we shall see in the next section, the definition of such interval function is motivated by an invariant of knots and links called the tunnel number. Before going into these details in Knot Theory, let us first refer to previous works in the literature concerning Graph Convexity.

It is important to emphasize that there are several interval functions defined over the vertices of a graph in the literature, leading to several graph convexities, e.g.: geodetic [4], P_3 [5], P_3^* [6], monophonic [7]. The graph convexities we should emphasize are the P_3 and P_3^*. Their interval functions are, respectively: $\mathrm{I_{P_3}}(C) = C \cup \{u \in V(G) \mid u \text{ has two neighbors in } C\}$ and $\mathrm{I^*_{P_3}}(C) = C \cup \{u \in V(G) \mid u \text{ has two } \textit{non-adjacent} \text{ neighbors in } C\}$. We mention that the P_3 convexity is also called *percolation* [8]; this concept has many applications in physics and they are usually interested in probabilistic results (e.g., what is the probability p such that if the vertices are initially chosen with probability p, then the the chosen set is a hull set - or percolates the entire graph). Observe that $\mathrm{hn_{P_3}}(G) = \mathrm{hn_{P_3^*}}(G)$ if G has no triangles. Also, because a hull-set in the P_3^*-convexity and in the cycle convexity is also a hull-set in the P_3-convexity, we get that $\mathrm{hn}_{\mathrm{P}_3}(G) \leq \mathrm{hn}_{\mathrm{P}_3^*}(G)$ and $\mathrm{hn}_{\mathrm{P}_3}(G) \leq \mathrm{hn}_{\mathrm{cc}}(G)$. And unless G is triangle-free, one cannot ensure a relationship between $\mathrm{hn}_{\mathrm{P}_3^*}(G)$ and $\mathrm{hn}_{\mathrm{cc}}(G)$. In Subsection 1.5, when presenting our results, we will comment about some known results on these convexities.

1.3. Knot Theory. In the sequel, we present basic notions of Knot Theory and describe a problem in this area which is related to the hull number (with respect to the interval functions of the previous section) of planar graphs arising from knots and link diagrams. Throughout this paper we work in the smooth category of manifolds.

Basics in knot theory. A *knot* is an embedding of a circle in the 3-dimensional sphere S^3. If $f : S^1 \to S^3$ is such an embedding, we think of the knot as the subset $K \subset S^3$ given by $K = f(S^1)$. A finite collection of disjoint knots in S^3 is called a *link*. Hence, knots are single component links. Two knots or links K_1, K_2 are said to be *equivalent* if they are *ambient isotopic*. An ambient isotopy $\phi : S^3 \times [0,1] \to S^3$ is a continous map such that each $\phi_t = \phi|_{S^3\times\{t\}} : S^3 \equiv S^3 \times \{t\} \to S^3$ is a homeomorphism and $\phi_0 = \mathrm{id}_{S^3}$. To say K_1 and K_2 are equivalent means that such a ϕ exists, satisfying $\phi_1(K_1) = K_2$.

A *polygonal knot* is a knot whose image is the union of a finite set of line segments. We shall only consider knots which are equivalent to polygonal knots. These are called *tame knots*.

Let $K \subset S^3$ be a knot or link. A useful way to visualize K is to consider its projection on a plane. It is a well known fact that, generically, such a projection is one-to-one, except at a finite number of double points where the projection crosses itself transversely. These projections correspond to finite graphs where all vertices are 4-regular. From these graphs one builds *diagrams* for K as follows: at each vertex we distinguish between the over-strand and the under-strand of K by creating a break in the strand going underneath. See Figure 1. Such a diagram is denoted by $D(K)$ and the corresponding 4-regular graph is denoted by $G_{D(K)}$.

We remark that the correspondence between the graphs and diagrams is not one-to-one. The same graph may be associated to several distinct diagrams. For example, it can be easily proved that any finite connected 4-regular graph is associated to a diagram of the trivial knot (the knot which bounds a disk embedded in S^3).

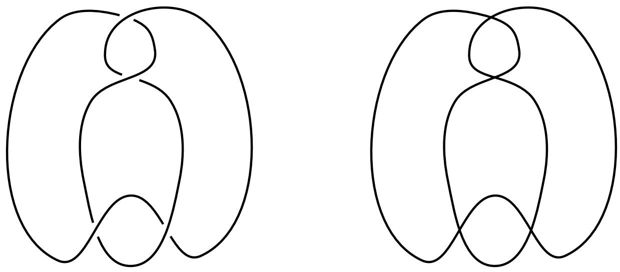

FIGURE 1. Left: a diagram of a knot; Right: the corresponding 4-regular graph.

Tunnel number. Given a knot or link K in S^3, a *tunnel* for K is a properly embedded arc in the exterior $E(K)$ of K. Given a collection of disjoint tunnels $\mathcal{T} = \{t_1, ..., t_m\}$ for K, we may think of $K \cup \mathcal{T}$ as a spatial graph. The collection $\mathcal{T}$ is called an *unknotting tunnel system* for K if a regular neighborhood of the graph $K \cup \mathcal{T}$ can be isotoped into a regular neighborhood of a plane graph in S^3. Every link K in S^3 admits an unknotting tunnel system. In fact, if we add one vertical tunnel at each crossing of a diagram $D(K)$, we obtain an unknotting tunnel system for K. The *tunnel number* of K is the minimal cardinality among all unknotting tunnel systems for K and we denote it by $t(K)$.

The tunnel number is also interpreted in the setting of Heegaard genus $g(M)$ of a 3-manifold M, which is the minimal genus of a surface splitting the 3-manifold into two compression bodies. (See for instance [9].) Note that the definition of an unknotting tunnel system $\mathcal{T} = \{t_1, ..., t_m\}$ is equivalent to say that the boundary of regular neighborhood of $K \cup \mathcal{T}$, which is a surface of genus $m + 1$, splits $E(K)$ into a handlebody and a compression body. We then have that $g(E(K)) = t(K) + 1$. The computation of the Heegaard genus of a 3-manifold is very difficult to compute in general. In fact, by [10], this problem is NP-hard.

In some situations the tunnel number can be computed exactly. For instance, if one single tunnel defines an unknotting tunnel system for a non-trivial knot, then the knot has tunnel number one. Also, it is known that tunnel number one knots are prime. So, if a composite knot has an unknotting tunnel system with two tunnels, then the tunnel number of the knot is 2. But determining if prime knots have tunnel number 2 or higher is a difficult task. In general, one works with upper bounds or lower bounds for the tunnel number. For instance, as observed above, the crossing number is

Crossing number	3	4	5	6	7	8	9	10	11	12
$\mathrm{hn}_{\mathrm{cc}}(G) = \mathrm{hn}_{\mathrm{fc}}(G) = 1$	1	1	2	3	7	12	24	45	91	176
$\mathrm{hn}_{\mathrm{cc}}(G) = \mathrm{hn}_{\mathrm{fc}}(G) = 2$	0	0	0	0	0	9	25	120	446	1952
$\mathrm{hn}_{\mathrm{cc}}(G) = \mathrm{hn}_{\mathrm{fc}}(G) = 3$	0	0	0	0	0	0	0	0	15	48
total	1	1	2	3	7	21	49	165	552	2176

TABLE 1. The number of prime knots K by crossing number and hull number of $G = G_{D(K)}$.

an upper bound for the tunnel number, and, from the Heegaard splitting of the knot exterior, the rank of the fundamental group of the knot exterior is a lower bound for the tunnel number plus one. For links the tunnels necessarily have to connect the link components. Hence, its tunnel number is at least the number of components minus one. This lower bound can be used to determine the tunnel number of a link in case one finds an unknotting tunnel system with that cardinality, as in [11]. Also in [12] the authors determine the tunnel number of a class of links by exploring the following upper bound: given a diagram $D(K)$ of K, consider vertical arcs added at certain crossings (see figure 2 below). What is the smallest unknotting tunnel system consisting of vertical arcs only? One of the results in [12] shows that this is at most the hull number associated to the interval function I_{fc} of the graph $G_{D(K)}$. We conjecture that the tunnel number is also bounded above by $\mathrm{hn}_{\mathrm{cc}}$. In fact, we developed a computer program to compute the hull number and the face hull number of every prime knot up to 12 crossings, whose results are described in Table 1. For every one of these 2977 knot diagrams $D(K)$, we obtained $\mathrm{hn}_{\mathrm{cc}}(G_{D(K)}) = \mathrm{hn}_{\mathrm{fc}}(G_{D(K)})$.

Hence, finding good upper bounds for the hull number of $G_{D(K)}$ would help us estimate $t(K)$.

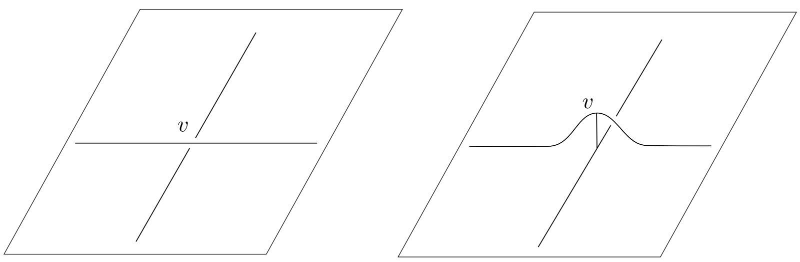

FIGURE 2. Adding a vertical arc at a crossing v of the diagram.

1.4. Relationship between hull number and tunnel number. The result of this subsection have appeared, in a more general setting, in [12]. We include it here for completeness.

Remark 1.1. Suppose one starts to add vertical arc and, at some point, there is a face f corresponding to one of the planar regions determined by the diagram $D(K)$ such that all, except one, of the crossings of f has a vertical arc. Let v represent this crossing. Then the crossing v of f may be removed. Let K_1 be the resulting 1-complex. This is described in Figure 3.

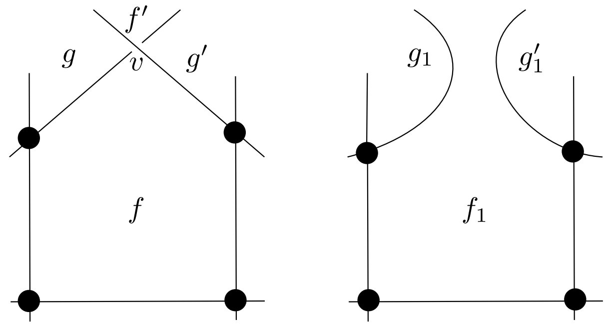

FIGURE 3. Removal of vertex

In the diagram $D(K)$, let f' be the face containing v, opposite to f. After v is removed, as above, the faces f and f' merge into a single face f_1 of $D(K_1)$. If necessary, we may keep adding vertical arcs. Whenever this procedure yields a face in which all crossings, except one, have a vertical arc, then the remaining crossing can be removed.

In what follows, we restrict to minimal crossing diagrams of knots and links. Consider the 4-regular graph $G_{D(K)}$ given by a diagram $D(K)$ of a link K, i.e., just ignore over/under information on $D(K)$.

Theorem 1.2. $t(K) \leq \mathrm{hn}_{\mathrm{fc}}(G_{D(K)})$.

We thus wish to find good upper bounds for $t(K)$ in terms of the *crossing number* of K. The proof of this theorem relies on Remark 1.1.

Let V_0 be a subset of vertices in $D(K)$ in which vertical arcs have been added. Let f be a face of $D(K)$ and suppose that in all, except one, vertices of f vertical arcs have been added. In the link K, let v represent the remaining crossing of f, as in the remark. Consider the 1-complex K_1 obtained by removing this crossing, according to the remark, and $D(K_1)$ the planar graph associated to K_1.

Lemma 1.3. *Percolation in the graph $D(K_1)$ is equivalent to the percolation in $D(K)$.*

Proof: The vertex (crossing) v is removed from $D(K)$. Let f_1 be the face of $D(K_1)$ obtained by merging the faces f and f'. Then the vertices of f_1 that do not have a vertical arc are exactly the vertices of f' that that do not have vertical arcs. Moreover, let g, g' be the faces of $D(K)$ adjacent to v other than f, f'. These yield two faces g_1, g_1' in $D(K_1)$, respectively. As before, the vertices of g_1, g_1' that do not a vertical arc are exactly those vertices of g, g' that do not a vertical arc. ■

Proof of theorem 1.2: The theorem will be proved by showing $\mathrm{hn}_{\mathrm{fc}}(D(K))$ is an upper bound for the number of vertical arcs needed to make the resulting 1-complex planar. Assume V_0 is a hull set for $D(K)$, i.e., coloring all vertices in V_0, then it percolates to the whole graph $D(K)$. We relate the percolation rule to Remark 1.1.

Place one vertical arc at each crossing corresponding to the vertices of V_0. Let v be the first crossing to be removed in $D(K)$. Then it belongs to a face f in which all other vertices were previously colored. By Remark 1.1, the crossing corresponding to v may be removed. Let $D(K_1)$ be the graph obtained by removing the crossing v. By Lemma 1.3, percolation in $D(K)$ is equivalent to percolation in $D(K_1)$. Then V_0 is a hull set for $D(K_1)$ also. Inductively, repeat this process to the remaining vertices (crossings). We end up with a diagram D' of 1-complex K' in which there are no crossings, i.e., K' is planar.

■

1.5. Our Results. The main initial motivation for this work has been to find good upper bounds for the hull number of 4-regular planar graphs, not necessarily simple, in order to obtain better bounds for the tunnel number of links. At the beginning of our studies, we believed that a version of Theorem 1.2 would also hold for $\mathrm{hn}_{\mathrm{cc}}(G)$. However, we have not been able to prove such a theorem and leave the following open question, whose positive answer would imply the desired result:

Question 1.4. Let G be a 4-regular planar graph. Does the following hold?

$$\mathrm{hn}_{\mathrm{fc}}(G) \leq \mathrm{hn}_{\mathrm{cc}}(G).$$

Because the face convexity depends on the embedding of the graph, it is of much harder approach. This is why we decided to investigate the cycle convexity nevertheless. First of all, we prove that we can consider only loopless graphs. We mention that we do not restrict our attention to simple graphs, as most convexity works do, because of our initial motivation on knot theory.

Proposition 1.5. *Let G be a graph, and $u \in V(G)$ be such that $\{u\}$ is a loop in G. Let H be obtained from G by removing u and making $N(u)$ be a clique in H. Then,* $\mathrm{hn}_{\mathrm{cc}}(H) = \mathrm{hn}_{\mathrm{cc}}(G)$.

Proof: We only need to prove that any hull-set of H is also hull-set of G, and vice-versa. But since $\{u\}$ is itself a cycle, we get $u \in \mathrm{Hull}(\emptyset)$, and one can verify that this implies the proposition. ■

This means that the problem is not harder if we have loops, and this is why from now on we consider only loopless graphs. If the graph has no loops, an equivalent way of defining the interval function for the cycle convexity would be:

$$\mathrm{I}_{\mathrm{cc}}(C) = C \cup \{u \in V(G) \mid d_H(u) \geq 2, \text{ for some component } H \text{ of } G[C]\}.$$

In Section 3, we show that the hull number is at most half of the vertices of G, when G is a 4-regular graph. This is a tight bound as can be seen by the example in Figure 4. It consists of the graph obtained from two disjoint cycles of length k, $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$, by adding edges of multiplicity two u_iv_i for each $i \in \{1, \ldots, k\}$. If $S \subseteq V(G)$ has size smaller than $k-1$, by the pigeonhole principle there exist $i, j \in \{1, \ldots, k\}$ such that $\{u_i, v_i, u_j, v_j\} \cap S = \emptyset$, which implies that $\{u_i, v_i, u_j, v_j\} \cap \mathrm{Hull}(S) = \emptyset$ since u_i, v_i, u_j, v_j don't have two neighbors in the same connected component of $G[S]$ (this is called a co-convex set and will be formally defined later). Therefore $\mathrm{hn}_{\mathrm{cc}}(G)/2k \geq \frac{k-1}{2k} \to \frac{1}{2}$ as $k \to +\infty$.

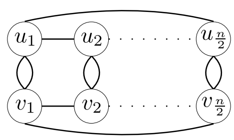

FIGURE 4. Tight example.

It is worth mentioning that we have not been able to find a *simple* 4-regular planar graph that needs half of the vertices to be contaminated. In fact, we believe that this could be improved to a third if G is a simple graph; hence we ask:

Question 1.6. What is the minimum c such that $\mathrm{hn}_{cc}(G) \leq c \cdot n$, for all simple 4-regular planar graphs G on n vertices?

It is also worth mentioning that, in general, planar graphs may need as many as all the vertices to be contaminated. As we will see in Section 4, if G is a chordal graph with p blocks, then $\mathrm{hn}_{cc}(G) \geq p + 1$. Because a tree T on p vertices is also a chordal planar graph with $p - 1$ blocks, we get $\mathrm{hn}_{cc}(T) = p$. Also, one can observe that a cycle C on n vertices is such that $\mathrm{hn}_{cc}(C) = n - 1$, which means that n cannot be improved even if G is a simple 2-connected planar graph, i.e., the 4-regularity in the question above is necessary.

Now, apart from the topological interest, the new graph convexity has an interest of its own. We recall the connection of our graph convexity to other previously known ones: the P_3 and P_3^* convexities. Concerning the P_3-convexity, in [13] the authors prove that computing the hull-number is NP-complete in general, and they give polynomial results for chordal graphs and cographs. In [6], the authors prove that computing the hull number remains NP-complete even when restricted to subgraphs of the grid, which implies that it is hard for bipartite graphs and planar graphs with bounded maximum degree; they also give polynomial results for P_4-sparse graphs. In addition, they introduce and investigate the P_3^*-convexity, and prove that computing the hull-number is NP-complete for bipartite graphs, and also polynomial for P_4-sparse graphs. A natural question is whether any of these results also apply to our convexity. In Section 2, we prove that computing $\mathrm{hn}_{cc}(G)$ is NP-complete even when G is a planar graph, and in Section 4 we give polynomial results for chordal graphs, P_4-sparse graphs and grids.

In particular, our NP-completeness proof does not limit the maximum degree of the graph, which means that we do not know what is the complexity of computing $\mathrm{hn}_{cc}(G)$ when G is a 4-regular planar graph, our main class of interest. We therefore pose the following question:

Question 1.7. Let G be a 4-regular planar graph. Can one compute $\mathrm{hn}_{cc}(G)$ in polynomial time?

Many other questions can be posed about all of these graph parameters, since very little is known. In particular, Table 2 shows the known results of the graph classes investigated so far.

Graph class	$\mathbf{P_3}$	$\mathbf{P_3^*}$	**cycle conv.**
chordal	P	?	P
P_4-sparse	P	P	P
bipartite	NP	NP	?
planar	NP	?	NP

Table 2. The complexity of computing the hull number of G in the given convexities. In the table, P stands for polynomial, NP stands for NP-complete and "?" stands for open.

2. NP-Completeness

In this section, we work only with simple graphs. Let us formally define the decision version of the problem we study:

Problem 2.1. Hull Number in Cycle Convexity
Input: A graph G and a positive integer k.
Question: Is $\mathrm{hn}_{\mathrm{cc}}(G) \leq k$?

The goal of this section is to prove that:

Theorem 2.2. Hull Number in Cycle Convexity *is* NP*-Complete, even when restricted to planar graphs.*

We shall reduce a variant of Planar 3-SAT to Hull Number in Cycle Convexity. We first explain which variant we use, so that our gadget construction becomes more natural.

Let φ be a boolean formula in 3-CNF having a set $\mathcal{C}$ of m clauses over the variables $\mathcal{X} = \{x_1, \ldots, x_n\}$. The *$\varphi$-graph* is the graph $G(\varphi)$ such that:

- $V(G(\varphi)) = \mathcal{X} \cup \mathcal{C} \cup \{\bar{x}_1, \ldots, \bar{x}_n\}$; and
- $E(G(\varphi)) = E_1 \cup E_2 \cup E_3$, where:
  - $E_1 = \{x_iC \mid x_i \text{ appears positively in } C\}$,
  - $E_2 = \{\bar{x}_iC \mid x_i \text{ appears negatively in } C\}$, and
  - $E_3 = \{x_1x_2, x_2x_3, \ldots, x_nx_1, x_1\bar{x}_1, \ldots, x_n\bar{x}_n\}$.

We say that φ is *linkable* if it is possible to add edges to $G(\varphi)$ in such a way that the obtained graph is planar and that the subgraph induced by $\mathcal{C}$ is connected. We call the obtained graph a *linked φ-graph*, and call the subgraph formed by the added edges a *link*. Observe that $G(\varphi)$ must be planar itself to start with, and that we can consider that the link only contains edges between vertices in $\mathcal{C}$ as otherwise we could remove some of the edges and still have a linked φ-graph.

The variation of the 3-SAT problem presented next can be proved to be NP-complete by making a few modifications in the proofs presented in [14, 15, 16]. Because the needed ideas are not in any way new, we restrain from presenting them here.

Problem 2.3. Linked Planar exactly 3-bounded 3-SAT

Input: A linkable boolean formula φ in 3-CNF such that each variable of φ appears exactly three times: twice positively and once negatively.

Question: Is φ satisfiable?

In what follows, we present constructions for edge, variable and clause gadgets that will replace the respective structures in a linked φ-graph, obtaining a planar graph H. Then, we prove that φ is satisfiable if and only if $\mathrm{hn}_{\mathrm{cc}}(H) \leq 4n + 4m - 4$, finishing the proof. .

Given a graph $G = (V, E)$ and a set of vertices $S \subseteq V(G)$, we define the *boundary* of S as $\partial(S) = \{v \in S \mid N(v) \nsubseteq S\}$. We say that S is *co-convex* if each $v \in \partial(S)$ has at most one neighbor in each connected component of $G \setminus S$. This means that S cannot be in the convex hull of a set that is disjoint from S, as proved in the lemma below.

Lemma 2.4. *If S is a co-convex set of G, then every hull-set of G contains at least one vertex of S. Furthermore, if $\partial(S)$ is a stable set, then every hull-set of G contains at least two vertices of S.*

Proof: By contradiction, let $C \subseteq V(G)$ be a hull-set of G such that $C \cap S = \emptyset$. Since $\mathrm{Hull}(C) = V(G)$, there exists $k \in \mathbb{N}^*$ such that $I^{k-1}(C) \cap S = \emptyset$ and $I^k(C) \cap S \neq \emptyset$ (consider that $I^0(C) = C$). Let $v \in S \cap I^k(C)$. Since $v \in I^k(C) \setminus I^{k-1}(C)$, v must have two neighbors $u, w \in I^{k-1}(C)$ such that u and w lie in the same connected component of $G[I^{k-1}(C)]$. Consequently, we deduce that $v \in \partial(S)$ and $u, w \in N(v) \setminus S$. This contradicts the fact that S is a co-convex set.

Now suppose that $\partial(S)$ is a stable set and that $S \cap \partial(S) = \{u\}$. Because $\partial(S)$ is a stable set, we know $\partial(S \setminus \{u\}) = (\partial(S) \setminus \{u\}) \cup N_S(u)$ and that $S \setminus \{u\}$ is also a co-convex set. The lemma thus follows from the first part. ∎

Let G be a connected graph and $H \subseteq G$ be an induced subgraph of G. We say that H is *pendant at $v \in V(H)$* if there are no edges between $V(H - v)$ and $V(G - H)$ (in other words, v separates H from the rest of the graph).

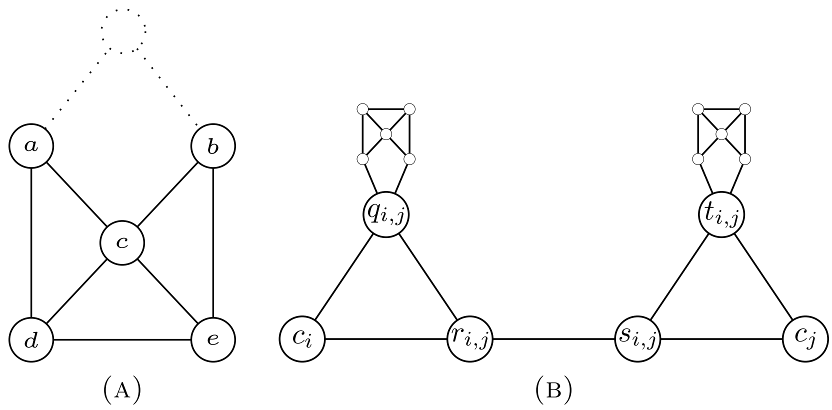

FIGURE 5. (A) Auxiliary gadget Λ; (B) Edge gadget Ξ_u^v.

To construct our variable and clause gadgets, we first need what we call *auxiliary gadget*. This is the graph Λ with vertex set $\{a, b, c, d, e, v\}$ and edge set $\{va, vb, ac, ad, bc, be, cd, ce, de\}$ as depicted in Figure 5.(a).

Now, consider a linked φ-graph L, with link T. We first explain how to construct the edge gadgets that will replace the edges in T. Given an edge $e = c_ic_j \in E(T)$, we denote the related *edge gadget* by Ξ_{ij} (observe Figure 5.(b) to follow the construction). Consider the graph with vertex set $\{c_i, c_j, q_{ij}, r_{ij}, s_{ij}, t_{ij}\}$ and having as edges $\{c_iq_{ij}, c_ir_{ij}, c_js_{ij}, c_jt_{ij}, r_{ij}s_{ij}, q_{ij}r_{ij}, s_{ij}t_{ij}\}$. To obtain Ξ_e we add two copies of the auxiliary graph, one pendant at q_{ij} and the other pendant at t_{ij}.

Finally, we define the *variable gadgets* (follow the construction by observing Figure 6). Let $i \in \{1, \ldots, n\}$, and define: $A_i = \{a_i^1, \ldots, a_i^8\}$ and $B_i = \{b_i^1, \ldots, b_i^8\}$. Let Υ_i be the variable gadget obtained as follows:

- Start with $V(\Upsilon_i) = A_i \cup B_i$ and $E(\Upsilon_i) = \emptyset$;
- Add to $V(\Upsilon_i)$ the vertices $v_i, x_i^1, x_i^2, x_i^a, \bar{x}_i^1, \bar{x}_i^2$;
- Add to $E(\Upsilon_i)$ the edges:
  - $v_ia_i^j$ and $v_ib_i^j$, for every $j \in \{1, 4, 5, 6\}$;
  - $a_i^1a_i^j$ and $b_i^1b_i^j$, for every $j \in \{2, 3, 4\}$;
  - $a_i^ja_i^{j+1}$ and $b_i^jb_i^{j+1}$, for every $j \in \{2, 3, 4, 5\}$;
  - $a_i^7a_i^j$ and $b_i^7b_i^j$, for every $j \in \{3, 4\}$;
  - $a_i^8a_i^j$ and $b_i^8b_i^j$, for every $j \in \{4, 5\}$;
  - $x_i^1a_i^j$ and $\bar{x}_i^1b_i^j$, for every $j \in \{2, 7\}$;
  - $x_i^2a_i^6$, $\bar{x}_i^2b_i^6$, $x_i^2a_i^8$, $\bar{x}_i^2b_i^8$, $x_i^1x_i^2$, $\bar{x}_i^1\bar{x}_i^2$, $x_i^1x_i^a$, $x_i^2x_i^a$, ;
- Add to Υ_i an auxiliary graph Λ_i such that Λ_i is pendant at $\bar{x}_i^1$;

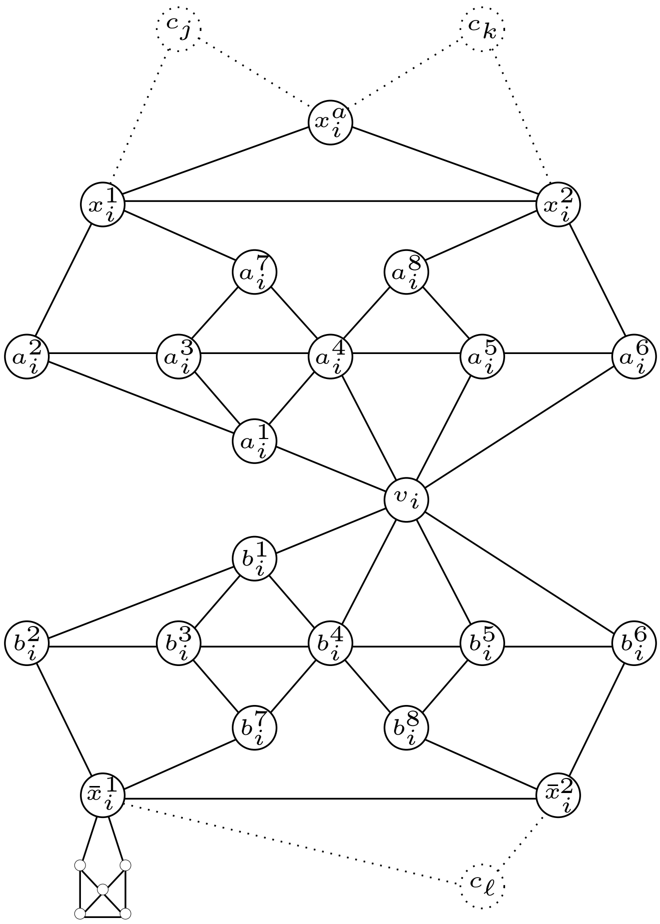

FIGURE 6. Variable gadget Υ_i.

We are now ready to prove Theorem 2.2, which we recall:

Theorem 2.2. HULL NUMBER IN CYCLE CONVEXITY *is* NP-*Complete, even when restricted to planar graphs.*

Proof: First, observe that HULL NUMBER IN CYCLE CONVEXITY is in NP, since for a given subset $S \subseteq V(G)$ one can compute $\mathrm{Hull}(S)$ in polynomial time and decide whether $\mathrm{Hull}(S) = V(G)$.

To prove the completeness of the problem, we reduce the LINKED PLANAR EXACTLY 3-BOUNDED 3-SAT to HULL NUMBER IN CYCLE CONVEXITY. Let φ be an instance of the considered 3-SATproblem, and let L be a linked φ-graph with link T. We can consider T to be a tree as otherwise we can simply remove some edges. We construct a graph L^* such that L^* is planar and prove that φ is satisfiable if, and only if, $\mathrm{hn}_{\mathrm{cc}}(L^*) \leq 4n + 4m - 4$. This graph is obtained from L as follows:

- For each variable x_i of φ, let c_j, c_k be the clauses in which x_i appears posivitively and c_ℓ be the clause in which x_i appears negatively. Replace vertices $x_i, \bar{x}_i$ by Υ_i, and add edges $\{c_j x_i^1, c_j x_i^a, c_k x_i^2, c_k x_i^a, c_\ell \bar{x}_i^1, c_\ell \bar{x}_i^2\}$;
- Replace each edge $c_i c_j$ in $E(T)$ by Ξ_{ij}.

One can verify that L^* is planar by construction. Moreover, observe that L^* has less than $27n+14m$ vertices, since each variable gadget has 27 vertices, each edge gadget has 14 new vertices, and since there are $m-1$ edges in T. Before we can present our proof, we still need some facts concerning hull-sets of L^*, presented below.

Claim 2.4.1. (1) *If $H \subseteq L^*$ is an induced subgraph isomorphic to Λ and is pendant on v, then $|V(H-v) \cap S| \geq 2$, for every hull-set S of L^*;*

(2) *For every variable x_i of φ, and every hull-set S of L^*, we have $|(A_i \cup B_i \cup \{v_i\}) \cap S| \geq 2$;*

(3) *For every edge $uv \in L^*[A_i \cup \{v_i\}]$, we have $A_i \cup \{v_i, x_i^1, x_i^2, x_i^a, c_j, c_k\} \subseteq \mathrm{Hull}(\{u,v\})$. The similar statement holds for B_i and the corresponding vertices; and*

(4) *Let $c_i c_j \in E(T)$, and let $S \subseteq V(L^*)$ not containing r_{ij}. If $c_i \notin \mathrm{Hull}(S)$, then $r_{ij} \notin \mathrm{Hull}(S)$. The analogous holds for c_j and s_{ij}.*

Proof: Itens (1) and (2) follow directly from Lemma 2.4 and the fact that the respective subsets are co-convex sets with stable boundaries. To verify item (3), first observe that $A_i \cup \{v_i\}$ induces a graph formed only by triangles and can be entirely infected by any of its edges. The item follows because $\{x_i^1, x_i^2, x_i^a\}$ is clearly contained in $\mathrm{Hull}(A_i \cup \{v_i\})$. Finally, let S be as in item (4), and suppose by contradiction that $c_i \notin \mathrm{Hull}(S)$ and $r_{ij} \in \mathrm{Hull}(S)$. Because $r_{ij} \notin S$, there exists $k \in \mathbb{N}$ such that $r_{ij} \in {\mathrm{I_{cc}}}^k(S) \setminus {\mathrm{I_{cc}}}^{k-1}(S)$. This means that there exists a component of $L^*[{\mathrm{I_{cc}}}^{k-1}(S)]$ containing q_{ij} and s_{ij}. This is a contradiction since neither c_i nor r_{ij} is in ${\mathrm{I_{cc}}}^{k-1}(S)$, and $\{c_i, r_{ij}\}$ separates q_{ij} from s_{ij}. ■

Let us now prove that φ is satisfiable if, and only if, $\mathrm{hn_{cc}}(L^*) \leq 4n+4m-4$.

Suppose first that φ is satisfiable. Let us construct a hull-set $S \subseteq V(L^*)$ such that $|S| = 4n+4m-4$. For every pendant auxiliary graph in L^*, add any two adjacent vertices of such auxiliary graph to S, say vertices d and e. Note that there are $n+2m-2$ such pendant auxiliary graphs. Finally, consider a truth assignment to the variables $x_1, \ldots, x_n$ satisfying φ. In case x_i is true, add to S the vertices v_i and a_i^1. Otherwise, add to S the vertices v_i and b_i^1. We then get $|S| = 2 \cdot (n+2m-2) + 2n = 4n+4m-4$. Observe that this is the smallest size of a hull-set by Claim 2.4.1, items 1 and 2. It remains to show that S is indeed a hull-set.

It is easy to verify that: (*) $H \cup \{v\} \subseteq \mathrm{Hull}(S)$ for every auxiliary graph H pendant at a vertex v. Also, by Claim 2.4.1 we get that: if x_i is true, then $A_i \cup \{v_i, x_i^1, x_i^2, x_i^a, c_j, c_k\} \subseteq \mathrm{Hull}(\{u, v\})$; while if x_i is false, then $B_i \cup \{v_i, \bar{x}_i^1, \bar{x}_i^2, c_\ell\} \subseteq \mathrm{Hull}(S)$. Because every gadget contains a truth literal, this means that $C \subseteq \mathrm{Hull}(S)$. This and (*) imply that $\mathrm{Hull}(S)$ contains every edge gadget, which means that: (**) C is contained in the same connected component of $\mathrm{Hull}(S)$. Now it remains to prove that the "untruth" side of the variable gadgets are also in $\mathrm{Hull}(S)$. So consider any variable x_i, and let c_j, c_k be the clauses that contain x_i positively, and c_ℓ be the clause containing the negation of x_i. First, suppose that x_i is true. Because $\{\bar{x}_i^1, c_\ell\} \subseteq \mathrm{Hull}(S)$, we get that $\bar{x}_i^2 \in \mathrm{Hull}(S)$. But in this case, by (**) and since $\{v_i, a_i^6, x_i^2, c_k\} \subseteq \mathrm{Hull}(\{u, v\})$, we get that $b_i^6 \in \mathrm{Hull}(S)$. It follows from Claim 2.4.1 that $B_i \subseteq \mathrm{Hull}(S)$. When x_i is false, by (**), we get that $\mathrm{Hull}(S)$ contains x_i^a and consequently x_i^2 and a_i^6; the rest follows similarly.

Now, let S be a hull-set of L^* such that $|S| \leq 4n + 4m - 4$. As before, we know that equality must occur by Claim 2.4.1, items 1 and 2. Let x_i be true if, and only if, $A_i \cap S \neq \emptyset$. One can verify that Claim 2.4.1, item 2, and the size of S tell us that at most one between A_i and B_i has non-empty intersection with S; hence, the assignment is well-defined. We prove that this assignment satisfies φ. For this, consider c_i to be the clause $(x_{i_1} \vee x_{i_2} \vee \bar{x}_{i_3})$ and suppose by contradiction that c_i is not satisfied. We prove that in this case $c_i \notin \mathrm{Hull}(S)$, thus getting a contradiction.

So, let $k \in \mathbb{N}^*$ be such that $c_i \in {\mathrm{I}_{\mathrm{cc}}}^k(S) \setminus {\mathrm{I}_{\mathrm{cc}}}^{k-1}(S)$, and let H be the component of L^* containing two neighbors of c_i, u and v. Observe that $c_j^a \notin \{u, v\}$ and that, by Claim 2.4.1, item 4, we also have that $r_{i,j} \notin \{u, v\}$ for every edge $c_i c_j$ of T. This means that c_i must depend on the vertices of some variable gadget to enter $\mathrm{Hull}(S)$. Now, for each $j \in \{1, 2, 3\}$, let c_1^j, c_2^j, c_3^j be the clauses containing the two positive and the one negative occurrence of x_{i_j}, respectively, with $c_1^1 = c_1^2 = c_3^3 = c_i$. Because c_i is not satisfied, it must occur that $A_{i_1} \cap S = \emptyset$, the same being valid for A_{i_2} and B_{i_3}. But note that the only vertex in $A'_{i_1} = A_{i_1} \cup \{x_{i_1}^1, x_{i_1}^2, x_{i_1}^a\}$ that has more than one neighbor outside of A'_1 is $x_{i_1}^a$. This means that it must be the first vertex of A'_{i_1} to enter $\mathrm{Hull}(S)$. However, this can only happen after time k, since it depends on c_i to be contaminated. The same argument trivially holds for

$A_{i_2} \cup \{x_{i_2}^1, x_{i_2}^2, x_{i_2}^a\}$ and a similar argument can be made for $B_{i_3} \cup \{\bar{x}_{i_3}^2\}$. We get a contradiction since in this case u, v cannot be in any of these sets. ■

3. Upper Bound for 4-Regular Planar Graphs

The main result of this section concerns 4-regular planar graphs, but our first lemmas actually hold for any 4-regular graph. The idea is to iteratively reduce the graph at the same time that we construct a hull-set of the original graph, and ensuring that for each vertex added to the hull-set, another vertex is gained for free. At the end, we prove that if G is planar, then it can be reduced to the graph containing just one edge of multiplicity 4, which implies that half of the vertices are enough to contaminate the original graph.

Let G be any 4-regular graph. Below, we present the reduction operations.

- M4 Let uv be such that $\mu(uv) = 4$. Remove u and v from G.
- M2 Let H be a component of the subgraph of G containing exactly the edges of multiplicity 2. Because G is 4-regular and $\mu(uv) = 2$ for every $uv \in E(H)$, we get that H is either a cycle or a path. Remove H from G and, if H is a path between vertices u, v, let $N_{G-H}(u) = \{x, x'\}$, and $N_{G-H}(v) = \{y, y'\}$. Then, add edges xx' and yy'.
- M3 Let $uv \in E(G)$ be such that $\mu(uv) = 3$. Also let $x \in N(u) \setminus \{v\}$ and $y \in N(v) \setminus \{u\}$. If $x \neq y$, remove vertices u and v and add edge xy to G. Otherwise, let $N(x) \setminus \{u, v\} = \{z, z'\}$. Remove vertices u, v, x and add edge zz'.
- T Let $T = (x, y, z)$ be any cycle of length 3 in G, and let $N_{G\setminus T}(x) = \{x_1, x_2\}$, $N_{G\setminus T}(y) = \{y_1, y_2\}$, and $N_{G\setminus T}(z) = \{z_1, z_2\}$. Remove T and add edge x_1x_2, a vertex w and edges wy_1, wy_2, wz_1, wz_2.

Let G' be obtained from G by exhaustively applying operations M4, M2, M3, and T, with this order priority, i.e., if at some point M4 and M2 can be applied, then we apply M4 and re-evaluate. We call G' the *reduction graph of* G. First, we prove the following simple lemma.

Lemma 3.1. *Let G be a 4-regular graph and G' be its reduction graph. Then, G' is a simple 4-regular graph with no cycles of length 3.*

Proof: Clearly, G' cannot have any edges of multiplicity bigger than 1, or any cycle of length 3, as otherwise we could still apply some of the operations. Thus, we just need to prove that these operations create no loops. Operation M4 clearly does not create any loop since it does not add any edges. In operation M2, the added edges xx' and yy' are not loops because otherwise

ux or vy would have multiplicity 2 and, therefore, should also be in H. In operation M3, when $x = y$ (only case in which we could have added a loop), we get that $z \neq z'$ as otherwise xz would be an edge of multiplicity 2 and we would have had applied M2 instead. A similar argument is applied to see that x_1x_2 is not a loop in operation T. ■

As we said before, we compute the reduction graph of G, and at the same time we construct a hull-set for G. In the first three operations, the vertex added to the hull-set is always u. But to explain how we choose a vertex in the third operation we need the tool lemma below.

Lemma 3.2. *If G is not the trivial graph and w is any vertex of G, then there exists some minimum hull-set of G that does not contain w.*

Proof: Let S be a minimum hull-set of G containing w, and let w' be the first vertex that needs w to be contaminated. More formally, there exists $k \in \mathbb{N}$ such that $w' \in {\mathrm{I}_{\mathrm{cc}}}^k(S) \setminus {\mathrm{I}_{\mathrm{cc}}}^{k-1}(S)$, and $N_H(w') = \{w, t\}$, where H is the component of $G[I^{k-1}(S)]$ containing $\{w, t\}$. Let $S' = (S \setminus \{w\}) \cup \{w'\}$. By the choice of w', we get that ${\mathrm{I}_{\mathrm{cc}}}^{k-1}(S \setminus \{w\}) \subseteq {\mathrm{I}_{\mathrm{cc}}}^{k-1}(S')$. Therefore, the subgraph $H \setminus \{w\}$ is contained in some component H' of $G[{\mathrm{I}_{\mathrm{cc}}}^{k-1}(S')]$. Because $t \in N_H(w')$, we get that $w' \in V(H')$, and because $N_H(w) \neq \emptyset$ and $w' \in N_{H'-H}(w)$, we get that $d_{H'}(w) \geq 2$, which implies that $w \in {\mathrm{I}_{\mathrm{cc}}}^k(S')$. Since $S \subseteq hull(S')$ and S is a hull-set, we get that S' is also a hull-set. ■

Theorem 3.3. *Let G be a 4-regular graph, and G' be the reduction graph of G. Then, $\mathrm{hn}_{\mathrm{cc}}(G) \leq \mathrm{hn}_{\mathrm{cc}}(G') + f$, where f is the number of operations applied to G in order to obtain G'.*

Proof: We can suppose that G is connected, as otherwise we just apply the argument to each of its connected components. We simply prove that if F is obtained from G by applying one of the operations above, then $\mathrm{hn}_{\mathrm{cc}}(G) \leq \mathrm{hn}_{\mathrm{cc}}(F) + 1$. For this, suppose that G is not an edge of multiplicity 4, nor a cycle of edges of multiplicity two, since in both cases we get that $V(F) = \emptyset$ and clearly $\mathrm{hn}_{\mathrm{cc}}(G) = 1$. So, we analyze only the remaining cases.

So, let S be a hull-set of F. If F is obtained by applying M2, by assumption the removed subgraph H is a path. Let u, v be its extremities, and let $N_{G\setminus H}(u) = \{x, x'\}$ and $N_{G\setminus H}(v) = \{y, y'\}$. Because $V(H) \subseteq \mathrm{Hull}(u)$, it is not hard to see that $S \cup \{u\}$ is a hull-set of G. Observe that by a similar argument, we get that if F is obtained by applying operation M3 on edge uv, then again $S \cup \{u\}$ is a hull-set of G.

Now, suppose that F is obtained by applying operation T on the triangle (x, y, z). We use the same notation as before. By Lemma 3.2, we can suppose that $w \notin S$. First, note that if S is a hull-set of $F - x_1x_2$, then $S \cup \{v\}$ is a hull-set of G for $v \in \{y, z\}$, depending on whether the cycle in $\mathrm{Hull}_{F-x_1x_2}(S)$ containing w intersects $\{y_1, y_2\}$ or $\{z_1, z_2\}$. So, consider the contrary and suppose, without loss of generality, that x_2 is contaminated after x_1, which means that x_2 depends on the edge x_1x_2 to be contaminated. Let k be such that $x_2 \in \mathrm{I_{cc}}^k_F(S) \setminus \mathrm{I_{cc}}^{k-1}_F(S)$, and denote $\mathrm{I_{cc}}^{k-1}_F(S)$ by S'. We prove that $S' \cup T \subseteq \mathrm{Hull}_G(S \cup \{v\})$, for some $v \in \{x, y, z\}$. Observe that this finishes the proof, since T connects anything that could be connected by w and x_1x_2. Recal that $w \notin S$ and let k' be such that $w \notin \mathrm{I_{cc}}^{k'}_F(S)$ and $d_H(w) \geq 2$, for a component H of $\mathrm{I_{cc}}^{k'}_F(S)$. We consider two cases. First, suppose that w does not depend on x_1x_2 to be contaminated, i.e., vertices x_1 and x_2 are not in $V(H)$. This means that $V(H) \subseteq \mathrm{Hull}_G(S)$. Let $v = y$ if $V(H) \cap \{y_1, y_2\} \neq \emptyset$, and let $v = z$ otherwise. Note that $V(H) \cup T \subseteq \mathrm{Hull}_G(S \cup \{v\})$, and that S' is also in $\mathrm{Hull}_G(S \cup \{v\})$ since T plays the role of x_1x_2. Now, suppose that w depend on x_1x_2, which means that $S' \subseteq V(H)$. Suppose that $y_1 \in V(H)$ (the other cases are analogous), and observe that $V(H) \subseteq \mathrm{Hull}_G(S \cup \{x\})$, since x plays the role of the edge x_1x_2. Now, let P be any path between y_1 and x_1 in H, and note that $P \cup \{x, y\}$ is a cycle whose vertices are all in $\mathrm{Hull}_G(S \cup \{x\})$, except y. Therefore, y is contaminated by $S \cup \{x\}$, and so does z. ■

The next lemma is the last tool needed in our proof.

Lemma 3.4. *Let G be a 4-regular plane graph, and denote by f_i the number of faces of degree i in G. Then,*

$$2f_2 + f_3 = 8 + \sum_{i \geq 4} (i - 4) f_i$$

Proof: Let $n = |V(G)|$, $m = |E(G)|$, and f denote the number of faces of G. Because G is 4-regular, we know that $m = 2n$. By Euler's Equation, we have:

$$n + f - m = 2 \Rightarrow n = f - 2 = \sum_{i \geq 2} f_i - 2$$

Since each edge is contained in exactly two faces, we also have:

$$\sum_{i \geq 2} i f_i = 2m = 4n = 4 \sum_{i \geq 2} f_i - 8$$

The lemma follows by rearrenging the above equation. ∎

The theorem below is an immediate consequence of the previous results.

Theorem 3.5. *$hn(G) \leq \frac{1}{2}|V(G)|$, for every 4-regular plane graph G.*

Proof: Let G' be the reduction graph of G. By Lemma 3.1, we get that G' has no multiple edges or cycles of length 3, which would contradict Lemma 3.4 unless $V(G') = \emptyset$. By Theorem 3.3 $\mathrm{hn_{cc}}(G) \leq f$, where f is the number of operations applied to G; the theorem follows because each operation removes at least two vertices from the graph. ∎

4. Polynomial Cases

In this section, we prove that one can compute $\mathrm{hn_{cc}}(G)$ in polynomial time for chordal graphs, P_4-sparse graphs and grids. The following lemma is a general one, so we present it first. It will be used in the proof of chordal graphs.

Lemma 4.1. *Let G be a connected graph, u be a cut-vertex of G, and C be the vertex set of some component of $G - u$. Also, let $G_1 = G[C \cup \{u\}]$ and $G_2 = G - C$. Then,*

$$\mathrm{hn_{cc}}(G_1) + \mathrm{hn_{cc}}(G_2) - 1 \leq \mathrm{hn_{cc}}(G) \leq \mathrm{hn_{cc}}(G_1) + \mathrm{hn_{cc}}(G_2).$$

Proof: The upper bound is trivial since the union of hull-sets of G_1 and G_2 clearly gives a hull-set of G. For the lower bound, consider a hull-set S of G and denote $(V(G_i) \cap S) \setminus \{u\}$ by S_i, for $i \in \{1, 2\}$. First, note that $S_i \cup \{u\}$ is a hull-set of G_i for $i = 1$ and $i = 2$; this means that $\mathrm{hn_{cc}}(G_i) \leq |S_i| + 1$, $i \in \{1, 2\}$. We consider the cases:

- $u \notin \mathrm{Hull}(S_i)$ for $i = 1$ and $i = 2$: then $u \in S$ since it cannot be contaminated by vertices x, y with $x \in V(G_1)$ and $y \in V(G_2)$. In this case,

$$\mathrm{hn_{cc}}(G) = |S| = 1 + |S_1| + |S_2| \geq \mathrm{hn_{cc}}(G_1) + \mathrm{hn_{cc}}(G_2) - 1.$$

- $u \in \mathrm{Hull}(S_1)$: then $u \notin S$ as otherwise $S \setminus \{u\}$ would be a smaller hull-set of G. Also, note that S_1 must be a minimum hull-set of G_1, i.e., $|S_1| = \mathrm{hn_{cc}}(G_1)$. Then, we get:

$$\mathrm{hn_{cc}}(G) = |S| = |S_1| + |S_2| \geq \mathrm{hn_{cc}}(G_1) + \mathrm{hn_{cc}}(G_2) - 1$$

∎

Corollary 4.2. *If G is a connected graph on $n \geq 2$ vertices with p blocks, then*

$$\mathrm{hn}_{\mathrm{cc}}(G) \geq p + 1.$$

Proof: We prove by induction on the number of blocks of G. If $p = 1$, it holds trivially since $n \geq 2$ and a single vertex cannot contaminate any vertex. Suppose it holds for every connected graph with less than p blocks, and let G_1 be a leaf block connected to the rest of the graph by vertex u. Let $G_2 = G - V(G_1 - u)$, and note that the number of blocks in G_2 equals $p - 1$. By induction hypothesis, we then get $\mathrm{hn}_{\mathrm{cc}}(G_2) \geq p$. The corollary follows by Lemma 4.1 and the fact that $\mathrm{hn}_{\mathrm{cc}}(G_1) \geq 2$. ■

Theorem 4.3. *If G is a chordal graph with p blocks, then* $\mathrm{hn}_{\mathrm{cc}}(G) = p + 1$.

Proof: In [13], the authors prove that if H is a 2-connected chordal graph, then $\{u, v\}$ is a hull-set in the P_3 convexity of H for any $uv \in E(H)$. We use this fact to construct a hull-set of G of size $p + 1$. The theorem follows by Corollary 4.2

Now, consider a sequence of subgraphs $G_0, \dots, G_q$ such that G_0 is any block of G, $G_q = G$, and, for each $i \in \{1, \dots, q\}$, we have that $G_i = G[V(G_{i-1}) \cup B_i]$, where B_i is the vertex set of a block of G_i separated from G_{i-1} by exactly one cut-vertex, x_i (in other words, G_i is obtained from G_{i-1} by appending a leaf block to vertex x_i). Let $S_0 = \{x_0, x_1\}$ for any $x_0 \in N_{G_0}(x_1)$ and, for each $i \in \{1, \dots, q\}$, let $S_i = S_{i-1} \cup \{x'_i\}$, where $x'_i \in N^{B_i}(x_i)$. We know that S_0 is a hull-set of G_0. Now if S_{i-1} is a hull-set of G_{i-1}, then $x_i \in \mathrm{Hull}(S_{i-1})$, and, since $B_i \subseteq \mathrm{Hull}(\{x_i, x'_i\})$, we get that S_i is a hull-set of G_i. Since $|S_q| = p+1$, the theorem follows. ■

Now, we investigate P_4-sparse graphs. It is well known that these graphs have a very nice decomposition, but before we present it, we need some new definitions. Given graphs G_1, G_2, the *union of* G_1, G_2 is simply the graph $G_1 \vee G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$, while the *join of* G_1, G_2 is the graph obtained from $G_1 \vee G_2$ by adding every possible edge between $V(G_1)$ and $V(G_2)$; it is denoted by $G_1 \wedge G_2$. A graph G is a *spider* if $V(G)$ can be partitioned into sets K, S, R such that $|K| = |S| \geq 2$, K is a clique, S is a stable set, R is complete to K and anti-complete to S, and the edges between K and S form either a perfect matching, in which case we say that G is a *thin spider*, or a perfect anti-matching, in which case we say that G is a *fat spider*.

Theorem 4.4. [17] *Let G be a non-trivial P_4-sparse graph. Then one of the following holds:*

(1) *G is the union of two P_4-sparse graphs;*
(2) *G is the join of two P_4-sparse graphs; or*
(3) *G is a spider with partition sets K, S, R such that $G[R]$ is a P_4-sparse graph.*

First, we investigate the union and the join of graphs.

Lemma 4.5. *Let G_1, G_2 be graphs, and let $|V(G_1)| \leq |V(G_2)|$ and p be the number of components of G_2. Then,*

$$\mathrm{hn_{cc}}(G_1 \vee G_2) = \mathrm{hn_{cc}}(G_1) + \mathrm{hn_{cc}}(G), \textit{ and}$$

$$\mathrm{hn_{cc}}(G_1 \wedge G_2) = \begin{cases} p+1 & , \textit{if } |V(G_1)| = 1, \\ 2 & , \textit{if } V(G_1) > 1 \textit{ and } E(G_2) \neq \emptyset, \textit{ and} \\ 3 & , \textit{if } E(G_1) = E(G_2) = \emptyset. \end{cases}$$

Proof: The first equation is straightforward; so consider $G = G_1 \wedge G_2$. If $V(G_1) = \{u\}$ and $p > 1$, then u is a cut-vertex of G. In any case, when $V(G_1) = \{u\}$, the 2-connected components of G are exactly the components of G_2 plus vertex u. Thus, by Corollary 4.2 we get that $\mathrm{hn_{cc}}(G) \geq p + 1$. Additionaly, because u is universal in G, for each component G' of G_2 and every $v \in V(G')$, we get that $V(G') \subseteq \mathrm{Hull}(\{u, v\})$. This means that if we pick one vertex of each component of G_2 together with vertex u we obtain a hull-set of G, i.e., $\mathrm{hn_{cc}}(G) \leq p + 1$. Now, suppose that $|V(G_1)| > 1$ and $E(G_2) \neq \emptyset$. Let $uv \in E(G_2)$, and note that $V(G_1) \subseteq \mathrm{Hull}(\{u, v\})$. Also, observe that ${\mathrm{I_{cc}}}^1(\{u, v\})$ is a connected subgraph of G, and because $|V(G_1)| > 1$, we get $V(G_2) \subseteq {\mathrm{I_{cc}}}^2(\{u, v\})$. Finally, suppose that $E(G_1) = E(G_2) = \emptyset$. An analogous argument works here, except that we need to pick two vertices from G_2 and one from G_1 in order to start with a connected subset. ∎

The next lemma is the last needed tool.

Lemma 4.6. *Let G be a spider with partition sets $\{K, S, R\}$ such that K is a clique and S is a stable set. Then,*

$$\mathrm{hn_{cc}}(G) = \begin{cases} 2 & , \textit{if } |K| \geq 3 \textit{ and } G \textit{ is a fat spider,} \\ 2 + |S| & , \textit{otherwise.} \end{cases}$$

Proof: If $|K| \geq 3$ and G is a fat spider, then every vertex of G has at least two neighbors in K; hence $\mathrm{hn_{cc}}(G) = 2$ clearly follows. Now, to prove the second part, we actually prove that if H is a graph and $u \in V(H)$ is a leaf of H (vertex of degree 1), then $\mathrm{hn_{cc}}(H) = \mathrm{hn_{cc}}(H-u)+1$. Clearly, if X is a hull-set of $H-u$, then $X \cup \{u\}$ is a hull-set of H. On the other hand, if X is a hull-set of H, then $u \in X$ since $d(u)=1$. Also, because u is separated from the rest of the graph by its neighbor, one can see that no vertex depends on u to be contaminated, i.e., $X-u$ is a hull-set of $G-u$. Because G is a thin spider, every $u \in S$ is a leaf of G and the lemma follows. ∎

Now, in [17], the authors also provide a linear time algorithm to compute a decomposition of G using the operations of Theorem 4.4. By the previous lemmas, we see that we need only to search the decomposition tree in a bottom-up way to compute $\mathrm{hn_{cc}}(G)$. Because the decomposition tree has also linear size, we get the following theorem.

Theorem 4.7. *If G is a P_4-sparse graph, then $\mathrm{hn_{cc}}(G)$ can be computed in linear time.*

Finally, we compute the hull number of a grid. Our proof is based on the following folklore result about the hull number on the P_3-convexity.

Proposition 4.8. *Let G be the $n \times n$ grid. Then, $\mathrm{hn_{P_3}}(G) = n$.*

The simplest explanation for the above proposition is as follows. Consider G as being the graph representation of the spaces on a chessboard H, i.e., each vertex of G represents a space in H and two vertices are adjacent if the spaces share a boundary (alternatively, G is obtained from the dual of a grid H by removing the vertex related to the outer face). One need only to observe that each application of the contamination rule on the faces of H cannot increase the perimeter of the contaminated area, since a space needs to be adjacent to at least two contaminated spaces in order to be contaminated. Observe that the same holds for the cycle convexity, except that we ask additionally that the boundaries containing these two spaces share an edge. Considering the related subgraph in G, let S be the starting set and observe that the perimeter of S in H equals $4|S| - \sum_{v \in S} d_G(v)$. Since $G[S]$ is connected and because the final perimeter must be $4n$, we get:

$$4n = 4|S| - 2|E(G[S])| \leq 4|S| - 2(|S|-1).$$

Therefore, the mininum size of a hull set on the cycle convexity must be $2n-1$. One can also verify that the first row and first column of G form a hull set of size exactly $2n-1$. Hence, $\mathrm{hn_{cc}}(G_{n\times n}) = 2n-1$. This argument can be easily generalized for rectangular grid.

Theorem 4.9. *Let m and n be positive integers. Then,*

$$\mathrm{hn}_{\mathrm{cc}}(G_{m\times n}) = m + n - 1.$$

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