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STAR-SHAPED ORDER FOR DISTRIBUTIONS WITH MULTIDIMENSIONAL PARAMETERS AND SOME APPLICATIONS

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ABSTRACT: In reliability theory the lifetimes of complex systems with heterogeneous components are well modelled by distributions indexed by more than one parameter. We extend a well known criterion for the star-shaped order to distributions with multidimensional parameters. This criterion is then applied to obtain star-shaped comparability within parallel and series system for several underlying components behaviour.

KEYWORDS: Star-shaped order, parallel systems, series systems, PHR models, PRHR models.
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1. Introduction

In many applications where two random variables represent the lifetime of different systems, it is of interest to study their ageing properties. This study will allow to determine which system is performing better with respect to some given property: the ageing rate, lifetime expectancy, skewness of lifetimes, etc. For this purpose, stochastic ordering between random variables provide a convenient way to describe such comparisons. These orderings may be defined through relations between distributions, survival or failure rate functions, of the relevant random variables. The monographs by Shaked and Shantikumar [16] or Marshal and Olkin [14], give a good account of various stochastic orders and their applications.

We will be interested in the star-shaped order, introduced by Barlow and Proschan [3] and defined by a monotonicity property on a suitable transformation on the distribution functions, as expressed by Definition 1 below. It can be easily seen that the definition is equivalent to allowing at most one crossing between the distribution functions of scaled lifetimes, as referred in

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Proposition C.11 of Marshal and Olkin [14] or 4.B.2 in Shaked and Shantikumar [16]. It follows from this characterization that the star-shaped order may thus be interpreted as a comparison of the lifetime ageing rate for systems that started functioning simultaneously. From the pratical point of view, since the distribution functions of the lifetime variables under comparison often do not have an explicit formula, it may be technically difficult to verify the star-shaped ordering. Thus, it becomes relevant to establish equivalent conditions for which the increasingness of the referred function holds. Using the sign technique referred in Marshal and Olkin [14] or Shaked and Shantikumar [16], Arab and Oliveira [1] analysed the ordering relationships within the Gamma and the Weibull families of distributions, later extended to the comparison of lifetimes of parallel systems in Arab et al. [2]. However, when the underlying distributions depend on a large number of parameters, this sign analysis becomes rather hard to control and often does not allow for a conclusion. An alternative approach may be based on a criterion proposed by Saunders and Moran [15] when the distribution functions depend on a single real parameter. This criterion turned out to be useful to exhibit order relations within parametric families of distributions (see Khaledi and Kochar [7], or Kochar and Xu[9, 10] among many others). As what concerns the lifetimes of more complex systems, the Saunders and Moran's criterion was used by Kochar and Xu [8] to obtain a characterization for parallel systems each one formed by two types of components with exponentially distributed lifetimes. More recently, Arab et al. [2] proved that the lifetimes of parallel systems with homogeneous and independent exponential components get smaller (or age faster) with respect to the star-shaped order, or age faster, as the number of components increases. We note that, for the case of series and parallel systems with more than two heterogeneous and, especially, non exponentially distributed components, not much work seems to have been done regarding the star-shaped comparability. Since these are complex models, depending on more than one parameter, the Saunders and Moran's [15] result cannot be used. Hence, it becomes natural to investigate possible extensions of Saunders and Moran's criterion for this type of models, which will be the main objective of this paper.

The paper is structured as follows. In Section 2, we present the extension of the Saunders and Moran's [15] criterion to families of distributions depending on multidimensional parameters. In Section 3, we discuss a few applications of the obtained criterion to complex systems with heterogeneous components, describing conditions on the parameters so that the star-shaped comparability holds when the lifetimes of the components satisfy suitable proportionality assumptions, including the popular PHR and PRHR models.

2. A criterion for the star-shaped order

Let $\mathcal{F}$ denote the family of distributions vanishing at 0 with support contained in $[0, +\infty)$. Let X be a nonnegative random variable with distribution function $F_X \in \mathcal{F}$, density function f_X, and survival function $\overline{F}_X$. In the following, we start by defining the star-shaped order relation, following Shaked and Shantikumar [16].

Definition 1. *Let X and Y be two nonnegative random variables with distribution functions $F_X, F_Y \in \mathcal{F}$, respectively. The random variable X (or its distribution F_X) is said to be smaller than Y (or its distribution F_Y) in the star-shaped order, denoted by $X \leq_* Y$ (or $F_X \leq_* F_Y$), if $\frac{1}{x}F_Y^{-1}(F_X(x))$ is increasing with $x > 0$ (or equivalently, $\frac{F_Y^{-1}(u)}{F_X^{-1}(u)}$ is increasing with $u \in (0,1)$).*

Remark 2. *Note that the star-shaped order is scale invariant, implying that in case of families of distributions that have a scale parameter, we are able to choose the parameter in a convenient way.*

The decision about the star-shaped order often relies on sign variations techniques, as follows from (4.B.2) from Shaked and Shantikumar [16]. Expectedly, the sign variation analysis raises technical difficulties, especially when dealing with distributions involving a large number of parameters, such as parallel systems, series systems or order statistics. Saunders and Moran [15] proved a more tractable condition for the star-shaped order to hold, providing a full characterization of such relation for the whole family of distributions.

Theorem 3. (Saunders and Moran [15]) *Let $\{F_a \colon a \in I \subseteq \mathbb{R}\}$ be a family of distributions such that $F_a \in \mathcal{F}$ with density f_a, which does not vanish on any subinterval of its support. Then $\frac{F_a^{-1}(\alpha)}{F_a^{-1}(\beta)}$ decreases (resp., increases) with respect to $a \in J \subseteq I$, for each fixed $\alpha > \beta$, if and only if $D(a,x) = \frac{1}{xf_a(x)}\frac{\partial F_a}{\partial a}(x)$ increases (resp., decreases) with respect to $x > 0$, for every fixed $a \in J \subseteq I$.*

The simple criterion for the star-shaped relationships that follows is derived as a direct consequence of the monotonicity result presented above.

Theorem 4. *Let $\{F_a\colon a \in I \subseteq \mathbb{R}\}$ be a family of distributions as in Theorem 3. Then $F_a \leq_* F_b$, for every $a \leq b$ such that $a, b \in J \subseteq I$ if and only if $D(a,x) = \frac{1}{xf_a(x)} \frac{\partial F_a}{\partial a}(x)$ is decreasing with $x > 0$, for every $a \in J \subseteq I$.*

Proof: Take $a, b \in J \subseteq I$, such that $a \leq b$. For $\alpha \geq \beta$, we have that

$$F_a \leq_* F_b \Leftrightarrow \frac{F_b^{-1}(\beta)}{F_a^{-1}(\beta)} \leq \frac{F_b^{-1}(\alpha)}{F_a^{-1}(\alpha)} \Leftrightarrow \frac{F_a^{-1}(\alpha)}{F_a^{-1}(\beta)} \leq \frac{F_b^{-1}(\alpha)}{F_b^{-1}(\beta)},$$

which is equivalent to $\frac{F_a^{-1}(\alpha)}{F_a^{-1}(\beta)}$ being increasing with respect to a. Taking into account Theorem 3, the conclusion follows. ■

Note that Theorem 4 states a necessary and sufficient condition for the star-shaped order to hold between distributions F_a, for every $a \in J$. However, in general, distributions may depend on more than one parameter, as happens for parallel or series systems with heterogeneous components, and, in general, coherent systems. Hence, it is natural to seek for extensions of Theorem 4 to families of distributions indexed by higher dimensional parameters.

To state our result, we need to introduce some notation. Let $\mu \in I \subseteq \mathbb{R}^n$ and $v \in \mathbb{R}^n$, and consider $\mu + tv$, $t \in \mathbb{R}$, the line that passes through μ and has direction vector v. We will denote by $L_{(\mu,v)} = \{\lambda_t \in I \subseteq \mathbb{R}^n\colon \lambda_t = \mu + tv, t \in \mathbb{R}\}$. Moreover, given a family of distributions F_λ, $\nabla F_\lambda(x)$ stands for the gradient of $F_\lambda(x)$ with respect to the parameter λ and by $\langle v, \nabla F_\lambda(x)\rangle$, we denote the inner product between v and $\nabla F_\lambda(x)$.

Theorem 5. *Let $\{F_\lambda\colon \lambda \in I \subseteq \mathbb{R}^n\}$ be a family of distributions such that $F_\lambda \in \mathcal{F}$ and has density function f_λ, which does not vanish on any subinterval of its support. Let $\mu \in I$, $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ and $J \subseteq I$. Then $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{(\mu,v)} \cap J$, for $t \leq t'$, if and only if $R(x) = \frac{\langle v, \nabla F_\lambda(x)\rangle}{xf_\lambda(x)}$ is decreasing with $x > 0$, for every $\lambda \in L_{(\mu,v)} \cap J$.*

Proof: We want to prove that $G_t \leq_* G_{t'}$, for $t \leq t'$, where $G_t(x) = F_{\lambda_t}(x)$, for every $x > 0$. By Theorem 4, this is equivalent to $\frac{1}{xg_t(x)} \frac{\partial G_t}{\partial t}(x)$ being decreasing with $x > 0$, where $g_t(x) = G_t'(x) = f_\lambda(x)$. Therefore, we may conclude that $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$ if and only if $\frac{\langle v, \nabla F_\lambda(x)\rangle}{xf_\lambda(x)}$ is decreasing with $x > 0$, for every $\lambda \in L_{(\mu,v)} \cap J$. ■

Remark 6. *Note that one could think, of comparing distributions whose parameters belong to some general parametric curve, instead of straight lines, which would probably lead to an obvious extension of Theorem 5.*

In the case of families of distributions with 2-dimensional parameters, the following version, using the slope of the line $L_{(\mu,v)}$, is convenient.

Proposition 7. *Let $\{F_\lambda\colon \lambda\in I\subseteq\mathbb{R}^2\}$ be a family of distributions such that $F_\lambda\in\mathcal{F}$ and has density function f_λ, which does not vanish on any subinterval of its support. Let $\mu\in I$, $v=(v_1,v_2)\in\mathbb{R}^2$ and $J\subseteq I$. If $v_1<0$ (resp., $v_1>0$), then $F_{\lambda_t}\leq_* F_{\lambda_{t'}}$, for every $\lambda_t,\lambda_{t'}\in L_{(\mu,v)}\cap J$, where $t\leq t'$, if and only if $Q(x)=\frac{1}{xf(x)}\left(\frac{\partial F_\lambda}{\partial\lambda_1}(x)+k\frac{\partial F_\lambda}{\partial\lambda_2}(x)\right)$ is increasing (resp., decreasing) with $x>0$, for every $\lambda\in L_{(\mu,v)}\cap J$, where $k=\frac{v_2}{v_1}$.*

Proof: According to Theorem 5, we have that, $F_{\lambda_t}\leq_* F_{\lambda_{t'}}$, for every $\lambda_t,\lambda_{t'}\in L_{(\mu,v)}\cap J$, where $t\leq t'$, if and only if $R(x)=\frac{1}{xf_\lambda(x)}\left(\frac{\partial F_\lambda}{\partial\lambda_1}(x)v_1+\frac{\partial F_\lambda}{\partial\lambda_2}(x)v_2\right)$ is decreasing with $x>0$, for every $\lambda\in L_{(\mu,v)}\cap J$. Factorizing $R(x)$ by v_1 and taking into account the sign of v_1, the conclusion follows. ■

3. Applications

We now apply the results proved in the previous section, to prove comparability, with respect to the star-shaped order, for some models that are popular in reliability theory. Throughout this section $X_1,\ldots,X_n$ will represent the lifetimes of the components of a complex system. The lifetime of a parallel system is $X_{(n)}=\max(X_1,\ldots,X_n)$, while the lifetime of a series system is given by $X_{(1)}=\min(X_1,\ldots,X_n)$.

3.1. Parallel systems with dependent components. First, we provide a condition for the star-shaped order to hold between parallel systems, for which their lifetime components are dependent and identically distributed. We say that the joint distribution of $(X_1,\ldots,X_n)$ follows an n-dimensional FGM (Farlie-Gumbel-Morgenstern, cf. [12]) distribution if

$$F_{(X_1,\ldots,X_n)}(x_1,\ldots,x_n)=\prod_{i=1}^n F(x_i)\left(1+\sum_{1\leq j<k\leq n}a_{jk}\overline{F}(x_j)\overline{F}(x_k)\right),\qquad(1)$$

where $\left|\sum_{1\leq j<k\leq n}a_{jk}\right|\leq 1$. Then, the distribution function of $X_{(n)}$ is given by

$$F_c(x)=F^n(x)(1+c\overline{F}^2(x)),\qquad(2)$$

where $c=\sum_{1\leq j<k\leq n}a_{jk}\in[-1,1]$. Note that the constant c describes the strength of dependence among the random variables, while its sign reveals

the direction of the dependence, i.e., if $c > 0$ ($c < 0$), the components are positively (negatively) dependent.

Proposition 8. *Let* $\{F_c\colon c \in [-1,1]\}$ *be a family of distributions defined as in* (2)*. Then* $F_a \leq_* F_b$*, whenever* $-1 \leq a < b \leq \frac{n}{n+2}$*, if* $Q_1(x) = \frac{F(x)\overline{F}(x)}{xf(x)}$ *is decreasing with* $x > 0$*.*

Proof: According to Theorem 4, we need to prove that $D(c,x) = \frac{1}{xf_c(x)}\frac{\partial F_c}{\partial c}(x)$ is decreasing with $x > 0$, for $-1 \leq c \leq \frac{n}{n+2}$. After simplifications we have $D(c,x) = Q_1(x)h(x)$, where $h(x) = \frac{\overline{F}(x)}{n(1+c)-2c(n+1)F(x)+c(n+2)F^2(x)}$. Since $\frac{\partial F_c}{\partial c}(x) \geq 0$, we have that $D(c,x) \geq 0$. Now, taking into account that obviously $Q_1(x) \geq 0$, it follows that $h(x) \geq 0$, for $x > 0$. If $c = 0$, h is decreasing and the conclusion follows. Finally, assume that $|c| \leq 1$ and $c \neq 0$. Given that F is increasing and nonnegative, the monotonicity of h is the same as the monotonicity of the companion function

$$\tilde{h}(x) = \frac{1-x}{n(1+c)-2c(n+1)x+c(n+2)x^2},$$

for $x \in (0,1)$. Differentiating, it is easily seen that $\tilde{h}'$ has the same sign as $N(x) = -n(1+c) + 2c(n+1) - 2c(n+2)x + c(n+2)x^2$. When $c > 0$, $N(x) \leq N(0) = c(n+2) - n \leq 0$, while when $c < 0$, $N(x) \leq N(1) \leq 0$. Thus, $\tilde{h}'(x) \leq 0$, implying that h is decreasing. Taking into account that Q_1 is a positive decreasing function, the conclusion follows. ∎

Remark 9. *The increasingness assumption about* Q_1 *in Proposition 8 is satisfied by many families of distributions, such as uniform, power, Gamma, Normal, Gumbell, Pareto, Weibull distributions. The actual verification may need using Lemma 8 in Arab and Oliveira* [1]*, if a closed form of the distribution function is not available.*

Remark 10. *Moreover, Proposition 8 implies that, with respect to systems with independent components, negatively dependent components results in faster ageing of parallel systems, while positive dependence mean slower ageing rates.*

3.2. Complex systems based on PHR and PRHR models. We now prove some ordering relationships for two models that have received extensive usage when modelling lifetime or survival time data: the proportional hazard rate model (PHR), introduced by Cox [4], and the proportional reversed hazard rate (PRHR) model, introduced by Gupta et al.[5]. The PRHR model was used, for example, by Tsodikov et al. [17] to describe a stochastic model of spontaneous carcinogenesis, for which the progression time of the tumor was modeled by a PRHR model, while Lane et al. [13] modelled bank failure through a PHR model. We recall the definition of these models: a PHR (resp., PRHR) model with baseline distribution F has distribution function satisfying $\overline{F}_a(x) = \overline{F}^a(x)$ (resp., $F_a(x) = F^a(x)$), for $a > 0$. Hence, the PHR and PRHR models introduce a family of distributions depending on one parameter. We first characterize the star-shaped ordering for each of these models as a straightforward consequence of the Saunders and Moran's criterion, Theorem 4.

Proposition 11. *Let $F \in \mathcal{F}$ with a density that does not vanish in any subinterval of its support, be some baseline distribution. Then $F_a \leq_* F_b$, for every $0 < a \leq b$ if and only if $\overline{g}(x) = \frac{\ln(\overline{F}(x))\overline{F}(x)}{xf(x)}$ is increasing with $x > 0$, in the case of the PHR model, or $g(x) = \frac{\ln(F(x))F(x)}{xf(x)}$ is decreasing with $x > 0$, in the case of the PRHR model.*

Proof: Consider the case of PHR model. According to Theorem 4, we have $D(a,x) = \frac{1}{xf_a(x)}\frac{\partial F_a}{\partial a}(x) = -\frac{1}{a}\overline{g}(x)$. Hence, since $a > 0$, the conclusion follows. The PRHR case follows analogously. ∎

Remark 12. *The sample maxima and minima from independent, identically distributed random variables are typical examples of PHR and PRHR models.*

3.3. Complex systems with heterogeneous components. Throughout this subsection we characterize the star-shaped ordering of a few different types of heterogeneous systems, looking both at parallel and series systems. A first model looks at parallel systems with components whose lifetimes distributions are subject to different scale changes. Let $F \in \mathcal{F}$ be a distribution function, with density function that does not vanish on any subinterval of its support and consider random variables X_i with distribution function $F_i(x) = F(\lambda_i x)$, where $\lambda_i > 0$, for $i = 1, \ldots, n$. The distribution function of $X_{(n)}$ is, with $\lambda = (\lambda_1, \ldots, \lambda_n)$,

$$F_\lambda(x) = \prod_{i=1}^{n} F(\lambda_i x). \tag{3}$$

First, recall the below definition that concerns a special class of functions.

Definition 13. (Karlin [6]) *A function* $f\colon \mathbb{R}^2 \to \mathcal{R}$ *is said to be totally positive of order* 2 *(TP2) if for every* $x_1 < x_2$ *and* $y_1 < y_2$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $f(x_1, y_2)f(x_2, y_1) \leq f(x_1, y_1)f(x_2, y_2)$.

Proposition 14. *Let* $\{F_\lambda\colon \lambda \in (0, +\infty)^n, \lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ *be a family of distributions defined as in* (3). *Let* $\mu, v \in (0, +\infty)^n$. *If* $G(a, x) = \frac{F(ax)}{f(ax)}$ *is TP2, for* $a > 0$, *then* $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$, *for every* $\lambda_t, \lambda_{t'} \in L_{(\mu,v)} \cap J$, *where* $t \leq t'$ *and* $J = \{\lambda \in (0, +\infty)^n\colon v_i\lambda_j - \lambda_i v_j \leq 0,\ i < j,\ i, j = 1, \ldots, n\}$.

Proof: According to Theorem 5, we need to prove that $R(x) = \frac{1}{x f_\lambda(x)} \sum_{i=1}^n v_i \frac{\partial F_\lambda}{\partial \lambda_i}(x)$ is decreasing with $x > 0$, where f_λ is the density function of F_λ. We have that

$$\frac{\partial F_\lambda}{\partial \lambda}(x) = x f(\lambda_i x) \prod_{\substack{j=1 \\ i \neq j}}^n F(\lambda_j x) \quad \text{and} \quad f_\lambda(x) = \sum_{i=1}^n \lambda_i f(\lambda_i x) \prod_{\substack{j=1 \\ i \neq j}}^n F(\lambda_j x).$$

Thus, $R(x) = \frac{\sum_{i=1}^n v_i P_i(x)}{\sum_{i=1}^n \lambda_i P_i(x)}$, where $P_i(x) = f(\lambda_i x) \prod_{\substack{j=1 \\ i \neq j}}^n F(\lambda_j x)$. Differentiating R, we get that the sign of R' is the same as the sign of

$$K(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n (v_i\lambda_j - \lambda_i v_j)(P_i'(x)P_j(x) - P_i(x)P_j'(x)).$$

Since, for $i < j$ and $i, j = 1, \ldots, n$, $v_i\lambda_j - \lambda_i v_j \leq 0$, we need to prove that $P_i'(x)P_j(x) - P_i(x)P_j'(x) \geq 0$, for $i < j$. The function $P_i'(x)P_j(x) - P_i(x)P_j'(x)$ is the numerator of the derivative of $L(x) = \frac{P_i(x)}{P_j(x)} = \frac{f(\lambda_i x)F(\lambda_j x)}{f(\lambda_j x)F(\lambda_i x)}$. Given that, for $i < j$, $\lambda_i < \lambda_j$ and $G(a, x)$ is TP2, for every $a > 0$, it follows that L is increasing. Hence, the proof is concluded. ∎

Remark 15. *The common families of distributions in reliability or ageing models, such as the Gamma, Weibull, Pareto or power verify the TP2 property assumed in Proposition 14.*

The following corollary complements the ordering result proved in Kochar and Xu [8], where only two types of components with exponential lifetimes were allowed in each parallel system.

Corollary 16. *Let* $\{F_\lambda\colon \lambda \in (0, +\infty)^n, \lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ *be a family of distributions defined as in* (3), *with* $F(x) = 1 - e^{-x}$. *Let* $\mu, v \in (0, +\infty)^n$. *Then* $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$, *for every* $\lambda_t, \lambda_{t'} \in L_{(\mu,v)} \cap J$, *where* $t \leq t'$ *and* $J = \{\lambda \in (0, +\infty)^n\colon v_i\lambda_j - \lambda_i v_j \leq 0,\ i, j = 1, \ldots, n,\ i < j\}$.

Proof: Taking into account Proposition 14, we only need to prove that $G(a,x)=\frac{F(ax)}{f(ax)}$ is TP2, for $a>0$. But this is equivalent to proving that, for $a<b$,

$$K(x)=\frac{f(ax)F(bx)}{f(bx)F(ax)}=\frac{e^{bx}-1}{e^{ax}-1}$$

is increasing with $x>0$, which is easily seen to be true. ■

We now have a look into complex systems based on components whose lifetimes follow a PRHR model. Assume that the lifetimes X_i have distribution function $F_i(x)=F^{\lambda_i}(x)$, for some baseline function $F\in\mathcal{F}$ with density f that does not vanish on any interval of the support of F, where $\lambda_i>0$, for every $i=1,\dots,n$. Then the distribution function of $X_{(1)}$ is, with $\lambda=(\lambda_1,\dots,\lambda_n)$,

$$F_\lambda(x)=1-\prod_{i=1}^{n}(1-F^{\lambda_i}(x)). \tag{4}$$

Proposition 17. *Let* $\{F_\lambda\colon \lambda\in(0,+\infty)^n, \lambda_1<\lambda_2<\cdots<\lambda_n\}$ *be a family of distributions defined as in* (4). *Let* $\mu,v\in(0,+\infty)^n$. *If* $g(x)=\frac{\ln(F(x))F(x)}{xf(x)}$ *is decreasing with* $x>0$, *then* $F_{\lambda_t}\leq_* F_{\lambda_{t'}}$, *for every* $\lambda_t,\lambda_{t'}\in L_{(\mu,v)}\cap J$, *where* $t\leq t'$ *and* $J=\{\lambda\in(0,+\infty)^n\colon v_i\lambda_j-\lambda_i v_j\leq 0,\ i<j,\ i,j=1,\dots,n\}$.

Proof: Taking into account Theorem 5 we need to prove that $R(x)=\frac{1}{xf_\lambda(x)}\sum_{i=1}^n v_i\frac{\partial F_\lambda}{\partial\lambda_i}(x)$ is decreasing for every $\lambda\in J$. We have that

$$\frac{\partial F_\lambda}{\partial\lambda_i}(x)=\ln(F(x))F^{\lambda_i}(x)\prod_{\substack{j=1\\ i\neq j}}^{n}(1-F^{\lambda_i}(x)),$$

and

$$f_\lambda(x)=f(x)\sum_{i=1}^{n}\lambda_i F^{\lambda_i-1}(x)\prod_{\substack{j=1\\ i\neq j}}^{n}(1-F^{\lambda_i}(x)).$$

Hence, $R(x)=g(x)h(x)$, where $h(x)=\frac{\sum_{i=1}^n v_iP_i(x)}{\sum_{i=1}^n\lambda_iP_i(x)}$, with $P_i(x)=F^{\lambda_i-1}(x)\prod_{\substack{j=1\\ i\neq j}}^{n}(1-F^{\lambda_j}(x))$. It is easily seen that the first term in $R'(x)=g'(x)h(x)+g(x)h'(x)$ is negative. Observe that, since $v_i>0$, for every $i=1,\dots,n$, it follows that $h(x)\geq 0$. Therefore, to prove that R is decreasing, it is enough to establish that h is increasing, given that g is a negative decreasing function. The sign of h' is easily seen to be the same as the sign of $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(v_i\lambda_j-\lambda_i v_j)(P_i'(x)P_j(x)-P_i(x)P_j'(x))$. Given that $v_i\lambda_j-\lambda_i v_j\leq 0$, it remains to prove that $P_i'(x)P_j(x)-P_i(x)P_j'(x)\leq 0$, for every $i,j=1,\dots,n$, $i<j$. Observe that $P_i'(x)P_j(x)-P_i(x)P_j'(x)$ is the numerator of the derivative of $K(x)=\frac{P_i(x)}{P_j(x)}=F^{\lambda_i-\lambda_j}(x)\frac{1-F^{\lambda_j}(x)}{1-F^{\lambda_i}(x)}$. Therefore, we need to prove that K is decreasing. Given that F is increasing and non-negative, the monotonicity of K will be the same as the monotonicity of the companion function

$$\tilde{K}(x)=x^{\lambda_i-\lambda_j}\frac{1-x^{\lambda_j}}{1-x^{\lambda_i}}=\frac{x^{-\lambda_j}-1}{x^{-\lambda_i}-1},\ \text{for } x\in(0,1),$$

which is easily seen to be decreasing on $(0,1)$. ■

Assuming the components follow a PHR model, we may derive a similar result about $X_{(n)}$. Consider that X_i has survival function $\overline{F}_i(x)=\overline{F}^{\lambda_i}(x)$, where $\lambda_i>0$, for every $i=1,\dots,n$ and F is a baseline distribution as above. Then the distribution function of $X_{(n)}$ is, with $\lambda=(\lambda_1,\dots,\lambda_n)$,

$$F_\lambda(x)=\prod_{i=1}^{n}(1-\overline{F}^{\lambda_i}(x)). \tag{5}$$

Proposition 18. *Let $\{F_\lambda\colon \lambda\in(0,+\infty)^n, \lambda_1<\lambda_2<\cdots<\lambda_n\}$ be a family of distributions defined as in (5). Let $\mu,v\in(0,+\infty)^n$. If $\overline{g}(x)=\frac{\ln(\overline{F}(x))\overline{F}(x)}{xf(x)}$ is increasing with $x>0$, then $F_{\lambda_t}\leq_* F_{\lambda_{t'}}$, for every $\lambda_t,\lambda_{t'}\in L_{(\mu,v)}\cap J$, where $t\leq t'$ and $J=\{\lambda\in(0,+\infty)^n\colon v_i\lambda_j-\lambda_i v_j\leq 0,\ i<j,\ i,j=1,\dots,n\}$.*

Proof: The proof is analogous to that of Proposition 17, taking into account that

$$\frac{\partial F_\lambda}{\partial\lambda_i}(x)=-\ln(\overline{F}(x))\overline{F}^{\lambda_i}(x)\prod_{\substack{j=1\\ i\neq j}}^{n}(1-\overline{F}^{\lambda_i}(x)),$$

$$f_\lambda(x)=f(x)\sum_{i=1}^{n}\lambda_i\overline{F}^{\lambda_i-1}(x)\prod_{\substack{j=1\\ i\neq j}}^{n}(1-\overline{F}^{\lambda_i}(x)),$$

where f is the density function of F, and $R(x)=-\overline{g}(x)h(x)$, where $h(x)=\frac{\sum_{i=1}^{n}v_iP_i(x)}{\sum_{i=1}^{n}\lambda_iP_i(x)}$, with $P_i(x)=\overline{F}^{\lambda_i-1}(x)\prod_{\substack{j=1\\ i\neq j}}^{n}(1-\overline{F}^{\lambda_j}(x))$. Thus, we now need to prove that $h'(x)\leq 0$, in order to conclude that $R'(x)\leq 0$. Since

$v_i\lambda_j - \lambda_i v_j \leq 0$, for $i,j = 1,\ldots,n$, $i \neq j$, this follows after proving that $P_i'(x)P_j(x) - P_i(x)P_j'(x) \geq 0$, for $i < j$, which is easily achieved following a similar approach to the one used in the proof of Proposition 17. ∎

The previous result is an extension of Theorem 3.3 in Kochar and Xu [11], where the authors considered one of the systems to be formed by homogeneous components.

Remark 19. *The conclusions in Propositions 17 and 18 may reverse the direction of the ordering, if the monotonicities assumed for g and $\overline{g}$ are reversed and we redefine the set as $J = \{\lambda \in (0,+\infty)^n \colon v_i\lambda_j - \lambda_i v_j \geq 0,\ i,j = 1,\ldots,n,\ i < j\}$.*

Remark 20. *It is easily verified that the functions g and $\overline{g}$ considered in Propositions 17 and 18, respectively, are monotone for several families of distributions popular in reliability or ageing models, such as the Gamma, Lomax, Weibull, Pareto or power.*

3.4. Parallel systems with homogeneous distributions. Arab et al [2] proved, in their Corollary 7.2, that parallel homogeneous systems with components that have exponential lifetimes age faster as the number of components increases. We may prove this also holds when the components have exponentiated Weibull lifetimes $X_1,\ldots,X_n$, whose distribution function is given by $F(x) = (1 - e^{-(\lambda x)^\beta})^\alpha$, for $x > 0$, where $\alpha, \beta > 0$ are shape parameters and $\lambda > 0$ is a scale parameter. The distribution function of $X_{(n)}$ is given by

$$F_X(x) = (1 - e^{-(\lambda x)^\beta})^a, \tag{6}$$

where $a = \alpha n$, for $n \geq 1$. Taking into account Remark 2, we may, without loss of generality, consider $\lambda = 1$.

Proposition 21. *Let $\{F_{(\beta,a)} \colon a > 0, \beta > 0\}$ be a family of distributions defined as in* (6)*. Let $(\beta', a'), (\tilde{\beta}, \tilde{a}) \in (0,+\infty)^2$, such that $\beta' \geq \tilde{\beta}$ and $v = (\tilde{\beta}, \tilde{a}) - (\beta', a')$. If $\beta' = \tilde{\beta}$, then $F_{a'} \leq_* F_{\tilde{a}}$, for every $a' \geq \tilde{a}$. If $\beta' > \tilde{\beta}$, then $F_{\lambda_t} \leq_* F_{\lambda_{t'}}$, for every $\lambda_t, \lambda_{t'} \in L_{((\beta',a'),v)}$, where $t \leq t'$, if and only if $\lambda_t, \lambda_{t'} \in J = \{(\beta, a) \in (0,+\infty)^2 \colon \frac{\tilde{a}-a'}{\tilde{\beta}-\beta'} \geq -\frac{a}{\beta}\}$.*

Proof: If $\beta' = \tilde{\beta}$, then the family of distributions $F_{(\beta,a)}$ follows a PRHR model. Thus, taking into account Proposition 11, since

$$\frac{\ln(F(x))F(x)}{xf(x)} = \frac{(e^x-1)\ln(1-e^{-x})}{x},$$

is increasing with $x > 0$, the conclusion follows. Consider now $\beta' > \tilde{\beta}$. By Proposition 7, we need to prove that $Q(x) = \frac{1}{xf_{(\beta,a)}(x)}\left(\frac{\partial F_{(\beta,a)}}{\partial \beta}(x) + k\frac{\partial F_{(\beta,a)}}{\partial a}(x)\right)$ is increasing with $x > 0$, for every $(\beta, a) \in L_{((\beta',a'),v)}$, if and only if $k = \frac{\tilde{a}-a'}{\tilde{\beta}-\beta'} \geq -\frac{a}{\beta}$. We begin by study the case where $k \geq 0$. Taking into account Lemma 8 in Arab and Oliveira [1], we need to prove that, for every $c \in \mathbb{R}$, $Q(x) - c$ changes sign at most once, as x goes from 0 to $+\infty$, and if the sign change occurs it is in the order "$-, +$". Note $Q(x)-c$ and $H(x) = (1-e^{-x^\beta})^{a-1}P(x)$, where

$$P(x) = \left(ae^{-x^\beta}x^\beta\ln(x) + k(1-e^{-x^\beta})\ln(1-e^{-x^\beta}) - ca\beta x^\beta e^{-x^\beta}\right).$$

have, for each $x > 0$, the same sign. Hence, it is enough to characterize the sign of P. We look at the sign of P', whose sign is, for $x > 0$, the same as the one of $V(x) = (-ax^\beta\ln(x) + a\ln(x) + \frac{a}{\beta} + k\ln(1-e^{-x^\beta}) + k - ca\beta + ca\beta x^\beta)$. Differentiating V, we have $V'(x) = x^{\beta-1}K(x)$, where $K(x) = -a\beta\ln(x) - a + \frac{a}{x^\beta} + \frac{k\beta}{e^{x^\beta}-1} + ca\beta^2$. Thus, $K'(x) = -\frac{a\beta}{x} - \frac{a\beta}{x^{\beta+1}} - \frac{k\beta^2x^{\beta-1}e^{x^\beta}}{(e^{x^\beta}-1)^2}$. Since $k \geq 0$, we have that $K'(x) \leq 0$, implying that K is decreasing. Given that, $\lim_{x\to0^+} K(x) = +\infty$ and $\lim_{x\to+\infty} K(x) = -\infty$, it follows that the sign variation of K, which is the same as sign variation of V', is "$+, -$". Therefore, V has monotonicity "$\nearrow\searrow$. Moreover, $\lim_{x\to0^+} V(x) = \lim_{x\to+\infty} V(x) = -\infty$, which implies that V has sign variation "$-, +, -$" or "$-$". In the first case, we have that P has monotonicity "$\searrow\nearrow\searrow$. Since $\lim_{x\to0^+} P(x) = \lim_{x\to+\infty} P(x) = 0$, the sign variation of P, which coincides with the sign variation of H, is "$-, +$". In the second cases, where $V(x) \leq 0$, then P would be decreasing. But this is impossible given the behaviour of P near 0 and at $+\infty$. Suppose now that $-\frac{-a}{\beta} \leq k = \frac{\tilde{a}-a'}{\tilde{\beta}-\beta'} \leq 0$. Differentiating Q, we obtain

$$Q'(x) = \frac{((\beta kx^\beta - \beta k)e^{x^\beta} + \beta k)\ln(1-e^{-x^\beta}) + (\beta k + a)x^\beta}{x^{\beta+1}a\beta}.$$

Some elementary calculus arguments show that $Q'(x) \geq 0$, implying that Q is increasing with $x > 0$. If $k \leq -\frac{a}{\beta}$, Q cannot be a monotone function, since $\lim_{x\to 0^+} Q(x) = \lim_{x\to+\infty} Q(x) = +\infty$. Therefore, the proof is concluded. ∎

Remark 22. *According to Proposition 21, given $(\tilde{\beta}, \tilde{a})$ and (β', a'), we do not only have that $F_{(\beta,a)} \leq_* F_{(\tilde{\beta},\tilde{a})}$, for $(\beta, a) \in J \cap L_{((\beta',a'),v)}$, but we also have that the distributions depending on the parameters in the set $J \cap L_{((\beta',a'),v)}$ are ordered, with respect to the star-shaped order. That is, if k is the slope of the line going through $(\tilde{\beta}, \tilde{a})$ and (β', a'), then every point (β, a) in this line and above the line $-k\beta'$ (condition given by the set J) defines distributions comparable with each other and with $F_{(\tilde{\beta},\tilde{a})}$. However, it does not allow us to decide about star-shaped comparability between $F_{(\beta,a)}$ and $F_{(\tilde{\beta},\tilde{a})}$, when $(\beta, a) \notin J$. Nevertheless, if we keep changing the value of k (and, therefore, the position of the point (β', a')), it follows from Proposition 21 that the set of points (β, a) for which we have $F_{(\beta,a)} \leq_* F_{(\tilde{\beta},\tilde{a})}$ is given by the set $\{(\beta, a) \in (0, +\infty)^2 \colon \beta \geq \tilde{\beta}$ and $a \geq \frac{\tilde{a}\beta}{\tilde{\beta}-2\beta}\}$.*

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