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AN ELEMENTARY APPROACH TO THE PROBLEM OF RANDOM FLIGHTS IN ODD DIMENSIONS

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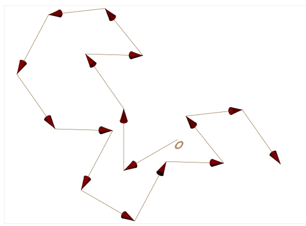
ABSTRACT: We give a procedure to compute the exact probability for that a particle starting from the origin of an odd-dimensional euclidean space will after n random jumps of unit length be encountered within a distance r from the start. Mathematica© code is included. The approach is significantly different and more elementary than the one chosen by García-Pelayo [G-P] and Borwein-Sinnamon [BS], although as yet a proof that the mentioned probabilities will always be piecewise polynomial functions can only be found in the works of those authors.

KEYWORDS: Random flights, odd dimensions, piecewise polynomial.
MATH. SUBJECT CLASSIFICATION (2010): 60G50.

0. Introduction



To motivate the investigation, we begin with a simple question. Assume a particle, at instant 0 at the origin of three dimensional euclidean space jumps at each tick of the clock exactly one unit from its current position into a random direction. (Here the directions are defined as position vectors to uniformly distributed points of the origin-centered unit sphere.)

Question: What is - as a function of r - the probability to encounter the particle after exactly n random jumps within the 0-centered ball $B = B(0, r)$ of radius r?

Thus if R_n is the distance of the particle from the starting place after n steps, viewn as a random variable, we want the probability distribution $r \mapsto \text{prob}(R_n \leq r)$ or, equivalently, the frequency function $r \mapsto f_{R_n}(r)$.

For this particular problem an elementary solution and solutions which use Fourier transforms and discontinuous factors are known and a further elementary solution was added by the present authors in [SK] where some of the history and ideas of the other solutions are also explained and many references cited. In the cited literature proofs are given for the fact that f_{R_n}

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is a piecewise polynomial function. The problem can of course be reasonably asked for euclidean spaces of arbitrary dimension and actually would make sense, suitably formulated, for many Riemannian manifolds and even for a given euclidean space interesting variants can be formulated, but the possbility of elementary or even 'closed form' solutions will be the exception.

In 2012 García-Pelayo [G-P] showed that the direct generalization of the problem to odd dimensional Euclidean space also leads to piecewise polynomial solutions for the functions f_{R_n}. His approach uses that the Fourier transform of a convolutional product is the product of the Fourier transforms of the factors, a result of Kingman of 1963 on the behaviour of convolutions under projections ; and a generalization of the Abel transform which is known as a tool to analyse radial (i.e. spherical symmetrical) functions. García-Pelayo stopped short of giving a simple explicit formulae. This was remedied by Borwein and Sinnamon [BS] who based their arguments on García-Pelayo's main result and who gave a formula valid for all n and d.

In this paper we propose an elementary approach to the same problem. The sophisticated techniques cited before will not be necessary, rather we need for our reasoning only standard formulas for the area of the spherical cap in dimension d and formulas from elementary probability theory. We present two conjectures which ought to have elementary proofs and which as well would imply that these formulas must always be piecewise polynomial. At the moment we are able to develop the theory to the point where we can write short computer code for Mathematica© by which we can obtain explicit exact formulas for f_{R_n}.

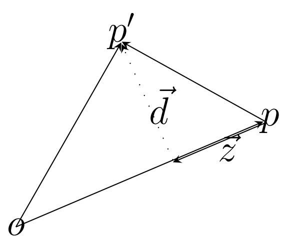
Thus this paper is for readers who wish elementary paths to formulae for f_{R_n} in odd dimensional euclidean space.

What concerns the organization of the paper it consists of essentially eight sections, mostly short which we thought better to leave largely untitled. A short description follows:

Section (or number/paragraph) 1 expresses the distance after $n+1$ jumps in function of the distance after n jumps; number 2 gives the formula for the area of a spherical cap of height h in d-dimensional space. The results are expressed it in probabilistic terms. In §3 we give a general formula for expressing the frequency function $f_{(R,S)}$ of a joint distribution (R,S) where $S = g(R,Z)$ in terms of $f_R, f_Z,$ and g and work it out in detail in §4 for the case $g(r,z) = \sqrt{r^2+2rz+1}$ (which is indeed the prominent formula of §1). In §5 we give a recurrence expressing $f_{R_{n+1}}$ via an integral over a function in which occurs f_{R_n} and do the same for simpler functions b_n which are related to f_{R_n} in a very simple way. With this we are able to write in §6 a very simple code that allows in principle the precise determination of the functions b_n (and hence f_{R_n}) with Mathematica. However running this code is somewhat more time consuming than would be expected. For this and more theoretical reasons as well we show in section 7 that the functions b_n have natural piecewise representations and this information can be used favorably for an implementation that accelerates the computations considerably. The paper is essentially concluded with section 8 where we mention a conjecture whose proof ought to be elementary and which would go a long way towards making the paper self-contained. Section 9 is an appendix in which we give for comparison the Borwein-Sinnamon formula (and a code to implement it) and some of the functions f_{R_n} explicitly for the dimensions 3,5 and 7.

1. We begin with the following lemma which can be found already in [SK] formulated for 3-space but valid in any Euclidean d-space $(E, \langle , \rangle)$. The 2-norm associated to the usual inner product $\langle , \rangle$ is denoted $|\cdot|$. Positions p are identified with the vector $\vec{op}$.

Lemma. *Let p and p′ be two points at distance 1 in Euclidean d-dimensional space $(E, \langle , \rangle)$ with origin denoted o. Let $\vec{z}$ be the orthogonal projection of vector $\vec{pp'}$ onto vector $\vec{op}$. Then $|\vec{p'}| = \sqrt{1 + |\vec{p}|^2 + 2\langle \vec{p}, \vec{z} \rangle}$.*



Proof. We may write $p' = p + \vec{pp'} = p + (\vec{z} + \vec{d})$ where $\vec{d} \perp \vec{z}$. It follows that

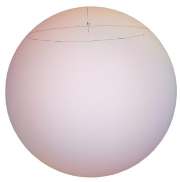
$$\begin{aligned}
|\vec{p'}|^2 &= \langle \vec{p'}, \vec{p'} \rangle \\
&= \langle \vec{p} + \vec{z} + \vec{d}, \vec{p} + \vec{z} + \vec{d} \rangle \\
&= |\vec{p}|^2 + |\vec{z}|^2 + |\vec{d}|^2 + 2\langle \vec{p}, \vec{z} \rangle + 2\langle \vec{p}, \vec{d} \rangle + 2\langle \vec{z}, \vec{d} \rangle \\
&= |\vec{p}|^2 + 1 + 2\langle \vec{p}, \vec{z} \rangle + 0 + 0
\end{aligned}$$

Here we used Phytagoras' theorem, and the parallelity or antiparallelity of $\vec{p}$ with $\vec{z}$ and the perpendicularity of $\vec{d}$ with respect to the latter two vectors. ■

We will use this later this way: Given n the particle is after n jumps at a certain position p; its distance from the origin $|p|$ is a random variable $R = R_n$. We can write $\langle \vec{p}, \vec{z} \rangle = |p| \langle \frac{\vec{p}}{|p|}, \vec{z} \rangle$. The $n+1$-st jump is also random variable and hence the inner product $\langle \frac{\vec{p}}{|p|}, \vec{z} \rangle$ is the random variable which expresses the length of the projection of the jump onto the line op. We designate this random variable by Z. We will have to consider, thus, the random variable $\sqrt{R^2 + 2RZ + 1}$.

From here on the further development differs significantly from the considerations in [SK].

2. In this section we aim to find the density function of the random variable Z referred above.



If $\vec{Z}$ is a d dimensional random vector with uniform distribution over the unit sphere $\mathbb{S}^{d-1}$, it defines a real random variable Z = d-th component of $\vec{Z}$. It will be clear that for reasons of symmetry, the random variable Z so defined is the same (i.e. has the same distribution) as the random variable Z defined in Section 1. We deduce the functions f_Z as the density functions associated to the distribution function prob$(Z \leq z)$. According to a paper by Chudkov [Ch], cited in Wikipedia [Wiki], the area $A(h)$ of the unit hyperspherical cap of height h, that is, the $d-1$-dimensional Lebesgue measure of the set $\{x \in \mathbb{R}^d : |x| = 1, x_d \geq 1-h\}$ can be computed via computing first A_d and G_{d-2}, as follows for $0 \leq h \leq 2$.

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}; \quad G_{d-2}(1-h) = \int_0^{1-h} (1-t^2)^{\frac{d-3}{2}} dt; \quad A(h) = A_d \cdot \tfrac{1}{2}\left(1 - \tfrac{G_{d-2}(1-h)}{G_{d-2}(1)}\right).$$

Since A_d is the area of S^{d-1} we see that the probability that $\vec{Z}$ falls into the spherical cap of height $h = 1-z$ is $A(h)/A_d$, so $\mathrm{Prob}(Z \leq z) = 1 - \mathrm{Prob}(Z \geq z) = 1 - A(h)/A_d = \frac{1}{2}\left(1 + \frac{G_{d-2}(z)}{G_{d-2}(1)}\right)$ and so for $-1 \leq z \leq 1$,

$$f_Z(z) = \tfrac{d}{dz}\mathrm{Prob}(Z \leq z) = \tfrac{1}{2G_{d-2}(1)} \tfrac{d}{dz} \int_0^z (1-t^2)^{\frac{d-3}{2}} dt = \tfrac{1}{2G_{d-2}(1)} (1-z^2)^{\frac{d-3}{2}}.$$

But for $z \notin [-1,1]$ it is clear that $f_Z(z) = 0$. It follows that a globally valid formula for the density f_Z is given by

$$f_Z(z) = \tfrac{1}{2G_{d-2}(1)} (1-z^2)^{\frac{d-3}{2}} \mathbb{1}_{[-1,1[}(z),\ z \in \mathbb{R}.$$

Note: Here and in many other cases later the arguments are indifferent to whether we take half open intervals like $[-1,1[$ or closed ones like $[-1,1]$. Sometimes however like in the representation theorem of section 7 we have to choose half open ones. We will use left closed, right open intervals wherever possible.

For odd dimensions $d = 2k + 1$ the mentioned site also tells us that

$$G_d(1-h) = \sum_{i=0}^{k}(-1)^i \binom{k}{i} \frac{(1-h)^{2i+1}}{2i+1},$$

a formula that will be easy to prove by induction.

This gives us the following table whose values we need for the multiplicative constant in $f_Z(z)$.

$$\begin{array}{ccccccccccccc} d: & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 \\ G_d(1) & 1 & \frac{2}{3} & \frac{8}{15} & \frac{16}{35} & \frac{128}{315} & \frac{256}{639} & \frac{1024}{3003} & \frac{2048}{6435} & \frac{32768}{109395} & \frac{65536}{230945} & \frac{262144}{969969} & \frac{524288}{2028117} \end{array}.$$

3. We explain in the next very short section how to find $f_{R_n}(r)$ by repeated marginalization. As a preparation we deal here with the following question. Assume R, Z are real random variables with known frequency functions and g is differentiable in an open set containing the range of the random vector (R, Z). Then $S = g(R, Z)$ is a further real random variable and by $f_{(R,S)}$ is denoted the frequency function of the joint variable (R, S).

How to find $f_{(R,S)}$ from f_R? We will answer this under mild conditions for g and f_R as implicit in the following reasoning.

The probability that $r-h \le R < r+h$ holds is given by the very definition of density functions, by $\int_{r-h}^{r+h} f_R(t)dt$, and hence assuming continuity of f_R, there is a $t_1 = t_1(h)$ such that $\mathrm{Prob}[r - h \le R < r + h] = 2hf_R(t_1)$.

Now take a $\dot r \in [r-h, r+h[$. Then $g_{\dot r}(Z) := g(\dot r, Z)$ is a real valued random variable and $z \mapsto g_{\dot r}(z)$ a differentiable function. We assume furthermore that $g_{\dot r}$ is strictly monotone in an interval containing the range of Z. By a known theorem, see [B], the density of the random variable $Y = g_{\dot r}(Z)$ is then given by $f_{g_{\dot r}(Z)}(y) = f_Z(g_{\dot r}^{-1}(y))|g_{\dot r}^{-1'}(y)|$, where $g_{\dot r}^{-1'}(y)$ is short for $(g_{\dot r}^{-1})'(y)$. So the probability that $g_{\dot r}(Z)$ assumes values in $[s - h, s + h[$ is given by $\int_{s-h}^{s+h} f_Z(g_{\dot r}^{-1}(y))|g_{\dot r}^{-1'}(y)|dy$ which in turn is equal to $2hf_Z(g_{\dot r}^{-1}(y_1))|g_{\dot r}^{-1'}(y_1)|$ for some $y_1 = y_1(h, \dot r) \in [s - h, s + h]$. Thus we find for any $h > 0$ and $\dot r \in [r-h, r+h[$, reals $t_1(h) \in [r-h, r+h[$ and $y_1(h, \dot r) \in [s-h, s+h[$ such that

$$\mathrm{Prob}[r-h \le R < r+h,\ s-h \le g(\dot r, Z) < s+h] = 4h^2 \overbrace{f_R(t_1) f_Z(g_{\dot r}^{-1}(y_1))|g_{\dot r}^{-1'}(y_1)|}^{\phi}.$$

The overbraced expression ϕ is a function of

$$(t_1, \dot r, y_1) \in [r-h, r+h[\times[r-h, r+h[\times[s-h, s+h[.$$

Supposing this function continuous we shall have

$$\lim_{h\downarrow 0}\phi(t_1,\dot r,y_1)=\phi(r,r,s)=f_R(r)f_Z(g_r^{-1}(s))|g_r^{-1\prime}(s)|.$$

At the other hand the density $f_{(R,S)}$ is a function which in the rectangle $A=[r-h,r+h[\times[s-h,s+h[$ satisfies

$$\int_A f_{(R,S)}d(r,s)=\mathrm{Prob}[r-h\le R<r+h,\ s-h\le g(R,Z)<s+h].$$

For every $h>0$ sufficiently small there exist - assuming continuity of $f_{(R,S)}$ - $(\dot r,\dot s)\in A$ such that the left hand side is $4h^2f_{(R,S)}(\dot r,\dot s)$. Letting h shrink to 0 we find from comparison of the expressions obtained that

$$f_{(R,S)}(r,s)=f_R(r)f_Z(g_r^{-1}(s))|g_r^{-1\prime}(s)|.$$

4. The above set-up will serve in many problems as theoretical foundation for random flights since g will in many cases have the properties necessary that make the above arguments work. For euclidean spaces we know from §2 that $f_Z(z)=\frac{1}{2G_{d-2}(1)}(1-z^2)^{\frac{d-3}{2}}\mathbb{1}_{[-1,1[}(z)$, and $g(r,z)=\sqrt{r^2+2rz+1}$; so $S=g(R,Z)=\sqrt{R^2+2RZ+1}$. It follows that $g_r^{-1}(s)=\frac{(s^2-r^2-1)}{2r}$ and so $g_r^{-1\prime}(s)=s/r$. We also compute $\mathbb{1}_{[-1,1[}(g_r^{-1}(s))$. Since $-1\le g_r^{-1}(s)<1$ iff $-2r\le s^2-r^2-1<2r$ iff $r^2-2r+1\le s^2<r^2+2r+1$ iff $|-1+r|\le s\le 1+r$, we get $\mathbb{1}_{[-1,1[}(g_r^{-1}(s))=\mathbb{1}_{[|-1+r|,1+r[}(s)$. Note that $[|-1+r|,1+r]\ne\emptyset$ iff $r\ge 0$ and that then $s\in[|-1+r|,1+r]$ iff $r\in[|-1+s|,1+s]$. Thus we find

$$\begin{aligned}f_{(R,S)}(r,s)&=f_R(r)\tfrac{|s/r|}{2G_{d-2}(1)}(1-(g_r^{-1}(s))^2)^{\frac{d-3}{2}}\mathbb{1}_{[|-1+r|,1+r]}(s)\\&=f_R(r)\tfrac{|s/r|}{2G_{d-2}(1)}(1-(\tfrac{s^2-r^2-1}{2r})^2)^{\frac{d-3}{2}}\mathbb{1}_{[|-1+s|,1+s]}(r).\end{aligned}$$

5. If one knows the density function $f_{(R,S)}$ of a joint distribution (R,S) one finds the density of random variable S by marginalization: $f_S(s)=\int_{-\infty}^{\infty}f_{(R,S)}(r,s)dr$. We know that the distribution of the random variable R_1, that is the distance of the particle after one jump is given, trivially, by $F_{R_1}(r)=\mathrm{prob}(R_1<r)=\mathbb{1}_{[1,\infty[}(r)$, that is, it is given by 0 if $r<1$ and 1 if $r\ge 1$. Therefore the probability density f_{R_1} is modelled by a shift of the Dirac delta function: $f_{R_1}(r)=\delta(1-r)$. Thus by the previous section we know $f_{(R_1,R_2)}$ and hence by marginalization (in principle) f_{R_2}. This then gives us $f_{(R_2,R_3)}$ and thus by marginalization f_{R_3}, etc. The density f_{R_2} is direct because if a function f satisfies some mild conditions then $\int_{-\infty}^{\infty} \delta(1-r)f(r)dr = f(1)$. So

$$\begin{aligned} f_{R_2}(s) &= \tfrac{1}{2^{d-2}G_{d-2}(1)} s^{d-2}(4-s^2)^{\frac{d-3}{2}} \mathbb{1}_{[0,2[}, \\ f_{R_{n+1}}(s) &= \textstyle\int_{-\infty}^{\infty} f_{R_n}(r) \tfrac{|s/r|}{2G_{d-2}(1)}(1-(\tfrac{s^2-r^2-1}{2r})^2)^{\frac{d-3}{2}} \mathbb{1}_{[|-1+s|,1+s[}(r)dr. \end{aligned}$$

In order to strip this formula down to the essentials note that in the inductive definition of the f_{R_n} each integration will introduce one more multiplication with $1/(2^{d-2}G_{d-2}(1))$. Furthermore $|s/r|$ can be substituted by s/r since we are only interested in values $s \geq 0$ and since $[|-1+s|, 1+s[\subseteq \mathbb{R}_{\geq 0}$ implies the integrations can be restricted to the realm of the nonnegative real numbers. We can then put s at the left of the integral sign, and divide both sides by s seing that we can formulate a recursion for the function $f_{R_n}(s)/s$. We introduce the functions $e(r,s)$, $e_2(r,s)$, and, by induction, b_n as follows:

$$\begin{aligned} e(r,s) &= (-1+r+s)(1-r+s)(1+r-s)(1+r+s) \\ &= -(r-(1-s))(r-(1+s))(r-(-1+s))(r-(-1-s)) \\ &= -(r^2-(1-s)^2)(r^2-(1+s)^2) \\ &= (-1+2r^2-r^4+2s^2+2r^2s^2-s^4) \\ e_2(r,s) &= e(r,s)/r^2. \end{aligned}$$

$$\begin{aligned} b_2(s) &= (s^2(4-s^2))^{\frac{d-3}{2}} \mathbb{1}_{[0,2[}(r) \\ b_{n+1}(s) &= \int\limits_{-\infty}^{\infty} b_n(r)(\tfrac{e(r,s)}{r^2})^{\frac{d-3}{2}} \mathbb{1}_{[|-1+s|,1+s[}(r)dr \\ &= \int\limits_{-\infty}^{\infty} b_n(r)(e_2(r,s))^{\frac{d-3}{2}} \mathbb{1}_{[|-1+s|,1+s[}(r)dr. \end{aligned}$$

Then the functions f_{R_n} and b_n are connected by the formula

$$f_{R_n}(s) = (1/(2^{d-2}G_{d-2}(1)))^{n-1} \cdot s \cdot b_n(s).$$

6. We have all pieces together to try to let Mathematica© do the work. First execute these three lines which put the functions $e_2(x,s)$, $\mathbb{1}_{[0,2]}$, and $b_2(r)$ into workspace

```
d = 5 ;(*d is user defined odd dimension*); counter = 2;
e2[x_, s_] := -((-(1 - s)^2 + x^2) (-(1 + s)^2 + x^2))/x^2;
indicator[x_, a_, b_] := UnitBox[1/(b - a) (x - (1/2) (a + b))];
b[r_] := (r^2 (4 - r^2))^((d - 3)/2) *indicator[r, 0, 2];
```

Now repeat the following lines:

```
b[s_]=Simplify[Integrate[ b[r]*e2(r,s)^((d-3)/2)*
          indicator[r,Abs[-1+s],1+s],{r,-Infinity,Infinity},
```

```
 Assumptions->{r>=0,s>=0} ] ];
++counter;
```

If `counter` has value `n` the associated `b` is the function b_n. Finally, if needed, via the lines

```
g = Sum[(-1)^i Binomial[(d-3)/2, i]*(2 i+1)^-1, {i,0,(d-3)/2}];
fRn = Simplify[(2^(d - 2) g)^(1 - counter) s*b[s]];
```

one recovers f_{R_n} in `fRn`.

Experimenting a little with this code one soon conjectures that the functions b_n (and hence the f_{R_n} are piecewise polynomial). Now the integration of polynomials in Mathematica© is in principle very fast. It could come as a surprise, then, that Mathematica slows down notably if n or d become a little larger. The authors suspect that a large part of the time is consumed treating the complicated codification and decodification of piecewise polynomial functions.

We can help Mathematica© using more insights which we have in our particular case. This is done in the next section.

7. We first prove a general piecewise representation theorem for the functions b_n and then apply it to give the code that accelerates Mathematica computations.

Theorem. *Assume $d \geq 3$ is an odd integer. Then the functions $b_n(s)$, inductively defined for $n \geq 2$ by*

$$b_2(s) = (s^2(4-s^2))^{\frac{d-3}{2}} \mathbb{1}_{[0,2[}(r), \quad b_{n+1}(s) = \int_{-\infty}^{\infty} b_n(r) e_2(r,s)^{\frac{d-3}{2}} \mathbb{1}_{[|-1+s|,1+s[}(r)dr;$$

admit piecewise representations

$$b_n(s) = \begin{cases} \textit{if } n = 2\dot{n}\textit{:} & \sum_{i=0}^{\dot{n}-1} \mathrm{pol}_i(s) \mathbb{1}_{[2i,2i+2[}(s) \\ \textit{if } n = 1+2\dot{n}\textit{:} & \sum_{i=0}^{\dot{n}} \tilde{\mathrm{pol}}_i(s) \mathbb{1}_{[(2i-1)^+,2i+1[}(s), \end{cases}$$

in which we can express the functions $\tilde{\mathrm{pol}}_i(s)$ from the functions pol_i by

$$\tilde{\mathrm{pol}}_i(s) = \begin{cases} \textit{if } i=0: \int\limits_{1-s}^{1+s} \mathrm{pol}_0(r) e_2(r,s)^{\frac{d-3}{2}} dr \\ \textit{if } i=1,...,\dot{n}-1: \int\limits_{-1+s}^{2i} \mathrm{pol}_{i-1}(r) e_2(r,s)^{\frac{d-3}{2}} dr + \int\limits_{2i}^{1+s} \mathrm{pol}_i(r) e_2(r,s)^{\frac{d-3}{2}} dr \\ \textit{if } i=\dot{n}: \int\limits_{-1+s}^{2\dot{n}} \mathrm{pol}_{\dot{n}-1}(r) e_2(r,s)^{\frac{d-3}{2}} dr; \end{cases}$$

and the functions pol_i *pertaining to the case* $n = 2(\dot{n}+1)$ *from the functions* $\tilde{\mathrm{pol}}_i(s)$ *pertaining to the case* $n = 1+2\dot{n}$ *by*

$$\mathrm{pol}_i(s) = \begin{cases} \textit{if } i=0: \int\limits_{|-1+s|}^{1} \tilde{\mathrm{pol}}_0(r) e_2(r,s)^{\frac{d-3}{2}} dr + \int\limits_{1}^{1+s} \tilde{\mathrm{pol}}_1(r) e_2(r,s)^{\frac{d-3}{2}} dr \\ \textit{if } i=1,...,\dot{n}-1: \int\limits_{s-1}^{2i+1} \tilde{\mathrm{pol}}_i(r) e_2(r,s)^{\frac{d-3}{2}} dr + \int\limits_{2i+1}^{s+1} \tilde{\mathrm{pol}}_{i+1}(r) e_2(r,s)^{\frac{d-3}{2}} dr \\ \textit{if } i=\dot{n}: \int\limits_{s-1}^{2\dot{n}+1} \tilde{\mathrm{pol}}_{\dot{n}}(r) e_2(r,s)^{\frac{d-3}{2}} dr. \end{cases}$$

(The index n on which pol_i and $\tilde{\mathrm{pol}}_i$ depend of course is for lightness of notation here suppressed.)

Proof. For the case $n = 2, \dot{n} = 1$, the representation given is evidently of the type claimed since then $\sum\limits_{i=0}^{\dot{n}-1} \mathrm{pol}_i(s) \mathbb{1}_{[2i,2i+2[}(s)$ collapses to $\mathrm{pol}_0(s)\mathbb{1}_{[0,2[}$, with $\mathrm{pol}_0(s) = (s^2(4-s^2))^{\frac{d-3}{2}}$. Now fix n even, $n = 2\dot{n}$, say, and compute from b_n the function b_{n+1} as defined above. We get

$$\begin{aligned} b_{n+1}(s) &= \int\limits_{-\infty}^{\infty} \sum\limits_{i=0}^{\dot{n}-1} \mathrm{pol}_i(r) \mathbb{1}_{[2i,2i+2[}(r)\ e_2(r,s)^{\frac{d-3}{2}}\ \mathbb{1}_{[|-1+s|,1+s]}(r) dr \\ &= \sum\limits_{i=0}^{\dot{n}-1} \int\limits_{-\infty}^{\infty} \mathrm{pol}_i(r) e_2(r,s)^{\frac{d-3}{2}} \mathbb{1}_{[2i,2i+2[\cap[|-1+s|,1+s[}(r) dr. \end{aligned}$$

We show first the formula for $\tilde{\mathrm{pol}}_0(s)$, which, by its definition, is to describe $b_{n+1} = b_{1+2\dot{n}}$ in the interval $[0,1[$. So assume $0 \le s < 1$. Then $[|-1+s|, 1+s[= [1-s, 1+s[\subseteq [0,2[$ and so of the intervals $\{[2i, 2i+2[\}_{i=0}^{\dot{n}-1}$ only one intersects $[|-1+s|, 1+s[$, namely the one associated to $i=0$. So we see for these s that $b_{n+1}(s) = \int_{1-s}^{1+s} \mathrm{pol}_0(r) e_2(r,s)^{\frac{d-3}{2}} dr$ proving the formula for $\tilde{\mathrm{pol}}_0(s)$.

Now consider $\tilde{\mathrm{pol}}_l$, with $l \in \{1,2,...,\dot{n}-1\}$. The range in which $\tilde{\mathrm{pol}}_l$ is to describe b_{n+1} is $[2l-1, 2l+1[$. So assume $2l-1 \le s < 2l+1$. Since $2l-1 \ge 1$, we see $|-1+s| = s-1$ and $[s-1, s+1[\subseteq [2l-2, 2l+2[= [2l-2, 2l[\uplus [2l, 2l+2[$, So that $[s-1, s+1[$ intersects typically only two adjacent of the intervals

$\{[2i,2i+2[\}_{i=0}^{\dot{n}-1}$, which figure in the defining formula for b_{n+1}, namely those pertaining to $i=l-1$ and $i=l$; the intersections yield the intervals in $[-1+s,2l[$ and $[2l,1+s[$. So for all $i\neq l-1,l,\quad \mathbb{1}_{[2i,2i+2[\cap[|-1+s|,1+s[}=0.$ and we get the claim concerning $\tilde{\mathrm{pol}}_l$.

Finally consider $\tilde{\mathrm{pol}}_{\dot{n}}$, that is, $s\in[2\dot{n}-1,2\dot{n}+1[$. In this case reasoning as before but taking into account that the interval $[s-1,s+1[\subseteq[2\dot{n}-2,2\dot{n}+2[$ intersects then only the last of the intervals figuring in the formula b_{n+1} gives the last of the formulas of the first part of the theorem.

The second part of the theorem follows largely the same pattern of reasoning. In this case we start from the assumption that $b_n=b_{1+2\dot{n}}$ is given as above and deduce via the defining formula the representation of $b_{2+2\dot{n}}=b_{2(1+2\dot{n})}$.

So now we have

$$\begin{aligned} b_{n+1}(s) &= \int_{-\infty}^{\infty}\sum_{i=0}^{\dot{n}}\tilde{\mathrm{pol}}_i(r)\mathbb{1}_{[(2i-1)^+,2i+1[}(r)e_2(r,s)^{\frac{d-3}{2}}\mathbb{1}_{[|-1+s|,1+s[}(r)dr \\ &= \sum_{i=0}^{\dot{n}}\int_{-\infty}^{\infty}\tilde{\mathrm{pol}}_i(r)e_2(r,s)^{\frac{d-3}{2}}\mathbb{1}_{[(2i-1)^+,2i+1[\cap[|-1+s|,1+s[}(r)dr. \end{aligned}$$

Now $\mathrm{pol}_0(s)$ is to represent $b_{n+1}(s)$ for s in the interval $[0,2[$. Of all the intervals $\{[(2i-1)^+,2i+1[\}_{i=0}^{\dot{n}}$ the only ones intersected by $[|-1+s|,1+s[\subseteq[0,3[$, are $[0,1[,[1,3[$ in $[|-1+s|,1[,[1,1+s[$, respectively. This gives the formula for $\mathrm{pol}_0(s)$. Now let $l\in 1,...,\dot{n}-1$. Then pol_l is to describe $b_{1+n}=b_{2(1+\dot{n})}$ for $s\in[2l,2l+2]$. Then $[|-1+s|,1+s[=[s-1,s+1[\subseteq[2l-1,2l+3[=[2l-1,2l+1[\uplus[2l+1,2l+3[$, so that $[s-1,s+1[$ intersects the two intervals $[2l-1,2l+1[,[2l+1,2l+3[$ in $[s-1,2l+1[,[2l+1,s+1[$, respectively. So from the defining formula above we get the formula for pol_l.

Finally, if $l=\dot{n}$, we have to look at $s\in[2\dot{n},2\dot{n}+2[$ Then $[|-1+s|,1+s[=[s-1,s+1[\subseteq[2\dot{n}-1,2\dot{n}+3[$ intersects among the intervals referred in b_n only $[2\dot{n}-1,2\dot{n}+1[$, namely in $[s-1,2\dot{n}+1[$. This yields the last formula of the second part. ∎

This result suggests to represent the functions b_n as lists; namely the functions $b_{2\dot{n}}$ as a list $\{\mathrm{pol}_0,\mathrm{pol}_1,...,\mathrm{pol}_{-1+\dot{n}}\}$ representing the function on $\{[0,2[,[2,4[,...,[2\dot{n}-2,2\dot{n}[\}$ respectively; and $b_{1+2\dot{n}}$ as a list $\{\tilde{\mathrm{pol}}_0,\tilde{\mathrm{pol}}_1,...,\tilde{\mathrm{pol}}_{\dot{n}}\}$ representing the function on $\{[0,1[,[1,3[,...,[2\dot{n}-1,2\dot{n}+1[\}$ respectively.

If these functions pol, $\tilde{\mathrm{pol}}$ happen to be symbolically easily integrable then we can save Mathematica© the work to manipulate piecewise functions and instead work with lists which is its fundamental data structure.

The following code named `process0` produces from a list containing in `lstofpols` the functions $\mathrm{pol}_i(s)$ pertaining to a function $b_{2\dot{n}}$ a new list `lstofpols` which contains the functions $\tilde{\mathrm{pol}}_i(s)$ pertaining to $b_{1+2\dot{n}}$. The code named `process1` produces from a list containing in `lstofpols` the functions $\tilde{\mathrm{pol}}_i(s)$ pertaining to a function $b_{1+2\dot{n}}$ a new list `lstofpols` which contains the functions $\mathrm{pol}_i(s)$ pertaining to $b_{2+2\dot{n}}$. The processes `process0` and `process1` are typically called alternatively (see below).

The whole code for the preparation of the computation has also to take care of defining the dimension, putting e_2 into workspace, and defining the initial list consisting of the only polynomial pertaining to b_2 (first three lines).

Concerning the workings of `process0`, `np` means 'n point' ($\dot{n}$), `lsofprims` abbreviates 'list of primitives', `lsofdefs` abbrviates 'list of definite integrals'. The second line translates the polynomials that come as functions with argument called `s` into ones with argument `r`, in line 3 primitivization with respect to variable `r` is carried through for all polynomials in `lsofpols`. In line 4 the integral $\int_{1-s}^{1+s} \mathrm{pol}_0(r) e_2(r,s)^{\frac{d-3}{2}} dr$ is computed - have in mind that the leftmost element of a list in Mathematica has position 1. The `For [ . . .]` loop computes the functions $\tilde{\mathrm{pol}}_i$ for $i = 1, ..., \dot{n} - 1$. The next two lines compute and add $\tilde{\mathrm{pol}}_{\dot{n}}$ to `lsofdefs`. The last line renames `lsofdefs` to `lsofpols` which then typically will be fed to `process1`. The structure of `process1` is very similar and will now not cause problems to the analysis by the reader.

```
  d= *  ; (*user defined odd dimension *)
  e2[x_,s_]:=-((-(1 - s)^2 + x^2) (-(1 + s)^2 + x^2))/x^2;
  lsofpols={(4s^2-s^4)^((d-3)/2)};

process0:=(np=Length[lsofpols];
lsofpols=lsofpols/.{s->r};
lsofprims=Map[Function[Integrate[#*e2[r,s]^((d-3)/2),r]],lsofpols];
lsofdefs={Simplify[(lsofprims[[1]]/.{r->1+s})-
                             (lsofprims[[1]]/.{r->1-s})]};
 For[i=1, i<=np-1, i++,
     tmp= (lsofprims[[i]]/.{r->2i})-(lsofprims[[i]]/.{r-> -1+s})+
     (lsofprims[[i+1]]/.{r->1+s}) - (lsofprims[[i+1]]/.{r->2i});
     tmp=Simplify[tmp];  AppendTo[lsofdefs,tmp];
        ];
```

```
tmp=(Last[lsofprims]/.{r-> 2np})-(Last[lsofprims]/.{r-> -1+s});
  tmp= Simplify[tmp];  AppendTo[lsofdefs, tmp];
  lsofpols=lsofdefs; )


process1:=(np=Length[lsofpols]-1;
 lsofpols=lsofpols /.{s -> r};
 lsofprims=Map[Function[Integrate[#*e2[r,s]^((d-3)/2),r]],lsofpols];
       tmp=(lsofprims[[1]]/.{r->1})-(lsofprims[[1]]/.{r->1-s}) +
            (lsofprims[[2]]/.{r->1+s})-(lsofprims[[2]]/.{r->1});
         tmp=Simplify[tmp];
         lsofdefs={tmp};
    For[i=1, i<=np-1, i++,
     tmp=lsofprims[[i+1]]/.{r->2i+1}-lsofprims[[i+1]]/.{r->-1+s}+
          lsofprims[[i+2]]/.{r->1+s}-lsofprims[[i+2]]/.{r->2i+1};
       tmp=Simplify[tmp];  AppendTo[lsofdefs,tmp];
        ];
  tmp=(Last[lsofprims]/.{r->2np+1})-(Last[lsofprims]/.{r->s-1});
  tmp=Simplify[tmp]; AppendTo[lsofdefs,tmp];
  lsofpols=lsofdefs;)
```

After this one has simply to execute alternatively `process0` and `process1` and gets upon execution of the command `lsofpols` the codification of $b_i(s)$ as a list of polynomials. The scheme $b_2 -$ `process0` $\to b_3 -$ `process1` $\to b_4 -$ `process0` $\to b_5$... should make this explanation clear. At the end one comes from b_n to f_{R_n} by multiplying b_n with $(2^{d-2}G_{d-2}(1))^{(1-n)}s$. That is, with the `g` defined in Section 6, `Simplify[(2^(d-2)g)^(1-n) s*lsofpols` gives f_{R_n} as a list.

The alert reader will note that in the definition of $\mathrm{pol}_0(s)$ in the theorem of Section 7 there lurks an integral $\int_{|-1+s|}^{1} \tilde{\mathrm{pol}}_0(r)$... which is exceptional in the sense that its lower limit uses a modulus. This modulus is not respected in `process1` but replaced by $1-s$, the value of $|-1+s|$ for $0 \le s \le 1$. As it happens in all our experiments $\tilde{\mathrm{pol}}_0$ coming out from `process0` is always an odd polynomial and for such polynomials it is not hard to see that the replacement of $|-1+s|$ by $1-s$ is justified.

8. Conclusions and Desiderata

Summarizing our arguments till here, we can say the following:

The process described of alternatingly applying `process0` and `process1` can be carried out and will compute the precise formulas for the functions b_n (or frequencies f_{R_n}) as long as the functions outputted, $\mathrm{pol}_i(r)$ and $\widetilde{\mathrm{pol}}_i(r)$ are polynomials and the polynomials $\widetilde{\mathrm{pol}}_0(r)$ are odd. As yet it has never happend that one of these conditions was violated. It thus can be expected that the method gives the correct functions b_n and f_{R_n} for all $n = 2, 3, 4, ...$ and odd dimensions d.

It is of course desirable to secure the conjectures implicit in this summary by proof. The results of García-Pelayo who proved that the functions f_{R_n} are piecewise polynomial (but is not very explicit about the partitions) can probably be construed with some work to yield such a detailed proof; and this is even more so the case if we use the sequel given by Borwein and Sinnamon. But our objective would be of course to get a self-contained proof. Along these lines we have currently the following proposition and conjecture.

Proposition. *For any integers $l, d \geq 0$, d odd and $l \notin L := \{1, 3, ..., d-4\}$, for the primitive $F(x, s)$ of $x^l e_2(x, s)^{\frac{d-3}{2}}$ w.r.t. x we have that the functions $F(1+s, s), F(1-s, s)$ and $F(r, s)$ with fixed $r \in \mathbb{R}_{>0}$ are polynomials in s. The first two polynomials have degree in s not higher than $l + d - 2$.*

Conjecture. *Under above conditions there will occur in $F(1+s, s)$ and $F(1-s, s)$ only monomials s^l with $l \notin L$*

The point of these two statements is that it is easily seen that if a polynomial pol_i, say, is a linear combination of powers s^l with $l \notin L$, then definite integrals as occurring in the theorem of §7 are real linear combinations of expressions of the form $F(1+s, s) - F(1-s, s)$, $F(2i, s) - F(-1+s, s)$ etc. and the conjecture guarantees that the integrations can be continued.

The results we have currently on these topics, although interesting, are too partial to merit further exposition here.

9. Appendix

A. The reader might be interested in the explicit formula given by Borwein and Sinnamon [BS, Theorem 5]. They derive the following theorem from the theory of García-Pelayo.

Theorem. *Let $n \geq 2$, and let the odd dimension be d, $d = 2m+1$. Define $C_m(x) = \sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!} x^k$, denote by H the Heaviside function, and let*

$Cfx^j(p)$ stand for the coefficient of x^j in polynomial p . Then the densities $f_{R_n}(x)$ (in BS-notation $p_n(m-1/2;x)$) are given by

$$f_{R_n}(x) = \left(\tfrac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n} \binom{n}{r}(-1)^{mr} H(n-2r+x) \sum_{k=1}^{m} (-2)^k \binom{m-1}{k-1} \tfrac{(2m-1-k)!}{(2m-1)!} x^k$$
$$\times \sum_{j=0}^{(m-1)n} \tfrac{(n-2r+x)^{(mn-1+j-k)}}{(mn-1+j-k)!} \cdot Cfx^j(C_m(x)^r C_m(-x)^{n-r}).$$

The following is a code implementing the above formula. The quantities m, n, a, b are user defined. In the example we are searching for the representation of the density f_{R_4} in 7-dimensional space on the interval $[a, b] = [0, 2]$

```
c[m_, x_]:= Sum[Divide[(m-1+k)!,2^k k!(m-1-k)!] x^k,{k,0,m - 1}];

m = 3; n = 4; a = 0; b = 2; (* user defined *)
Divide[Gamma[2 m], 2^m Gamma[m]]^n*
  Sum[Binomial[n, r] (-1)^(m*r) HeavisideTheta[n - 2 r + x]*
    Sum[(-2)^k Binomial[m - 1, k - 1]*
      Divide[(2 m - 1 - k)!, (2 m - 1)!] x^k*
      Sum[Divide[(n - 2 r + x)^(m*n-1+j-k), (m*n-1+j-k)!]*
        If[j==0,1, Coefficient[c[m,x]^r*c[m,-x]^(n-r), x^j]],
                 {j, 0, (m - 1)*n}], {k, 1, m}], {r, 0, n}];
  Simplify[%, a < x < b]
```

This code is here for the convenience of the reader. It solves the problem to find the coefficient of x^0, that is, the constant term of a polynomial in x with the line `If[j==0, . . .]` , using that $C_m(0) = 1$.

B. Finally here are some formulas which one obtains by our codes of sections 6 or the more rapid procedures `process0` and `process1` after editing; or by the Borwein-Sinnamon formula.

Formulas for f_{R_n} obtained for dimensions 3, 5, 7 and small n.

Dimension 3: $d = 3$.

$f_{R1}(s) = \delta(1-s)$

$f_{R_2}(s) = \begin{cases} \frac{s}{2} & 0 \le s \le 2 \end{cases}$

$f_{R_3}(s) = \begin{cases} \frac{s^2}{2} & 0 < s < 1 \\ -\frac{1}{4}(s-3)s & 1 \le s < 3 \end{cases}$

$$f_{R_4}(s)=\begin{cases}\frac{1}{16}(8-3s)s^2 & 0<s\le 2\\ \frac{1}{16}(s-4)^2 s & 2<s<4\end{cases}$$

$$f_{R_5}(s)=\begin{cases}-\frac{1}{16}s^2\left(s^2-5\right) & 0<s\le 1\\ \frac{1}{48}s\left(2s^3-15s^2+30s-5\right) & 1<s<3\\ -\frac{1}{96}(s-5)^3 s & 3\le s<5\end{cases}$$

Dimension 5: $d=5$.

$f_{R1}(s)=\delta(1-s)$

$$f_{R_2}(s)=\begin{cases}(3/4s^3-3/16s^5) & 0\le r\le 2\end{cases}$$

$$f_{R_3}(s)=\begin{cases}\frac{3}{560}s^4\left(s^4-42s^2+105\right) & 0<s<1\\ -\frac{3}{1120}(s-3)^3 s\left(s^4+9s^3+12s^2-3s-3\right) & 1\le s<3\end{cases}$$

$$f_{R_4}(s)=\begin{cases}\frac{3}{358400}s^4\left(3s^7-420s^5+640s^4+8400s^3-25088s^2+40960\right) & 0<s<2\\ -\frac{3}{358400}(s-4)^5 s\left(s^5+20s^4+100s^3+80s^2-80s-64\right) & 2\le s<4\end{cases}$$

The function f_{R_5} is represented on $[0,1],[1,3],[3,5]$, respectively, by the following list of polynomials:

$$-\frac{3}{51251200}s^4(s^{10}-325s^8+21450s^6-250250s^4+1406405s^2-3502785),$$

$$\frac{1}{51251200}s(2s^{13}-650s^{11}+2145s^{10}+42900s^9-285285s^8-42900s^7+4826250s^6-13843830s^5+10045750s^4+4800510s^3+1487525s^2-55905),$$

$$-\frac{1}{102502400}(s-5)^7 s(s^6+35s^5+375s^4+1270s^3+275s^2-1785s-1275)$$

Dimension 7: $d=7$.

$$f_{R_2}(s)=\begin{cases}\frac{15}{256}s^5\left(s^2-4\right)^2 & 0\le s\le 2\end{cases}$$

$$f_{R_3}(s)=\begin{cases}\frac{15s^6\left(s^8-52s^6+1430s^4-6292s^2+9009\right)}{256256} & 0<s<1\\ -\frac{15(s-3)^5 s\left(s^8+15s^7+83s^6+165s^5+80s^4-75s^3-37s^2+15s+9\right)}{512512} & 1\le s<3\end{cases}$$

The function f_{R_4} is represented on $[0,2],[2,4]$ by the two polynomials

$$-\frac{5}{3673686016}s^6(3s^{13}-468s^{11}+41184s^9-43008s^8-1153152s^7+1916928s^6+16144128s^5-51544064s^4+122683392s^2-176160768),$$

$$\frac{5}{3673686016}(s-4)^8 s(s^{10}+32s^9+420s^8+2688s^7+8352s^6+9216s^5-3456s^4-12288s^3-768s^2+8192s+4096).$$

The function f_{R_5} is represented on $[0,1],[1,3],[3,5]$, respectively, by the following list of polynomials:

$-((1/191042693890048)(5s^6(-4582287247475+2598198958875s^2-697855744300s^4+115805480700s^6-12482280090s^8+775364730s^{10}-16702140s^{12}+202860s^{14}-1035s^{16}+3s^{18}))),$

$(1/191042693890048)(5s(-452473025+11192926425s^2-285911544100s^4+5885109305950s^5-3041586030700s^6+1831027487250s^7-4220106373630s^8+3269691596600s^9-872285321250s^{10}-61249133400s^{11}+69564413100s^{12}-4593945020s^{13}-3080806300s^{14}+516909820s^{15}+36216375s^{16}-11134760s^{17}-260015s^{18}+135240s^{19}-690s^{21}+2s^{23})),$

$-((1/382085387780096)(5(-5+s)^{11}s(1571125+3456475s+1766925s^2-1643125s^3-1751465s^4+166350s^5+989170s^6+509710s^7+123995s^8+16775s^9+1305s^{10}+55s^{11}+s^{12})))$

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