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A MULTIVARIATE IDENTITY INVOLVING STIRLING NUMBERS OF THE SECOND KIND AND SOME NON-COMBINATORIAL APPLICATIONS

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ABSTRACT: An apparently new type of multivariate identity involving Stirling numbers of the second kind is proven. From this, for integers $m \geq 0$ and a certain large family of integers $l \geq 0$, detailed information concerning the primitives $\int x^{l-2m}((-1+x+s)(1-x+s)(1+x-s)(1+x+s))^m dx$ and associated definite integrals is deduced. This is then used to complete an elementary proof given in an authors' earlier paper that the density functions of unit step random flights in odd dimensions are piecewise polynomial.

KEYWORDS: Stirling Numbers, Multivariate Identity, Primitivization, Random Flights.
MATH. SUBJECT CLASSIFICATION (2020): 11B37, 05A19.

0. Introduction

Following notation proposed in [GKP], given nonnegative integers n, k, and an indeterminate x, denote by $\left\{ {n \atop k} \right\}$ the number of partitions of $\{1, 2, ..., n\}$ into k nonempty blocks (Stirling numbers of the second kind) and by $x^{\underline{n}}$ the polynomial $x(x-1)\cdots(x-n+1)$.

Recently, when working on the problem to find an elementary proof for a result by García-Pelayo [G-P] according to which a certain probability density associated to random flights in odd dimensions is piecewise polynomial, the authors discovered the following identity which in spite of much searching they could not find in the literature in any form equivalent to it.

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Theorem 1. *Denote by $z_1, z_2, ..., z_n$ complex numbers of sum 0 and by $\alpha_1, \alpha_2, ..., \alpha_n$ and M nonnegative integers. By understanding the sums below as over nonnegative integers $a_1, ..., a_n$ (of sum M), there holds*

$$(1) \quad \sum_{a_1+\cdots+a_n=M} z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \frac{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}}{a_1! a_2! \cdots a_n!} = \sum_{a_1+\cdots+a_n=M} z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \left\{ \begin{matrix} \alpha_1 \\ a_1 \end{matrix} \right\} \left\{ \begin{matrix} \alpha_2 \\ a_2 \end{matrix} \right\} \cdots \left\{ \begin{matrix} \alpha_n \\ a_n \end{matrix} \right\}.$$

This result helped us to prove in succession the following two results. Theorem 2 seems to about as good as possible. In a number of experiments with triples $(m, \dot{s}, \ddot{s})$ violating the conditions i and ii, the sum was always nonzero.

Theorem 2. *Assume nonnegative integers $m, \dot{s}, s, \ddot{s}$ satisfy* i or ii*:*

i. $m \geq \dot{s} + \lceil \frac{1+s}{2} \rceil + \ddot{s}$. ii. s is odd and $\ddot{s} = 0$.

Then:

$$(2): \quad \sum_{\substack{a, b, c \geq 0 \\ a+b+c=m}} \frac{(-2)^b}{a!b!c!} b^{\dot{s}} (a-c)^s c^{\ddot{s}} = 0.$$

Theorem 3. *Let $e(x, s) = (-1 + x + s)(1 - x + s)(1 + x - s)(1 + x + s)$ and $e_2(x, s) = e(x, s)/x^2$. For integers $m \geq 0$ and $l \in \{0, 2, 4, ..., 2m - 2\} \cup Z_{\geq 2m}$ there exists a primitive $F_{l,m}(x, s) \in \int x^l(-e_2(x, s))^m dx$, so that:*

a. *$F_{l,m}(x, s)$ is element of $\mathbb{R}[x, x^{-1}, s]$ with $\deg_s F_{l,m} \leq 4m$; and:*

i. *$F_{l,m}(x, s)$ is even in s. So $F_{l,m}(x, s) = F_{l,m}(x, -s)$.*

ii. *For (admitted) odd l, $F_{l,m}(x, s)$ is an even polynomial also in x.*

iii. *For even l, $F_{l,m}(x, s)$ is odd in x.*

b. *Furthermore $F_{l,m}(1 + s, s)$ has these properties:*

i. *It is a polynomial in s of degree $\leq 1 + l + 2m$.*

ii. *It has $(1 + s)^{1+l}$ as a factor.*

iii. *It is reciprocal: written $F_{l,m}(1 + s, s) = \sum_{\nu=0}^{1+l+2m} a_\nu s^\nu$ one finds $a_\nu = a_{1+l+2m-\nu}$ for $\nu = 0, 1, ..., 1 + l + 2m$.*

iv. *The coefficients a_ν with $\nu \in \{1, 3, ..., 2m - 1\} \cup \{l + 2, l + 4, ..., l + 2m\}$ are zeros. In particular the exponents which occur in $F_{l,m}(1 + s, s)$ again lie all in $\{0, 2, 4, ..., 2m - 2\} \cup Z_{\geq 2m}$.*

The degrees mentioned above will be usually equal to $4m$ and $1 + l + 2m$, respectively, but we will not need this. Of all the claims made the by far hardest to establish is b.iv.

It is the purpose of this article to prove in Section 1 Theorem 1 and to deduce certain often used facts for univariate polynomials as corollaries, and as a particular example another identity which has some similarities with Theorem 2 but is 'universal' and easier to prove; to give in Section 2 a bijective proof for a certain expression in the Stirling numbers of the second kind being zero and to deduce from this, and Theorem 1, Theorem 2. Section 3 deduces Theorem 3. In Section 4 we add a brief explanation as to what the main result of García-Pelayo says, what tools he used, and how we achieve the same result using apart of some probabilistic considerations essentially the results laid down in the present paper.

1. Proof of Theorem 1 and some corollaries

For an indeterminate x and integers $a, i, n \geq 0$ one has that $x^n = \sum_k \left\{ n \atop k \right\} x^{\underline{k}}$, see [GKP, (6.10)], and finds by inspection

$$\frac{a^{\underline{i}}}{a!} = \frac{a(a-1)\cdots(a-i+1)}{a!} = \begin{cases} 0 & \text{if } i > a \\ 1/(a-i)! & \text{if } i \leq a. \end{cases}$$

Therefore

lhs(1) =

$$= \sum_{a_1+\cdots+a_n=M} \frac{z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}}{a_1! a_2! \cdots a_n!} \sum_{i_1} \left\{ \alpha_1 \atop i_1 \right\} a_1^{\underline{i_1}} \sum_{i_2} \left\{ \alpha_2 \atop i_2 \right\} a_2^{\underline{i_2}} \cdots \sum_{i_n} \left\{ \alpha_n \atop i_n \right\} a_n^{\underline{i_n}}$$

$$= \sum_{a_1+\cdots+a_n=M} z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \sum_{i_1, i_2, \ldots, i_n} \left\{ \alpha_1 \atop i_1 \right\} \frac{a_1^{\underline{i_1}}}{a_1!} \left\{ \alpha_2 \atop \underline{i_2} \right\} \frac{a_1^{\underline{i_2}}}{a_2!} \cdots \left\{ \alpha_n \atop i_n \right\} \frac{a_n^{\underline{i_n}}}{a_n!}$$

$$= \sum_{a_1+\cdots+a_n=M} z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \sum_{i_1 \leq a_1, \ldots, i_n \leq a_n} \left\{ \alpha_1 \atop i_1 \right\} \frac{1}{(a_1-i_1)!} \left\{ \alpha_2 \atop i_2 \right\} \frac{1}{(a_2-i_2)!} \cdots \left\{ \alpha_n \atop i_n \right\} \frac{1}{(a_n-i_n)!}$$

$$= \sum_{i_1, i_2, \ldots, i_n} \left\{ \alpha_1 \atop i_1 \right\} \left\{ \alpha_2 \atop i_2 \right\} \cdots \left\{ \alpha_n \atop i_n \right\} \sum_{\substack{a_1 \geq i_1, a_2 \geq i_2, \ldots, a_n \geq i_n \\ a_1 + a_2 + \cdots + a_n = M}} \frac{z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}}{(a_1-i_1)!(a_2-i_2)! \cdots (a_n-i_n)!}$$

$$= \sum_{i_1,i_2,\dots,i_n} z_1^{i_1} z_2^{i_2}\cdots z_n^{i_n} \left\{ {\alpha_1 \atop i_1} \right\} \left\{ {\alpha_2 \atop i_2} \right\} \cdots \left\{ {\alpha_n \atop i_n} \right\} \sum_{\substack{\dot{a}_1 \geq 0, \dot{a}_2 \geq 0, \cdots, \dot{a}_n \geq 0 \\ \dot{a}_1 + \dot{a}_2 + \cdots + \dot{a}_n = M - i_1 - i_2 - \cdots i_n}} \frac{z_1^{\dot{a}_1} z_2^{\dot{a}_2} \cdots z_n^{\dot{a}_n}}{\dot{a}_1! \dot{a}_2! \cdots \dot{a}_n!}$$

$$= \sum_{i_1,i_2,\dots,i_n} z_1^{i_1} z_2^{i_2}\cdots z_n^{i_n} \left\{ {\alpha_1 \atop i_1} \right\} \left\{ {\alpha_2 \atop i_2} \right\} \cdots \left\{ {\alpha_n \atop i_n} \right\} \delta_{M, i_1+i_2+\cdots+i_n}$$

$$= \mathrm{rhs}(1).$$

Here the penultimate equality sign is justified as follows: if $M = i_1 + \cdots + i_n$, then the inner sum reduces to the one term $\frac{z_1^0 \cdots z_n^0}{0!0!\cdots 0!} = 1$. If $M - i_1 - \cdots - i_n > 0$, then the inner sum is by the multinomial theorem and the hypothesis of the theorem equal to $\frac{(z_1+\cdots+z_n)^{M-i_1-\cdots-i_n}}{(M-i_1-\cdots-i_n)!} = 0$. ■

So the two polynomial expressions lhs(1) and rhs(1) of the theorem define on the complex hyperplane $z_1 + z_2 + \cdots + z_n = 0$ the same polynomial map and their difference is a polynomial which is divisible by $z_1 + z_2 + \cdots + z_n$.

We next note some corollaries, the last two of which (at least) are known.

Corollary 1.1. *As before assume $\sum_{i=1}^n z_i = 0$. Let $p \in \mathbb{C}[x_1, x_2, ..., x_n]$ be a polynomial in n variables of degree$\leq d$. Let p_d be its homogeneous component of degree d. Then*

$$\sum_{a_1+\cdots+a_n=d} z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \frac{p(a_1, ..., a_n)}{a_1! a_2! \cdots a_n!} = p_d(z_1, z_2, ..., z_n).$$

Proof: In the case that $\alpha_1 + \cdots + \alpha_n = d$ there exists exactly one non-negative n-uple $(a_1, ..., a_n)$ among those of sum d for which the product $\left\{ {\alpha_1 \atop a_1} \right\} \left\{ {\alpha_2 \atop a_2} \right\} \cdots \left\{ {\alpha_n \atop a_n} \right\}$ is nonzero; namely $(\alpha_1, ..., \alpha_n)$. In this case the mentioned product is 1, because by the definition of Stirling numbers of the second kind, all factors are 1. In the case $\alpha_1 + \cdots + \alpha_n < d$, there exists for each of the considered $(a_1, ..., a_n)$ an i such that $\alpha_i < a_i$ and then the product of the Stirling numbers must be 0. We can now write $p = p_{<d} + p_d$ where $p_{<d}$ is the sum of the homogeneous components of degree $< d$ of p. Since $p_{<d}(a_1, ..., a_n)$ is a sum of terms of form $c a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$ with $c \in \mathbb{C}$ and $\sum_i \alpha_i < d$ it is clear that putting in the above sum $p_{<d}$ in place of p we

get by Theorem 1 value 0; and putting p_d in place of p, each of its terms $c\, a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$ contributes $c\, z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ and therefore we get the claim. ■

The following result is quite often used in work with binomial coefficients. Usually it is proved using the forward difference operator, see [GKP, formula (5.42)].

Corollary 1.2. *Let $p \in \mathbb{C}[x]$ be a polynomial of degree $\leq d$. Then*

$$\sum_{k=0}^{d} (-1)^k \binom{d}{k} p(k) = (-1)^d d! \cdot \text{coefficient of } x^d \text{ in } p.$$

Proof: Write $p(x) = \sum_{i=0}^{d} a_i x^i$ as a 2-variable polynomial $\hat{p}(x, y) = \sum\limits_{i=0}^{d} a_i x^i y^0$. Evidently, for any integers k, l, $\hat{p}(k, l) = p(k)$ with the convention here that $0^0 = 1$. We now have, using the previous corollary for the case $n = 2, z_1 = -1, z_2 = 1$,

$$\text{lhs} = d! \sum_{k+l=d} (-1)^k 1^l \frac{\hat{p}(k, l)}{k!l!} = d! \hat{p}_d(-1, 1) = d!(-1)^d a_d = \text{rhs}. \qquad ■$$

Corollary 1.3. [GKP, 6.19] *For integers $m, n \geq 0$ there holds*

$$\sum_{k=0}^{m} \binom{m}{k} k^n (-1)^{m-k} = m! \left\{ {n \atop m} \right\}.$$

Proof: Choose in our general identity $n = 2, M = m$, write $a_1 = k$ so that $a_2 = m - k$, and put $z_1 = 1, z_2 = -1$. Then the identity becomes

$$\sum_{k=0}^{m} (-1)^{m-k} \frac{k^{\alpha_1} (m-k)^{\alpha_2}}{k!(m-k)!} = \sum_{k=0}^{m} (-1)^{m-k} \left\{ {\alpha_1 \atop k} \right\} \left\{ {\alpha_2 \atop m-k} \right\}.$$

Now put $\alpha_1 = n$ and $\alpha_2 = 0$. Then the right hand side turns because of $\left\{ {0 \atop j} \right\} = \delta_{0,j}$ into $\left\{ {n \atop m} \right\}$. Finally multiply both sides with $m!$ and use the definition of binomial coefficients. ■

As a foretaste to Theorem 2, the following corollary can serve.

Corollary 1.4. For $m, s \geq 0$ integers there holds

$$\sum_{\substack{a,b,c\geq 0\\ a+b+c=m}} \frac{(-2)^b}{a!b!c!}(a+c)^s = 2^m \left\{ {s \atop m} \right\}.$$

Proof:

$$\begin{aligned}
\text{lhs} &= \sum_{a+b+c=m} \frac{(-2)^b}{a!b!c!} \sum_{\mu=0}^{s} \binom{s}{\mu} a^\mu b^0 c^{s-\mu} \\
&= \sum_{\mu=0}^{s} \binom{s}{\mu} \sum_{a+b+c=m} \frac{(-2)^b}{a!b!c!} a^\mu b^0 c^{s-\mu} \\
&\overset{\text{Th1}}{=} \sum_{\mu=0}^{s} \binom{s}{\mu} \sum_{a+b+c=m} (-2)^b \left\{ {\mu \atop a} \right\} \left\{ {0 \atop b} \right\} \left\{ {s-\mu \atop c} \right\} \\
&= \sum_{\mu=0}^{s} \binom{s}{\mu} \sum_{a+c=m} \left\{ {\mu \atop a} \right\} \left\{ {s-\mu \atop c} \right\} = \sum_{a+c=m} \sum_{\mu=0}^{s} \binom{s}{\mu} \left\{ {\mu \atop a} \right\} \left\{ {s-\mu \atop c} \right\} \\
&\overset{2}{=} \sum_{a+c=m} \left\{ {s \atop a+c} \right\} \binom{a+c}{a} = \sum_{a=0}^{m} \left\{ {s \atop m} \right\} \binom{m}{a} = 2^m \left\{ {s \atop m} \right\}.
\end{aligned}$$

Here we used Theorem 1 as indicated and in '$\overset{2}{=}$' an identity that can be found as number 28 in a table of Stirling identities in [GKP]. ∎

2. Proof of Theorem 2

We shall need the following proposition.

Proposition 2.1. Assume nonnegative integers $m, s, \ddot{s}$ satisfy $m \geq \ddot{s} + \lceil \frac{1+s}{2} \rceil$. Then

$$*_2\colon \sum_{\substack{\mu=0\\ \mu \text{ even}}}^{s} \binom{s}{\mu} \sum_{\substack{a,c\geq 0\\ a+c=m}} \left\{ {\mu \atop a} \right\} \left\{ {s+\ddot{s}-\mu \atop c} \right\} = \sum_{\substack{\mu=0\\ \mu \text{ odd}}}^{s} \binom{s}{\mu} \sum_{\substack{a,c\geq 0\\ a+c=m}} \left\{ {\mu \atop a} \right\} \left\{ {s+\ddot{s}-\mu \atop c} \right\}.$$

Proof: We give a bijective proof. Having in mind the combinatorial significance of the Stirling numbers $\left\{ {n \atop k} \right\}$ we see that $\left\{ {\mu \atop a} \right\} \left\{ {s+\ddot{s}-\mu \atop c} \right\}$ is the number of pairs (π_1, π_2) where π_1 is a partition of $\{1, ..., \mu\}$ into a (nonempty) parts and π_2 is a partition of $\{1+\mu, 2+\mu, ..., s+\ddot{s}\}$ into c parts. Therefore the inner sum

$$\sum_{\substack{a,c \geq 0 \\ a+c=m}} \left\{ {\mu \atop a} \right\} \left\{ {s+\ddot{s}-\mu \atop c} \right\}$$

is the number of partitions π of $\{1, ..., s+\ddot{s}\}$ into m parts, where every part pertains either to the set $\{1, ..., \mu\}$ or to the set $\{1+\mu, ..., s+\ddot{s}\}$. Since $\binom{s}{\mu}$ is the number of subsets U of $\{1, ..., s\}$ of cardinality μ, we see for the left hand side of $(*_2)$ that:

lhs$(*_2)$ =number of pairs (U, π) where U is a subset of $\{1, ..., s\}$ of even cardinality and π is a partition of $\{1, ..., s+\ddot{s}\}$ into m parts such that each part pertains either to U or to $\{1, ..., s+\ddot{s}\} \setminus U$.

The rhs$(*_2)$ is analougously defined with the difference that the subsets U are of odd cardinality. From the hypothesis on $s, \ddot{s}, m$ we get the following:

Claim: A partition π of $\{1, ..., s+\ddot{s}\}$ into m parts has a singleton block whose element lies in $\{1, ..., s\}$.

$\triangleright$ Assume otherwise; that is every part of π which is subset of $\{1, ..., s\}$ has two or more elements. Then clearly the number of blocks in $\{1, ..., s\}$ is $< \lceil \frac{1+s}{2} \rceil$. The number of blocks in $\{1+s, ..., \ddot{s}+s\}$ is of course $\leq \ddot{s}$. So the total number of blocks of π is $< \ddot{s} + \lceil \frac{1+s}{2} \rceil$, contradicting the hypothesis. $\triangleleft$

Fix a partition π of $\{1, ..., s+\ddot{s}\}$ into m parts. Let $\mathcal{E}_\pi = \{U \subseteq \{1, ..., s\} : |U|$ is even and each part of π lies either in U or in $[s+\ddot{s}] \setminus U\}$. Similarly define $\mathcal{O}_\pi = \{U \subseteq \{1, ..., s\} : |U|$ is odd and each part of π lies either in U or in $[s+\ddot{s}] \setminus U\}$.

For U in $\mathcal{E}_\pi \uplus \mathcal{O}_\pi$ and m_π :=smallest k such that $\{k\}$ is a block of π, define

$$\iota(U) = \begin{cases} U \setminus \{m_\pi\} & \text{if } m_\pi \in U \\ U \uplus \{m_\pi\} & \text{if } m_\pi \notin U \end{cases}.$$

It is easy to see that $\iota_{|\mathcal{E}_\pi} : \mathcal{E}_\pi \to \mathcal{O}_\pi$ and that ι is injective. Similarly $\iota_{|\mathcal{O}_\pi} : \mathcal{O}_\pi \to \mathcal{E}_\pi$ is injective. In consequence, $|\mathcal{E}_\pi| = |\mathcal{O}_\pi|$. Now we have lhs$(*_2) = \sum_\pi |\mathcal{E}_\pi| = \sum_\pi |\mathcal{O}_\pi| =$ rhs$(*_2)$, where both sums are over the partitions π of $\{1, ..., s+\ddot{s}\}$ into m parts. ■

Proof of Theorem 2. The case of assumption ii is trivial: if (a, b, c) is a triple of nonnegative integers of sum m, so is (c, b, a) and the term the latter triple produces in the sum is evidently the negative of that produced by (a, b, c) and the result follows. For the case of assumption i, we compute

$$\begin{aligned}
\text{lhs}(2) &= \sum_{\substack{a,b,c\geq 0\\ a+b+c=m}} \frac{(-2)^b}{a!b!c!}\sum_{\mu=0}^{s}(-1)^{s-\mu}\binom{s}{\mu}a^{\mu}b^{\dot{s}}c^{s+\ddot{s}-\mu}\\
&= \sum_{\mu=0}^{s}(-1)^{s-\mu}\binom{s}{\mu}\sum_{\substack{a,b,c\geq 0\\ a+b+c=m}}\frac{(-2)^b}{a!b!c!}a^{\mu}b^{\dot{s}}c^{s+\ddot{s}-\mu}\\
&\overset{1}{=} \sum_{\mu=0}^{s}(-1)^{s-\mu}\binom{s}{\mu}\sum_{\substack{a,b,c\geq 0\\ a+b+c=m}}(-2)^b\left\{\begin{matrix}\mu\\ a\end{matrix}\right\}\left\{\begin{matrix}\dot{s}\\ b\end{matrix}\right\}\left\{\begin{matrix}s+\ddot{s}-\mu\\ c\end{matrix}\right\}\\
&= \sum_{b=0}^{m}(-2)^b\left\{\begin{matrix}\dot{s}\\ b\end{matrix}\right\}\sum_{\mu=0}^{s}(-1)^{s-\mu}\binom{s}{\mu}\sum_{\substack{a,c\geq 0\\ a+c=m-b}}\left\{\begin{matrix}\mu\\ a\end{matrix}\right\}\left\{\begin{matrix}s+\ddot{s}-\mu\\ c\end{matrix}\right\}.
\end{aligned}$$

Here '$\overset{1}{=}$' follows by using Theorem 1 with $z_1 = z_3 = 1, z_2 = -2$, and $(\alpha_1, \alpha_2, \alpha_3) = (\mu, \dot{s}, s + \ddot{s} - \mu)$. We have $m > \dot{s}$ from where it follows that the outer sum $\sum_{b=0}^{m} \ldots$ can be written $\sum_{b=0}^{\dot{s}} \ldots$ so it can be assumed that $b \leq \dot{s}$. The hypothesis of the theorem then guarantees $m - b \geq \ddot{s} + \lceil\frac{1+s}{2}\rceil$. The previous proposition implies that the inner double sum is zero. Hence the theorem follows. ■

3. Proof of Theorem 3

Expanding e_2 gives $-e_2(x, s) = x^2 - 2(1 + s^2) + (1 - s^2)^2x^{-2}$ and the multinomial theorem yields

$$x^l(-e_2(x, s))^m = \sum_{a,b,c\geq 0}(-2)^b\binom{m}{a, b, c}(1 + s^2)^b x^{l+2(a-c)}(1 - s^2)^{2c}.$$

By definition of multinomial coefficients, in this sum occur only terms associated to nonnegative integer triples (a, b, c) for which $a + b + c = m$. Therefore $2(a - c)$ is an even integer in the interval $[-2m, 2m]$, while, if l is an integer satisfying the hypothesis, then $l \in \{$even integers in $[0, 2m - 2]\} \cup \mathbb{Z}_{\geq 2m}$. Thus $l + 2(a - c) \neq -1$. Consequently the indefinite integral

$\int x^l(-e_2(x,s))^m dx$ has no logarithmic term and one of the primitive functions of $x^l(-e_2(x,s))^m$ is given by

$$F_{l,m}(x,s) = \sum_{a,b,c\geq 0} \frac{(-2)^b\binom{m}{a,b,c}}{(1+l+2(a-c))}(1+s^2)^b x^{1+l+2(a-c)}(1-s^2)^{2c}.$$

This is evidently a polynomial in even powers of s (and coefficients that are Laurent polynomials in x). Its degree in s is not larger than the maximal value which $2b+4c$ assumes when (a,b,c) range over nonnegative integers of sum m. This value is evidently $4m$. Theorem 3a is now evident. Substituting x by $(1+s)$ and again using the definition of multinomial coefficients we get

$$F_{l,m}(1+s,s) = m! \sum_{\substack{a,b,c\geq 0\\ a+b+c=m}} \frac{(-2)^b}{a!b!c!(1+l+2(a-c))} \underbrace{(1+s^2)^b(1+s)^{1+l+2(a-c)}(1-s^2)^{2c}}_{=:\varphi(s)}.$$

Writing $(1-s^2)^{2c} = (1-s)^{2c}(1+s)^{2c}$, we find

$$\varphi(s) = (1+s)^{1+l}(1+s^2)^b(1+s)^{2a}(1-s)^{2c}$$

which has leading term $s^{1+l+2b+2a+2c} = s^{1+l+2m}$. Also since $(1-s)^{2c} = (1-2s+s^2)^c$ is reciprocal, $\varphi(s)$ is a product of reciprocal polynomials and therefore reciprocal. Independent of a,b,c, $\varphi(s)$ has constant term 1 and leading coefficient 1 and (as seen) degree $1+l+2m$. Thus $F_{l,m}(1+s,s)$ as a linear combination of reciprocal polynomials $\varphi(s)$ is reciprocal in the sense of b.iii. Now b.ii is also clear.

We still need to prove iv. From the original definition of $\varphi(s)$ we find

$$\varphi(s) = \sum_{k\geq 0}\binom{b}{k}s^{2k}\sum_{i\geq 0}\binom{1+l+2(a-c)}{i}s^i\sum_{j\geq 0}\binom{2c}{j}(-1)^j s^{2j}.$$

By writing binomial coefficients via falling factorials, we see

$$\begin{aligned}
&(\text{coefficient of } s^t \text{ in } \varphi(s)) = \\
&= \sum_{i+2(k+j)=t}\binom{b}{k}\binom{1+l+2(a-c)}{i}\binom{2c}{j}(-1)^j \\
&\overset{\text{if t odd}}{=} (1+l+2(a-c)) \sum_{i+2(j+k)=t} (i!j!k!)^{-1}(l+2(a-c))^{\underline{i-1}}\, b^{\underline{k}}\, (2c)^{\underline{j}}.
\end{aligned}$$

Here the ‘if t is odd’ condition guarantees that $i-1\geq 0$. Thus, if t is odd, (coefficient of s^t in $F_{l,m}(1+s,s)$)=

$$= m! \sum_{\substack{a,b,c\geq 0 \\ a+b+c=m}} \frac{(-2)^b}{a!b!c!} \sum_{i+2(j+k)=t} (i!j!k!)^{-1}(l+2(a-c))^{\underline{i-1}}\, b^{\underline{k}}\, (2c)^{\underline{j}}$$

$$= m! \sum_{i+2(j+k)=t} (i!j!k!)^{-1} \sum_{\substack{a,b,c\geq 0 \\ a+b+c=m}} \frac{(-2)^b}{a!b!c!}(l+2(a-c))^{\underline{i-1}}\, b^{\underline{k}}\, (2c)^{\underline{j}}$$

Now evidently $(l+2(a-c))^{\underline{i-1}}\, b^{\underline{k}}\, (2c)^{\underline{j}}$ can be expanded into a linear combination of products $b^{\dot{s}}(a-c)^s c^{\ddot{s}}$ in each of which we will have $(\dot{s}, s, \ddot{s}) \leq (k, i-1, j)$ in componentwise order. Now assume additionally $t \leq 2m-1$. Since t, and hence i, are odd we then get

$$\dot{s} + \lceil \frac{1+s}{2} \rceil + \ddot{s} \leq k + \lceil \frac{i}{2} \rceil + j = (t+1)/2 \leq m.$$

So the claim that the coefficient of an s^t with $t \in \{1, 3, ..., 2m-1\}$ in $F_{l,m}(1+s,s)$ is 0 follows from the preceding theorem. That the monomials s^t with $t \in \{l+2, l+4, ..., l+2m\}$ cannot occur either is then a consequence of the reciprocality of $F_{l,m}(1+s,s)$. ∎

We conclude this section with examples of the objects we are speaking of:

$$\begin{aligned}
F_{2,3}(1+s,s) &= \tfrac{1024}{63} - \tfrac{1024s^2}{35} - \tfrac{1024s^7}{35} + \tfrac{1024s^9}{63} \\
&= \tfrac{1024}{315}(1+s)^3\left(5 - 15s + 21s^2 - 23s^3 + 21s^4 - 15s^5 + 5s^6\right) \\
F_{2,3}(2,s) &= \tfrac{11491}{504} - \tfrac{8297s^2}{140} + \tfrac{459s^4}{8} - \tfrac{141s^6}{2} + \tfrac{163s^8}{8} + \tfrac{13s^{10}}{4} - \tfrac{s^{12}}{24} \\
F_{4,5}(1+s,s) &= -\tfrac{262144}{2145} + \tfrac{524288s^2}{1287} - \tfrac{262144s^4}{693} - \tfrac{262144s^{11}}{693} + \tfrac{524288s^{13}}{1287} - \tfrac{262144s^{15}}{2145} \\
F_{2,3}(x,s) &= (\tfrac{x^9}{9} - \tfrac{6x^7}{7} + 3x^5 - \tfrac{20x^3}{3} - \tfrac{1}{3x^3} + 15x + \tfrac{6}{x})s^0 \\
&\quad +(-\tfrac{6x^7}{7} + \tfrac{18x^5}{5} - 4x^3 + \tfrac{2}{x^3} - 12x - \tfrac{18}{x})s^2 \\
&\quad +(3x^5 - 4x^3 - \tfrac{5}{x^3} - 6x + \tfrac{12}{x})s^4 \\
&\quad +(-\tfrac{20x^3}{3} + \tfrac{20}{3x^3} - 12x + \tfrac{12}{x})s^6 \\
&\quad +(-\tfrac{5}{x^3} + 15x - \tfrac{18}{x})s^8 + (\tfrac{2}{x^3} + \tfrac{6}{x})s^{10} + (-\tfrac{1}{3x^3})s^{12}.
\end{aligned}$$

4. Connection to Random flights.

Consider in and odd-dimensional euclidean space a particle at instant 0 at the origin which at each tick of the clock jumps exactly one unit from its current position into a random direction (with uniform distribution). Let R_n be the distance of the particle from the starting place after n steps viewn as a random variable. Determine the character of the density function f_{R_n}.

This problem was first solved in 2012 by R. García - Pelayo [G-P] who showed that f_{R_n} is piecewise polynomial function. He uses that the Fourier transform of a convolutional product is the product of the Fourier transforms of the factors, a result of Kingman of 1963 on the behaviour of convolutions under projections, and a multivariate generalization of the the Abel transform which is known as a tool to analyse radial functions.

In the preprint [SK2] the authors presented an elementary approach to this problem. After a few pages they get the result that the function

$$f_{R_n}(s) = C_d^{n-1} \cdot s \cdot b_n(s),$$

where C_d is a certain explicitly known rational which depends only on the dimension d and the functions b_n can be inductively defined via the formulas given in the first lines of the following theorem. These formulas can directly be used for a Mathematica implementation using the `Piecewise` command but it was shown that if we treat the functions b_n as lists of functions on different intervals then we can speed up computation considerably. Experiments then show that these lists seem to consist only of polynomial functions; and in fact that this must be so is the main result of García - Pelayo' s paper. We can now rectify the insactisfactory state of affairs that we have an in principle elementary algorithm but the correctness proof for it relies on sofisticated methods.

In [SK2] the following is shown:

Theorem. *Assume $d \geq 3$ is an odd integer and let $m = \frac{d-3}{2}$. Then the functions $b_n(s)$, inductively defined for $n \geq 2$ by*

$$b_2(s) = (s^2(4-s^2))^m \mathbb{1}_{[0,2[}(r), \quad b_{n+1}(s) = \int_{-\infty}^{\infty} b_n(r) e_2(r,s)^m \mathbb{1}_{[|-1+s|,1+s[}(r)dr;$$

admit depending on the parity of index n the piecewise representations

$$b_{2\dot{n}} = \sum_{i=0}^{\dot{n}-1} \mathrm{pol}_i(s) \mathbb{1}_{[2i,2i+2[}(s)$$

$$b_{1+2\dot{n}} = \sum_{i=0}^{\dot{n}} \widetilde{\mathrm{pol}}_i(s) \mathbb{1}_{[(2i-1)^+,2i+1[}(s),$$

in which we can express the functions $\widetilde{\mathrm{pol}}_i(s)$ pertaining to indices $1+2\dot{n}$ from the functions pol_i pertaining to index $2\dot{n}$ by

$$\widetilde{\mathrm{pol}}_i(s)=\begin{cases} \textit{if } i=0: \int\limits_{1-s}^{1+s}\mathrm{pol}_0(r)e_2(r,s)^m dr\\ \textit{if } i=1,...,\dot n-1: \int\limits_{-1+s}^{2i}\mathrm{pol}_{i-1}(r)e_2(r,s)^m dr+\int\limits_{2i}^{1+s}\mathrm{pol}_i(r)e_2(r,s)^m dr\\ \textit{if } i=\dot n: \int\limits_{-1+s}^{2\dot n}\mathrm{pol}_{\dot n-1}(r)e_2(r,s)^m dr;\end{cases}$$

and the functions pol_i *pertaining to the case* $n=2(\dot n+1)$ *from the functions* $\widetilde{\mathrm{pol}}_i(s)$ *pertaining to the case* $n=1+2\dot n$ *by*

$$\mathrm{pol}_i(s)=\begin{cases} \textit{if } i=0: \int\limits_{|-1+s|}^{1}\widetilde{\mathrm{pol}}_0(r)e_2(r,s)^m dr+\int\limits_{1}^{1+s}\widetilde{\mathrm{pol}}_1(r)e_2(r,s)^m dr\\ \textit{if } i=1,...,\dot n-1: \int\limits_{s-1}^{2i+1}\widetilde{\mathrm{pol}}_i(r)e_2(r,s)^m dr+\int\limits_{2i+1}^{s+1}\widetilde{\mathrm{pol}}_{i+1}(r)e_2(r,s)^m dr\\ \textit{if } i=\dot n: \int\limits_{s-1}^{2\dot n+1}\widetilde{\mathrm{pol}}_{\dot n}(r)e_2(r,s)^m dr.\end{cases}$$

The missing piece was to show that the functions pol_i and $\widetilde{\mathrm{pol}}_i$ are polynomials. This will follow from the observations $\underline{0},\underline{1},\underline{2},\underline{3},\underline{4}$, below which rely heavily on the fact that by Theorem 3a, we have for positive reals a,b and $l\in\{2,4,...,2m-2\}\cup\mathbb{Z}_{\geq 2m}$, that the definite integral $\int_a^b x^l(-e_2(x,s))^m dx = F_{l,m}(b,s)-F_{l,m}(a,s)$; hence if $p(x)$ is a polynomial in which only monomials x^l with exponents in $l\in\{2,4,...,2m-2\}\cup\mathbb{Z}_{\geq 2m}$, occur, then $\int_a^b p(x)e_2(x,s)^m dx$ is a real linear combination of expressions $F_{l,m}(b,s)-F_{l,m}(a,s)$ with such l.

Looking at the definition of $b_2(s)$, we find:

$\underline{0}$. The expression pol_0 associated to $\dot n=1$ is the polynomial $(s^2(4-s^2))^m$ and hence is even and has only monomials s^t with $t\geq 2m$.

$\underline{1}$. If pol_0, associated to $b_{2\dot n}$ is a polynomial all whose exponents l are in $\{2,4,...,2m-2\}\cup\mathbb{Z}_{\geq 2m}$, then $\widetilde{\mathrm{pol}}_0$ associated to $b_{1+2\dot n}$ is an odd polynomial of underdegree $\geq 2m$.

Proof: For an l as admitted we have that by its definition, $\widetilde{\mathrm{pol}}_0(s)$ is a linear combination of differences

$$F_{l,m}(1+s,s)-F_{l,m}(1-s,s)=F_{l,m}(1+s,s)-F_{l,m}(1-s,-s)$$

for different l. Here we used that by Theorem 3a.i, $F(x,s)=F(x,-s)$. We see now that these differences turn into their negatives as we replace s by $-s$, so they are odd polynomials. By Theorem 3b.iv, the exponents $\leq 2m-1$ occurring in $F_{l,m}(1+s,s)$ are even; so the same holds for $F_{l,m}(1-s,-s)$. So the exponents occurring in the differences all must be $\geq 2m$. This yields the claim. ■

$\underline{2}$. If $l \geq 2m$ is odd, then $\int_{|-1+s|}^{1} x^l e_2(x,s)^m dx = \int_{1-s}^{1} x^l e_2(x,s)^m dx$.

Proof: Recall that $e_2(x,s) = -(1-s^2)^2 x^{-2} + 2(1+s^2) - x^2$. So in $e_2(x,s)^m$ the powers of x occurring are all even and$\geq -2m$. Hence $x^l e_2^m$ has only odd powers which are all ≥ 0. Now if x^o is such an odd power of x, then $\int_{|-1+s|}^{1} x^o dx = (1+o)^{-1}(1-|-1+s|^{1+o}) = (1+o)^{-1}(1-(1-s)^{1+o}) = \int_{1-s}^{1} x^o dx$, since $1+o$ is even. The claim follows. ■

$\underline{3}$. All the expressions $F_{l,m}(a,s)$ with a being one of the s-independent lower or upper bounds in the integrals occurring for the $\mathrm{pol}_i, \widetilde{\mathrm{pol}}_i$ are (for admitted l) even polynomials in s.

Proof: This is immediate from Theorem 3a.i. ■

$\underline{4}$. Concerning the integrals below at the left which occur in the Theorem of [SK2] for the computation of the functions pol_0 associated to $b_2, b_4, b_6, ...$, there holds

$$\int_{|-1+s|}^{1} \widetilde{\mathrm{pol}}_0(r) e_2(r,s)^m dr = \int_{1-s}^{1} \widetilde{\mathrm{pol}}_0(r) e_2(r,s)^m dr.$$

Proof: The expression $\mathrm{pol}_0(s)$ associated to b_2 is by observation 0 even and has underdegree $\geq 2m$. It satisfies hence the hypothesis of observation $\underline{1}$ for $\dot{n} = 1$ and that observation yields that $\widetilde{\mathrm{pol}}_0$ associated to b_3 is an odd polynomial of underdegree $\geq 2m$. Thus by observation $\underline{2}$, we may replace the integral $\int_{|-1+s|}^{1} ...$ by $\int_{1-s}^{1} ...$. If we do so we get that the expression pol_0 pertaining to b_4 is given by $\mathrm{pol}_0(s) = \int_{1-s}^{1} \mathrm{pol}_0(r) e_2(r,s)^m dr + \int_{1}^{1+s} \widetilde{\mathrm{pol}}_1(r) e_2(r,s)^m dr$. This expression is hence a linear combination of differences $F_{l,m}(1,s) - F_{l,m}(1-s,s)$ and $F_{l,m}(1+s,s) - F_{l,m}(1,s)$ for various ls and so by $\underline{3}$ and Theorem 3b.iv we see this $\mathrm{pol}_0(s)$ has only exponents in $\{2,4,...,2m-2\} \cup \mathbb{Z}_{\geq 2m}$. Then $\underline{1}$ gives us that $\widetilde{\mathrm{pol}}_0(s)$ associated to $b_5 = b_{1+2\cdot 1}$ is odd and has underdegree $\geq 2m$. This in turn, as above, by $\underline{2}$ allows us again a replacement of $\int_{|-1+s|}^{1} ...$ by $\int_{1-s}^{1} ...$ and so for the calculation of pol_0 associated to b_6 to write the same formula as before we did for the pol_0 associated to b_4. Repeating this type of reasoning we see by induction that indeed we can write the formulas for pol_0 for all polynomials $b_{2\dot{n}}$ with integrals $\int_{1-s}^{1} \ldots$ instead of $\int_{|-1+s|}^{1} \cdots$. ■

Conclusion of the proof: We realize that all the integrals figuring in the computations of the $\mathrm{pol}_i(s)$ and $\widetilde{\mathrm{pol}}_i(s)$ can be written with upper and lower bounds which are reals $a > 0$ (independent of s) or are of the forms $s-1, 1-s, 1+s$. Thanks to the observations done up till here, we find that indeed the expressions $\mathrm{pol}_i(s)$ and $\widetilde{\mathrm{pol}}_i(s)$ are polynomials and we are done. ■

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