Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 21–39

ON CLASSES OF LOCALIC MAPS DEFINED BY THEIR BEHAVIOR ON ZERO SUBLOCALES

ANA BELÉN AVILEZ

ABSTRACT: We study classes of localic maps defined by conditions on the behavior of their images and preimages on zero sublocales. Relations between them and closed and open localic maps are presented. These maps are then used to characterize normality, and weaker forms of normality, in a manner akin to the characterization of normal locales that states that a locale is normal iff every closed localic embedding in it is a C-map.

KEYWORDS: Frame, locale, sublocale, zero sublocale, completely separated sublocales, z- w- and n-maps, variants of normality.

2020 MATHEMATICS SUBJECT CLASSIFICATION: 06D22, 18F70, 54C20, 54C30, 54C45.

Introduction

This paper may be considered as a sequel to [2], a paper devoted to the study of special localic embeddings like z-embeddings. In the present paper, we consider general localic z-maps, an extension of z-embeddings, and other related classes of localic maps. These notions can be nicely motivated via the familiar image/preimage adjunction in the category of locales.

Let $f\colon L \to M$ be a localic map. For any sublocale S of L, its set theoretic image $f[S]$ is a sublocale of M. On the other hand, the set theoretic preimage $f^{-1}[T]$ of a sublocale T of M may not be a sublocale of L. But since f is meet preserving, $f^{-1}[T]$ is closed under meets and thus there exists the largest sublocale of L contained in $f^{-1}[T]$, usually denoted as $f_{-1}[T]$ ([18, III.4]). This is the *localic preimage* of T that provides the image/preimage Galois adjunction

$$\begin{tikzcd} \mathsf{S}(L) \arrow[r, bend left, "{f[-]}"] \arrow[r, phantom, "\perp"] & \mathsf{S}(M) \arrow[l, bend left, "{f_{-1}[-]}"] \end{tikzcd}$$

between coframes $\mathsf{S}(L)$ and $\mathsf{S}(M)$ of sublocales of L and sublocales of M, respectively. The right adjoint $f_{-1}[-]$ is a coframe homomorphism that preserves complements while $f[-]$ is a colocalic map ([18, III.9]).

Received October 21, 2021.

Localic preimages of closed (resp. open) sublocales are closed (resp. open). More specifically, denoting by f^* the frame homomorphism left adjoint to f, we have

$$f_{-1}[\mathfrak{o}_M(a)] = \mathfrak{o}_L(f^*(a)) \quad \text{and} \quad f_{-1}[\mathfrak{c}_M(a)] = \mathfrak{c}_L(f^*(a))$$

for any $a \in M$.

Since frame homomorphisms preserve cozero elements, the preimage map $f_{-1}[-]$ restricts to maps

$$f^z_{-1}[-]\colon \mathsf{ZS}(M) \to \mathsf{ZS}(L) \quad \text{and} \quad f^{coz}_{-1}[-]\colon \mathsf{CoZS}(M) \to \mathsf{CoZS}(L)$$

in the sub-σ-coframe $\mathsf{ZS}(M)$ (resp. sub-σ-frame $\mathsf{CoZS}(M)$) of zero sublocales of M (resp. cozero sublocales of M).

The *zero map* $f^z_{-1}[-]$ is a σ-coframe homomorphism while the *cozero map* $f^{coz}_{-1}[-]$ is a σ-frame homomorphism. Clearly, the former is surjective if and only if the latter is surjective. In this case, we say that f is a *z-map*.

These maps are the right adjoints of the coz-onto frame homomorphisms of [10, 3] and, in the particular case when the embedding of a sublocale S of L is a z-map, one gets precisely the notion of a z-embedding, treated in [2] (in this case one refers to S as *z-embedded* in L). Hence z-maps are an extension of z-embeddings to general localic maps.

Inspired by classical results of Weir [21], these and other related classes of localic maps are treated in this paper (e.g. when the image of any zero sublocale is closed we speak of z-closed localic maps). In particular, we extend results from [9, 10, 15, 17].

Here is a brief outlay of the paper. We collect some background notions and notation in Section 1. Following that, we study, in Section 2, the classes of z-maps, z-dense maps and almost z-dense localic maps, and, in Section 3, the classes of z-open and z-closed maps and variants of them. Then, in Section 4, we characterize normality and some weaker forms of it in terms of z-embeddings. In particular, we identify sufficient conditions under which the characterizations hold for certain variants of normality. In Section 5, we introduce n- and w-maps and present a similar unifying study in terms of those maps. The last section (Section 6) is devoted to a further type of maps, the wz-maps .

1. Notation and terminology

1.1. The categories of frames and locales. Our notation and terminology for frames and locales is that of [18]. We recall here some of the basic notions involved. A *frame* L is a complete lattice in which

$$a \wedge \bigvee S = \bigvee \{a \wedge b \mid b \in S\} \quad \text{for any } a \in L \text{ and } S \subseteq L. \tag{1.1.1}$$

A *frame homomorphism* preserves all joins (in particular, the bottom element 0 of the lattice) and all finite meets (in particular, the top element 1).

In any frame L, the mappings $(x \mapsto (a \wedge x)) \colon L \to L$ preserve suprema and hence they have right Galois adjoints $(y \mapsto (a \to y)) \colon L \to L$, satisfying

$$a \wedge x \leq y \quad \text{iff} \quad x \leq a \to y$$

and making L a complete Heyting algebra. The *pseudocomplement* of $a \in L$ is the element $a^* = a \to 0 = \bigvee \{x \mid x \wedge a = 0\}$. A *regular element* of L is an element of the form a^* for some a (equivalently, an element a such that $a^{**} = a$).

The *rather below relation* $\prec$ in L is defined by $b \prec a$ iff $b^* \vee a = 1$. A frame is *regular* if $a = \bigvee \{b \in L \mid b \prec a\}$ for every $a \in L$, and it is *normal* if for any $a, b \in L$ such that $a \vee b = 1$, there is a pair $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = b \vee v = 1$.

The category of locales and localic maps is the dual category of frames and frame homomorphisms. Thus a *locale* is a frame and *localic maps* can be represented by the (uniquely defined) right adjoints $f = h_* \colon L \to M$ of frame homomorphisms $h \colon M \to L$. They are precisely the meet preserving maps $f \colon L \to M$ such that $f(h(a) \to b) = a \to f(b)$ and $f(a) = 1 \Rightarrow a = 1$. A localic map is *dense* if $f(0) = 0$.

1.2. The coframe of sublocales of a locale. A *sublocale* of a locale L is a subset $S \subseteq L$ closed under arbitrary meets such that

$$\forall x \in L \;\; \forall s \in S \;\; (x \to s \in S).$$

They are precisely the subsets of L for which the embedding $S \subseteq L$ is a localic map. The system $\mathsf{S}(L)$ of all sublocales of L, partially ordered by inclusion, is a *coframe*, that is, its dual lattice is a frame [18, Thm. III.3.2.1]. Infima and suprema are given by

$$\bigwedge_{i \in J} S_i = \bigcap_{i \in J} S_i, \qquad \bigvee_{i \in J} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i \in J} S_i\}.$$

The least element is the *void sublocale* $\mathsf{O} = \{1\}$ and the greatest element is the entire locale L. Since $\mathsf{S}(L)$ is a coframe, there is a co-Heyting operator $S \smallsetminus T$ given by the formula $\bigcap\{A \mid S \subseteq T \vee A\}$ and characterized by the condition

$$S \smallsetminus T \subseteq A \;\Leftrightarrow\; S \subseteq T \vee A.$$

In particular, the co-pseudocomplement (usually called *supplement*) of S is the sublocale $S^{\#} = L \smallsetminus S$ and we have that for any $S, T \in \mathsf{S}(L)$,

$$S \vee T = L \;\Leftrightarrow\; S^{\#} \subseteq T \quad \text{and} \quad S \cap T = \mathsf{O} \;\Rightarrow\; S \subseteq T^{\#} \tag{1.2.1}$$

(we refer to [11] for more information on supplements in $\mathsf{S}(L)$). Furthermore, any complemented S satisfies the special distributivity law $S \cap \bigvee_{i \in J} T_i = \bigvee_{i \in J}(S \cap T_i)$ for every $\{T_i\}_{i \in J} \subseteq \mathsf{S}(L)$.

For each $a \in L$, the sublocales

$$\mathfrak{c}_L(a) = {\uparrow}a = \{x \in L \mid x \geq a\} \quad \text{and} \quad \mathfrak{o}_L(a) = \{a \to b \mid b \in L\}$$

are the *closed* and *open* sublocales of L, respectively (that we shall denote simply by $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ when there is no danger of confusion). For each $a \in L$, $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathsf{S}(L)$ and satisfy the identities

$$\bigcap_{i \in J} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i \in J} a_i), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b), \tag{1.2.2}$$

$$\bigvee_{i \in J} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i \in J} a_i) \quad \text{and} \quad \mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b). \tag{1.2.3}$$

Open sublocales have a further distributivity property: for every $S \in \mathsf{S}(L)$, $S \cap \bigvee_{i \in J} \mathfrak{o}(a_i) = \bigvee_{i \in J}(S \cap \mathfrak{o}(a_i))$.

Let j_S^* be the left adjoint of the localic embedding $j_S \colon S \hookrightarrow L$, given by $j_S^*(a) = \bigwedge\{s \in S \mid s \geq a\}$. The closed (resp. open) sublocales $\mathfrak{c}_S(a)$ (resp. $\mathfrak{o}_S(a)$) of S ($a \in S$) are precisely the intersections $\mathfrak{c}(a) \cap S$ (resp. $\mathfrak{o}(a) \cap S$) and we have, for any $a \in L$,

$$\mathfrak{c}(a) \cap S = \mathfrak{c}_S(j_S^*(a)) \quad \text{and} \quad \mathfrak{o}(a) \cap S = \mathfrak{o}_S(j_S^*(a)). \tag{1.2.4}$$

The *closure* $\overline{S}$ of a sublocale S is the smallest closed sublocale containing S, and the *interior* S° is the largest open sublocale contained in S. There is a particularly simple formula for the closure:

$$\overline{S} = \mathfrak{c}(\bigwedge S). \tag{1.2.5}$$

Hence $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and, consequently, $\mathfrak{c}(a)^{\circ} = \mathfrak{o}(a^*)$. More generally, for every sublocale S, $\overline{S^{\#}} = (S^{\circ})^{\#}$ [11, Section 4].

1.3. Open and closed localic maps. A localic map $f\colon L \to M$ is said to be *open* (resp. *closed*) if the image $f[S]$ of each open (resp. closed) sublocale $S \subseteq L$ is open (resp. closed). By the celebrated Joyal-Tierney Theorem, f is open iff its left adjoint f^* is a complete Heyting homomorphism. In particular, f^* has a left adjoint ϕ and $f[\mathfrak{o}(a)]$ is precisely the open $\mathfrak{o}(\phi(a))$ for every $a \in L$.

We will also need the following alternative characterization ([18, III.7.2.1]): a localic $f\colon L \to M$ is open iff f^* has a left adjoint ϕ such that

$$f(a \to f^*(b)) = \phi(a) \to b \quad \text{for all } a \in L \text{ and } b \in M. \tag{1.3.1}$$

In particular, for $b = 0$, $f(a^*) = \phi(a)^*$.

Closed maps are much easier to characterize: a localic $f\colon L \to M$ is closed iff $f(a \vee f^*(b)) = f(a) \vee b$ for every $a \in L$ and $b \in M$.

1.4. Continuous real-valued functions. Recall the frame of reals $\mathfrak{L}(\mathbb{R})$ from [4]. Here we define it, equivalently, as the frame presented by generators $(r,\text{—})$ and $(\text{—},r)$ for all rationals r, and relations

(r1) $(p,\text{—}) \wedge (\text{—},q) = 0$ if $q \le p$,
(r2) $(p,\text{—}) \vee (\text{—},q) = 1$ if $p < q$,
(r3) $(p,\text{—}) = \bigvee_{r>p}(r,\text{—})$,
(r4) $(\text{—},q) = \bigvee_{s<q}(\text{—},s)$,
(r5) $\bigvee_{p\in\mathbb{Q}}(p,\text{—}) = 1$,
(r6) $\bigvee_{q\in\mathbb{Q}}(\text{—},q) = 1$.

Note that $(\text{—},q)^* = (q,\text{—})$ and $(p,\text{—})^* = (\text{—},p)$. For each $p < q$ in $\mathbb{Q}$, the element $(p,\text{—}) \wedge (\text{—},q)$ in $\mathfrak{L}(\mathbb{R})$ is denoted by (p,q).

The ℓ-ring $\mathcal{R}(L)$ of *continuous real-valued functions* [4] on a frame L is the set of all frame homomorphisms $f\colon \mathfrak{L}(\mathbb{R}) \to L$. Each element of $\mathcal{R}(L)$ is uniquely determined by a map defined on the generators of $\mathfrak{L}(\mathbb{R})$ that turns relations (r1)-(r6) into identities in L.

Scales are a useful tool to define continuous real functions on a frame L. A *scale* in L is a family $(a_p)_{p\in\mathbb{Q}} \subseteq L$ such that

(S1) $p < q \;\Rightarrow\; a_q \prec a_p$.
(S2) $\bigvee_{p\in\mathbb{Q}} a_p = 1 = \bigvee_{p\in\mathbb{Q}} a_p^*$.

(If the a_p's are complemented then $a_p \prec a_p$ for any p and condition (S1) amounts only to $p < q \;\Rightarrow\; a_q \le a_p$.)

Proposition 1.4.1. ([18, XIV.5.2.2]) *Let* $(a_p)_{p\in\mathbb{Q}}$ *be a scale in* L. *Then the formulas*

$$f(p,\text{—}) = \bigvee_{r>p} a_r \quad \textit{and} \quad f(\text{—},q) = \bigvee_{s<q} a_s^*$$

define a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$.

1.5. Zero sublocales. A *cozero element* [4] of L is an element of the form $f((\text{—},0)\vee(0,\text{—}))$ for some frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$, usually denoted as $\operatorname{coz}(f)$. They form a σ-frame $\operatorname{Coz} L \subseteq L$, that is, a lattice in which all *countable* subsets have a join such that the distributivity law (1.1.1) holds for any countable S. Furthermore, $\operatorname{Coz} L$ is a *normal* σ-frame, that is, $a \vee b = 1$ $(a, b \in \operatorname{Coz} L)$ implies there exist c and d in $\operatorname{Coz} L$ such that $a \vee c = 1 = b \vee d$ and $c \wedge d = 0$ [5].

Cozero elements can be described without reference to the frame of reals as follows. An $a \in L$ is a cozero element iff $a = \bigvee_{n=1}^{\infty} a_n$ for some $a_n \prec\!\prec a$ $(n = 1, 2, \ldots)$ where the *completely below relation* $\prec\!\prec$ is the interpolative modification of the rather below relation: $b \prec\!\prec a$ if and only if there exists a subset $\{a_q \mid q \in [0,1]\cap\mathbb{Q}\} \subseteq L$ with $a_0 = b$ and $a_1 = a$ such that $a_p \prec a_q$ whenever $p < q$. Recall also that a frame L is said to be *completely regular* if $a = \bigvee\{b \in L \mid b \prec\!\prec a\}$ for every $a \in L$.

The *zero sublocales* (resp. *cozero sublocales*) are the $\mathfrak{c}(a)$ (resp. $\mathfrak{o}(a)$) with $a \in \operatorname{Coz} L$. We denote by

$$\mathsf{ZS}(L) \quad \text{and} \quad \mathsf{CoZS}(L)$$

the classes of zero and cozero sublocales, respectively[1]. The former class is a sub-σ-coframe of $\mathsf{S}(L)$ while the latter is a sub-σ-frame.

1.5.1. We know from [15, 5.6.1] that any $Z \in \mathsf{ZS}(L)$ can be written as $\bigcap_{n=1}^{\infty} \mathfrak{c}(a_n)$ where for each n there are a zero sublocale Z_n and a cozero sublocale C_n such that $Z \subseteq Z_n^\circ \subseteq Z_n \subseteq C_n \subseteq \mathfrak{c}(a_n)$ (and, moreover, this characterizes the sublocales in $\mathsf{ZS}(L)$). Similarly, a sublocale C is in $\mathsf{CoZS}(L)$ iff it can be written as $\bigvee_{n=1}^{\infty} \mathfrak{o}(a_n)$ where for each n there are a zero sublocale Z_n and a cozero sublocale C_n such that $\mathfrak{o}(a_n) \subseteq Z_n \subseteq C_n \subseteq \overline{C_n} \subseteq C$. By [18, XIV.6.2.4] and [15, 5.4.2], we may consider in the meet (resp. join) above that every a_n is in $\operatorname{Coz} L$ and $\mathfrak{c}(a_{n+1}) \subseteq \mathfrak{c}(a_n)$ (resp. $\mathfrak{o}(a_n) \subseteq \mathfrak{o}(a_{n+1})$) for

[1] In [2] the authors, following the notation of [15], assumed zero sublocales (resp. cozero sublocales) to be the $\mathfrak{o}(a)$ (resp. $\mathfrak{c}(a)$) with $a \in \operatorname{Coz} L$. The present terminology, less confusing (in agreement with the classical fact that zero subspaces are closed), has been adopted in recent papers by T. Dube *et al.*

every n. Furthermore, a sublocale S is both a zero and a cozero sublocale iff it is closed and open.

1.6. General real-valued functions. Consider the assembly frame of L, that is, the dual frame $\mathsf{S}(L)^{op}$ of the coframe of sublocales of L. By the identities in (1.2.2), the set of all closed sublocales of L form a subframe of $\mathsf{S}(L)^{op}$ isomorphic to the given L. Hence the ℓ-ring $\mathcal{R}(\mathsf{S}(L)^{op})$ is an extension of $\mathcal{R}(L)$, regarded as the ring of general real functions on L and denoted simply by $\mathsf{F}(L)$ (see [13, 15] for motivation and more information). It is partially ordered by

$$f \leq g \;\text{ iff }\; \forall r \in \mathbb{Q},\; f(\text{—}, r) \subseteq g(\text{—}, r) \;\text{ iff }\; \forall r \in \mathbb{Q},\; g(r, \text{—}) \subseteq f(r, \text{—}).$$

Note that a scale in the frame $\mathsf{S}(L)^{op}$ is a family $(S_p)_{p\in\mathbb{Q}}$ of sublocales of L satisfying

(S1) $p < q \;\Rightarrow\; S_q \prec S_p$ (i.e. $S_q^{\#} \cap S_p = \mathsf{O}$), and

(S2) $\bigcap_{p\in\mathbb{Q}} S_p = \mathsf{O} = \bigcap_{p\in\mathbb{Q}} S_p^{\#}$.

The extension $\mathsf{F}(L)$ of $\mathcal{R}(L)$ allows to deal with more general types of real functions. In particular, an $f \in \mathsf{F}(L)$ is continuous if $f(p, \text{—})$ and $f(\text{—}, q)$ are closed sublocales for every rationals p, q. The subring of all continuous members of $\mathsf{F}(L)$, denoted by $\mathsf{C}(L)$, is an isomorphic copy of $\mathcal{R}(L)$ inside $\mathsf{F}(L)$. We will work always in $\mathsf{F}(L)$ and regard $\mathcal{R}(L)$ as the subring $\mathsf{C}(L)$ of $\mathsf{F}(L)$.

For any $f \in \mathsf{C}(L)$ and $r \in \mathbb{Q}$, both $f(\text{—}, r)$ and $f(r, \text{—})$ are zero sublocales ([15, 5.3.1]).

For each $r \in \mathbb{Q}$, $(S_p^r \mid p \in \mathbb{Q})$, defined by $S_p^r = \mathsf{O}$ if $p < r$ and $S_p^r = L$ if $p \geq r$, is a scale in $\mathsf{S}(L)^{op}$. The corresponding function in $\mathsf{C}(L)$, the *constant function* $\boldsymbol{r}$, is given by

$$\boldsymbol{r}(p, \text{—}) = \begin{cases} \mathsf{O} & \text{if } p < r \\ L & \text{if } p \geq r \end{cases} \quad \text{and} \quad \boldsymbol{r}(\text{—}, q) = \begin{cases} L & \text{if } q \leq r \\ \mathsf{O} & \text{if } q > r. \end{cases}$$

The *bounded part* $\mathsf{C}^*(L)$ of $\mathsf{C}(L)$ consists of all $f \in \mathsf{C}(L)$ such that $\boldsymbol{p} \leq f \leq \boldsymbol{q}$, that is, $f(\text{—}, p) \cap f(q, \text{—}) = L$, for some pair $p < q$ in $\mathbb{Q}$.

By the isomorphism between $\mathcal{R}(L)$ and $\mathsf{C}(L)$ every zero sublocale is of the form $\mathfrak{c}(a) = f(0, \text{—}) \cap f(\text{—}, 0)$ for some $f \in \mathsf{C}(L)$ (which, furthermore, can always be considered to be bounded); so we can always assume that a zero sublocale is of the form $f(0, \text{—})$ for some continuous f satisfying $\mathbf{0} \leq f \leq \mathbf{1}$.

We will need the following result:

Proposition 1.6.1. ([3, 5.1.2]) *For any disjoint zero sublocales $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ of L, there exists an $f \in \mathsf{C}(L)$, with $\mathbf{0} \leq f \leq \mathbf{1}$, such that $\mathfrak{c}(a) = f(0, \text{—})$ and $\mathfrak{c}(b) = f(\text{—}, 1)$.*

1.7. Completely separated sublocales. Two sublocales S and T of L are said to be *completely separated in L* ([12]) if

$$S \subseteq f(0, \text{—}) \quad \text{and} \quad T \subseteq f(\text{—}, 1)$$

for some $f \in \mathsf{C}(L)$ such that $\mathbf{0} \leq f \leq \mathbf{1}$.

By 1.6.1, any pair of disjoint zero sublocales are completely separated. Consequently, two sublocales are completely separated iff they are contained in disjoint zero sublocales. Note further that if two sublocales are completely separated so are their closures. We refer to [2] and [15] for more information about completely separated sublocales. E.g., sublocales $\mathfrak{c}(a)$ and $\mathfrak{o}(b)$ are completely separated iff $b \prec\!\!\prec a$ [15, Lemma 5.4.2]. It follows from this result (and the normality of Coz L) that whenever S and T are completely separated sublocales of L there are completely separated sublocales $\mathfrak{c}(a) \in \mathsf{ZS}(L)$ and $\mathfrak{o}(b) \in \mathsf{CoZS}(L)$ such that $S \subseteq \mathfrak{c}(a)$ and $T \subseteq \mathfrak{o}(b)$.

2. Classes of localic maps defined by the behavior of their preimages on zero sublocales

2.1. Localic z-maps. Let $f : L \to M$ be a localic map and consider the image/preimage Galois adjunction

$$\begin{tikzcd} \mathsf{S}(L) \arrow[r, bend left, "{f[-]}"] \arrow[r, phantom, "\perp"] & \mathsf{S}(M) \arrow[l, bend left, "{f_{-1}[-]}"] \end{tikzcd}$$

described in the Introduction. Since frame homomorphisms preserve cozero elements, the preimage map $f_{-1} : \mathsf{S}(M) \to \mathsf{S}(L)$ restricts to maps

$$f^{z}_{-1}[-] : \mathsf{ZS}(M) \to \mathsf{ZS}(L) \quad \text{and} \quad f^{coz}_{-1}[-] : \mathsf{CoZS}(M) \to \mathsf{CoZS}(L).$$

The former is a σ-coframe homomorphism and the latter is a σ-frame homomorphism. Whenever $f^{z}_{-1}[-]$ is surjective, we say that f is a *z-map*. These maps are the right adjoints of the coz-onto frame homomorphisms of [10]. Note that when L is completely regular, a z-map is always injective because a completely regular frame is join-generated by its cozero σ-frame ([18, XIV.6.2.5]).

In the particular case where the embedding $j\colon S \hookrightarrow L$ of a sublocale S of L is a z-map, one refers to S as *z-embedded* in L ([2]). In this case, since $j_{-1}[T] = S \cap T$ we have immediately the following:

Proposition 2.1.1. *A sublocale S of L is z-embedded if and only if for each zero sublocale Z of S there is a zero sublocale T of L such that $T \cap S = Z$. Equivalently, for every cozero sublocale C of S there is a cozero sublocale T of L such that $T \cap S = C$.*

Remark 2.1.2. If $f\colon L \to M$ is a z-map, then $f[L]$ is a z-embedded sublocale of M. Indeed, consider the standard factorization (see [18, IV.1.4])

$$\begin{tikzcd} L \arrow[r, two heads, "\phi"'] \arrow[rr, bend left, "f"] & f[L] \arrow[r, hook, "j"'] & M \end{tikzcd}$$

with ϕ onto and j one-to-one. Since the correspondence $L \mapsto \mathsf{ZS}(L)$ is functorial ([4, Section 8]), we get the following commutative diagram:

$$\begin{tikzcd} \mathsf{ZS}(L) & \mathsf{ZS}(f[L]) \arrow[l, "\phi^z_{-1}[-]"] & \mathsf{ZS}(M) \arrow[l, "j^z_{-1}[-]"] \arrow[ll, bend right, "f^z_{-1}[-]"'] \end{tikzcd}$$

Then, for any $\mathfrak{c}(a) \in \mathsf{ZS}(f[L])$, $\phi^z_{-1}[\mathfrak{c}(a)] \in \mathsf{ZS}(L)$. Since f is a z-map,

$$\phi^z_{-1}[\mathfrak{c}(a)] = f^z_{-1}[\mathfrak{c}(b)] = \phi^z_{-1}[j^z_{-1}[\mathfrak{c}(b)]]$$

for some $\mathfrak{c}(b) \in \mathsf{ZS}(M)$. This means that $\mathfrak{c}(\phi^* j^*(b)) = \mathfrak{c}(\phi^*(a))$. Finally, since ϕ^* is injective (it is the left adjoint of an onto localic map), $j^*(b) = a$ and thus $\mathfrak{c}(b) \cap f[L] = j_{-1}[\mathfrak{c}(b)] = \mathfrak{c}(j^*(b)) = \mathfrak{c}(a)$.

2.2. z-Dense maps. We say that a localic map $f\colon L \to M$ is *z-dense* if

$$f^z_{-1}[Z] = \mathsf{O} \;\Rightarrow\; Z = \mathsf{O}.$$

Remarks 2.2.1. (1) Equivalently, f is z-dense if the cozero map $f^{coz}_{-1}[-]$ is a codense σ-frame homomorphism, that is, $f^{coz}_{-1}[C] = L \Rightarrow C = M$. The z-dense localic maps are the right adjoints of the coz-codense frame homomorphisms of [3].

(2) For a sublocale S of L, the embedding $j\colon S \hookrightarrow L$ is z-dense iff $Z \cap S = \mathsf{O}$ implies $Z = \mathsf{O}$ for every $Z \in \mathsf{ZS}(L)$.

2.3. Almost z-dense maps. Furthermore, we say that an f is *almost z-dense* if for every $Z \in \mathsf{ZS}(M)$ such that $f^z_{-1}[Z] = \mathsf{O}$, there exists a $Z' \in \mathsf{ZS}(M)$ such that $f^z_{-1}[Z'] = L$ and $Z \cap Z' = \mathsf{O}$.

Remarks 2.3.1. (1) Almost z-dense maps are the right adjoints of the almost coz-codense frame homomorphisms of [10, 3]. Clearly, any codense localic map is z-dense, and any z-dense localic map is almost z-dense.

(2) If f is dense and almost z-dense, then it is z-dense.

(3) For each sublocale S of L, the embedding $j\colon S \hookrightarrow L$ is almost z-dense iff for every $Z \in \mathsf{ZS}(L)$ such that $Z \cap S = \mathsf{O}$, there exists a $Z' \in \mathsf{ZS}(L)$ such that $S \subseteq Z'$ and $Z \cap Z' = \mathsf{O}$ (i.e., S is completely separated from every zero sublocale disjoint from it).

The following result characterizes almost z-dense maps. It was proved in [3, 7.2.1] for coz-codense frame homomorphisms.

Proposition 2.3.2. *A localic map $f\colon L \to M$ is almost z-dense if and only if for every $Z \in \mathsf{ZS}(M)$ such that $f_{-1}[Z] = \mathsf{O}$ there exists a bounded $g \in \mathsf{C}(M)$ such that $Z \subseteq g(0, \text{—})$ and $f_{-1}[g(\text{—}, 1)] = L$.*

Proof: The implication '$\Rightarrow$' follows from 1.6.1. The converse is clear since $g(0, \text{—})$ and $g(\text{—}, 1)$ are disjoint zero sublocales for every bounded $g \in \mathsf{C}(M)$. ∎

2.4. f-Separation. Let $f\colon L \to M$ be a localic map. We say that two sublocales S and T of M are *f-separated* whenever there exist $Z_1, Z_2 \in \mathsf{ZS}(M)$ such that

$$S \subseteq Z_1, \quad T \subseteq Z_2, \quad \text{and} \quad f^z_{-1}[Z_1] \cap f^z_{-1}[Z_2] = \mathsf{O}.$$

In particular, for a localic embedding $j\colon R \subseteq L$, the sublocales S and T are j-separated iff there exist $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that $S \subseteq Z_1$, $T \subseteq Z_2$ and $Z_1 \cap Z_2 \cap R = \mathsf{O}$. In this case we say that S and T are *R-separated.*

Remarks 2.4.1. (1) If S and T are f-separated sublocales of M and f is z-dense, then S and T are completely separated in M.

(2) Any two completely separated sublocales of M are f-separated for any $f\colon L \to M$.

2.5. More on z-maps. Next result is a characterization of z-maps that appears in [10, Prop. 3.3] in terms of coz-onto frame homomorphisms.

Proposition 2.5.1. *The following are equivalent for any localic map $f\colon L \to M$:*

(i) *f is a z-map.*
(ii) *For any $C \in \mathsf{CoZS}(L)$ and $Z \in \mathsf{ZS}(L)$ such that $C \subseteq Z$, there exist $C' \in \mathsf{CoZS}(M)$ and $Z' \in \mathsf{ZS}(M)$ with $C' \subseteq Z'$ such that $f_{-1}^{coz}[C'] = C$ and $f_{-1}^{z}[Z'] = Z$.*
(iii) *For any $C \in \mathsf{CoZS}(L)$ and $Z \in \mathsf{ZS}(L)$ such that $C \subseteq Z$, there exist $C' \in \mathsf{CoZS}(M)$ and $Z' \in \mathsf{ZS}(M)$ with $C' \subseteq Z'$ such that $C \subseteq f_{-1}^{coz}[C'] \subseteq f_{-1}^{z}[Z'] \subseteq Z$.*

Proof: (i)$\Rightarrow$(ii): Let f be a z-map. Consider $Z \in \mathsf{ZS}(L)$ and $C \in \mathsf{CoZS}(L)$ such that $C \subseteq Z$. Since f is a z-map we have

$$f_{-1}^{z}[\mathfrak{c}(a)] = Z \quad \text{and} \quad f_{-1}^{coz}[\mathfrak{o}(b)] = C$$

for some $a, b \in \mathrm{Coz}(M)$. Since $\mathfrak{c}(a)$ is a zero sublocale of L, we get from 1.5.1 that $\mathfrak{c}(a) = \bigcap_{n=1}^{\infty} \mathfrak{c}(a_n)$ for some $a_n \in \mathrm{Coz}M$ such that, for every $n \in \mathbb{N}$,

$$\mathfrak{c}(a) \subseteq \mathfrak{o}(x_n) \subseteq \mathfrak{c}(a_n) \quad \text{and} \quad \mathfrak{c}(a_{n+1}) \subseteq \mathfrak{c}(a_n) \tag{$*$}$$

for some $x_n \in \mathrm{Coz}M$. Analogously, $\mathfrak{o}(b) = \bigvee_{n=1}^{\infty} \mathfrak{o}(b_n)$ for some $b_n \in \mathrm{Coz}M$ and

$$\mathfrak{o}(b_n) \subseteq \mathfrak{c}(y_n) \subseteq \mathfrak{o}(b) \quad \text{and} \quad \mathfrak{o}(b_n) \subseteq \mathfrak{o}(b_{n+1}) \tag{$**$}$$

for some $y_n \in \mathrm{Coz}M$. Now, let

$$Z' = \bigcap_{n=1}^{\infty} (\mathfrak{c}(a_n) \vee \mathfrak{c}(y_n)) \quad \text{and} \quad C' = \bigvee_{n=1}^{\infty} (\mathfrak{o}(b_n) \cap \mathfrak{o}(x_n)).$$

Clearly, $Z' \in \mathsf{ZS}(M)$ and $C' \in \mathsf{CoZS}(M)$. Fix an $m \in \mathbb{N}$. From ($*$) and ($**$) we know that

$$\begin{aligned} &\mathfrak{o}(b_m) \cap \mathfrak{o}(x_m) \subseteq \mathfrak{o}(b_m) \subseteq \mathfrak{c}(y_n) \subseteq \mathfrak{c}(y_n) \vee \mathfrak{c}(a_n) \quad \forall n \geq m, \\ \text{and} \quad &\mathfrak{o}(b_m) \cap \mathfrak{o}(x_m) \subseteq \mathfrak{o}(x_m) \subseteq \mathfrak{c}(a_n) \subseteq \mathfrak{c}(y_n) \vee \mathfrak{c}(a_n) \quad \forall n \leq m. \end{aligned}$$

Hence, $\mathfrak{o}(b_m) \vee \mathfrak{o}(x_m) \subseteq Z'$ for every $m \in \mathbb{N}$, and $C' \subseteq Z'$. Moreover,

$$\mathfrak{c}(a) = \bigcap_{n=1}^{\infty} \mathfrak{c}(a_n) \subseteq Z' \subseteq \bigcap_{n=1}^{\infty} \mathfrak{c}(a_n) \vee \mathfrak{o}(b) = \mathfrak{c}(a) \vee \mathfrak{o}(b)$$

and $Z = f^{z}_{-1}[\mathfrak{c}(a)] \subseteq f^{z}_{-1}[Z'] \subseteq f^{z}_{-1}[\mathfrak{c}(a)] \vee f^{coz}_{-1}[\mathfrak{o}(b)] = Z \vee C = Z$. Similarly,

$$\mathfrak{o}(b) \cap \mathfrak{c}(a) = \bigvee_{n=1}^{\infty} \mathfrak{o}(b_n) \cap \mathfrak{c}(a) \subseteq C' \subseteq \bigvee_{n=1}^{\infty} \mathfrak{o}(b_n) = \mathfrak{o}(b).$$

Finally, $C = C \cap Z = f^{coz}_{-1}[\mathfrak{o}(b)] \cap f^{z}_{-1}[\mathfrak{c}(a)] \subseteq f^{coz}_{-1}[C'] \subseteq f^{coz}_{-1}[\mathfrak{o}(b)] = C$ as required.

(ii) $\Rightarrow$ (iii) is trivial.

(iii)$\Rightarrow$ (i): Let $\mathfrak{c}(a) = \bigcap_{n=1}^{\infty} \mathfrak{c}(a_n)$ be a zero sublocale with $a_n \in \mathrm{Coz}L$ such that for each natural n there is a cozero sublocale $\mathfrak{o}(x_n)$ satisfying $\mathfrak{c}(a) \subseteq \mathfrak{o}(x_n) \subseteq \mathfrak{c}(a_n)$ (recall 1.5.1). By hypothesis, there exist zero and cozero sublocales $\mathfrak{c}(b_n)$ and $\mathfrak{o}(d_n)$ in M, such that

$$\mathfrak{o}(d_n) \subseteq \mathfrak{c}(b_n) \quad \text{and} \quad \mathfrak{o}(x_n) \subseteq f_{-1}[\mathfrak{o}(d_n)] \subseteq f_{-1}[\mathfrak{c}(b_n)] \subseteq \mathfrak{c}(a_n)$$

for every n. We claim that $\mathfrak{c}(a) = f_{-1}\left[\bigcap_{n=1}^{\infty} \mathfrak{c}(b_n)\right]$. Indeed,

$$\mathfrak{c}(a) \subseteq \bigcap_{n=1}^{\infty} \mathfrak{o}(x_n) \subseteq \bigcap_{n=1}^{\infty} f_{-1}[\mathfrak{o}(d_n)] \subseteq \bigcap_{n=1}^{\infty} f_{-1}[\mathfrak{c}(b_n)] \subseteq \bigcap_{n=1}^{\infty} \mathfrak{c}(a_n) = \mathfrak{c}(a)$$

and $\bigcap_{n=1}^{\infty} f_{-1}[\mathfrak{c}(b_n)] = f_{-1}\left[\bigcap_{n=1}^{\infty} \mathfrak{c}(b_n)\right]$, where $\bigcap_{n=1}^{\infty} \mathfrak{c}(b_n)$ is clearly a zero sublocale of M. ∎

Remark 2.5.2. Conditions (ii) and (iii) above can be equivalently written as follows:

(ii)' For any disjoint pair of cozero sublocales $C_1, C_2 \in \mathsf{CoZS}(L)$, there exist disjoint $C_1', C_2' \in \mathsf{CoZS}(M)$ such that $f^{coz}_{-1}[C_1'] = C_1$ and $f^{coz}_{-1}[C_2'] = C_2$.

(ii)" For any pair of zero sublocales $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that $Z_1 \vee Z_2 = L$, there exist $Z_1', Z_2' \in \mathsf{ZS}(M)$ such that $D_1 \vee D_2 = M$, $f^{z}_{-1}[Z_1'] = Z_1$ and $f^{z}_{-1}[Z_2'] = Z_2$.

(iii)' For any disjoint pair of cozero sublocales $C_1, C_2 \in \mathsf{CoZS}(L)$, there exist disjoint $C_1', C_2' \in \mathsf{CoZS}(M)$ such that $C_1 \subseteq f^{coz}_{-1}[C_1']$ and $C_2 \subseteq f^{coz}_{-1}[C_2']$.

(iii)" For any pair of zero sublocales $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that $Z_1 \vee Z_2 = L$, there exist $Z_1', Z_2' \in \mathsf{ZS}(M)$ such that $Z_1' \vee Z_2' = M$, $f^{z}_{-1}[Z_1'] \subseteq Z_1$ and $f^{z}_{-1}[Z_2'] \subseteq Z_2$.

Proposition 2.5.1 applied to the case of a sublocale embedding $S \hookrightarrow L$ yields immediately the following corollary:

Corollary 2.5.3. *The following are equivalent for any sublocale S of L:*

(i) *S is z-embedded.*

(ii) *For any* $C \in \mathsf{CoZS}(S)$ *and* $Z \in \mathsf{ZS}(S)$ *such that* $C \subseteq Z$*, there exist* $C' \in \mathsf{CoZS}(L)$ *and* $Z' \in \mathsf{ZS}(L)$ *with* $C' \subseteq Z'$ *such that* $S \cap C' = C$ *and* $S \cap Z' = Z$*.*
(iii) *For any* $C \in \mathsf{CoZS}(S)$ *and* $Z \in \mathsf{ZS}(S)$ *such that* $C \subseteq Z$*, there exist* $C' \in \mathsf{CoZS}(L)$ *and* $Z' \in \mathsf{ZS}(L)$ *with* $C' \subseteq Z'$ *such that* $C \subseteq S \cap C' \subseteq S \cap Z' \subseteq Z$*.*
(iv) *For any pair of disjoint cozero sublocales* $C_1, C_2 \in \mathsf{CoZS}(S)$*, there exist disjoint* $C_1', C_2' \in \mathsf{CoZS}(L)$ *such that* $S \cap C_1' = C_1$ *and* $S \cap C_2' = C_2$*.*
(v) *For any pair of zero sublocales* $Z_1, Z_2 \in \mathsf{ZS}(S)$ *such that* $Z_1 \vee Z_2 = S$*, there exist* $Z_1', Z_2' \in \mathsf{ZS}(L)$ *such that* $Z_1' \vee Z_2' = L$*,* $S \cap Z_1' = Z_1$ *and* $S \cap Z_2' = Z_2$*.*

We have a further characterization of z-maps in terms of complete separation and f-separation:

Proposition 2.5.4. *The following are equivalent for any localic map* $f\colon L \to M$*:*

(i) f *is a* z*-map.*
(ii) *If* S *and* T *are completely separated sublocales of* L*, then there exists* $Z \in \mathsf{ZS}(M)$ *such that* $S \subseteq f_{-1}[Z]$ *and* $T \subseteq f_{-1}[Z^{\#}] = f_{-1}[Z]^{\#}$*.*
(iii) *If* S *and* T *are completely separated sublocales of* L*, then* $f[S]$ *and* $f[T]$ *are* f*-separated.*

Proof: (i) $\Rightarrow$ (ii): Let S and T be completely separated sublocales of L. There exists a $Z \in \mathsf{ZS}(L)$ such that $S \subseteq Z$ and $T \subseteq Z^{\#}$. Then, since f is a z-map, $Z = f_{-1}[Z']$ where $Z' \in \mathsf{ZS}(M)$.

(ii) $\Rightarrow$ (iii): It suffices to show condition (iii) for disjoint zero sublocales (recall 1.7). Consider $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that $Z_1 \cap Z_2 = \mathsf{O}$. By assumption there exists $Z \in \mathsf{ZS}(M)$ such that $Z_1 \subseteq f_{-1}[Z]$ and $Z_2 \subseteq f_{-1}[Z^{\#}]$. Then

$$Z_2 \cap f_{-1}[Z] \subseteq f_{-1}[Z^{\#}] \cap f_{-1}[Z] = f_{-1}[Z \cap Z^{\#}] = f_{-1}[\mathsf{O}_M] = \mathsf{O}_L,$$

and thus Z_2 and $f_{-1}[Z]$ are completely separated in L. We apply once again (ii) to obtain $Z_2 \subseteq f_{-1}[Z']$ and $f_{-1}[Z] \subseteq f_{-1}[Z'^{\#}]$ for some $Z' \in \mathsf{ZS}(M)$. Then

$$f_{-1}[Z'] \cap f_{-1}[Z] \subseteq f_{-1}[Z'] \cap f_{-1}[Z'^{\#}] = f_{-1}[Z' \cap Z'^{\#}] = f_{-1}[\mathsf{O}_M] = \mathsf{O}_L.$$

Finally, by the image/preimage adjunction, we get $f[Z_1] \subseteq Z$ and $f[Z_2] \subseteq Z'$. Hence, $f[Z_1]$ and $f[Z_2]$ are f-separated.

(iii) $\Rightarrow$ (i): In order to show that f is a z-map, let $Z \in \mathsf{ZS}(L)$. By 1.5.1, $Z = \bigcap_{n=1}^{\infty} \mathfrak{c}(a_n)$ where for each n there exist zero and cozero sublocales Z_n and C_n such that $Z \subseteq Z_n \subseteq C_n \subseteq \mathfrak{c}(a_n)$. In particular, Z and $\mathfrak{o}(a_n)$ are completely separated sublocales in L. Then, by assumption, there are $F'_n, F_n \in \mathsf{ZS}(M)$ such that

$$f[Z] \subseteq F_n, \quad f[\mathfrak{o}(a_n)] \subseteq F'_n \quad \text{and} \quad f_{-1}[F_n] \cap f_{-1}[F'_n] = \mathsf{O}.$$

Clearly, $\bigcap_{n=1}^{\infty} F_n \in \mathsf{ZS}(M)$ and $Z \subseteq f_{-1}\left[\bigcap_{n=1}^{\infty} F_n\right]$. For the other inclusion, since $\mathfrak{o}(a_n) \subseteq f_{-1}[F'_n]$, we have

$$f_{-1}\left[\bigcap_{n=1}^{\infty} F_n\right] = \bigcap_{n=1}^{\infty} f_{-1}[F_n] \subseteq \bigcap f_{-1}[F'_n]^{\#} \subseteq \bigcap_{n=1}^{\infty} \mathfrak{c}(a_n) = Z. \qquad \blacksquare$$

The following corollary (which is Prop. 7.3 of [2]) is the application of Proposition 2.5.4 to the case of a sublocale embedding $S \hookrightarrow L$.

Corollary 2.5.5. *The following are equivalent for a sublocale S of L:*

(i) *S is z-embedded in L.*
(ii) *If T and R are completely separated sublocales of S, then there exists $Z \in \mathsf{ZS}(L)$ such that $T \subseteq Z$ and $R \subseteq Z^{\#}$.*
(iii) *If T and R are completely separated sublocales of S, then they are S-separated.*

2.6. More on almost z-dense maps. Similarly to the extension of z-embedded sublocales to z-maps, we can generalize the notions of C- and C^*-embedded sublocales of [2, 3].

We say that a localic map $f\colon L \to M$ is a C-map (resp. C^*-map) if for every continuous (resp. bounded and continuous) real function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ there exists a continuous (resp. bounded and continuous) function $\overline{f}\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(M)^{op}$ such that the diagram

$$\begin{tikzcd} \mathfrak{L}(\mathbb{R}) \arrow[r, "f"] \arrow[dr, "\overline{f}"'] & \mathsf{S}(L)^{op} \\ & \mathsf{S}(M)^{op} \arrow[u, "f_{-1}[-]"'] \end{tikzcd}$$

commutes.

Remarks 2.6.1. (1) Every C-map is a C^*-map, and every C^*-map is a z-map.

(2) If $f\colon L \to M$ is a C-map (resp. C^*-map), then $f[L]$ is a C-embedded (resp. C^*-embedded) sublocale of L. In particular, a sublocale S of L is C-embedded (resp. C^*-embedded) iff the embedding $S \subseteq L$ is a C-map (resp. C^*-map).

Some of the results of [3, 2] can be immediately generalized to C- and C^*-maps. E.g., mimicking the proofs of Theorems 6.1 and 6.2 of [2] we get:

Proposition 2.6.2. *Every C-map is almost z-dense.*

Proposition 2.6.3. *A localic map $f\colon L \to M$ is a C-map if and only if it is an almost z-dense C^*-map.*

Proposition 2.6.4. *A localic map $f\colon L \to M$ is a C^*-map if and only if for every pair of disjoint zero sublocales $Z_1, Z_2 \in \mathsf{ZS}(L)$ there are disjoint sublocales $Z_1', Z_2' \in \mathsf{ZS}(M)$ such that $Z_1 \subseteq f_{-1}^{z}[Z_1']$ and $Z_2 \subseteq f_{-1}^{z}[Z_2']$.*

Moreover, we have:

Proposition 2.6.5. *A localic map $f\colon L \to M$ is a C-map if and only if it is an almost z-dense z-map.*

Proof: If f is a C-map it is a z-map, and it is almost z-dense by 2.6.2. Conversely, assume f is an almost z-dense z-map. To show that it is a C-map it suffices, by 2.6.3, to check that f is a C^*-map. We will do that using 2.6.4. Consider a pair of disjoint sublocales $Z_1, Z_2 \in \mathsf{ZS}(L)$. Since f is a z-map, $f[Z_1]$ and $f[Z_2]$ are f-separated (by 2.5.4), that is, there exist $Z_1', Z_2' \in \mathsf{ZS}(M)$ such that $f[Z_1] \subseteq Z_1'$, $f[Z_2] \subseteq Z_2'$ and $f_{-1}^{z}[Z_1'] \cap f_{-1}^{z}[Z_2'] = \mathsf{O}$. Then by almost z-density, there exists an $F \in \mathsf{ZS}(M)$ such that $f_{-1}^{z}[F] = L$ and $Z_1' \cap Z_2' \cap F = \mathsf{O}$. Then Z_1' and $Z_2' \cap F$ are disjoint zero sublocales of M such that $Z_1 \subseteq f_{-1}^{z}[Z_1']$ and $Z_2 \subseteq f_{-1}^{z}[Z_2' \cap F]$. ■

Theorem 2.6.6. *The following assertions are equivalent for a locale M:*

(i) *Every closed localic map with codomain M is a z-map.*
(ii) *Every closed localic map with codomain M is a C^*-map.*
(iii) *Every closed localic map with codomain M is a C-map.*

Proof: (iii)⇒(ii)⇒(i) is trivial because

$$C\text{-map} \Rightarrow C^*\text{-map} \Rightarrow z\text{-map}.$$

(i)⇒(iii): Consider a closed localic map $f\colon L \to M$. By assumption it is a z-map; we will use Prop. 2.6.5 in order to prove that it is a C-map. It suffices to show that f is almost z-dense so consider $Z \in \mathsf{ZS}(M)$ such that $f_{-1}[Z] = \mathsf{O}$. Since $f[L]$ is a closed sublocale of M we have

$$f[L] \cap Z^{\#} \overset{(*)}{=} f[L] \smallsetminus Z \overset{(**)}{=} f[L \smallsetminus f_{-1}[Z]] = f[L \smallsetminus \mathsf{O}] = f[L]$$

($(*)$ holds because Z and $f[L]$ are closed, hence complemented; in $(**)$ we use the fact that $f[-]$ is a colocalic map).

The equality above shows that $f[L] \subseteq Z^{\#}$, hence $f[L] \cap Z = \mathsf{O}$. Consider the closed sublocale $T = f[L] \vee Z$ of M. The localic embedding $j : T \hookrightarrow M$ is closed and, by hypothesis, it is a z-map. Therefore T is z-embedded in M. Note that $T \cap f[L] = (f[L] \vee Z) \cap f[L] = f[L]$ and

$$T \cap Z^{\#} = (f[L] \vee Z) \cap Z^{\#} = f[L] \cap Z^{\#} = f[L].$$

This means that $f[L]$ is both closed and open in T. Consequently, by 1.5.1, $f[L]$ is both a zero and a cozero sublocale of T. Since T is z-embedded in M, there exists $Z' \in \mathsf{ZS}(M)$ such that $T \cap Z' = f[L]$. Then $f[L] \subseteq Z'$, that is, $L = f_{-1}[Z']$. Moreover, $Z' \cap Z = Z' \cap (Z \cap T) = (Z' \cap T) \cap Z = f[L] \cap Z = \mathsf{O}$, which shows that f is almost z-dense. ■

3. Classes of localic maps defined by the behavior of their images on zero sublocales

So far we have discussed classes of localic maps defined by conditions on the behavior of their preimages on zero and cozero sublocales. In this section, inspired by [21], we introduce similar classes of localic maps defined by conditions on the behavior of their images on zero and cozero sublocales.

Definition 3.1. Let $f : L \to M$ be a localic map. We say that f is

(a) *z-closed* if $f[Z]$ is a closed sublocale of M for every $Z \in \mathsf{ZS}(L)$;
(b) *coz-open* if $f[C]$ is an open sublocale of M for every $C \in \mathsf{CoZS}(L)$;
(c) *z-open* if $\overline{f[Z]} \subseteq f[C]^{\circ}$ for every $Z \in \mathsf{ZS}(L)$ and every $C \in \mathsf{CoZS}(L)$ such that $Z \subseteq C$;
(d) *z-preserving* (resp. *coz-preserving*) if the image of every zero (resp. cozero) sublocale is a zero (resp. cozero) sublocale.

Remarks 3.2. (1) The σ-coframe homomorphism $f^{z}_{-1}[-] : \mathsf{ZS}(M) \to \mathsf{ZS}(L)$ has a left adjoint if and only if f is z-preserving. On the other hand the σ-frame homomorphism $f^{coz}_{-1}[-] : \mathsf{CoZS}(M) \to \mathsf{CoZS}(L)$ has a right adjoint if and only if f is *coz*-preserving.

(2) Clearly, any open or *coz*-preserving map is *coz*-open. Similarly, any closed or z-preserving localic map is z-closed.

(3) If f is z-closed and *coz*-open, then it is z-open.

(4) Recall from 1.2 that $\overline{S^{\#}} = (S^{\circ})^{\#}$ for every sublocale S. Hence, $Z \subseteq C \Leftrightarrow Z \cap C^{\#} = \mathsf{O}$ and $\overline{f[Z]} \subseteq f[C]^{\circ} \Leftrightarrow \overline{f[Z]} \cap (f[C]^{\circ})^{\#} = \overline{f[Z]} \cap \overline{f[C]^{\#}} = \mathsf{O}$ for any $Z \in \mathsf{ZS}(L)$ and $C \in \mathsf{CoZS}(L)$. Therefore, f is z-open if and only if for any disjoint $Z_1, Z_2 \in \mathsf{ZS}(L)$, the sublocales $\overline{f[Z_1]}$ and $\overline{f[Z_2^{\#}]^{\#}}$ are also disjoint.

In [15, 6.3.2] it is proved that for any localic map $f\colon L \to M$, if L is completely regular and f is *coz*-open, then it is open. We have a similar result for z-open maps.

Proposition 3.3. *Let $f\colon L \to M$ be a localic map. If L is completely regular and f is z-open, then f is open.*

Proof: Let $\mathfrak{o}(a)$ be an open sublocale of L. By complete regularity, $\mathfrak{o}(a) = \bigvee_{b \prec\!\!\prec a} \mathfrak{o}(b)$. Moreover, by [15, 5.4.2], sublocales $\mathfrak{c}(a)$ and $\mathfrak{o}(b)$ are completely separated, that is, there exist $Z_b \in \mathsf{ZS}(L)$ and $C_b \in \mathsf{CoZS}(L)$ such that $\mathfrak{o}(b) \subseteq Z_b \subseteq C_b \subseteq \mathfrak{o}(a)$. Hence, by the z-openness of f,

$$f[\mathfrak{o}(b)] \subseteq f[Z_b] \subseteq \overline{f[Z_b]} \subseteq f[C_b]^{\circ} \subseteq f[C_b] \subseteq f[\mathfrak{o}(a)].$$

Finally, taking joins we obtain

$$f[\mathfrak{o}(a)] = f\left[\bigvee_{b \prec\!\!\prec a} \mathfrak{o}(b)\right] = \bigvee_{b \prec\!\!\prec a} f[\mathfrak{o}(b)] \subseteq \bigvee_{b \prec\!\!\prec a} f[C_b]^{\circ} \subseteq f[\mathfrak{o}(a)],$$

which shows that $f[\mathfrak{o}(a)]$ is a join of open sublocales of M, hence open. ∎

Summing up, we have the following diagram depicting the relations between the mentioned classes of maps

$$\begin{tikzcd}
z\text{-open} \arrow[dr, Rightarrow, "\text{CR}"] & & \\
& \text{open} \arrow[r, Rightarrow, shift left] & coz\text{-open} \arrow[l, Rightarrow, shift left, "\text{CR}"] \arrow[ull, Rightarrow, bend right, "+z\text{-closed}"'] \\
coz\text{-preserving} \arrow[ur, Rightarrow, "\text{CR}"'] \arrow[urr, Rightarrow, bend right] & &
\end{tikzcd} \tag{3.3.1}$$

(where CR indicates that we need to assume complete regularity in the domain).

Proposition 3.4. *Let $f\colon L \to M$ be a localic map. If for any completely separated sublocales S and T, $f[S]$ and $f[T^{\#}]^{\#}$ are completely separated in M, then f is z-open.*

Proof: Let Z_1 and Z_2 be disjoint zero sublocales of L. By assumption, $f[Z_1]$ and $f[Z_2^{\#}]^{\#}$ are completely separated in M. In particular, $\overline{f[Z_1]}$ and $\overline{f[Z_2^{\#}]^{\#}}$ are disjoint sublocales hence f is z-open by 3.2(4). ∎

The converse holds under complete regularity:

Theorem 3.5. *The following are equivalent for a localic map $f\colon L \to M$ with completely regular domain:*

(i) *f is z-open.*
(ii) *If S and T are completely separated sublocales of L, then $f[S]$ and $f[T^{\#}]^{\#}$ are completely separated in M.*

Proof: Let f be a z-open map. If S and T are completely separated sublocales of L, they are contained in disjoint zero sublocales Z_1 and Z_2. Clearly, in order to show that $f[S]$ and $f[T^{\#}]^{\#}$ are completely separated, it suffices to show that so are $f[Z_1]$ and $f[Z_2^{\#}]^{\#}$.

By 1.6.1, there exists a continuous $g'\colon \mathfrak{L}(\mathbb{R}) \to S(L)^{op}$, with $\mathbf{0} \le g' \le \mathbf{1}$ such that $Z_1 = g'(0,\text{—})$ and $Z_2 = g'(\text{—},1)$. From 3.3, we know that f is open, hence f^* has a left adjoint ϕ such that $f[\mathfrak{o}(a)] = \mathfrak{o}(\phi(a))$ for any $a \in L$. We have the following diagram

$$
\begin{tikzcd}
\mathfrak{L}(\mathbb{R}) \arrow[r, "g"] \arrow[dr, "g'"'] & L \arrow[r, bend left, "\phi"] \arrow[r, bend right, "f"'] \arrow[d, "\mathfrak{c}_L{[-]}"] & M \arrow[l, "f^*"] \arrow[d, "\mathfrak{c}_M{[-]}"] \\
 & S(L)^{op} \arrow[r, shift left, "f{[-]}"] & S(M)^{op} \arrow[l, shift left, "f_{-1}{[-]}"]
\end{tikzcd}
$$

where the left triangle and the square $f_{-1}[-]\mathfrak{c}_M[-] = \mathfrak{c}_L[-]f^*$ commute.

For each $r \in \mathbb{Q}$ let

$$C_r = g'(-,r)^{\#} = \mathfrak{o}_L(g(-,r)) \quad \text{and} \quad F_r = g'(r,-) = \mathfrak{c}_L(g(r,\text{—})).$$

Clearly, $C_r \in \mathsf{CoZS}(L)$ and $F_r \in \mathsf{ZS}(L)$. If $r < s$ then $g'(r,\text{—}) \vee g'(\text{—},r) = L$ and $g'(r,\text{—}) \cap g'(\text{—},s) = \mathsf{O}$. Hence, by 1.2.1, $C_r \subseteq F_r \subseteq C_s$, and, since f is

z-open,

$$\overline{f[C_r]} \subseteq \overline{f[F_r]} \subseteq f[C_s]^\circ = f[C_s] \tag{3.5.1}$$

for every $r < s$.

Consider further the following family of open sublocales of M $(r \in \mathbb{Q})$:

$$U_r = \begin{cases} \mathsf{O} & \text{if } r < 0 \\ f[C_r] & \text{if } 0 \le r \le 1 \\ M & \text{if } r > 1. \end{cases}$$

Since every U_r is complemented and $U_r \subseteq U_s$ for every $r < s$, it is easy to see that $\{U_r\}_{r\in\mathbb{Q}}$ is a scale in $S(M)^{op}$ (recall 1.6). By 1.4.1, it generates a frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to S(M)^{op}$ given by

$$\varphi(p,\text{—}) = \bigcap_{r>p} f[C_r] \quad \text{and} \quad \varphi(\text{—},q) = \bigcap_{s<q} f[C_s]^{\#}.$$

Clearly, $\varphi(\text{—},q)$ is closed for every $q \in \mathbb{Q}$ and, by (3.5.1)

$$\varphi(p,\text{—}) = \bigcap_{r'>p} f[C_{r'}] \subseteq \bigcap_{r'>p} \overline{f[C_{r'}]} \subseteq \bigcap_{r>p} f[C_r] = \varphi(p,\text{—}).$$

So $\varphi(p,\text{—})$ is closed for every $p \in \mathbb{Q}$ and φ is continuous. Notice φ is also bounded since $\varphi(\text{—},0) \cap \varphi(1,\text{—}) = M$. We claim that φ completely separates $f[Z_1]$ and $f[Z_2^{\#}]^{\#}$ in M. Indeed:

$$\begin{aligned}\varphi(\text{—},1) &= \bigcap_{s<1} f[C_s]^{\#} = \bigcap_{s<1} f[\mathfrak{o}_L(g(\text{—},s))]^{\#} = \bigcap_{s<1} \mathfrak{o}_M(\phi(g(\text{—},s)))^{\#} = \\ &= \bigcap_{s<1} \mathfrak{c}_M(\phi(g(\text{—},s))) = \mathfrak{c}_M\left(\phi\left(g\left(\bigvee_{s<1}(\text{—},s)\right)\right)\right) = \mathfrak{c}_M(\phi(g(\text{—},1))) = \\ &= \mathfrak{o}_M(\phi(g(\text{—},1)))^{\#} = f[\mathfrak{o}_L(g(\text{—},1))]^{\#} = f[g'(\text{—},1)^{\#}]^{\#} = f[Z_2^{\#}]^{\#}.\end{aligned}$$

Notice that we are using the fact that ϕ and g, being left adjoints, preserve arbitrary joins. Moreover,

$$\begin{aligned}\varphi(0,\text{—}) &= \bigcap_{0<r} f[C_r] \overset{(*)}{\supseteq} \bigcap_{0<s} \overline{f[C_s]} = \bigcap_{0<s} \overline{\mathfrak{o}_M(\phi(g(\text{—},s)))} = \bigcap_{0<s} \mathfrak{c}_M(\phi(g(\text{—},s))^*) \\ &\overset{(**)}{=} \bigcap_{0<s} \mathfrak{c}_M(f(g(\text{—},s)^*)) = \mathfrak{c}_M\left(f\left(\bigvee_{0<s} g(\text{—},s)^*\right)\right) \overset{(***)}{\supseteq} \mathfrak{c}_M(f(g(0,\text{—}))) \\ &\supseteq f[\mathfrak{c}_L(g(0,\text{—}))] = f[g'(0,\text{—})] = f[Z_1]\end{aligned}$$

where $(*)$ follows from (3.5.1), $(**)$ from (1.3.1), and $(***)$ holds since $g(\text{—},s)^* \le g(0,\text{—})$ (because $g(0,\text{—}) \vee g(\text{—},s) = 1$) for every $s > 0$. ■

We get a similar result by replacing the condition on the domain with the normality of codomain. For proving it we need to recall from [4, Prop. 5] that in normal locales, disjoint closed sublocales are always completely separated.

Proposition 3.6. *The following are equivalent for a localic map $f\colon L \to M$ with normal M:*

(i) *f is z-open.*
(ii) *If S and T are completely separated sublocales of L, then $f[S]$ and $f[T^{\#}]^{\#}$ are completely separated in M.*

Proof: Assume that f is z-open. As in 3.5, it suffices to show (ii) for Z_1 and Z_2 disjoint zero sublocales of L. Since $Z_1 \subseteq Z_2^{\#}$, we have, by hypothesis, $\overline{f[Z_1]} \subseteq f[Z_2^{\#}]^{\circ}$. Hence $\overline{f[Z_1]}$ and $(f[Z_2^{\#}]^{\circ})^{\#} = \overline{f[Z_2^{\#}]^{\#}}$ are disjoint closed sublocales and since M is normal, they are completely separated. In particular, $f[Z_1]$ and $f[Z_2^{\#}]^{\#}$ are completely separated. ■

In pointfree topology, the role of classical T_1-axiom is usually taken by the so called subfit axiom (see [19])[2]. One speaks of a *subfit locale* whenever

$$a \nleq b \;\Rightarrow\; \exists c,\; a \vee c = 1 \neq b \vee c.$$

Normality in conjunction with subfitness implies complete regularity (see [18] or [19]).

Proposition 3.7. *Let $f\colon L \to M$ be a localic map between subfit locales, with L normal. Then f is z-open if and only if it is open and closed.*

Proof: By (3.3.1), if f is open and closed then it is z-open. Conversely, let f be z-open. Since L is completely regular, we know by 3.3 that f is open. To prove that it is also closed, let $\mathfrak{c}_L(a) \subseteq L$ and let $\mathfrak{c}_M(b) = \overline{f[\mathfrak{c}_L(a)]}$. It suffices to show $\mathfrak{c}_M(b) \subseteq f[\mathfrak{c}_L(a)]$. We will proceed by contradiction.

If $\mathfrak{c}_M(b) \nsubseteq f[\mathfrak{c}_L(a)]$, then, by (1.2.1), $f[\mathfrak{c}_L(a)] \vee \mathfrak{o}_M(b) \neq M$, and since M is subfit there would exist some $\mathfrak{c}_M(d) \neq \mathsf{O}_M$ in M such that $(f[\mathfrak{c}_L(a)] \vee \mathfrak{o}_M(b)) \cap \mathfrak{c}_M(d) = \mathsf{O}_M$ (see [18, V.1.4]). Then, $f[\mathfrak{c}_L(a)] \subseteq f[\mathfrak{c}_L(a)] \vee \mathfrak{o}_M(b) \subseteq \mathfrak{o}_M(d)$ and, consequently, $\mathfrak{c}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(d)] = \mathfrak{o}_L(f^*(d))$. This would mean that $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(f^*(d))$ would be disjoint closed sublocales hence completely separated (by the normality of L). It then would follow, by 3.5, the existence of $Z_1, Z_2 \in \mathsf{ZS}(M)$ such that

$$f[\mathfrak{c}_L(a)] \subseteq Z_1, \quad f[\mathfrak{o}_L(f^*(d))]^{\#} \subseteq Z_2, \quad \text{and} \quad Z_1 \cap Z_2 = \mathsf{O}_M.$$

[2] In spaces, the subfit property is in fact slightly weaker than T_1.

Indeed, $\mathfrak{c}_M(d) \subseteq f[\mathfrak{o}_L(f^*(d))]^\# \subseteq Z_2$ since

$$\mathfrak{c}_M(d) \cap f[\mathfrak{o}_L(f^*(d))] = \mathfrak{c}_M(d) \cap \mathfrak{o}_M(\phi(f^*(d))) \subseteq \mathfrak{c}_M(d) \cap \mathfrak{o}_M(d) = \mathsf{O}_M$$

(where ϕ denotes the left adjoint of f^* provided by the openness of f). Moreover, $f[\mathfrak{c}_L(a)] \subseteq \mathfrak{c}_M(b) \subseteq Z_1 \subseteq Z_2^\# \subseteq \mathfrak{o}_M(d)$. Then we would get $M = \mathfrak{o}_M(b) \vee \mathfrak{c}_M(b) \subseteq \mathfrak{o}_M(d)$, contradicting the fact that $\mathfrak{c}_M(d)$ is nonempty. ■

4. Forms of normality and z-embeddings

As is well known, normality can be characterized in terms of C-embeddings and C^*-embeddings (see [3]) as well as in terms of z-embeddings (see [2, Theorem 7.10]). Similar results hold for some weaker forms of normality as e.g. mild normality. We will present now some general results that cover and unify all such characterizations under a single proof. Besides, the setting will allow us to identify general conditions under which this kind of characterizations may hold.

In what follows $\mathfrak{s}$ denotes a function which assigns to each locale L a subset $\mathfrak{s}L$ of L and $\mathfrak{S}$ denotes the function which assigns to each L the set of closed sublocales

$$\mathfrak{S}(L) = \{\mathfrak{c}(a) \mid a \in \mathfrak{s}L\}.$$

We call $\mathfrak{S}$ a (*closed*) *selection function* and we say that the sublocales in $\mathfrak{S}(L)$ are the $\mathfrak{S}$-*closed* sublocales of L. Accordingly, we say that L is *completely separated* $\mathfrak{S}$-*normal* (briefly, *c. s.* $\mathfrak{S}$-*normal*) if every two disjoint $\mathfrak{S}$-closed sublocales of L are completely separated in L.

The standard examples for $\mathfrak{S}$ are given by selecting respectively all elements, regular elements, cozero elements, δ-elements and δ-regular elements[3]. In the following, these will be denoted as

$$\mathfrak{S}_1,\ \mathfrak{S}_{\mathrm{reg}},\ \mathfrak{S}_{\mathrm{coz}},\ \mathfrak{S}_\delta,\ \mathfrak{S}_{\delta\mathrm{reg}}$$

respectively.

Given a closed selection $\mathfrak{S}$, a locale L is called $\mathfrak{S}$-*normal* [14] whenever $a \vee b = 1$ for $a, b \in \mathfrak{s}L$ implies the existence of $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$ and $a \vee u = 1 = v \vee b$. It will be useful for the exposition to introduce the following (formally) weaker variant of this notion: we say that a locale L is *weakly* $\mathfrak{S}$-*normal* whenever $a \vee b = 1$ for $a, b \in \mathfrak{s}L$ implies the existence of

[3] An $a \in L$ is a δ-*element* [17] if $a = \bigvee\{x \in L \mid x \text{ is regular}, x \leq a\}$; it is a δ-*regular element* if $a = \bigvee_{n=1}^\infty a_n$ for some $a_n \prec a$ (we may assume that each a_n is regular since $a_n \prec a$ implies $a_n^{**} \prec a$, hence any δ-regular element is a δ-element); see [14] for more information.

$u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = v \vee b$. Clearly, any c. s. $\mathfrak{S}$-normal locale is weakly $\mathfrak{S}$-normal.

Examples 4.1. For the selection $\mathfrak{S} = \mathfrak{S}_1$, $\mathfrak{S}$-normality and weak $\mathfrak{S}$-normality are just standard normality, while for $\mathfrak{S}_{\text{reg}}$, weak $\mathfrak{S}$-normality is *mild normality* ([17]). In this case, as well as in any case where $\mathfrak{s}L$ contains all regular elements, $\mathfrak{S}$-normality coincides with weak $\mathfrak{S}$-normality because in any frame (more generally, any distributive pseudocomplemented algebra [14, Prop. 1.4]), $u \wedge v = 0$ iff $u^{**} \wedge v^{**} = 0$. This is also the case of $\mathfrak{S}_\delta$.

For $\mathfrak{S} = \mathfrak{S}_{\text{coz}}$, $\mathfrak{S}$-normality is a property satisfied by any locale.

The fact that $\prec$ interpolates in normal locales, and thus $\prec = \prec\!\!\prec$, plays an important role in the proof that a locale is normal iff every pair of disjoint closed sublocales is completely separated. Certainly, the following conditions on a locale L might also play some role if we want to obtain similar results for other variants of normality:

(I) For every $a, b \in \mathfrak{s}L$, if $a \prec b$ then there is a $c \in \mathfrak{s}L$ such that $a \prec c \prec b$.

(II) For every $a \in L$ and $b \in \mathfrak{s}L$, if $a \prec b$ then there is a $c \in \mathfrak{s}L$ such that $a \prec c \prec b$.

(wI) For every $a, b \in \mathfrak{s}L$, if $a \prec b$ then $a \prec\!\!\prec b$.

(wII) For every $a \in L$ and $b \in \mathfrak{s}L$, if $a \prec b$ then $a \prec\!\!\prec b$.

(wI') For every $a, b \in \mathfrak{s}L$, if $a \prec b$ then there is a $c \in \mathrm{Coz}L$ such that $a \prec c \prec b$.

(wII') For every $a \in L$ and $b \in \mathfrak{s}L$, if $a \prec b$ then there is a $c \in \mathrm{Coz}L$ such that $a \prec c \prec b$.

Clearly, we have:

$$
\begin{array}{ccccc}
\text{(II)} & \Longrightarrow & \text{(wII)} & \Longrightarrow & \text{(wII')} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{(I)} & \Longrightarrow & \text{(wI)} & \Longrightarrow & \text{(wI')}
\end{array}
$$

Remark 4.2. If L is a $\mathfrak{S}$-normal (resp. weakly $\mathfrak{S}$-normal) locale and satisfies (wI) (resp. (wII)) then L is c. s. $\mathfrak{S}$-normal. Indeed, if $a, b \in \mathfrak{s}L$ are such that $a \vee b = 1$ then there are $u, v \in \mathfrak{s}L$ (resp. in L) such that $u \wedge v = 1$ and $a \vee u = 1 = v \vee b$. This implies $a \vee v^* = 1$ meaning $v \prec a$. By (wI) (resp. (wII)), $v \prec\!\!\prec a$. Thus, from [15, 5.4.2], $\mathfrak{c}(a)$ is completely separated from $\mathfrak{o}(v)$. Because $\mathfrak{c}(b) \subseteq \mathfrak{o}(v)$, then $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ are also completely separated in L.

Summing up, we have

$$\begin{tikzcd} & \text{weakly } \mathfrak{S}\text{-normal} \arrow[dd, Rightarrow, shift left, "\text{under (wII)}"] \\ \mathfrak{S}\text{-normal} \arrow[ur, Rightarrow, bend left] \arrow[dr, Rightarrow, bend right, "\text{under (wI)}"] & \\ & \text{c. s. } \mathfrak{S}\text{-normal} \arrow[uu, Rightarrow, shift left] \end{tikzcd}$$

Consider now the following further conditions on a selection $\mathfrak{S}$:

(s1) If $a, b \in \mathfrak{s}L$ then $a \wedge b \in \mathfrak{s}L$.

(s2) If $a \in \mathfrak{s}L$ and $b \in \mathfrak{s}\mathfrak{c}_L(a)$, then $b \in \mathfrak{s}L$.

(s3) If $a, b \in \mathrm{Coz}L$ are such that $a \vee b = 1$, then there are $u, v \in \mathfrak{s}L$ such that $v \leq a$, $u \leq b$ and $u \vee v = 1$.

When a selection function $\mathfrak{S}$ satisfies all them we say that $\mathfrak{S}$ is an *adequate selection*. E.g. $\mathfrak{S}_1$ and $\mathfrak{S}_{\mathrm{reg}}$ are examples of adequate selections.

Proposition 4.3. *Let $\mathfrak{S}$ be a selection with property (s2). If L is a weakly $\mathfrak{S}$-normal locale, then $\mathfrak{c}_L(a)$ is weakly $\mathfrak{S}$-normal for every $a \in \mathfrak{s}L$.*

Proof: Let S and T be disjoint $\mathfrak{S}$-closed sublocales of $\mathfrak{c}_L(a)$ for some $a \in \mathfrak{s}L$. By (s2), S and T are $\mathfrak{S}$-closed sublocales of L. By assumption, there are open sublocales $\mathfrak{o}_L(x)$ and $\mathfrak{o}_L(y)$ of L such that $S \subseteq \mathfrak{o}_L(x)$ and $T \subseteq \mathfrak{o}_L(y)$. Thus, $S \subseteq \mathfrak{o}_L(x) \cap \mathfrak{c}_L(a)$ and $T \subseteq \mathfrak{o}_L(y) \cap \mathfrak{c}_L(a)$ where $\mathfrak{o}(x)_L \cap \mathfrak{c}_L(a)$ and $\mathfrak{o}(y)_L \cap \mathfrak{c}_L(a)$ are open sublocales of $\mathfrak{c}_L(a)$. ■

Proposition 4.4. *Let $\mathfrak{S}$ be a selection with properties (s2) and (s3). If L is completely separated $\mathfrak{S}$-normal, then every $\mathfrak{S}$-closed sublocale of L is C^*-embedded in L.*

Proof: Let $\mathfrak{c}_L(a)$ be a $\mathfrak{S}$-closed sublocale of L. We will use 2.6.4 (more precisely, the particular case of it for sublocale embeddings) to show that $\mathfrak{c}_L(a)$ is C^*-embedded. Let Z_1 and Z_2 be disjoint zero sublocales of $\mathfrak{c}_L(a)$. By (s3) there are disjoint $\mathfrak{S}$-closed sublocales D_1 and D_2 of $\mathfrak{c}_L(a)$ such that $Z_1 \subseteq D_1$ and $Z_2 \subseteq D_2$. Since (s2) holds, D_1 and D_2 are $\mathfrak{S}$-closed in L. Because L is c. s. $\mathfrak{S}$-normal, then D_1 and D_2 are completely separated in L, and so are Z_1 and Z_2. ■

Proposition 4.5. *Let $\mathfrak{S}$ be a selection with properties (s2) and (s3). Consider the following statements for a locale L:*

(a) *For every pair of disjoint $\mathfrak{S}$-closed sublocales $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ of L there is a zero sublocale Z such that $\mathfrak{c}(a) \subseteq Z$ and $\mathfrak{c}(b) \subseteq Z^{\#}$.*

(b) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded in L.*

Then (a) $\Rightarrow$ (b).

Proof: Let $\mathfrak{c}_L(a)$ be a $\mathfrak{S}$-closed sublocale. We will use (ii) of 2.5.5 to prove that $\mathfrak{c}_L(a)$ is z-embedded. It suffices to take disjoint zero sublocales instead of general completely separated sublocales (recall 1.7). Let Z_1 and Z_2 be disjoint zero sublocales of $\mathfrak{c}_L(a)$. By (s3) there are disjoint $\mathfrak{S}$-closed sublocales D_1 and D_2 of $\mathfrak{c}(a)$ such that $Z_1 \subseteq D_1$ and $Z_2 \subseteq D_2$. Since (s2) holds, D_1 and D_2 are $\mathfrak{S}$-closed in L. By assumption, there is a zero sublocale Z of L such that $Z_1 \subseteq Z$ and $Z_2 \subseteq Z^{\#}$. ■

Corollary 4.6. *Let $\mathfrak{S}$ be a selection satisfying (s2) and (s3). If L is completely separated $\mathfrak{S}$-normal, then every $\mathfrak{S}$-closed sublocale of L is z-embedded in L.*

The following proposition gives a sufficient condition for weak $\mathfrak{S}$-normality that only requires property (s1); hence it covers also the selections $\mathfrak{S}_\delta$ and $\mathfrak{S}_{\delta\mathrm{reg}}$.

Proposition 4.7. *Let $\mathfrak{S}$ be a selection with property (s1). If L is a locale in which every $\mathfrak{S}$-closed sublocale is z-embedded, then L is weakly $\mathfrak{S}$-normal.*

Proof: Let $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ be disjoint $\mathfrak{S}$-closed sublocales of L. Consider the sublocale $M = \mathfrak{c}_L(a) \vee \mathfrak{c}_L(b) = \mathfrak{c}_L(a \wedge b)$. By (s1), M is $\mathfrak{S}$-closed in L. Then the sublocales $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are clopen in M; indeed $\mathfrak{c}_L(a) = \mathfrak{c}_L(a) \cap M$, $\mathfrak{c}_L(b) = \mathfrak{c}_L(b) \cap M$ and, since $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b) = \mathsf{O}$,

$$\mathfrak{c}_L(a) = \mathfrak{o}_L(b) \cap \mathfrak{c}_L(a) = \mathfrak{o}_L(b) \cap (\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)) = \mathfrak{o}_L(b) \cap M,$$

$$\mathfrak{c}_L(b) = \mathfrak{o}_L(a) \cap \mathfrak{c}_L(b) = \mathfrak{o}_L(a) \cap (\mathfrak{c}_L(b) \vee \mathfrak{c}_L(a)) = \mathfrak{o}_L(a) \cap M.$$

Consequently (recall 1.5.1), $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are disjoint cozero sublocales of M. By assumption, M is z-embedded so from (iv) of 2.5.3 we know that there are disjoint cozero sublocales $\mathfrak{o}_L(v)$ and $\mathfrak{o}_L(u)$ in L such that

$$\mathfrak{o}_L(u) \cap \mathfrak{o}_L(v) = \mathsf{O}, \quad \mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(u) \quad \text{and} \quad \mathfrak{c}_L(b) \subseteq \mathfrak{o}_L(v),$$

as required. ■

Proposition 4.8. *Let $\mathfrak{S}$ be a selection with properties (s2) and (s3). If L is a weakly $\mathfrak{S}$-normal locale and (wII') holds, then every $\mathfrak{S}$-closed sublocale of L is z-embedded.*

Proof: To prove that every $\mathfrak{S}$-closed sublocale is z-embedded we will show that condition (a) of 4.5 holds. Let $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ be disjoint $\mathfrak{S}$-closed sublocales. Then $a \vee b = 1$. Since L is weakly $\mathfrak{S}$-normal, there are $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$. This implies $v \prec a$. By (wII'), there is a $c \in \mathrm{Coz}L$ such that $v \prec c \prec a$. In particular, $c \leq a$, which means $\mathfrak{c}(a) \subseteq \mathfrak{c}(c)$. Furthermore, $v^* \vee c = 1$ so $v \leq v^{**} \leq c$. Hence, $1 = v \vee b \leq c \vee b$, that is, $\mathfrak{c}(b) \subseteq \mathfrak{o}(c) = \mathfrak{c}(c)^{\#}$ as required. ■

Corollary 4.9. *Let $\mathfrak{S}$ be an adequate selection. If (wII') holds on a locale L, then L is weakly $\mathfrak{S}$-normal if and only if every $\mathfrak{S}$-closed sublocale is z-embedded.*

Mimicking the proof of Proposition 4.8 we can show a similar result for $\mathfrak{S}$-normality:

Proposition 4.10. *Let $\mathfrak{S}$ be a selection with properties (s2) and (s3). If L is an $\mathfrak{S}$-normal locale and (wI') holds, then every $\mathfrak{S}$-closed sublocale is z-embedded.*

Putting together all the results above we obtain the following theorems:

Theorem 4.11. *Let $\mathfrak{S}$ be an adequate selection. Consider the following statements for a locale L:*

(a) *Any pair of disjoint $\mathfrak{S}$-closed sublocales of L are completely separated in L (i.e. L is completely separated $\mathfrak{S}$-normal).*
(b) *Every $\mathfrak{S}$-closed sublocale of L is C^*-embedded.*
(c) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded.*
(d) *L is weakly $\mathfrak{S}$-normal.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Theorem 4.12. *Let $\mathfrak{S}$ be an adequate selection. The following statements are equivalent for any locale L with property (wII):*

(i) *Any pair of disjoint $\mathfrak{S}$-closed sublocales of L are completely separated in L (i.e. L is completely separated $\mathfrak{S}$-normal).*
(ii) *Every $\mathfrak{S}$-closed sublocale of L is C^*-embedded.*
(iii) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded.*
(iv) *L is weakly $\mathfrak{S}$-normal.*

Theorem 4.13. *Let $\mathfrak{S}$ be an adequate selection. Consider the following statements for a locale L with property (wI):*

(a) *L is $\mathfrak{S}$-normal.*
(b) *Any pair of disjoint $\mathfrak{S}$-closed sublocales of L are completely separated in L (i.e. L is completely separated $\mathfrak{S}$-normal).*
(c) *Every $\mathfrak{S}$-closed sublocale of L is C^*-embedded.*
(d) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded.*
(e) *L is weakly $\mathfrak{S}$-normal.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e).

In the standard example $\mathfrak{S} = \mathfrak{S}_1$, an important fact is that $\mathrm{Coz}L \subseteq \mathfrak{s}L$. For a general $\mathfrak{S}$, we do not have necessarily $\mathrm{Coz}L \subseteq \mathfrak{s}L$, but we need cozero elements to behave "normally" with respect to $\mathfrak{s}L$ in order to get the converses. For this we consider the following condition on L:

(D) For every $a, b \in \mathrm{Coz}L$ such that $a \vee b = 1$ there are $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$.

Note that if (D) holds and L is c. s. $\mathfrak{S}$-normal , then L is $\mathfrak{S}$-normal. Hence:

Theorem 4.14. *Let $\mathfrak{S}$ be an adequate selection. The following statements are equivalent for any locale L with properties (wII) and (D):*

(i) *L is $\mathfrak{S}$-normal.*
(ii) *Any pair of disjoint $\mathfrak{S}$-closed sublocales of L are completely separated in L (i.e. L is completely separated $\mathfrak{S}$-normal).*
(iii) *Every $\mathfrak{S}$-closed sublocale of L is C^*-embedded.*
(iv) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded.*
(v) *L is weakly $\mathfrak{S}$-normal.*

Remarks 4.15. (1) If $\mathrm{Coz}L \subseteq \mathfrak{s}L$, then clearly (D) holds. Furthermore, in this case we can add one more equivalent statement to 4.14, namely:

Every $\mathfrak{S}$-closed sublocale of L is C-embedded.

Indeed, let $\mathfrak{c}(a)$ be a $\mathfrak{S}$-closed sublocale. If L is c. s. $\mathfrak{S}$-normal then $\mathfrak{c}(a)$ is C^*-embedded. From [2, 6.2] it suffices to show that $\mathfrak{c}(a)$ is completely separated from every zero sublocale disjoint form it, but this is immediate since $\mathrm{Coz}L \subseteq \mathfrak{s}L$ and L is c. s. $\mathfrak{S}$-normal.

This assertion can also be added to theorems 4.11 (in between statements (a) and (b)) and 4.12 whenever $\mathrm{Coz}L \subseteq \mathfrak{s}L$.

(2) On the other hand, the property that $\mathfrak{s}L$ contains the set L^* of regular elements is equivalent to the following condition (by the property that $u \wedge v = 0$ iff $u^{**} \wedge v^{**} = 0$):

(DC) For every $a, b \in L$ such that $a \wedge b = 0$ there are $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$, $a \leq u$ and $b \leq v$.

(DC) is stronger than (D): Indeed, if (DC) holds then for every $a, b \in \mathrm{Coz}L$ with $a \wedge b = 0$ there are $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$. This together with the fact that $\mathrm{Coz}L$ is a normal σ-frame yields (D).

This means that if L is c. s. $\mathfrak{S}$-normal then it is $\mathfrak{S}$-normal. Furthermore, if L is $\mathfrak{S}$-normal then (I) holds. Indeed, if $a, b \in \mathfrak{s}L$ are such that $a \prec b$ then $a^* \vee b = 1$, and since $a^* \in L^* \subseteq \mathfrak{s}L$, there are $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$ and $a^* \vee u = 1 = b \vee v$; thus, $a \prec u$ and $u \prec b$, as required. Hence, L is c. s. $\mathfrak{S}$-normal. Together with 4.1 this means that whenever $\mathfrak{s}L$ contains all regular elements, the notions of c. s. $\mathfrak{S}$-normality, weak $\mathfrak{S}$-normality and $\mathfrak{S}$-normality are equivalent.

5. Localic n- and w-maps

In this section, we present a similar study for the localic counterpart of the continuous N-maps and WN-maps introduced by Woods in [22]. Both notions can be mimicked in the category of locales. The latter were already studied by Dube [8] in terms of frame homomorphisms, referred to as W-maps; here we call them w-maps.

Definition 5.1. A localic map $f\colon L \to M$ is

(a) a *w-map* if whenever an open sublocale $\mathfrak{o}_L(a)$ is completely separated from $f_{-1}[Z]$ where $Z \in \mathsf{ZS}(M)$, there exists an open sublocale $\mathfrak{o}_M(b)$ such that $\mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(b)]$, and $\mathfrak{o}_M(b)$ is completely separated from Z.

(b) an *n-map* if whenever an open sublocale $\mathfrak{o}_L(a)$ is completely separated from $f_{-1}[\mathfrak{c}_M(b)]$ where $b \in M$, there exists an open sublocale $\mathfrak{o}_M(d)$ such that $\mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(d)]$, and $\mathfrak{o}_M(d)$ is completely separated from $\mathfrak{c}_M(b)$.

Clearly, every n-map is a w-map.

We may unify both notions by defining the concept in terms of a selection function $\mathfrak{S}$ on locales. We can also make our results more general by extending the notion of a closed selection to an arbitrary selection $\mathfrak{S}$ where $\mathfrak{S}$ selects for each L a set $\mathfrak{S}(L)$ of sublocales of L (not necessarily closed). In the next section we will deal with examples where the sublocales of the selection may not even be complemented.

Definition 5.2. We say then that a locale L is $\mathfrak{S}$-*normally separated* if every $S \in \mathfrak{S}(L)$ is completely separated from every closed sublocale of L disjoint from it. A localic map $f\colon L\to M$ is an $\mathfrak{S}$-*map* if whenever an open sublocale $\mathfrak{o}_L(a)$ of L is completely separated from $f_{-1}[T]$ with $T\in\mathfrak{S}(M)$, there exists an open sublocale $\mathfrak{o}_M(d)$ of M such that $\mathfrak{o}_L(a)\subseteq f_{-1}[\mathfrak{o}_M(d)]$, and $\mathfrak{o}_M(d)$ is completely separated from T.

Of course, for $\mathfrak{S}=\mathfrak{S}_1$, $\mathfrak{S}$-normally separated is just normality and $\mathfrak{S}$-maps are precisely the n-maps.

The definition of a δ-*normally separated* frame was introduced in [9, 3.13]. Rephrasing it in terms of sublocales and localic maps we have that a locale L is δ-*normally separated* if every zero sublocale is completely separated from every closed sublocale disjoint from it (that is, if the embedding $\mathfrak{c}_L(a)\hookrightarrow L$ is almost z-dense for every $a\in L$). Hence, for $\mathfrak{S}=\mathfrak{S}_{\mathrm{coz}}$, $\mathfrak{S}$-normally separated is precisely δ-normally separated while $\mathfrak{S}$-maps are w-maps.

Our first result shows that in $\mathfrak{S}$-normally separated locales M (and only on them), any closed, z-closed, or proper map with codomain M is an $\mathfrak{S}$-map (recall that a *proper* localic map is a closed map that preserves directed joins [20]).

Theorem 5.3. *Let $\mathfrak{S}$ be a selection on locales and let M be a locale such that every $T\in\mathfrak{S}(M)$ is complemented. The following are equivalent:*

(i) *M is $\mathfrak{S}$-normally separated.*
(ii) *Every z-closed localic map $f\colon L\to M$ is an $\mathfrak{S}$-map.*
(iii) *Every closed localic map $f\colon L\to M$ is an $\mathfrak{S}$-map.*
(iv) *Every proper localic map $f\colon L\to M$ is an $\mathfrak{S}$-map.*

Proof: (i)⇒(ii): Let $f\colon L\to M$ be a z-closed localic map and take $\mathfrak{o}_L(a)$ and $f_{-1}[S]$ with $S\in\mathfrak{S}(M)$ such that they are completely separated in L. Then there exists a zero sublocale $\mathfrak{c}_L(d)$ such that $\mathfrak{o}_L(a)\subseteq\mathfrak{c}_L(d)$ and $f_{-1}[S]\cap\mathfrak{c}_L(d)=\mathsf{O}_L$ (which implies that $\mathfrak{c}_L(d)\subseteq f_{-1}[S]^{\#}$). Taking images we obtain

$$f[\mathfrak{o}_L(a)]\subseteq f[\mathfrak{c}_L(d)]\subseteq f[f_{-1}[S]^{\#}]=f[f_{-1}[S^{\#}]]\subseteq S^{\#}.$$

Note that the equality above holds because preimages preserve complements, and that the last inclusion holds due to the adjunction between image and preimage. Hence, since S is complemented, $f[\mathfrak{c}_L(d)]\cap S=\mathsf{O}$ and $f[\mathfrak{c}_L(d)]$ is closed because f is z-closed. Then, since M is $\mathfrak{S}$-normally separated, $f[\mathfrak{c}_L(d)]$ and S are completely separated in M and, using the fact that $\mathrm{Coz}M$ is a normal σ-frame, there is a cozero sublocale $\mathfrak{o}_M(x)$, completely separated from S, such that $f[\mathfrak{c}_L(d)]\subseteq\mathfrak{o}_M(x)$ (recall 1.7). In particular, $f[\mathfrak{o}_L(a)]\subseteq\mathfrak{o}_M(x)$. Hence $\mathfrak{o}_L(a)\subseteq f_{-1}[\mathfrak{o}_M(x)]$, as required.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial since every closed map is z-closed and every proper map is closed.

(iv)$\Rightarrow$(i): Let $\mathfrak{c}_M(a)$ and $S \in \mathfrak{S}(M)$ be disjoint sublocales of M. To prove that M is $\mathfrak{S}$-normally separated we will show they are completely separated in M. Consider the embedding $j\colon \mathfrak{c}_M(a) \hookrightarrow M$. Since j is a proper map, by assumption, it is an $\mathfrak{S}$-map. Consider $\mathfrak{c}_M(a)$ (which is open in $\mathfrak{c}_M(a)$) and $j_{-1}[S]$; they are completely separated in $\mathfrak{c}_M(a)$ because $j_{-1}[S] = S \cap \mathfrak{c}_M(a) = \mathsf{O}_{\mathfrak{c}_M(a)}$. Thus, there exists $\mathfrak{o}_M(d)$ such that $\mathfrak{c}_M(a) \subseteq j_{-1}[\mathfrak{o}_M(d)]$ and $\mathfrak{o}_M(d)$ is completely separated from S in M. In particular, since $\mathfrak{c}_M(a) = j[\mathfrak{c}_M(a)] \subseteq \mathfrak{o}_M(d)$, $\mathfrak{c}_M(a)$ is completely separated from S in M. ■

Hence, we may conclude when M is a normal locale that any closed, z-closed or proper map with codomain M is an n-map:

Corollary 5.4. *The following are equivalent for a locale M:*

(i) *M is normal.*
(ii) *Every z-closed localic map $f\colon L \to M$ is an n-map.*
(iii) *Every closed localic map $f\colon L \to M$ is an n-map.*
(iv) *Every proper localic map $f\colon L \to M$ is an n-map.*

On the other hand, for the selection $\mathfrak{S} = \mathfrak{S}_{\mathrm{coz}}$ we get:

Corollary 5.5. *The following are equivalent for a locale M:*

(i) *M is δ-normally separated.*
(ii) *Every z-closed localic map $f\colon L \to M$ is a w-map.*
(iii) *Every closed localic map $f\colon L \to M$ is a w-map.*
(iv) *Every proper localic map $f\colon L \to M$ is a w-map.*

Examples 5.6. We briefly describe now another type of examples of w- and n-maps, inspired by an example in classical topology from [23, Section 2]. The details of the construction and the proofs of the results presented below, rather long and technical, will appear in the author's PhD dissertation (in preparation).

We will first build a frame P_a. Consider a frame L, an $a \in L$ and the onto frame homomorphism $p_a\colon L \to \mathfrak{c}(a)$ given by $x \mapsto x \vee a$. Let $\mathbf{2}$ be the two-element frame $\{0, 1\}$: there is a unique frame homomorphism $\iota\colon \mathbf{2} \to \mathfrak{c}(a)$.

The frame P_a is given by the pullback

$$\begin{tikzcd} P_a \arrow[r, "k"] \arrow[d, "h"'] & \mathbf{2} \arrow[d, "\iota"] \\ L \arrow[r, "p_a"] & \mathfrak{c}(a) \end{tikzcd}$$

in the category of frames, of morphisms ι and p_a. Since the pullback is the equalizer of

$$L \times \mathbf{2} \xrightarrow{p_L} L \xrightarrow{p_a} \mathfrak{c}(a) \quad \text{and} \quad L \times \mathbf{2} \xrightarrow{p_2} \mathbf{2} \xrightarrow{\iota} \mathfrak{c}(a)$$

(where p_L and p_2 are the product projections), P_a is the subframe of $L \times \mathbf{2}$ given by

$$P_a = \{(x, 0) \in L \times \mathbf{2} \mid x \leq a\} \cup \{(x, 1) \in L \times \mathbf{2} \mid x \vee a = 1\},$$

and $h = p_L i$ and $k = p_2 i$ (being i the subframe inclusion $P_a \subseteq L \times \mathbf{2}$).

It can be shown that the cozero elements of P_a are precisely the $(x, 0) \in P_a$ with $x \in \mathrm{Coz}L$ and the $(x, 1) \in P_a$ with $x \in \mathrm{Coz}L$ such that $\mathfrak{c}(x)$ is completely separated from $\mathfrak{c}(a)$ in L. It is also easy to see that if $\mathfrak{c}(a)$ is completely separated from every zero sublocale disjoint from it, then

$$\mathrm{Coz}P_a = \{(x, y) \in P_a \mid x \in \mathrm{Coz}L\}.$$

Consequently, L is δ-normally separated if and only if $\mathrm{Coz}P_a = \{(x, y) \in P_a \mid x \in \mathrm{Coz}L\}$ for every $a \in L$.

Regarding separation properties, P_a is normal (resp. subfit) whenever L is normal (resp. subfit). However, the corresponding result for complete regularity is not true. Nevertheless, it can be shown that:

(1) If P_a is completely regular then $\mathfrak{c}(a)$ is completely separated in L from every closed sublocale disjoint from it. Thus, if P_a is completely regular for every $a \in L$, then L is normal (recall that a frame is normal if and only if every pair of disjoint closed sublocales in it are completely separated).

(2) If L is completely regular and $\mathfrak{c}(a)$ is completely separated from every closed sublocale of L disjoint from it, then P_a is completely regular. Hence, for a completely regular L, L is normal if and only if P_a is completely regular for every $a \in L$.

Finally, for the localic map $h_*\colon L \to P_a$ (the right adjoint of h), we have:

Proposition. (a) *h_* is always a w-map.*

(b) *If P_a is completely regular then h_* is an n-map.*

We end with one more unifying result that shows that under some assumptions on L and M, n-maps $f\colon L \to M$ are $\mathfrak{S}$-*closed*, that is, $f[S]$ is closed for every $S \in \mathfrak{S}(L)$. Of course, for $\mathfrak{S} = \mathfrak{S}_1$, $\mathfrak{S}$-closed maps are the closed localic maps; for $\mathfrak{S} = \mathfrak{S}_{\mathrm{coz}}$, $\mathfrak{S}$-closed maps are the z-closed maps.

Theorem 5.7. *Let $f\colon L \to M$ be an n-map. If L is $\mathfrak{S}$-normally separated and M is subfit, then f is $\mathfrak{S}$-closed.*

Proof: Let $S \in \mathfrak{S}(L)$ and consider $\mathfrak{c}_M(b) = \overline{f[S]}$. Clearly, $f[S] \subseteq \mathfrak{c}_M(b)$. To prove that f is $\mathfrak{S}$-closed it suffices to show that $\mathfrak{c}_M(b) \subseteq f[S]$. Suppose $\mathfrak{c}_M(b) \nsubseteq f[S]$. Then $\mathfrak{o}_M(b) \vee f[\mathfrak{c}_L(a)] \neq L$, by (1.2.1), and, by a well-known characterization of subfit locales ([18, V.1.4]), there is a closed sublocale $\mathfrak{c}_M(d) \neq \mathsf{O}_M$ such that

$$(\mathfrak{o}_M(b) \vee f[S]) \cap \mathfrak{c}_M(d) = \mathsf{O}_M. \tag{5.6.1}$$

Then $f[S] \subseteq (\mathfrak{o}_M(b) \vee f[S]) \subseteq \mathfrak{o}_M(d)$. Taking preimages we obtain

$$S \subseteq f_{-1}[f[S]] \subseteq f_{-1}[\mathfrak{o}_M(d)] = \mathfrak{o}_L(f^*(d)),$$

from which it follows that $S \cap \mathfrak{c}_L(f^*(d)) = \mathsf{O}_L$. Since L is $\mathfrak{S}$-normally separated, S and $\mathfrak{c}_L(f^*(d))$ are completely separated in L. Therefore there are $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that

$$S \subseteq Z_1, \quad \mathfrak{c}_L(f^*(d)) \subseteq Z_2 \quad \text{and} \quad Z_1 \cap Z_2 = \mathsf{O}$$

In fact, since $\mathrm{Coz}L$ is a normal σ-frame, there is a cozero sublocale $\mathfrak{o}_L(y)$ such that $S \subseteq Z_1 \subseteq \mathfrak{o}_L(y)$, and $\mathfrak{o}_L(y)$ is completely separated from Z_2 (recall 1.7). In particular, $\mathfrak{o}_L(y)$ is completely separated from $\mathfrak{c}_L(f^*(d)) = f_{-1}[\mathfrak{c}_M(d)]$. Since f is an n-map, there is $\mathfrak{o}_M(z)$ such that

$$S \subseteq Z_1 \subseteq \mathfrak{o}_L(y) \subseteq f_{-1}[\mathfrak{o}_M(z)] \tag{5.6.2}$$

and $\mathfrak{o}_M(z)$ is completely separated from $\mathfrak{c}_M(d)$ in M. Taking images in (5.6.2) we deduce that $f[S] \subseteq f[f_{-1}[\mathfrak{o}_M(z)]] \subseteq \mathfrak{o}_M(z)$. So, in fact, $f[S]$ is completely separated from $\mathfrak{c}_M(d)$ in M. In particular, $\overline{f[S]}$ is completely separated from $\mathfrak{c}_M(d)$. Thus $\mathfrak{c}_M(b) = \overline{f[S]} \subseteq \mathfrak{o}_M(d)$, and it follows from (5.6.1) that $\mathfrak{o}_M(b) \subseteq (\mathfrak{o}_M(b) \vee f[S]) \subseteq \mathfrak{o}_M(d)$. Consequently, $M = \mathfrak{c}_M(b) \vee \mathfrak{o}_M(b) \subseteq \mathfrak{o}_M(d)$, which contradicts the fact that $\mathfrak{c}_M(d) \neq \mathsf{O}_M$. Hence, $\overline{f[S]} = \mathfrak{c}_M(b) \subseteq f[S]$, as required. ∎

In particular, we have:

Corollary 5.8. *Let $f\colon L \to M$ be a localic n-map with M a subfit locale.*

(a) *If L is normal then f is closed.*

(b) *If L is δ-normally separated then f is z-closed.*

6. Localic wz-maps

In this final section, we discuss a different type of selections that is still covered by the results in the preceding section.

Recall that the *points* of a locale L are the *prime* elements, that is, the $p \in L \smallsetminus \{1\}$ such that $p = a \wedge b$ implies $p = a$ or $p = b$. We denote by $\mathsf{Pr}L$ the set of all prime elements of L. A special kind of points are the *covered prime* elements of L that satisfy the condition $p = \bigwedge S \Rightarrow p \in S$ for any $S \subseteq L$ ([6]).

For each $a \in L$, the *boolean sublocale*

$$\mathfrak{b}(a) = \{x \to a \mid x \in L\}$$

is the least sublocale containing a ([18, III.10.2]). Let $\mathfrak{S}_p$ denote the sublocale selection given by

$$\mathfrak{S}_p(L) = \{\mathfrak{b}(p) \mid p \in \mathsf{Pr}(L)\}\,.$$

Note that for any $p \in \mathsf{Pr}L$ and $x \in L$, $p = (x \vee p) \wedge (x \to p)$ and, therefore, $p = x \vee p$ or $p = x \to p$. Hence $\mathfrak{b}(p) = \{1, p\}$ (these are the *one-point sublocales* [18]).

The $\mathfrak{S}_p$-maps will be called *wz-maps*: they are the point-free counterparts of the WZ-maps of Zenor [23].

We will show now that any completely regular locale is $\mathfrak{S}_p$-normally separated. To simplify terminology, we say that a point p is *completely separated* from a sublocale T whenever sublocales $\mathfrak{b}(p)$ and T are completely separated.

Proposition 6.1. *In a locale L, every point is completely separated from every zero sublocale disjoint from it.*

Proof: Let $\mathfrak{c}(a) \in \mathsf{ZS}(L)$ such that $\mathfrak{b}(p) \cap \mathfrak{c}(a) = \mathsf{O}$. Then $\mathfrak{b}(p) \subseteq \mathfrak{o}(a)$. By 1.5.1,

$$\mathfrak{o}(a) = \bigvee_{n=1}^{\infty} \mathfrak{o}(a_n) = \bigvee_{n=1}^{\infty} \mathfrak{c}(b_n)$$

where $\mathfrak{o}(a_n) \subseteq \mathfrak{c}(b_n)$ and $b_n \in \mathrm{Coz}L$ for every $n \in \mathbb{N}$. Since $\mathfrak{b}(p) \subseteq \mathfrak{o}\left(\bigvee_{n=1}^{\infty} a_n\right)$, we have

$$p = \left(\bigvee_{n=1}^{\infty} a_n\right) \to p = \bigwedge_{n=1}^{\infty} (a_n \to p)\,.$$

As remarked above, since p is prime, $a_n \to p = 1$ or $a_n \to p = p$, so there is a $k \in \mathbb{N}$ such that $a_k \to p = p$ and, therefore, $\mathfrak{b}(p) \subseteq \mathfrak{o}(a_k) \subseteq \mathfrak{c}(b_k) \subseteq \mathfrak{o}(a)$, as required. ∎

In other words,

> *the localic embedding $\mathfrak{b}(p) \hookrightarrow L$ is almost z-dense for every $p \in \mathsf{Pr}L$.*

Corollary 6.2. *In a completely regular locale L, every point is completely separated from every closed sublocale disjoint from it. That is, any completely regular locale is $\mathfrak{S}_p$-normally separated.*[4]

Proof: Let $\mathfrak{c}(a)$ be a sublocale of L such that $\mathfrak{b}(p) \cap \mathfrak{c}(b) = \mathsf{O}$. Since L is completely regular, $\mathfrak{c}(a) = \bigcap\{Z \in \mathsf{ZS}(L) \mid \mathfrak{c}(a) \subseteq Z\}$ (see [15, 5.5]). Consequently, since $p \notin \mathfrak{c}(a)$, there is a $Z \in \mathsf{ZS}(L)$ such that $\mathfrak{c}(a) \subseteq Z$ and $Z \cap \mathfrak{b}(p) = \mathsf{O}$. By 6.1, $\mathfrak{c}(a)$ and $\mathfrak{b}(p)$ are completely separated. ∎

Remark 6.3. Hence, in any completely regular locale L, for each $p \in \mathsf{Pr}L$ and $a \in L$ such that $a \not\leq p$, there is a continuous real-valued function $f\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{0} \leq f \leq \mathbf{1}$, $\mathfrak{b}(p) \subseteq f(0,\text{—})$ and $\mathfrak{c}(a) \subseteq f(\text{—},1)$.

A one-point sublocale $\mathfrak{b}(p)$ is complemented iff p is a covered prime ([11, Prop. 10.2]). Moreover, in regular locales every prime is covered ([11, Prop. 10.3]). Hence, $\mathfrak{S}_p$ satisfies the assumptions of Theorem 5.3 whenever codomain M is regular and we have:

Corollary 6.4. *The following assertions are equivalent for a regular locale M:*

(i) *M is $\mathfrak{S}_p$-normally separated.*
(ii) *Every z-closed localic map $f\colon L \to M$ is a wz-map.*
(iii) *Every closed localic map $f\colon L \to M$ is a wz-map.*
(iv) *Every proper localic map $f\colon L \to M$ is a wz-map.*

Corollary 6.5. *Let $f\colon L \to M$ be a localic n-map with M a subfit locale. If L is $\mathfrak{S}_p$-normally separated, then f is a wz-map.*

Let us consider also the general (boolean) selection $\mathfrak{S}_{\mathfrak{b}}$ defined by

$$\mathfrak{S}_{\mathfrak{b}}(L) = \{\mathfrak{b}(x) \mid x \in L\}.$$

Proposition 6.6. *Each $\mathfrak{S}_{\mathfrak{b}}$-normally separated locale is normal and subfit.*

[4] The converse cannot hold since there are pointless locales that are not completely regular.

Proof: Let L be a $\mathfrak{S}_{\mathfrak{b}}$-normally separated locale. If $\mathfrak{c}(a) \cap \mathfrak{c}(b) = \mathsf{O}$ then $\mathfrak{b}(a) \cap \mathfrak{c}(b) \subseteq \mathfrak{c}(a) \cap \mathfrak{c}(b) = \mathsf{O}$. By assumption, $\mathfrak{b}(a)$ and $\mathfrak{c}(b)$ are completely separated. By (1.2.5),

$$\overline{\mathfrak{b}(a)} = \uparrow\!\left(\bigwedge \mathfrak{b}(a)\right) = \uparrow\!\left(\bigwedge_{x \in L} (x \to a)\right) = \uparrow\!\left(\left(\bigvee_{x \in L} x\right) \to a\right) = \uparrow(1 \to a) = \mathfrak{c}(a).$$

Hence $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ are also completely separated (recall 1.7). This means that every pair of disjoint closed sublocales are completely separated, which characterizes normality.

Regarding subfitness, consider $a, b \in L$ such that $a \not\leq b$ (equivalently, $a \to b \neq 1$). Given $d = a \to b \geq b$, consider $\mathfrak{b}(d)$ and $\mathfrak{c}(a)$. Notice that if $x \in \mathfrak{b}(d) \cap \mathfrak{c}(a)$, then $x \geq a$ and $x = (x \to d) \to d$. Hence

$$\begin{aligned} a \leq (x \to d) \to d &\Leftrightarrow (x \to d) \wedge a \leq d \\ &\Leftrightarrow (x \to d) \leq a \to d = a \to (a \to b) = a \to b = d \\ &\Leftrightarrow 1 \leq (x \to d) \to d = x. \end{aligned}$$

Thus, $\mathfrak{b}(d) \cap \mathfrak{c}(a) = \mathsf{O}$. Then, since L is $\mathfrak{S}_{\mathfrak{b}}$-normally separated, $\mathfrak{b}(d)$ and $\mathfrak{c}(a)$ are completely separated: there exist $x, y \in \mathrm{Coz}L$ such that

$$\mathfrak{b}(d) \subseteq \mathfrak{c}(x), \quad \mathfrak{c}(a) \subseteq \mathfrak{c}(y) \quad \text{and} \quad \mathfrak{c}(x) \cap \mathfrak{c}(y) = \mathsf{O}.$$

This means that $\mathfrak{c}(a) \subseteq \mathfrak{o}(x)$, that is, $a \vee x = 1$. Moreover, $x \leq d$ and $b \leq d$, hence $x \vee b \leq d \neq 1$ and L is subfit. ■

Moreover,

each normal locale is $\mathfrak{S}_p$-normally separated.[5]

Summing up, since each subfit normal locale is completely regular we have:

$$\begin{tikzcd} & & \text{normal} \arrow[dr, Rightarrow] & \\ \mathfrak{S}_{\mathfrak{b}}\text{-norm. sep.} \arrow[r, Rightarrow] & \text{normal} + \text{subfit} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \mathfrak{S}_p\text{-norm. sep.} \\ & & \text{c. regular} \arrow[ur, Rightarrow] & \end{tikzcd}$$

We end with examples of z-embedded sublocales that are also C-embedded.

[5]Again, use the fact that a locale is normal if and only if every two disjoint closed sublocales are completely separated and note that if $p \in L$ is a point and $\mathfrak{b}(p) \cap \mathfrak{c}(a) = \mathsf{O}$, then $\mathfrak{c}(p) \cap \mathfrak{c}(a) = \mathsf{O}$.

Proposition 6.7. *Let $f\colon L \to M$ be a z-closed localic map with L and M completely regular locales, and let $p \in \mathsf{Pr}M$. If $f_{-1}[\mathfrak{b}(p)]$ is z-embedded, then it is C-embedded.*

Proof: Applying 2.6.5 and 2.3.1 to the localic embedding $f_{-1}[\mathfrak{b}(p)] \hookrightarrow L$, it suffices to show that $f_{-1}[\mathfrak{b}(p)]$ is completely separated from every zero sublocale disjoint from it. So consider $\mathfrak{c}_L(a) \in \mathsf{ZS}(L)$ such that $f_{-1}[\mathfrak{b}(p)] \cap \mathfrak{c}_L(a) = \mathsf{O}_L$. By the regularity of M, $\mathfrak{b}(p)$ is complemented, hence $f_{-1}[\mathfrak{b}(p)]$ is also complemented (since preimages preserve complements). Then

$$f_{-1}[\mathfrak{b}(p)] \cap \mathfrak{c}_L(a) = \mathfrak{c}_L(a) \smallsetminus f_{-1}[\mathfrak{b}(p)]^{\#} = \mathfrak{c}_L(a) \smallsetminus f_{-1}[\mathfrak{b}(p)^{\#}]$$

because $f_{-1}[-]$ is a coframe homomorphism. Furthermore, $f[-]$ is a colocalic map hence

$$\begin{aligned}\mathsf{O}_M &= f[f_{-1}[\mathfrak{b}(p)] \cap \mathfrak{c}_L(a)] = f[\mathfrak{c}_L(a) \smallsetminus f_{-1}[\mathfrak{b}(p)^{\#}]] = f[\mathfrak{c}_L(a)] \smallsetminus \mathfrak{b}(p)^{\#} = \\ &= f[\mathfrak{c}_L(a)] \cap \mathfrak{b}(p) = \mathfrak{c}_M(f(a)) \cap \mathfrak{b}(p)\end{aligned}$$

(where the last equality follows from f being z-closed). By 6.2, $\mathfrak{b}(p)$ and $\mathfrak{c}_M(f(a))$ are completely separated in M. Then, $f_{-1}[\mathfrak{b}(p)]$ and $f_{-1}[\mathfrak{c}_M(f(a))]$ are completely separated in L. Since $\mathfrak{c}_L(a) \subseteq f_{-1}[\mathfrak{c}_M(f(a))]$, this completes the proof. ■

Acknowledgements

Financial assistance from FCT (PhD grant PD/BD/150353/2019) and from the Centre for Mathematics of the University of Coimbra (UIDB/00324/2020, funded by the Portuguese Government through FCT) is gratefully acknowledged.

The author is also grateful to her PhD supervisor Jorge Picado for the time invested in corrections and suggestions and his insightful advice.

References

[1] R. Alò and H. Shapiro, *Normal Topological Spaces*, Cambridge, 1974.

[2] A. B. Avilez and J. Picado, Continuous extensions of real functions on arbitrary sublocales and C-, C^*- and z-embeddings, *J. Pure Appl. Algebra* 225 (2021) art. n. 106702.

[3] R. N. Ball and J. Walters-Wayland, C- and C^*-quotients in pointfree topology, *Dissertationes Mathematicae* (*Rozprawy Mat.*) 412 (2002) 1-62.

[4] B. Banaschewski, *The real numbers in pointfree topology*, Textos de Matemática, vol. 12, Universidade de Coimbra, 1997.

[5] B. Banaschewski and C. Gilmour, Cozero bases of frames, *J. Pure Appl. Algebra* 157 (2001) 1–22.

[6] B. Banaschewski and A. Pultr, On covered prime elements and complete homomorphisms of frames, *Quaest. Math.* 37 (2014) 451–454.
[7] R. L. Blair, Spaces in which special sets are z-embedded, *Can. J. Math.* 28 (1976) 673–690.
[8] T. Dube, Contracting the socle in rings of continuous functions, *Rendiconti del Seminario Matematico della Universita di Padova* 123 (2010) 37–54.
[9] T. Dube and M. Matlabyana, Concerning som variants of C-embedding in pointfree topology, *Topology Appl.* 158 (2011) 2307–2321.
[10] T. Dube and J. Walters-Wayland, Coz-onto frame maps and some applications, *Appl. Categ. Structures* 15 (2007) 119-133.
[11] M. J. Ferreira, J. Picado and S. Pinto, Remainders in pointfree topology, *Topology Appl.* 245 (2018) 21-45.
[12] J. Gutiérrez García and T. Kubiak, General insertion and extension theorems for localic real functions, *J. Pure Appl. Algebra* 215 (2011) 1198–1204.
[13] J. Gutiérrez García, T. Kubiak and J. Picado, Localic real functions: a general setting, *J. Pure Appl. Algebra* 213 (2009) 1064–1074.
[14] J. Gutiérrez García and J. Picado, On the parallel between normality and extremal disconnectedness, *J. Pure Appl. Algebra* 218 (2014) 784–803.
[15] J. Gutiérrez García, J. Picado and A. Pultr, Notes on point-free real functions and sublocales, in: *Categorical Methods in Algebra and Topology*, Textos de Matemática, DMUC, vol. 46, pp. 167-200, 2014.
[16] T. Isiwata, Mappings and spaces, *Pacific J. Math.* 20 (1967) 455–480; correction 23 (1967) 630–631.
[17] M. Matlabyana, *Coz-related and other special quotients in frames*, PhD thesis, University of South Africa, Pretoria, 2012.
[18] J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in Mathematics, vol. 28, Springer, Basel, 2012.
[19] J. Picado and A. Pultr, *Separation in point-free topology*, Birkhäuser/Springer, Cham, 2021.
[20] J. J. C. Vermeulen, Proper maps of locales *J. Pure Appl. Algebra* 92 (1994) 79–107.
[21] M. D. Weir, *Hewitt-Nachbin spaces*, North-Holland Mathematics Studies, vol.17. North-Holland Publishing Co., Amsterdam-Oxford, 1975.
[22] R. G. Woods, Maps that characterize normality properties and pseudocompactness, *J. London Math. Soc.* 7 (1973) 453–461.
[23] P. Zenor, A note on Z-maps and WZ-maps, *Proc. Amer. Math. Soc.* 23 (1969) 273–275.

Ana Belén Avilez
University of Coimbra, CMUC, Department of Mathematics, 3001-501 Coimbra, Portugal
E-mail address: anabelenag@ciencias.unam.mx