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# On matrices of closed balls

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**Abstract**

Motivated by interval mathematics, the research on an arithmetic for closed balls in $\mathbb{R}^n$ is continued. In this sense, more algebraic properties of certain operations on closed balls in $\mathbb{R}^n$, some of which related either to the Hadamard product of vectors or to the 2-fold vector cross product when $n \in \{3, 7\}$, are established. Furthermore, an arithmetic for matrices of closed balls is pursued, and algebraic properties of certain operations on matrices of closed balls, some related to the mentioned operations on closed balls, are studied. In addition, metric properties for closed balls in $\mathbb{R}^n$ as well as for matrices of closed balls, convergence in particular, are also presented.

**Key words.** closed ball, ball matrix, operation, convergence, 2-fold vector cross product, Hadamard product of vectors.

**AMS subject classifications.** 15A69, 15A72, 15B99, 08A40, 65G30.

## 1 Motivation, and structure

According to Pedrycz [14], there is an interest in computing with sets of numbers, and in identifying associated algebraic structures or systems [3, 8], for error control purposes. Reference [14] contains research on interval mathematics, involved in granular computing – a computing paradigm of information processing. More research on interval mathematics, that makes

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use of closed intervals and of closed balls, is due to Gargantini and Henrici [10], Alefeld and Herzberger [1], and Petković and Petković [15]; Mayer, in [13], and Farhadsefat, Rohn and Lotfi, in [9], approached, respectively, convergence and norms of real interval matrices. Recently, Johansson [12] presented ball arithmetic as a tool for rigorous algebraic computation with real numbers, and Hoeven [11] stated his reasoned preference for balls over intervals for complex computations (e. g., inversion of a matrix) and most applications.

Motivated by interval mathematics, namely the previously cited references, an arithmetic for closed balls in $\mathbb{R}^n$ was pursued by Beites, Nicolás and Vitória in [5]. Concretely, the properties of certain operations on closed balls in $\mathbb{R}^n$, some of which related either to the Hadamard product of vectors, [4, 16] – for us called "algebraic-Hadamard way" ($\circ_{\mathcal{B},r}$), "interval arithmetic-Hadamard way" ($\circ_{\mathcal{B},c}$) – or to the 2-fold vector cross product when $n \in \{3, 7\}$, [4, 7] – for us called "algebraic-cross way" ($\times_{\mathcal{B},r}$), "interval arithmetic-cross way" ($\times_{\mathcal{B},c}$) –, were studied. In particular, known results for operations on closed balls in $\mathbb{C}$, which can be identified with $\mathbb{R}^2$, were extended to closed balls in $\mathbb{R}^n$. More recently, the properties of possible multiplications for closed balls in $\mathbb{C}^n$ were considered in [6], and certain equations involving these operations were solved.

The present work is structured as follows. In section 2, the research on an arithmetic for closed balls in $\mathbb{R}^n$ started in [5] is continued. In this sense, more algebraic properties of the operations $\times_{\mathcal{B},r}$, $\times_{\mathcal{B},c}$, $\circ_{\mathcal{B},r}$ and $\circ_{\mathcal{B},c}$ are established. In section 3, an arithmetic for ball matrices is pursued, and algebraic properties of certain operations on ball matrices, some of them related to $\times_{\mathcal{B},r}$, $\times_{\mathcal{B},c}$, $\circ_{\mathcal{B},r}$ and $\circ_{\mathcal{B},c}$, are studied. Inclusion monotonicity, which according to Alefeld and Herzberger in [1] is the foundation for many applications of interval arithmetic, is satisfied by some of the mentioned operations on closed balls, as observed in [5], and on ball matrices, as proved in the current work. Furthermore, metric properties for closed balls as well as for ball matrices, convergence in particular, are respectively presented in sections 2 and 3.

# 2 Closed balls, revisited

The present section is devoted to operations on closed balls, and their properties.

Consider the Euclidean vector space $\mathbb{R}^n$, and denote the Euclidean norm of a vector by $\|\cdot\|_2$. Let us recall some definitions on closed balls.

**Definition 2.1.** *[5] Let $c \in \mathbb{R}^n$ and let $r \in \mathbb{R}_0^+$. The closed ball in $\mathbb{R}^n$ with* *center a and radius r is*

$$\boldsymbol{a} = \langle c; r \rangle = \{x \in \mathbb{R}^n : \|x - c\|_2 \le r\}.$$

*The set of closed balls in* $\mathbb{R}^n$ *is denoted by* $\mathcal{B}$, *and by* $\mathcal{B}^+$ *or* $\mathcal{B}^0$ *if, respectively,* $r \in \mathbb{R}^+$ *or* $r = 0$.

**Definition 2.2.** *[5] Let* $\boldsymbol{a} = \langle c_1; r_1 \rangle, \boldsymbol{b} = \langle c_2; r_2 \rangle \in \mathcal{B}$. *The closed balls* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are equal* ($\boldsymbol{a} = \boldsymbol{b}$) *if there is set-theoretic equality between them, that is,* $c_1 = c_2$ *and* $r_1 = r_2$. *The closed ball* $\boldsymbol{a}$ *is contained in* $\boldsymbol{b}$ ($\boldsymbol{a} \subseteq \boldsymbol{b}$) *if set-theoretic inclusion holds.*

Let $\|\cdot\|_\infty$, $\circ$ and, with $n \in \{3, 7\}$, $\times$ respectively denote the $\infty$-norm of a vector, the Hadamard product and the 2-fold vector cross product of vectors.

**Definition 2.3.** *[5] The binary operation* $+_\mathcal{B} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, *hereinafter called addition* $+_\mathcal{B}$, *is given by*

$$\boldsymbol{a} +_\mathcal{B} \boldsymbol{b} = \langle c_1; r_1 \rangle +_\mathcal{B} \langle c_2; r_2 \rangle := \langle c_1 + c_2; r_1 + r_2 \rangle.$$

*The binary operation* $\times_{\mathcal{B},r} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, *hereinafter called multiplication* $\times_{\mathcal{B},r}$, *is given by*

$$\boldsymbol{a} \times_{\mathcal{B},r} \boldsymbol{b} = \langle c_1; r_1 \rangle \times_{\mathcal{B},r} \langle c_2; r_2 \rangle := \langle c_1 \times c_2 + r_2 c_1 + r_1 c_2; r_1 r_2 \rangle.$$

*The binary operation* $\times_{\mathcal{B},c} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, *hereinafter called multiplication* $\times_{\mathcal{B},c}$, *is given by*

$$\boldsymbol{a} \times_{\mathcal{B},c} \boldsymbol{b} = \langle c_1; r_1 \rangle \times_{\mathcal{B},c} \langle c_2; r_2 \rangle := \langle c_1 \times c_2; r_2\|c_1\|_2 + r_1\|c_2\|_2 + r_1 r_2 \rangle.$$

*The binary operation* $\circ_{\mathcal{B},r} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, *hereinafter called multiplication* $\circ_{\mathcal{B},r}$, *is given by*

$$\boldsymbol{a} \circ_{\mathcal{B},r} \boldsymbol{b} = \langle c_1; r_1 \rangle \circ_{\mathcal{B},r} \langle c_2; r_2 \rangle := \langle c_1 \circ c_2 + r_2 c_1 + r_1 c_2; r_1 r_2 \rangle.$$

*The binary operation* $\circ_{\mathcal{B},c} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, *hereinafter called multiplication* $\circ_{\mathcal{B},c}$, *is given by*

$$\boldsymbol{a} \circ_{\mathcal{B},c} \boldsymbol{b} = \langle c_1; r_1 \rangle \circ_{\mathcal{B},c} \langle c_2; r_2 \rangle := \langle c_1 \circ c_2; r_2\|c_1\|_\infty + r_1\|c_2\|_\infty + r_1 r_2 \rangle.$$

We now introduce one more operation on closed balls, a multiplication by a scalar.

**Definition 2.4.** *The binary operation* $\cdot_\mathcal{B} : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$, *hereinafter called scalar multiplication* $\cdot_\mathcal{B}$, *is given by*

$$\alpha \cdot_\mathcal{B} \boldsymbol{a} = \alpha \cdot_\mathcal{B} \langle c_1; r_1 \rangle := \langle \alpha c_1; |\alpha| r_1 \rangle.$$

Next result gives an alternative description of a closed ball obtained by scalar multiplication.

**Theorem 2.5.** *Let $\boldsymbol{a} \in \mathcal{B}$. Let $\alpha \in \mathbb{R}$. Then $\alpha \cdot_{\mathcal{B}} \boldsymbol{a} = \{\alpha x : x \in \boldsymbol{a}\}$.*

*Proof.* It is clear that the result holds when $\alpha = 0$. Assume now that $\alpha \neq 0$.

($\supseteq$) Let $\boldsymbol{a} = \langle c_1; r_1 \rangle \in \mathcal{B}$, and let $x \in \boldsymbol{a}$. Let $\alpha \in \mathbb{R}$. As $x \in \boldsymbol{a}$ then

$$\|\alpha x - \alpha c_1\|_2 = |\alpha| \|x - c_1\|_2 \leq |\alpha| r_1.$$

Thus, $\alpha x \in \alpha \cdot_{\mathcal{B}} \boldsymbol{a}$.

($\subseteq$) Let $\boldsymbol{a} = \langle c_1; r_1 \rangle \in \mathcal{B}$. Let $\alpha \in \mathbb{R}$. Let $w \in \alpha \cdot_{\mathcal{B}} \boldsymbol{a}$. Then

$$\|w - \alpha c_1\|_2 \leq |\alpha| r_1.$$

If $r_1 = 0$ then $w = \alpha c_1$ with $c_1 \in \boldsymbol{a}$. If $r_1 \neq 0$ then $w = \alpha \frac{1}{\alpha} w$, with $\frac{1}{\alpha} w \in \boldsymbol{a}$. In fact, $\left\| \frac{1}{\alpha} w - c_1 \right\|_2 = \frac{1}{|\alpha|} \|w - \alpha c_1\|_2 \leq r_1$. □

By [5, Corollary 3.2], $\mathcal{B}^0$ is the set of elements of $\mathcal{B}$ which possess reciprocal relative to $+_{\mathcal{B}}$. In this sense, we consider an alternative definition for the subtraction of elements in $\mathcal{B}$.

**Definition 2.6.** *The binary operation $-_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, hereinafter called subtraction $-_{\mathcal{B}}$, is given by*

$$\boldsymbol{a} -_{\mathcal{B}} \boldsymbol{b} = \langle c_1; r_1 \rangle -_{\mathcal{B}} \langle c_2; r_2 \rangle := \langle c_1 - c_2; |r_1 - r_2| \rangle.$$

Also for the reason stated in the paragraph previous to this definition, $(\mathcal{B}, +_{\mathcal{B}}, \cdot_{\mathcal{B}})$ is not a real linear space. However, we have all the properties required for the following algebraic structure introduced by Aseev in [2].

**Theorem 2.7.** *$(\mathcal{B}, \subseteq, +_{\mathcal{B}}, \cdot_{\mathcal{B}})$ is a real quasilinear space.*

*Proof.* Invoking [5, Corollary 2.2.], $\subseteq$ is easily seen to be a partial order relation defined on $\mathcal{B}$. Furthermore, by the same result, it is straightforward that, for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathcal{B}$ and for all $\alpha \in \mathbb{R}$, if $\boldsymbol{a} \subseteq \boldsymbol{b}$ then $\alpha \cdot_{\mathcal{B}} \boldsymbol{a} \subseteq \alpha \cdot_{\mathcal{B}} \boldsymbol{b}$, and if $\boldsymbol{a} \subseteq \boldsymbol{b}$, $\boldsymbol{c} \subseteq \boldsymbol{d}$ then $\boldsymbol{a} +_{\mathcal{B}} \boldsymbol{c} \subseteq \boldsymbol{b} +_{\mathcal{B}} \boldsymbol{d}$.

From [5, Theorem 3.1], $+_{\mathcal{B}}$ is commutative and associative, and $\boldsymbol{o} = \langle 0; 0 \rangle$ is the neutral element relative to $+_{\mathcal{B}}$. Now let $\boldsymbol{a} = \langle c_1; r_1 \rangle \in \mathcal{B}$, and let $\alpha, \beta \in \mathbb{R}$. For all $\boldsymbol{a} = \langle c_1; r_1 \rangle, \boldsymbol{b} = \langle c_2; r_2 \rangle \in \mathcal{B}$ and for all $\alpha, \beta \in \mathbb{R}$,

$$(\alpha\beta) \cdot_{\mathcal{B}} \boldsymbol{a} = \langle \alpha\beta c_1; |\alpha\beta| r_1 \rangle = \langle \alpha\beta c_1; |\alpha||\beta| r_1 \rangle = \alpha \cdot_{\mathcal{B}} (\beta \cdot_{\mathcal{B}} \boldsymbol{a}),$$

$$1 \cdot_{\mathcal{B}} \boldsymbol{a} = \boldsymbol{a},$$
$$\alpha \cdot_{\mathcal{B}} (\boldsymbol{a} + \boldsymbol{b}) = \langle \alpha(c_1 + c_2); |\alpha|(r_1 + r_2) \rangle = \alpha \cdot_{\mathcal{B}} \boldsymbol{a} + \alpha \cdot_{\mathcal{B}} \boldsymbol{b},$$
$$0 \cdot_{\mathcal{B}} \boldsymbol{a} = \boldsymbol{o}.$$

Moreover,

$$(\alpha+\beta) \cdot_{\mathcal{B}} \boldsymbol{a} = \langle (\alpha+\beta)c_1; |\alpha+\beta| r_1 \rangle, \alpha \cdot_{\mathcal{B}} \boldsymbol{a} + \beta \cdot_{\mathcal{B}} \boldsymbol{a} = \langle (\alpha+\beta)c_1; (|\alpha|+|\beta|)r_1 \rangle,$$

which are concentric balls. As $|\alpha + \beta| r_1 \leq (|\alpha| + |\beta|) r_1$, then, from [5, Corollary 2.2.],

$$(\alpha + \beta) \cdot_{\mathcal{B}} \boldsymbol{a} \subseteq \alpha \cdot_{\mathcal{B}} \boldsymbol{a} + \beta \cdot_{\mathcal{B}} \boldsymbol{a}$$

and the reverse inclusion does not hold.

□

We now consider a metric concept for closed balls, and the corresponding notation.

**Definition 2.8.** *Let* $\boldsymbol{a} = \langle c_1; r_1 \rangle \in \mathcal{B}$*. The absolute value of* $\boldsymbol{a}$ *is* $|\boldsymbol{a}| = \|c_1\|_2 + r_1$.

Next results give an alternative description for the absolute value of a closed ball.

**Lemma 2.9.** *The function* $|||\cdot||| : \mathcal{B} \to \mathbb{R}$*, defined by*

$$|||\boldsymbol{a}||| = \sup\{\|x\|_2 : x \in \boldsymbol{a}\},$$

*is a norm on* $\mathcal{B}$*.*

*Proof.* First of all, as each $\boldsymbol{a} \in \mathcal{B}$ is a bounded set of $\mathbb{R}^n$, $|||\cdot|||$ is well-defined. Now let $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{B}$, and let $\alpha \in \mathbb{R}$. For all $x \in \boldsymbol{a}$, $|||\boldsymbol{a}||| \geq \|x\|_2 \geq 0$. Also, $|||\boldsymbol{a}||| = 0$ iff, for all $x \in \boldsymbol{a}$, $\|x\|_2 = 0$ iff $\boldsymbol{a} = \boldsymbol{o}$, where $\boldsymbol{o} = \langle 0; 0 \rangle$. By Theorem 2.5, we obtain

$$|||\alpha \cdot_{\mathcal{B}} \boldsymbol{a}||| = \sup\{\|\alpha x\|_2 : x \in \boldsymbol{a}\} = |\alpha| \sup\{\|x\|_2 : x \in \boldsymbol{a}\} = |\alpha| \; |||\boldsymbol{a}|||.$$

From [5, Lemma 3.3], we get

$$\begin{aligned} |||\boldsymbol{a} + \boldsymbol{b}||| &= \sup\{\|u\|_2 : u \in \boldsymbol{a} +_{\mathcal{B}} \boldsymbol{b}\} \\ &= \sup\{\|x + y\|_2 : x \in \boldsymbol{a} \wedge y \in \boldsymbol{b}\} \\ &\leq \sup\{\|x\|_2 + \|y\|_2 : x \in \boldsymbol{a} \wedge y \in \boldsymbol{b}\} \\ &\leq \sup\{\|x\|_2 : x \in \boldsymbol{a}\} + \sup\{\|y\|_2 : y \in \boldsymbol{b}\} \\ &= |||\boldsymbol{a}||| + |||\boldsymbol{b}|||. \end{aligned}$$

□

**Theorem 2.10.** *Let* $\boldsymbol{a} \in \mathcal{B}$. *Then* $|\boldsymbol{a}| = \max\{\|x\|_2 : x \in \boldsymbol{a}\}$.

*Proof.* Let $\boldsymbol{a} = \langle c_1; r_1\rangle \in \mathcal{B}$. Let $x \in \boldsymbol{a}$. By the reverse triangular inequality, we have $\|x\|_2 - \|c_1\|_2 \leq \|x - c_1\|_2 \leq r_1$. Hence,

$$\|x\|_2 \leq \|c_1\|_2 + r_1,$$

that is, $\|c_1\|_2 + r_1$ is an upper bound of $\{\|x\|_2 : x \in \boldsymbol{a}\}$. Now suppose that, for some $y \in \mathbb{R}^n$, $\|y\|_2$ is another upper bound of $\{\|x\|_2 : x \in \boldsymbol{a}\}$, that is, for all $x \in \boldsymbol{a}$, $\|x\|_2 \leq \|y\|_2$. Consider the line that passes through $c_1$ and $y$, intersecting the border of $\boldsymbol{a}$ at a point $x$ such that $\|x\|_2 = \|c_1\|_2+r_1+\|x-y\|_2$. Once again by the reverse triangular inequality,

$$\|y\|_2 \geq \|x\|_2 - \|x - y\|_2 = \|c_1\|_2 + r_1.$$

Thus, $\|c_1\|_2 + r_1$ is the least upper bound of $\{\|x\|_2 : x \in \boldsymbol{a}\}$. Therefore,

$$\sup\{\|x\|_2 : x \in \boldsymbol{a}\} = \|c_1\|_2 + r_1.$$

Since every $\boldsymbol{a}$ is a compact set of $\mathbb{R}^n$ and $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ is continous, by the Weierstrass Theorem, the supremum in the definition of $|||\cdot|||$ is attained as the maximum of $\{\|x\|_2 : x \in \boldsymbol{a}\}$. $\square$

Properties of the absolute value of a closed ball are collected in the following results.

**Theorem 2.11.** *Let* $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{B}$. *Let* $\alpha \in \mathbb{R}$. *Then:*

$$|\boldsymbol{a}| \geq 0, |\boldsymbol{a}| = 0 \text{ iff } \boldsymbol{a} = \boldsymbol{o},\ \textit{where } \boldsymbol{o} = \langle 0; 0\rangle; \tag{1}$$

$$|\boldsymbol{a} \pm_{\mathcal{B}} \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|; \tag{2}$$

$$|\alpha \cdot_{\mathcal{B}} \boldsymbol{a}| = |\alpha||\boldsymbol{a}|; \tag{3}$$

*for* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \times_{\mathcal{B},c}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}\backslash\{\circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$,

$$|\boldsymbol{a} *_{\mathcal{B}} \boldsymbol{b}| \leq |\boldsymbol{a}||\boldsymbol{b}|. \tag{4}$$

*Proof.* Property (1) is straightforward. Now let $\boldsymbol{a} = \langle c_1; r_1\rangle, \boldsymbol{b} = \langle c_2; r_2\rangle \in \mathcal{B}$. Respectively for (2) and (3), we get

$$|\boldsymbol{a} \pm_{\mathcal{B}} \boldsymbol{b}| = \|c_1 \pm c_2\|_2 + |r_1 \pm r_2| \leq \|c_1\|_2 + \|c_2\|_2 + r_1 + r_2 = |\boldsymbol{a}| + |\boldsymbol{b}|,$$

$$|\alpha \cdot_{\mathcal{B}} \boldsymbol{a}| = \|\alpha c_1\|_2 + |\alpha|r_1 = |\alpha|(\|c_1\|_2 + r_1) = |\alpha||\boldsymbol{a}|.$$

Now consider (4). We have

$$\begin{aligned}
|\boldsymbol{a} \times_{\mathcal{B},r} \boldsymbol{b}| &= |\langle c_1 \times c_2 + r_2 c_1 + r_1 c_2; r_1 r_2\rangle| \\
&= \|c_1 \times c_2 + r_2 c_1 + r_1 c_2\|_2 + r_1 r_2 \\
&\leq \|c_1\|_2 \|c_2\|_2 + r_2 \|c_1\|_2 + r_1 \|c_2\|_2 + r_1 r_2 \\
&= (\|c_1\|_2 + r_1)(\|c_2\|_2 + r_2) \\
&= |\boldsymbol{a}||\boldsymbol{b}|,
\end{aligned}$$

$$\begin{aligned}
|\boldsymbol{a} \times_{\mathcal{B},c} \boldsymbol{b}| &= |\langle c_1 \times c_2; r_2 \|c_1\|_2 + r_1 \|c_2\|_2 + r_1 r_2\rangle| \\
&= \|c_1 \times c_2\|_2 + r_2 \|c_1\|_2 + r_1 \|c_2\|_2 + r_1 r_2 \\
&\leq \|c_1\|_2 \|c_2\|_2 + r_2 \|c_1\|_2 + r_1 \|c_2\|_2 + r_1 r_2 \\
&= |\boldsymbol{a}||\boldsymbol{b}|.
\end{aligned}$$

We also get

$$\begin{aligned}
|\boldsymbol{a} \circ_{\mathcal{B},r} \boldsymbol{b}| &= |\langle c_1 \circ c_2 + r_2 c_1 + r_1 c_2; r_1 r_2\rangle| \\
&= \|c_1 \circ c_2 + r_2 c_1 + r_1 c_2\|_2 + r_1 r_2 \\
&\leq \|c_1 \circ c_2\|_2 + r_2 \|c_1\|_2 + r_1 \|c_2\|_2 + r_1 r_2,
\end{aligned}$$

$$\begin{aligned}
|\boldsymbol{a} \circ_{\mathcal{B},c} \boldsymbol{b}| &= |\langle c_1 \circ c_2; r_2 \|c_1\|_\infty + r_1 \|c_2\|_\infty + r_1 r_2\rangle| \\
&= \|c_1 \circ c_2\|_2 + r_2 \|c_1\|_\infty + r_1 \|c_2\|_\infty + r_1 r_2 \\
&\leq \|c_1 \circ c_2\|_2 + r_2 \|c_1\|_2 + r_1 \|c_2\|_2 + r_1 r_2.
\end{aligned}$$

As, for some $c_1, c_2 \in \mathbb{R}^n$, $\|c_1 \circ c_2\|_2 \geq \|c_1\|_2 \|c_2\|_2$, then (4) does not hold for $* \in \{\circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$. □

**Corollary 2.12.** *The function* $|\cdot| : \mathcal{B} \to \mathbb{R}$*, defined by* $\boldsymbol{a} \mapsto |\boldsymbol{a}|$*, is a norm on* $\mathcal{B}$*.*

*Proof.* A consequence of Theorem 2.11 (1)-(3). □

Notice that the function $q : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$, defined by

$$(\boldsymbol{a}, \boldsymbol{b}) \mapsto q(\boldsymbol{a}, \boldsymbol{b}) = |\boldsymbol{a} -_{\mathcal{B}} \boldsymbol{b}|,$$

is the metric associated with the norm $|\cdot|$, which turns $\mathcal{B}$ into a metric space.

We now consider one more metric concept for closed balls, and the corresponding notation.

**Definition 2.13.** *Let* $\boldsymbol{a} = \langle c_1; r_1 \rangle \in \mathcal{B}$. *The width of* $\boldsymbol{a}$ *is* $d(\boldsymbol{a}) = 2r_1$.

Properties of the width of a closed ball are presented in the following results.

**Theorem 2.14.** *Let* $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{B}$. *Let* $\alpha \in \mathbb{R}$. *Then:*

$$d(\boldsymbol{a}) \geq 0; \tag{5}$$

$$d(\boldsymbol{a} \pm_{\mathcal{B}} \boldsymbol{b}) \leq d(\boldsymbol{a}) + d(\boldsymbol{b}); \tag{6}$$

$$d(\alpha \cdot_{\mathcal{B}} \boldsymbol{a}) = |\alpha| d(\boldsymbol{a}); \tag{7}$$

*for* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \times_{\mathcal{B},c}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\} \backslash \{\times_{\mathcal{B},c}, \circ_{\mathcal{B},c}\}$,

$$d(\boldsymbol{a} *_{\mathcal{B}} \boldsymbol{b}) \leq d(\boldsymbol{a}) |\boldsymbol{b}|. \tag{8}$$

*Proof.* Properties (5)-(7) are straightforward. Now let $\boldsymbol{a} = \langle c_1; r_1 \rangle, \boldsymbol{b} = \langle c_2; r_2 \rangle \in \mathcal{B}$. Concerning (8), observe that

$$d(\boldsymbol{a}) |\boldsymbol{b}| = 2r_1 (\|c_2\|_2 + r_2)$$

and

$$d(\boldsymbol{a} \times_{\mathcal{B},r} \boldsymbol{b}) = d(\boldsymbol{a} \circ_{\mathcal{B},r} \boldsymbol{b}) = 2r_1 r_2 \leq d(\boldsymbol{a}) |\boldsymbol{b}|,$$
$$d(\boldsymbol{a} \times_{\mathcal{B},c} \boldsymbol{b}) = 2(r_2 \|c_1\|_2 + r_1 \|c_2\|_2 + r_1 r_2) \geq d(\boldsymbol{a}) |\boldsymbol{b}|,$$
$$d(\boldsymbol{a} \circ_{\mathcal{B},c} \boldsymbol{b}) = 2(r_2 \|c_1\|_\infty + r_1 \|c_2\|_\infty + r_1 r_2) \leq 2r_2 \|c_1\|_2 + d(\boldsymbol{a}) |\boldsymbol{b}|.$$

□

**Corollary 2.15.** *The function* $d : \mathcal{B} \to \mathbb{R}$, *defined by* $\boldsymbol{a} \mapsto d(\boldsymbol{a})$, *is a seminorm on* $\mathcal{B}$.

*Proof.* Although we have (5)-(7) from Theorem 2.14, observe that $d(\boldsymbol{a}) = 0$ for $\boldsymbol{a} = \langle c_1; 0 \rangle \neq \boldsymbol{o}$. □

We now recall the (right) powers of a closed ball relative to $\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}$ and $\circ_{\mathcal{B},c}$, these defined except for $\times_{\mathcal{B},c}$ due to the lack of neutral element.

**Definition 2.16.** *[5] Let* $\boldsymbol{a} = \langle c_1; r_1 \rangle \in \mathcal{B}$. *Let us denote* $(1, \dots, 1)$ *by* $c_1^{\circ 0}$ *and* $c_1^{\circ(k-1)} \circ c_1$ *by* $c_1^{\circ k}$ *for* $k \in \mathbb{N}$.

*The powers of* $\boldsymbol{a}$ *relative to* $\times_{\mathcal{B},r}$ *are defined by*

$$\boldsymbol{a}^{\times_{\mathcal{B},r} 0} = \langle 0; 1 \rangle \text{ and } \boldsymbol{a}^{\times_{\mathcal{B},r} k} = \boldsymbol{a}^{\times_{\mathcal{B},r} (k-1)} \times_{\mathcal{B},r} \boldsymbol{a} \text{ for } k \in \mathbb{N}.$$

*The powers of* $\boldsymbol{a}$ *relative to* $\circ_{\mathcal{B},r}$ *are defined by*

$$\boldsymbol{a}^{\circ_{\mathcal{B},r} 0} = \langle 0; 1 \rangle \text{ and } \boldsymbol{a}^{\circ_{\mathcal{B},r} k} = \boldsymbol{a}^{\circ_{\mathcal{B},r} (k-1)} \circ_{\mathcal{B},r} \boldsymbol{a} \text{ for } k \in \mathbb{N}.$$

*The powers of* $\boldsymbol{a}$ *relative to* $\circ_{\mathcal{B},c}$ *are defined by*

$$\boldsymbol{a}^{\circ_{\mathcal{B},c} 0} = \langle (1, \dots, 1); 0 \rangle \text{ and } \boldsymbol{a}^{\circ_{\mathcal{B},c} k} = \boldsymbol{a}^{\circ_{\mathcal{B},c} (k-1)} \circ_{\mathcal{B},c} \boldsymbol{a} \text{ for } k \in \mathbb{N}.$$

Next new notion requires the powers of a closed ball relative to $\times_{\mathcal{B},r}$, $\circ_{\mathcal{B},r}$ and $\circ_{\mathcal{B},c}$.

**Definition 2.17.** *Let* $\boldsymbol{a} = \langle c_1; r_1\rangle \in \mathcal{B}$. *For each* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$, $\boldsymbol{a}$ *is* $*_{\mathcal{B}}$*-convergent (to zero) if the sequence*

$$\{\boldsymbol{a}^{*_{\mathcal{B}}k}\}_{k=0}^{\infty} = \{< c_1^{*_{\mathcal{B}}\{k\}}; r_1^{*_{\mathcal{B}}\{k\}} >\}_{k=0}^{\infty}$$

*of the powers of* $\boldsymbol{a}$ *relative to* $*_{\mathcal{B}}$ *converges to the closed ball* $\boldsymbol{o} = \langle 0; 0\rangle$ *with respect to the norm* $|\cdot|$, *that is,* $\lim_{k\to\infty} c_1^{*_{\mathcal{B}}\{k\}} = 0$ *and* $\lim_{k\to\infty} r_1^{*_{\mathcal{B}}\{k\}} = 0$. *When* $\boldsymbol{a}$ *is not* $*_{\mathcal{B}}$*-convergent,* $\boldsymbol{a}$ *is said to be* $*_{\mathcal{B}}$*-divergent.*

We finish the present section with the subsequent example, which will be useful at the end of the forthcoming section.

**Example 2.18.** *Let* $\boldsymbol{b} = \langle(\frac{1}{2}, \ldots, \frac{1}{2}); \frac{2}{3}\rangle \in \mathcal{B}$. *From [5, Theorem 3.10], for* $k \in \mathbb{N}$,

$$\boldsymbol{b}^{\times_{\mathcal{B},r}k} = \left\langle k\left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{2}, \ldots, \frac{1}{2}\right); \left(\frac{2}{3}\right)^{k}\right\rangle.$$

*As* $\lim_{k\to+\infty}\left(\frac{2}{3}\right)^{k} = 0 = \lim_{k\to+\infty}\left(k\left(\frac{2}{3}\right)^{k-1}\frac{1}{2}\right)$, $\boldsymbol{b}$ *is* $\times_{\mathcal{B},r}$*-convergent.*

# 3 Ball matrices, visited

The present section is devoted to operations on ball matrices, and their properties.

Consider the set $M_{m\times n} = M_{m\times n}(\mathbb{R})$ of $m \times n$ matrices over $\mathbb{R}$. Consider also the set $\mathcal{M}_{m\times n} = M_{m\times n}(\mathcal{B})$ of $m \times n$ matrices over $\mathcal{B}$, where each element is hereinafter called matrix of closed balls or, simply, ball matrix. The notation $\mathcal{M}$ is used when we refer to ball matrices of appropriate sizes.

**Definition 3.1.** *Let* $\boldsymbol{A} = [\boldsymbol{a_{ij}}], \boldsymbol{B} = [\boldsymbol{b_{ij}}] \in \mathcal{M}_{m\times n}$. *The ball matrices* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are equal (*$\boldsymbol{A} = \boldsymbol{B}$*) if, for all* $i \in \{1, \ldots, m\}$ *and for all* $j \in \{1, \ldots, n\}$, $\boldsymbol{a_{ij}} = \boldsymbol{b_{ij}}$. *The ball matrix* $\boldsymbol{A}$ *is contained in* $\boldsymbol{B}$ *(*$\boldsymbol{A} \subseteq \boldsymbol{B}$*) if, for all* $i \in \{1, \ldots, m\}$ *and for all* $j \in \{1, \ldots, n\}$, $\boldsymbol{a_{ij}} \subseteq \boldsymbol{b_{ij}}$.

We now introduce, through the operations previously defined on closed balls, operations on ball matrices.

**Definition 3.2.** *The binary operations* $\pm_{\mathcal{M}} : \mathcal{M}_{m\times n} \times \mathcal{M}_{m\times n} \to \mathcal{M}_{m\times n}$, *hereinafter respectively called addition* $+_{\mathcal{M}}$ *and subtraction* $-_{\mathcal{M}}$, *are given by*

$$\boldsymbol{A} \pm_{\mathcal{M}} \boldsymbol{B} = [\boldsymbol{a_{ij}}] \pm_{\mathcal{M}} [\boldsymbol{b_{ij}}] := [\boldsymbol{a_{ij}} \pm_{\mathcal{B}} \boldsymbol{b_{ij}}],\ i = 1, \ldots, m,\ j = 1, \ldots, n.$$

The following result shows that the addition $+_{\mathcal{M}}$ is inclusion monotonic.

**Theorem 3.3.** *Let $\boldsymbol{A}^{(t)}, \boldsymbol{B}^{(t)} \in \mathcal{M}_{m\times n}$, $t \in \{1, 2\}$. If $\boldsymbol{A}^{(t)} \subseteq \boldsymbol{B}^{(t)}$, $t \in \{1, 2\}$, then*

$$\boldsymbol{A}^{(1)} +_{\mathcal{M}} \boldsymbol{B}^{(1)} \subseteq \boldsymbol{A}^{(2)} +_{\mathcal{M}} \boldsymbol{B}^{(2)}.$$

*Proof.* Let $\boldsymbol{A}^{(t)} = [\boldsymbol{a}_{\boldsymbol{ij}}^{(t)}], \boldsymbol{B}^{(t)} = [\boldsymbol{b}_{\boldsymbol{ij}}^{(t)}] \in \mathcal{M}_{m\times n}$ such that $\boldsymbol{A}^{(t)} \subseteq \boldsymbol{B}^{(t)}$, $t \in \{1, 2\}$. By [5, Theorem 3.4], $+_{\mathcal{B}}$ is inclusion monotonic. Hence,

$$\boldsymbol{A}^{(1)} +_{\mathcal{M}} \boldsymbol{B}^{(1)} = [\boldsymbol{a}_{\boldsymbol{ij}}^{(1)} +_{\mathcal{B}} \boldsymbol{b}_{\boldsymbol{ij}}^{(1)}] \subseteq [\boldsymbol{a}_{\boldsymbol{ij}}^{(2)} +_{\mathcal{B}} \boldsymbol{b}_{\boldsymbol{ij}}^{(2)}] = \boldsymbol{A}^{(2)} +_{\mathcal{M}} \boldsymbol{B}^{(2)}.$$

□

We continue with some more properties of the addition $+_{\mathcal{M}}$.

**Theorem 3.4.** *Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{M}_{m\times n}$. Then*

$$\boldsymbol{A} +_{\mathcal{M}} \boldsymbol{B} = \boldsymbol{B} +_{\mathcal{M}} \boldsymbol{A}, \tag{9}$$

$$(\boldsymbol{A} +_{\mathcal{M}} \boldsymbol{B}) +_{\mathcal{M}} \boldsymbol{C} = \boldsymbol{A} +_{\mathcal{M}} (\boldsymbol{B} +_{\mathcal{M}} \boldsymbol{C}), \tag{10}$$

$$\boldsymbol{A} +_{\mathcal{M}} \boldsymbol{0} = \boldsymbol{A} = \boldsymbol{0} +_{\mathcal{M}} \boldsymbol{A},\ \text{where } \boldsymbol{0} = [\boldsymbol{o}_{ij}] \text{ with } \boldsymbol{o}_{ij} = \boldsymbol{o} = \langle 0; 0\rangle. \tag{11}$$

*Proof.* From [5, Theorem 3.1], $+_{\mathcal{B}}$ is commutative, associative and $\boldsymbol{o} = \langle 0; 0\rangle$ is the neutral element of $(\mathcal{B}, +_{\mathcal{B}})$, properties from where (9)-(11) follow in an entrywise manner. □

We now introduce one more operation on ball matrices, a multiplication by a scalar.

**Definition 3.5.** *The binary operation $\cdot_{\mathcal{M}} : \mathbb{R} \times \mathcal{M}_{m\times n} \to \mathcal{M}_{m\times n}$, hereinafter called scalar multiplication $\cdot_{\mathcal{M}}$, is given by*

$$\alpha \cdot_{\mathcal{M}} \boldsymbol{A} = \alpha \cdot_{\mathcal{M}} [\boldsymbol{a_{ij}}] := [\alpha \cdot_{\mathcal{B}} \boldsymbol{a_{ij}}],\ i = 1, \ldots, m,\ j = 1, \ldots, n.$$

Next result gives an alternative description of a ball matrix obtained by scalar multiplication.

**Theorem 3.6.** *Let $\boldsymbol{A} = [\boldsymbol{a}_{ij}] \in \mathcal{M}_{m\times n}$, and let $\alpha \in \mathbb{R}$. Then $\alpha \cdot_{\mathcal{M}} \boldsymbol{A} = [\{\alpha x : x \in \boldsymbol{a_{ij}}\}]$.*

*Proof.* A straightforward consequence of Theorem 2.5. □

Through the multiplications on closed balls, we define multiplications on ball matrices.

**Definition 3.7.** *For each* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \times_{\mathcal{B},c}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$*, the binary operation* $*_{\mathcal{M},*_{\mathcal{B}}} : \mathcal{M}_{m\times r} \times \mathcal{M}_{r\times n} \to \mathcal{M}_{m\times n}$*, hereinafter called multiplication* $*_{\mathcal{M},*_{\mathcal{B}}}$*, is given by*

$$\boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{B} = [\boldsymbol{a_{ij}}] *_{\mathcal{M},*_{\mathcal{B}}} [\boldsymbol{b_{ij}}] := \left[\sum_{k=1}^{r} \boldsymbol{a_{ik}} *_{\mathcal{B}} \boldsymbol{b_{kj}}\right], \, i = 1, \dots, m, \, j = 1, \dots, n.$$

Before presenting properties of the multiplications $*_{\mathcal{M},*_{\mathcal{B}}}$, we define matrices that depend on the neutral element $\mathbf{1}_{*_{\mathcal{B}}}$, in [5], relative to $*_{\mathcal{B}}$.

**Definition 3.8.** *Let*

$$\mathbf{1}_{*_{\mathcal{B}}} = \begin{cases} \langle 0; 1\rangle & \text{if } *_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}\} \\ \langle (1, \dots, 1); 0\rangle & \text{if } *_{\mathcal{B}} = \circ_{\mathcal{B},c} \end{cases}.$$

*For* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$*, the identity matrix of order* $t$ *relative to* $*_{\mathcal{B}}$ *is* $\boldsymbol{I}_{t,*_{\mathcal{B}}} \in \mathcal{M}_{t\times t}$ *given by*

$$(\boldsymbol{I}_{t,*_{\mathcal{B}}})_{ij} = \begin{cases} \mathbf{1}_{*_{\mathcal{B}}} & \text{if } i = j \\ \boldsymbol{o} & \text{if } i \neq j \end{cases}, \text{ where } \boldsymbol{o} = \langle 0; 0\rangle.$$

**Theorem 3.9.** *Let* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$*. Let* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{M}$*. Then*

$$(\boldsymbol{A} *_{\mathcal{M},\circ_{\mathcal{B},r}} \boldsymbol{B}) *_{\mathcal{M},\circ_{\mathcal{B},r}} \boldsymbol{C} = \boldsymbol{A} *_{\mathcal{M},\circ_{\mathcal{B},r}} (\boldsymbol{B} *_{\mathcal{M},\circ_{\mathcal{B},r}} \boldsymbol{C}); \tag{12}$$

$$\boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{I}_{n,*_{\mathcal{B}}} = \boldsymbol{A} = \boldsymbol{I}_{m,*_{\mathcal{B}}} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A} \text{ if } \boldsymbol{A} \in \mathcal{M}_{m\times n}; \tag{13}$$

*for* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}\}$,

$$\boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} (\boldsymbol{B} +_{\mathcal{M}} \boldsymbol{C}) = \boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{B} +_{\mathcal{M}} \boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{C}, \tag{14}$$

$$(\boldsymbol{B} +_{\mathcal{M}} \boldsymbol{C}) *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A} = \boldsymbol{B} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A} +_{\mathcal{M}} \boldsymbol{C} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A}; \tag{15}$$

*for* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},c}, \circ_{\mathcal{B},c}\}$,

$$\boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} (\boldsymbol{B} +_{\mathcal{M}} \boldsymbol{C}) \subseteq \boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{B} +_{\mathcal{M}} \boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{C}, \tag{16}$$

$$(\boldsymbol{B} +_{\mathcal{M}} \boldsymbol{C}) *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A} \subseteq \boldsymbol{B} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A} +_{\mathcal{M}} \boldsymbol{C} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A}. \tag{17}$$

*Proof.* Let $\boldsymbol{A} \in \mathcal{M}_{m\times n}, \boldsymbol{B} \in \mathcal{M}_{n\times p}, \boldsymbol{C} \in \mathcal{M}_{p\times q}$. As, by [5, Theorem 3.16], $\circ_{\mathcal{B},r}$ is associative, then, for (12), we get

$$\begin{aligned}
(\boldsymbol{A} *_{\mathcal{M},\circ_{\mathcal{B},r}} \boldsymbol{B}) *_{\mathcal{M},\circ_{\mathcal{B},r}} \boldsymbol{C} &= \left[\sum_{k=1}^{p}\sum_{s=1}^{n} (\boldsymbol{a_{is}} \circ_{\mathcal{B},r} \boldsymbol{b_{sk}}) \circ_{\mathcal{B},r} \boldsymbol{c_{kj}}\right] \\
&= \left[\sum_{k=1}^{p}\sum_{s=1}^{n} \boldsymbol{a_{is}} \circ_{\mathcal{B},r} (\boldsymbol{b_{sk}} \circ_{\mathcal{B},r} \boldsymbol{c_{kj}})\right] \\
&= \left[\sum_{s=1}^{n} \boldsymbol{a_{is}} \circ_{\mathcal{B},r} \left(\sum_{k=1}^{p} \boldsymbol{b_{sk}} \circ_{\mathcal{B},r} \boldsymbol{c_{kj}}\right)\right] \\
&= \boldsymbol{A} *_{\mathcal{M},\circ_{\mathcal{B},r}} (\boldsymbol{B} *_{\mathcal{M},\circ_{\mathcal{B},r}} \boldsymbol{C}).
\end{aligned}$$

Let $\boldsymbol{A} \in \mathcal{M}_{m\times n}$. From [5, Theorem 3.6, Theorem 3.16, Theorem 3.22], for $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$, $\mathbf{1}_{*_{\mathcal{B}}}$ in Definition 3.8 is the neutral element of $(\mathcal{B}, *_{\mathcal{B}})$. Then, for (13), we obtain

$$(\boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{I}_{n,*_{\mathcal{B}}})_{ij} = \sum_{k=1}^{n} \boldsymbol{a_{ik}} *_{\mathcal{B}} (\boldsymbol{I}_{n,*_{\mathcal{B}}})_{kj} = \boldsymbol{a_{ij}} *_{\mathcal{B}} (\boldsymbol{I}_{n,*_{\mathcal{B}}})_{jj} = \boldsymbol{a_{ij}} *_{\mathcal{B}} \mathbf{1}_{*_{\mathcal{B}}} = \boldsymbol{a_{ij}}.$$

An analogous reasoning provides the proof of $\boldsymbol{I}_{m,*_{\mathcal{B}}} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A} = \boldsymbol{A}$.

Now let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{M}$. By [5, Theorem 3.12, Theorem 3.21], $\times_{\mathcal{B},r}$ and $\circ_{\mathcal{B},r}$ are distributive relative to $+_{\mathcal{B}}$, and, from [5, Theorem 3.1], $+_{\mathcal{B}}$ is commutative and associative. Hence, for $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}\}$, we obtain

$$\begin{aligned}
\boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} (\boldsymbol{B} +_{\mathcal{M}} \boldsymbol{C}) &= \left[\sum_{k=1}^{r} \boldsymbol{a_{ik}} *_{\mathcal{B}} (\boldsymbol{b}_{kj} +_{\mathcal{B}} \boldsymbol{c_{kj}})\right] \\
&= \left[\sum_{k=1}^{r} \boldsymbol{a_{ik}} *_{\mathcal{B}} \boldsymbol{b}_{kj} +_{\mathcal{B}} \sum_{k=1}^{r} \boldsymbol{a_{ik}} *_{\mathcal{B}} \boldsymbol{c_{kj}}\right] \\
&= \boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{B} +_{\mathcal{M}} \boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{C},
\end{aligned}$$

and (14) holds. Property (15) follows in the same way. As, by [5, Theorem 3.15, Theorem 3.28], $\times_{\mathcal{B},c}$ and $\circ_{\mathcal{B},c}$ are subdistributive relative to $+_{\mathcal{B}}$, and, by [5, Theorem 3.4], $+_{\mathcal{B}}$ is inclusion monotonic, then, for $*_{\mathcal{B}} \in \{\times_{\mathcal{B},c}, \circ_{\mathcal{B},c}\}$ we get

$$\begin{aligned}
(\boldsymbol{B} +_{\mathcal{M}} \boldsymbol{C}) *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A} &= \left[\sum_{k=1}^{r} (\boldsymbol{b_{ik}} +_{\mathcal{B}} \boldsymbol{c_{ik}}) *_{\mathcal{B}} \boldsymbol{a_{kj}}\right] \\
&\subseteq \left[\sum_{k=1}^{r} \boldsymbol{b_{ik}} *_{\mathcal{B}} \boldsymbol{a_{kj}} +_{\mathcal{B}} \sum_{k=1}^{r} \boldsymbol{c_{ik}} *_{\mathcal{B}} \boldsymbol{a_{kj}}\right] \\
&= \boldsymbol{B} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A} +_{\mathcal{M}} \boldsymbol{C} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A},
\end{aligned}$$

and (17) is valid. Property (16) follows in a similar way. □

The following result shows for which $*_{\mathcal{B}}$ the multiplication $*_{\mathcal{M},*_{\mathcal{B}}}$ is inclusion monotonic.

**Theorem 3.10.** *Let $\boldsymbol{A}^{(t)}, \boldsymbol{B}^{(t)} \in \mathcal{M}$, $t \in \{1, 2\}$. If $\boldsymbol{A}^{(t)} \subseteq \boldsymbol{B}^{(t)}$, $t \in \{1, 2\}$, then, for $*_{\mathcal{M},*_{\mathcal{B}}}$ with $*_{\mathcal{B}} \in \{\times_{\mathcal{B},c}, \circ_{\mathcal{B},c}\}$,*

$$\boldsymbol{A}^{(1)} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{B}^{(1)} \subseteq \boldsymbol{A}^{(2)} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{B}^{(2)}.$$

*Proof.* Let $\boldsymbol{A}^{(t)} = [\boldsymbol{a}_{\boldsymbol{ij}}^{(t)}], \boldsymbol{B}^{(t)} = [\boldsymbol{b}_{\boldsymbol{ij}}^{(t)}] \in \mathcal{M}$ such that $\boldsymbol{A}^{(t)} \subseteq \boldsymbol{B}^{(t)}$, $t \in \{1, 2\}$. By [5, Theorem 3.4, Theorem 3.14, Theorem 3.27], $+_{\mathcal{B}}$, $\times_{\mathcal{B},c}$ and $\circ_{\mathcal{B},c}$ are inclusion monotonic. Thus, for $*_{\mathcal{M},*_{\mathcal{B}}}$ with $*_{\mathcal{B}} \in \{\times_{\mathcal{B},c}, \circ_{\mathcal{B},c}\}$, we get

$$\boldsymbol{A}^{(1)} *_{\mathcal{M}} \boldsymbol{B}^{(1)} = \left[\sum_{k=1}^{r} \boldsymbol{a}_{\boldsymbol{ik}}^{(1)} *_{\mathcal{B}} \boldsymbol{b}_{\boldsymbol{kj}}^{(1)}\right] \subseteq \left[\sum_{k=1}^{r} \boldsymbol{a}_{\boldsymbol{ik}}^{(2)} *_{\mathcal{B}} \boldsymbol{b}_{\boldsymbol{kj}}^{(2)}\right] = \boldsymbol{A}^{(2)} *_{\mathcal{M}} \boldsymbol{B}^{(2)}.$$

□

We now consider a metric concept for ball matrices, and the corresponding notation.

**Definition 3.11.** *Let $\boldsymbol{A} = [\boldsymbol{a}_{\boldsymbol{ij}}] \in \mathcal{M}_{m\times n}$. The absolute value (matrix) of $\boldsymbol{A}$ is $|\boldsymbol{A}| = [|\boldsymbol{a}_{\boldsymbol{ij}}|] \in M_{m\times n}$.*

Next result gives an alternative description for the absolute value of a ball matrix.

**Theorem 3.12.** *Let $\boldsymbol{A} = [\boldsymbol{a}_{\boldsymbol{ij}}] \in \mathcal{M}_{m\times n}$. Then $|\boldsymbol{A}| = [\max\{\|x\|_2 : x \in \boldsymbol{a}_{\boldsymbol{ij}}\}]$.*

*Proof.* A direct consequence of Theorem 2.10. □

Let $X = [x_{ij}], Y = [y_{ij}] \in M_{m\times n}$ and recall the partial order relation $\leq$ defined on $M_{m\times n}$ by writing $X \leq Y$ iff $x_{ij} \leq y_{ij}$.

**Theorem 3.13.** *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}$. Let $\alpha \in \mathbb{R}$. Then:*

$$|\boldsymbol{A}| \geq 0_{m\times n}, |\boldsymbol{A}| = 0_{m\times n} \text{ iff } \boldsymbol{A} = \boldsymbol{0}, \text{ where } \boldsymbol{0} = [\boldsymbol{o}_{\boldsymbol{ij}}] \text{ with } \boldsymbol{o}_{\boldsymbol{ij}} = \boldsymbol{o} = \langle 0; 0\rangle; \tag{18}$$

$$|\boldsymbol{A} \pm_{\mathcal{M}} \boldsymbol{B}| \leq |\boldsymbol{A}| + |\boldsymbol{B}|; \tag{19}$$

$$|\alpha \cdot_{\mathcal{M}} \boldsymbol{A}| = |\alpha||\boldsymbol{A}|; \tag{20}$$

*for $*_{\mathcal{M},*_{\mathcal{B}}} \in \{*_{\mathcal{M},\times_{\mathcal{B},r}}, *_{\mathcal{M},\times_{\mathcal{B},c}}\}$,*

$$|\boldsymbol{A} *_{\mathcal{M}} \boldsymbol{B}| \leq |\boldsymbol{A}||\boldsymbol{B}|. \tag{21}$$

*Proof.* The proof follows in an entrytwise manner from Theorem 2.11. □

In the case of square matrices, the absolute value of a matrix is a norm, as can be seen in the following result.

**Corollary 3.14.** *For $*_{\mathcal{M},*_{\mathcal{B}}} \in \{*_{\mathcal{M},\times_{\mathcal{B},r}}, *_{\mathcal{M},\times_{\mathcal{B},c}}\}$, the function $|\cdot| : \mathcal{M}_{n\times n} \to M_{n\times n}$, defined by $\boldsymbol{A} \mapsto |\boldsymbol{A}| = [|\boldsymbol{a}_{\boldsymbol{ij}}|]$, is a matricial norm on $\mathcal{M}_{n\times n}$.*

*Proof.* A consequence of Theorem 3.13 (18)-(21). □

We now consider one more metric concept for ball matrices, and the corresponding notation.

**Definition 3.15.** *Let* $\boldsymbol{A} = [\boldsymbol{a_{ij}}] \in \mathcal{M}_{m\times n}$*. The width (matrix) of* $\boldsymbol{A}$ *is* $d(\boldsymbol{A}) = [d(\boldsymbol{a_{ij}})] \in M_{m\times n}$*.*

Properties of the width of a ball matrix are collected in the following result.

**Theorem 3.16.** *Let* $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}$*. Then:*

$$d(\boldsymbol{A}) \geq 0_{m\times n}; \tag{22}$$

$$d(\boldsymbol{A} \pm_{\mathcal{M}} \boldsymbol{B}) \ \leq \ d(\boldsymbol{A}) + d(\boldsymbol{B}); \tag{23}$$

$$d(\alpha \cdot_{\mathcal{M}} \boldsymbol{A}) = |\alpha| d(\boldsymbol{A}); \tag{24}$$

*for* $*_{\mathcal{M},*_{\mathcal{B}}} \in \{*_{\mathcal{M},\times_{\mathcal{B},r}} *_{\mathcal{M},\circ_{\mathcal{B},r}}\}$*,*

$$d(\boldsymbol{A} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{B}) \leq d(\boldsymbol{A})|\boldsymbol{B}|. \tag{25}$$

*Proof.* The proof follows in an entrywise manner from Theorem 2.14. □

We now define the (right) powers of a ball matrix relative to $*_{\mathcal{M},*_{\mathcal{B}}}$ except, due to the lack of neutral element, for $*_{\mathcal{B}} = \times_{\mathcal{B},c}$.

**Definition 3.17.** *Let* $\boldsymbol{A} \in \mathcal{M}_{n\times n}$*. For* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$*, the powers of* $\boldsymbol{A}$ *relative to* $*_{\mathcal{M},*_{\mathcal{B}}}$ *are defined by*

$$\boldsymbol{A}^{*_{\mathcal{M},*_{\mathcal{B}}}0} = \boldsymbol{I}_{n,*_{\mathcal{B}}} \text{ and } \boldsymbol{A}^{*_{\mathcal{M},*_{\mathcal{B}}}k} = \boldsymbol{A}^{*_{\mathcal{M},*_{\mathcal{B}}}(k-1)} *_{\mathcal{M},*_{\mathcal{B}}} \boldsymbol{A}.$$

Next new notion requires the powers of a ball matrix relative to $*_{\mathcal{M},*_{\mathcal{B}}}$, with $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$.

**Definition 3.18.** *Let* $\mathbf{A} \in \mathcal{M}_{n\times n}$*. For each* $*_{\mathcal{B}} \in \{\times_{\mathcal{B},r}, \circ_{\mathcal{B},r}, \circ_{\mathcal{B},c}\}$*,* $\boldsymbol{A}$ *is* $*_{\mathcal{M},*_{\mathcal{B}}}$*-convergent (to zero) if the sequence*

$$\{\boldsymbol{A}^{*_{\mathcal{M},*_{\mathcal{B}}}k}\}_{k=0}^{\infty} = \left\{\left[\boldsymbol{a_{ij}}^{*_{\mathcal{M},*_{\mathcal{B}}}\{k\}}\right]\right\}_{k=0}^{\infty}$$

*of the powers of* $\boldsymbol{A}$ *relative to* $*_{\mathcal{M},*_{\mathcal{B}}}$ *converges to the ball matrix* $\mathbf{0}$ *with respect to the matricial norm* $|\cdot|$*, where* $\mathbf{0} = [\boldsymbol{o}_{ij}]$ with $\boldsymbol{o}_{ij} = \boldsymbol{o} = \langle 0; 0\rangle$*, that is, for each* $k$ *and for* $i, j \in \{1, \ldots, n\}$*,* $\boldsymbol{a_{ij}}^{*_{\mathcal{M},*_{\mathcal{B}}}\{k\}}$ *is* $*_{\mathcal{B}}$*-convergent (to zero). When* $\boldsymbol{A}$ *is not* $*_{\mathcal{M},*_{\mathcal{B}}}$*-convergent,* $\boldsymbol{A}$ *is said to be* $*_{\mathcal{M},*_{\mathcal{B}}}$*-divergent.*

For the following theorem, using standard notation, let $\rho(\cdot)$ denote the spectral radius of a certain matrix.

**Lemma 3.19.** *For $k \in \mathbb{N}$, $|\boldsymbol{A}^{\times_{\mathcal{B},r} k}| \leq |\boldsymbol{A}|^k$.*

*Proof.* We proceed by induction on $k$. The inequality clearly holds for $k = 1$, and, by Theorem 3.13 (21), it is also valid for $k = 2$. As for the induction step, invoking again Theorem 3.13 (21), we have

$$|\boldsymbol{A}^{\times_{\mathcal{B},r}(k+1)}| = |\boldsymbol{A}^{\times_{\mathcal{B},r} k} *_{\mathcal{M},\times_{\mathcal{B},r}} \boldsymbol{A}| \leq |\boldsymbol{A}^{\times_{\mathcal{B},r} k}||\boldsymbol{A}| \leq |\boldsymbol{A}|^k|\boldsymbol{A}| = |\boldsymbol{A}|^{k+1}.$$

□

**Theorem 3.20.** *Let $\mathbf{A} \in \mathcal{M}_{n\times n}$. If $\rho(|\mathbf{A}|) < 1$ then $\mathbf{A}$ is $\times_{\mathcal{B},r}$-convergent.*

*Proof.* Let $\mathbf{A} \in \mathcal{M}_{n\times n}$. Suppose that $\rho(|\mathbf{A}|) < 1$. From [17, Theorem 1.4],

$$|\boldsymbol{A}| \text{ is convergent (to zero)},$$

that is, the sequence $\{|\boldsymbol{A}|^k\}_{k=1}^{\infty}$ of the powers of $|\boldsymbol{A}|$ converges to the null matrix $0_{n\times n}$. By Lemma 3.19,

$$|\boldsymbol{A}^{\times_{\mathcal{B},r} k}| \leq |\boldsymbol{A}|^k.$$

Hence, $\boldsymbol{A}$ is $\times_{\mathcal{B},r}$-convergent. □

A natural question is whether Theorem 3.20 is reversible; the following example shows that the answer is negative.

**Example 3.21.** *Let*

$$\mathbf{B} = \begin{bmatrix} \boldsymbol{b} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{o} \end{bmatrix} \in \mathcal{M}_{2\times 2},$$

*where $\boldsymbol{b} = \langle (\frac{1}{2}, \ldots, \frac{1}{2}); \frac{2}{3} \rangle$. We get*

$$\mathbf{B}^{\times_{\mathcal{B},r} k} = \begin{bmatrix} \boldsymbol{b}^{\times_{\mathcal{B},r} k} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{o} \end{bmatrix},$$

*and, from Example 2.18, $\boldsymbol{b}$ is $\times_{\mathcal{B},r}$-convergent. Hence, $\mathbf{B}$ is $\times_{\mathcal{B},r}$-convergent too. However, $\rho(|\boldsymbol{B}|) = \frac{1}{2}\sqrt{n} + \frac{2}{3} > 1$ since*

$$|\mathbf{B}| = \begin{bmatrix} |\boldsymbol{b}| & |\boldsymbol{o}| \\ |\boldsymbol{o}| & |\boldsymbol{o}| \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{n} + \frac{2}{3} & 0 \\ 0 & 0 \end{bmatrix}.$$

Another natural question is whether Theorem 3.20 is reversible adding some assumption(s), as done by Mayer, in [13, Theorem 2], for an interval matrix. In this sense, one might think of irreducibility of $\boldsymbol{A}$ and $d(\boldsymbol{A}) \neq 0_{n\times n}$, but the following result sheds light on why this does not seem to be the case. Similarly to [13], we would need, for all $k \in \mathbb{N}$ and for all $\mathbf{A} \in \mathcal{M}_{n\times n}$, $d(\boldsymbol{A}^{\times_{\mathcal{B},r}(k+1)}) \geq d(\boldsymbol{A})|\boldsymbol{A}|^k$, but this does not hold. In fact, from Theorem 3.16 (25), for all $\mathbf{A} \in \mathcal{M}_{n\times n}$, $d(\boldsymbol{A}^{\times_{\mathcal{B},r}2}) \leq d(\boldsymbol{A})|\boldsymbol{A}|$.

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## Declaration of competing interest

There is no competing interest.

## References

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