Free extensivity via distributivity

Fernando Lucatelli Nunes, Rui Prezado, and Matthijs Vákár

Abstract. We consider the canonical pseudodistributive law between various free limit completion pseudomonads and the free coproduct completion pseudomonad. When the class of limits includes pullbacks, we show that this consideration leads to notions of extensive categories. More precisely, we show that *extensive categories with pullbacks* and *infinitary lextensive categories* are the pseudoalgebras for the pseudomonads resulting from two of these pseudodistributive laws. Moreover, we introduce the notion of *doubly-infinitary lextensive category*, and we establish that the freely generated ones are cartesian closed. From this result, we further deduce that, in freely generated infinitary lextensive categories, the objects with a finite number of connected components are exponentiable. We conclude our work with remarks on examples, descent theoretical aspects of this work, results concerning non-canonical isomorphisms, and relationship with other work.

Introduction

Two-dimensional monad theory [5, 7, 28, 31] is the categorical approach to bidimensional universal algebra, which mainly deals with the problem of understanding *algebraic structures*, in a suitable sense, over objects in a 2-category.

Focusing on the case where the base 2-category is the 2-category of categories **CAT**, this leads to the systematic study of categories with additional *(algebraic)* structures (or properties) [5,26,31]. The 2-categories of interest usually arise as 2-categories of pseudoalgebras or lax algebras of a given pseudomonad – we refer, for instance, to [29, 32] for the definitions of these concepts.

There are many well-known examples of such 2-categories of interest, namely:

– the 2-category of monoidal categories, monoidal functors and monoidal natural transformations is the 2-category of pseudoalgebras for the free monoid 2-monad (also known as the *list* 2-monad) on **CAT**, *e.g.* [5, 21, 35];

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- the 2-category of monads is given by the 2-category of lax algebras w.r.t. the identity 2-monad on **CAT**, *e.g.* [34, pag. 33] and [32];
- 2-categories of pseudofunctors and pseudonatural transformations between two suitable 2-categories with weighted bicolimits is given by the 2-category of pseudoalgebras w.r.t. a pseudomonad induced by a suitable pseudo-Kan extension, *e.g.* [31, 33];
- the 2-category of categories with Φ-(co)limits and Φ-(co)limit preserving functors is the 2-category of pseudoalgebras and pseudomorphisms w.r.t. a suitable free (co)limit completion pseudomonad on **CAT**, *e.g.* [26, 35, 40, 45].

The framework of two-dimensional monad theory is well-suited for studying the age-old problem of distributivity between limits and colimits of a given category. Specifically, our focus lies on the canonical *pseudodistributive law* [41, 42] between various sorts of *free limit completion* pseudomonads and the *free coproduct completion* pseudomonad. Previous considerations of such distributivity properties include (infinitary) distributive categories [9], completely distributive categories [43], and doubly-infinitary distributive categories [39]. In this paper, we show that a similar analysis gives rise to well-known and novel notions of *extensive categories*.

Recall that, if $\mathbb{C}$ has (in)finite coproducts, $\mathbb{C}$ is said to be an (*infinitary*) *extensive category* [9] if the canonical functor

$$\prod_{i\in I} \mathbb{C}\downarrow X_i \xrightarrow{\coprod} \mathbb{C}\downarrow \coprod_{i\in I} X_i$$

is an equivalence of categories for every (in)finite family $(X_i)_{i\in I}$ of objects in $\mathbb{C}$. It has been observed in [10] and [48, Section 7] that "(infinitary) extensivity" can be viewed as a distributivity condition of pullbacks over (infinitary) coproducts.

The present work, which is a sequel to [39], aims to study categories with a given class of limits, small coproducts, and a (pseudo)distributive law between them. More precisely, given a class Φ of diagrams, we remark that there is a canonical pseudodistributive law between the free Φ-limit completion pseudomonad and the free coproduct completion, denoted by **Fam** [27, 40, 54]. We show that the pseudoalgebras for the composite pseudomonad can be easily described; namely, they can be given as the categories with Φ-limits and coproducts such that the coproduct functor

$$\coprod : \mathbf{Fam}(\mathbb{C}) \to \mathbb{C} \tag{0.1}$$

preserves Φ-limits.

Our key contribution is the observation that various flavors of infinitary extensive categories are pseudoalgebras for such composites of pseudomonads. More precisely, assuming that $\mathbb{C}$ is a category with coproducts:

- if $\mathbb{C}$ has pullbacks, and (0.1) preserves them, then $\mathbb{C}$ is *infinitary extensive with pullbacks*;
- if $\mathbb{C}$ has finite limits, and (0.1) preserves them, then $\mathbb{C}$ is *infinitary lextensive*;
- if $\mathbb{C}$ has small limits, and (0.1) preserves them, we say that the category $\mathbb{C}$ is *doubly-infinitary lextensive*. We observe that $\mathbb{C}$ satisfies such properties if and only if $\mathbb{C}$ is simultaneously a *doubly-infinitary distributive* category [39] as well as a *lextensive* category.

The observations presented above, coupled with the findings of [38], contribute to the understanding of extensive categories and distributive categories through the prism of 2-dimensional universal algebra, adding to the comparison of these notions originally started in [9].

In [39], it was demonstrated that freely generated doubly-infinitary distributive categories are cartesian closed. Furthermore, this investigation extended to encompass the study of exponentials in freely generated infinitary distributive categories. More generally, in [37], a comprehensive analysis was conducted, yielding general results concerning exponentiability and cartesian closedness of Grothendieck constructions. Notably, these results are applicable to a wide array of contexts, including freely generated categorical structures.

Motivated by [37, 38], we further study the exponentiable objects of the free pseudoalgebras for the pseudomonads we considered; namely, we find that:

- in a freely generated infinitary lextensive category, objects with a finite number of connected components are exponentiable;
- freely generated doubly-infinitary extensive categories are *cartesian closed*.

Outline: We revisit the notion of free Φ-colimit completions for a class Φ of diagrams (small categories) in Section 1. Several authors have worked on free (co)limit completions; namely, we have [1, 20, 52] for ordinary categories, and [3, 25] in the context of enriched category theory. We also have the accounts [27, 40, 45, 54] which study free Φ-(co)limit completions from the perspective of 2-dimensional monad theory [5, 31, 32, 35], which is the approach we employ, so some familiarity with these methods is assumed. We focus specifically on four classes of free (co)limit completions:

- the free *coproduct* completion, denoted **Fam**,
- the free *finite limit* completion, denoted $\mathcal{L}_{\mathsf{fin}}$,
- the free *pullback* completion, denoted $\mathcal{L}_{\mathsf{pb}}$,
- the free *small limit* completion, denoted $\mathcal{L}$.

In Section 2, we study the distributivity of Φ-limits over coproducts. Similar work has been carried out in [2, 18, 43] and in the prequel [39]. After recalling the necessary concepts pertaining to pseudodistributive laws [41, 42, 53], we confirm that there is a pseudodistributive law between any free Φ-limit completion pseudomonad and the free coproduct completion pseudomonad **Fam** (Lemma 2.4). Instantiating this result with each of the aforementioned free limit completions, we obtain the composite pseudomonads $\mathbf{Fam} \circ \mathcal{L}_{\mathsf{fin}}$, $\mathbf{Fam} \circ \mathcal{L}_{\mathsf{pb}}$, and $\mathbf{Fam} \circ \mathcal{L}$.

The study of these pseudomonads and their pseudoalgebras have given us novel characterizations of (infinitary) extensivity. More specifically, we prove that:

- $(\mathbf{Fam} \circ \mathcal{L}_{\mathsf{fin}})$-pseudoalgebras are precisely the lextensive categories (Theorem 2.6),
- $(\mathbf{Fam} \circ \mathcal{L}_{\mathsf{pb}})$-pseudoalgebras are precisely the extensive categories with pullbacks (Theorem 2.8).

Moreover, in Section 2.3, we introduce the notion of *doubly-infinitary lextensive categories*: these are the $(\mathbf{Fam} \circ \mathcal{L})$-pseudoalgebras. Finally, we prove in Theorem 2.9 that doubly-infinitary lextensive categories correspond to lextensive categories that are also doubly-infinitary distributive as introduced in [39].

Mainly motivated by [37, 39], in the present work, our study exponentiable objects in freely generated categorical structures is the content of Section 3. This includes our main results, which respectively state that:

- freely generated doubly-infinitary lextensive categories are *cartesian closed* (Theorem 3.4),
- in freely generated infinitary lextensive categories, *finite coproducts of connected objects* are exponentiable (Theorem 3.7).

In Section 4, we discuss examples of (doubly)-infinitary lextensive categories. Finally, in Section 5, we show that analogous results also hold for the *free finite coproduct completion* pseudomonad, leading to similar characterisations of (finitely) extensive categories. Further, we discuss possible avenues for future work, descent theoretical considerations of our findings, and we note a result on non-canonical isomorphisms, as a direct consequence of the work of [35].

1. Free colimit completions

Let **CAT** be the 2-category of locally small (**Set**-enriched) categories. Any other category considered in this work is assumed to be an object of **CAT**.

Let Φ be a class of small categories. We say that a category $\mathbb{C}$ has Φ-colimits if any functor $D\colon \mathbb{J} \to \mathbb{C}$ with $\mathbb{J} \in \Phi$ has a colimit in $\mathbb{C}$. Moreover, if $F\colon \mathbb{C} \to \mathbb{D}$ is a functor between categories with Φ-colimits, we have a morphism

$$\operatorname{colim} FD \to F(\operatorname{colim} D) \tag{1.1}$$

which is natural in $D\colon \mathbb{J} \to \mathbb{C}$ for $\mathbb{J} \in \Phi$. We say that F preserves Φ-colimits if (1.1) is a natural isomorphism.

We let **Φ-Colim** be the 2-category of categories with Φ-colimits, Φ-colimit preserving functors and natural transformations. We have a forgetful 2-functor

$$\mathbf{\Phi\text{-}Colim} \longrightarrow \mathbf{CAT} \tag{1.2}$$

which is pseudomonadic – we let $\mathcal{P}_\Phi$ be the left biadjoint to (1.2), as well as the induced pseudomonad by the biadjunction – the *free Φ-colimit completion* pseudomonad. We can justify this abuse of notation, by noting that a category $\mathbb{C}$ has Φ-colimits if and only if the (fully faithful) unit of $\mathcal{P}_\Phi$ at $\mathbb{C}$, denoted by $\mathfrak{y}\colon \mathbb{C} \to \mathcal{P}_\Phi(\mathbb{C})$, has a left adjoint [3]. Thus, being a $\mathcal{P}_\Phi$-pseudoalgebra is a *property* of the category $\mathbb{C}$, as opposed to structure [26]. In other words, $\mathcal{P}_\Phi$ is a *lax idempotent pseudomonad* [13, 27, 30, 40, 45] (also known as *Kock-Zöberlein pseudomonad*), and, hence, a property-like pseudomonad [26, 33].

Dually, we say that a category $\mathbb{C}$ has Φ-limits whenever $\mathbb{C}^{\mathrm{op}}$ has Φ-colimits, and we say that a functor $F\colon \mathbb{C} \to \mathbb{D}$ between categories with Φ-limits preserves Φ-limits if $F^{\mathrm{op}}\colon \mathbb{C}^{\mathrm{op}} \to \mathbb{D}^{\mathrm{op}}$ preserves Φ-colimits. We denote by **Φ-Lim** the 2-category of categories with Φ-limits, Φ-limit preserving functors and natural transformations. We also have a pseudomonadic 2-functor

$$\mathbf{\Phi\text{-}Lim} \longrightarrow \mathbf{CAT} \tag{1.3}$$

whose left biadjoint and induced pseudomonad are denoted by $\mathcal{L}_\Phi$, so that we have a biequivalence $\mathcal{L}_\Phi$-**PsAlg** ≃ **Φ-Lim**. In fact, we note that $\mathcal{L}_\Phi(\mathbb{C}) = \mathcal{P}_\Phi(\mathbb{C}^{\mathrm{op}})^{\mathrm{op}}$. We likewise denote the (fully faithful) unit at a category $\mathbb{C}$ by $\mathfrak{y}\colon \mathbb{C} \to \mathcal{L}_\Phi(\mathbb{C})$. This unit has a right adjoint if and only if $\mathbb{C}$ has Φ-limits.

Remark 1.1. In [3, 25], the notions of Φ-colimits and Φ-colimit completions were worked out in the more general setting of enriched category theory, where Φ is taken to be a class of small weights instead (that is, functors $\mathbb{J}^{\mathrm{op}} \to \mathbb{V}$ with $\mathbb{J}$ small), where $\mathbb{V}$ is the base monoidal category.

In our setting, the notions we provided correspond to the classes Φ of weights that are constant functors to the terminal object. We leave the consideration of our results in an enriched setting for future work.

As argued in [3, 25], the free Φ-colimit completion $\mathcal{P}_\Phi(\mathbb{C})$ of a category $\mathbb{C}$ is most succinctly described as the smallest full subcategory of $\mathbf{CAT}(\mathbb{C}^{\mathrm{op}}, \mathbf{Set})$ that has Φ-colimits. Dually, $\mathcal{L}_\Phi(\mathbb{C})$ is the smallest full subcategory of $\mathbf{CAT}(\mathbb{C}, \mathbf{Set})^{\mathrm{op}}$ that has Φ-limits. With this, we can obtain an expression for the hom-sets of Φ-(co)limit completions:

Lemma 1.2. *Let Φ be a class of small categories, let C be an object of $\mathbb{C}$, and let $E\colon \mathbb{K} \to \mathcal{P}_\Phi(\mathbb{C})$ be a diagram with $\mathbb{K} \in \Phi$. We have a natural isomorphism*

$$\mathcal{P}_\Phi(\mathbb{C})\big(\mathfrak{y}(C), \operatorname*{colim}_{k\in\mathbb{K}} Ek\big) \cong \operatorname*{colim}_{k\in\mathbb{K}} \mathcal{P}_\Phi(\mathbb{C})\big(\mathfrak{y}(C), Ek\big), \tag{1.4}$$

and dually, for a diagram $F\colon \mathbb{K} \to \mathcal{L}_\Phi(\mathbb{C})$,

$$\mathcal{L}_\Phi(\mathbb{C})\big(\lim_{k\in\mathbb{K}} Fk, \mathfrak{y}(C)\big) \cong \operatorname*{colim}_{k\in\mathbb{K}} \mathcal{L}_\Phi(\mathbb{C})\big(Fk, \mathfrak{y}(C)\big). \tag{1.5}$$

Proof. We have

$$\begin{aligned}
&\mathcal{P}_\Phi(\mathbb{C})\big(\mathfrak{y}(C), \operatorname*{colim}_{k\in\mathbb{K}} Ek\big) \\
&\qquad \cong \mathbf{CAT}(\mathbb{C}^{\mathsf{op}}, \mathbf{Set})\big(\mathbb{C}(-, C), \operatorname*{colim}_{k\in\mathbb{K}} Ek\big) \\
&\qquad \cong \big(\operatorname*{colim}_{k\in\mathbb{K}} Ek\big)(C) && \text{Yoneda lemma,} \\
&\qquad \cong \operatorname*{colim}_{k\in\mathbb{K}} \big((Ek)C\big) && \text{componentwise colimits,} \\
&\qquad \cong \operatorname*{colim}_{k\in\mathbb{K}} \mathbf{CAT}(\mathbb{C}^{\mathsf{op}}, \mathbf{Set})\big(\mathbb{C}(-, C), Ek\big) && \text{Yoneda lemma,} \\
&\qquad \cong \operatorname*{colim}_{k\in\mathbb{K}} \mathcal{P}_\Phi(\mathbb{C})\big(\mathfrak{y}(C), Ek\big).
\end{aligned}$$

∎

This leads to the following formulas for the sets of morphisms (hom-sets), based on the observation that representable functors preserve limits.

Corollary 1.3. *Let Φ be a class of small categories. If $\mathbb{J}, \mathbb{K} \in \Phi$, and $F\colon \mathbb{J} \to \mathbb{C}$, $G\colon \mathbb{K} \to \mathcal{P}_\Phi(\mathbb{C})$, then*

$$\mathcal{P}_\Phi(\mathbb{C})\big(\operatorname*{colim}_{j\in\mathbb{J}} Fj, \operatorname*{colim}_{k\in\mathbb{K}} Gk\big) \cong \lim_{j\in\mathbb{J}} \operatorname*{colim}_{k\in\mathbb{K}} \mathcal{P}_\Phi(\mathbb{C})(Fj, Gk), \tag{1.6}$$

and dually, if $H : \mathbb{K} \to \mathcal{L}_\Phi(\mathbb{C})$, then

$$\mathcal{L}_\Phi(\mathbb{C})\big(\lim_{k\in\mathbb{K}} Hk, \lim_{j\in\mathbb{J}} Fj\big) \cong \lim_{j\in\mathbb{J}} \operatorname*{colim}_{k\in\mathbb{K}} \mathcal{L}_\Phi(\mathbb{C})(Hk, Fj), \tag{1.7}$$

where we identify an object of $\mathbb{C}$ with its image in $\mathcal{P}_\Phi(\mathbb{C})$ and $\mathcal{L}_\Phi(\mathbb{C})$.

Alternatively, one may constuct $\mathcal{P}_\Phi(\mathbb{C})$, and, dually, $\mathcal{L}_\Phi(\mathbb{C})$, via transfinite induction [3,25], by iteratively adjoining (co)limits of diagrams with domain in Φ, and taking unions at limit ordinals. In certain important cases, such as those of small (or finite) (co)limit or (co)product completions (see below), the induction stabilises after only one step.

Therefore, if Φ is a class of small categories such that the transfinite construction converges in one step, every object in $\mathcal{P}_\Phi(\mathbb{C})$ is obtained as the Φ-colimit of a diagram in $\mathbb{C}$, from which we obtain the following characterisation of the Φ-(co)limit completion of $\mathbb{C}$; $\mathcal{P}_\Phi(\mathbb{C})$ consists of

– diagrams $F\colon \mathbb{J} \to \mathbb{C}$ with $\mathbb{J} \in \Phi$ as objects,
– hom-sets given by the formula[1]

$$\mathcal{P}_\Phi(\mathbb{C})(F, G) = \lim_{j\in\mathbb{J}} \operatorname{colim}_{k\in\mathbb{K}} \mathbb{C}(Fj, Gk) \tag{1.8}$$

for diagrams $F\colon \mathbb{J} \to \mathbb{C}$, $G\colon \mathbb{K} \to \mathbb{C}$ with $\mathbb{J}, \mathbb{K} \in \Phi$.

Dually, in case every object in $\mathcal{L}_\Phi(\mathbb{C})$ is obtained as the Φ-limit of a diagram in $\mathbb{C}$, the free limit completion of a category $\mathbb{C}$ is given by $\mathcal{L}_\Phi(\mathbb{C}) = \mathcal{P}_\Phi(\mathbb{C}^{\mathsf{op}})^{\mathsf{op}}$. Explicitly, it consists of

– diagrams $F\colon \mathbb{J} \to \mathbb{C}$ with $\mathbb{J} \in \Phi$ as objects,
– hom-sets given by the formula

$$\mathcal{L}_\Phi(\mathbb{C})(F, G) = \lim_{k\in\mathbb{K}} \operatorname{colim}_{j\in\mathbb{J}} \mathbb{C}(Fj, Gk) \tag{1.9}$$

for diagrams $F\colon \mathbb{J} \to \mathbb{C}$, $G\colon \mathbb{K} \to \mathbb{C}$ with $\mathbb{J}, \mathbb{K} \in \Phi$.

Such a characterisation is appropriate, for example, when Φ consists of the class of all small (resp. finite) discrete categories, yielding small (resp. finite) coproduct and product completions, or if Φ consists of the class of all small (resp. finite) categories, yielding small (resp. finite) colimit and limit completions.

Free coproduct completion: If Φ is the class of discrete small categories (sets), then **Φ-Colim** is the 2-category of categories with coproducts, coproduct-preserving functors and all natural transformations. In this case, we write **Fam** $= \mathcal{P}_\Phi$.

We can explicitly describe the objects of $\mathbf{Fam}(\mathbb{C})$ – these are given by set-indexed families of objects $(X_i)_{i\in I}$, with $X_i \in \mathbb{C}$. Using the representation coming out of Corollary 1.3, we can also describe the hom-sets of morphisms from $(X_i)_{i\in I}$ to $(Y_j)_{j\in J}$ as

$$\prod_{i\in I}\coprod_{j\in J}\mathbb{C}(X_i, Y_j).$$

There is a wealth of literature studying free coproduct completions and their properties. For instance, we refer the reader to [1, 9], [6, Chapter 6], [48, Section 7], and [37].

[1] See [52, Section 1], and compare with (1.6).

Free (co)limit completion: When Φ consists of all small categories, Φ-**Colim** is the 2-category of categories with small colimits and small-colimit preserving functors.

Given a category $\mathbb{C}$, its *free colimit completion* $\mathcal{P}(\mathbb{C})$ is the full subcategory of $\mathbf{CAT}(\mathbb{C}^{\mathsf{op}}, \mathbf{Set})$ consisting of the *essentially small* or *accessible* functors [25]. When $\mathbb{C}$ is itself essentially small, we have $\mathcal{P}(\mathbb{C}) \simeq \mathbf{CAT}(\mathbb{C}^{\mathsf{op}}, \mathbf{Set})$.

Alternatively, as noted above, we can characterise $\mathcal{P}(\mathbb{C})$ as the category with diagrams $F\colon \mathbb{J} \to \mathbb{C}$ with $\mathbb{J}$ small as objects and homsets

$$\mathcal{P}(\mathbb{C})(F, G) = \lim_{j \in \mathbb{J}} \operatorname{colim}_{k \in \mathbb{K}} \mathbb{C}(Fj, Gk)$$

for diagrams $F\colon \mathbb{J} \to \mathbb{C}$, $G\colon \mathbb{K} \to \mathbb{C}$ with $\mathbb{J}, \mathbb{K}$ small.

Free finite limit completion: We consider the class $\Phi = \mathsf{fin}$ of all finite categories, in which case Φ-**Lim** is the 2-category of categories with finite limits and the functors that preserve them. We denote the free finite limit completion pseudomonad by $\mathcal{L}_{\mathsf{fin}}$.

For any given category $\mathbb{C}$, the category $\mathcal{L}_{\mathsf{fin}}(\mathbb{C})$ also admits a description as a category of diagrams, similar to $\mathcal{L}(\mathbb{C})$.

Free pullback completion: We consider the class $\Phi = \mathsf{pb}$ consisting of a single element, the cospan category: $\cdot \to \cdot \leftarrow \cdot$

The 2-category Φ-**Lim** is the 2-category of categories with pullbacks and pullback preserving functors between them, and we denote the free pullback completion pseudomonad by $\mathcal{L}_{\mathsf{pb}}$.

Unlike previous examples, not every object in $\mathcal{L}_{\mathsf{pb}}(\mathbb{C})$ can be obtained by taking the pullback of a diagram in $\mathcal{L}_{\mathsf{pb}}(\mathbb{C})$ of objects in the essential image of $\mathfrak{y}\colon \mathbb{C} \to \mathcal{L}_{\mathsf{pb}}(\mathbb{C})$, so we cannot recover any formulae analogous to (1.9); we refer the interested reader to [3, Section 7] for further details.

2. Three pseudomonads

Let $\mathcal{T}$ be a pseudomonad on **CAT**. We consider the following instance of the main result from [53]:

Lemma 2.1. *The following are equivalent:*

(i) **Fam** *lifts to a (lax idempotent) pseudomonad* $\mathbf{Fam}_{\mathcal{T}}$ *on* $\mathcal{T}$-**PsAlg**.

(ii) *There exists a pseudodistributive law* $\delta\colon \mathcal{T} \circ \mathbf{Fam} \to \mathbf{Fam} \circ \mathcal{T}$.

Proof. Since **Fam** is a lax idempotent pseudomonad [27], we may instantiate [53, Theorem 35] with $\mathcal{P} = \mathbf{Fam}$. ∎

In the presence of a pseudodistributive law $\delta\colon \mathcal{T}\circ\mathbf{Fam}\to\mathbf{Fam}\circ\mathcal{T}$, the composite $\mathbf{Fam}\circ\mathcal{T}$ also has the structure of a pseudomonad on **CAT** [41, Section 5]. We also recall the following result from [42, Section 6]:

Lemma 2.2. *We have a biequivalence* $\mathbf{Fam}_{\mathcal{T}}\text{-}\mathbf{PsAlg}\simeq(\mathbf{Fam}\circ\mathcal{T})\text{-}\mathbf{PsAlg}$.

In [42] we also find a description of the $\mathbf{Fam}_{\mathcal{T}}$-pseudoalgebras; they are the categories $\mathbb{C}$ together with

- a $\mathcal{T}$-pseudoalgebra structure $\Lambda\colon\mathcal{T}(\mathbb{C})\to\mathbb{C}$ on $\mathbb{C}$,
- a **Fam**-pseudoalgebra structure $\coprod\colon\mathbf{Fam}(\mathbb{C})\to\mathbb{C}$ on $\mathbb{C}$ – in other words, $\mathbb{C}$ is a category with coproducts,
- The coproduct functor $\coprod\colon\mathbf{Fam}(\mathbb{C})\to\mathbb{C}$ lifts to a $\mathcal{T}$-pseudomorphism.

Moreover, a $\mathbf{Fam}_{\mathcal{T}}$-pseudomorphism $F\colon\mathbb{C}\to\mathbb{D}$ is a functor F that preserves coproducts and is a $\mathcal{T}$-pseudomorphism in a compatible way (up to natural isomorphism).

Our work focuses on pseudomonads $\mathcal{T}$ that are free Φ-limit completions for a class Φ of small categories. For simplicity, we introduce the following terminology:

Definition 2.3. For a class Φ of small categories, we say that the $(\mathbf{Fam}\circ\mathcal{L}_\Phi)$-pseudoalgebras are the Φ-*coproduct distributive categories*.

In this setting, we have the following result.

Lemma 2.4. *For a class* Φ *of small categories,* **Fam** *lifts to a pseudomonad* $\mathbf{Fam}_{\mathcal{L}_\Phi}$ *on* Φ-**Lim**. *Consequently,* $\mathbf{Fam}_{\mathcal{L}_\Phi}\text{-}\mathbf{PsAlg}$ *is biequivalent to* $(\mathbf{Fam}\circ\mathcal{L}_\Phi)\text{-}\mathbf{PsAlg}$.

Proof. Since $\mathbf{Fam}(\mathbb{C})$ has whichever Φ-limits that $\mathbb{C}$ has and $\mathbf{Fam}(F)$ is Φ-limit preserving whenever F is [19, Section 4], we conclude that **Fam** lifts to an endo-2-functor on Φ-**Lim**, and $\mathfrak{y}\colon\mathbb{C}\to\mathbf{Fam}(\mathbb{C})$ preserves Φ-limits. Moreover, since we have a fully faithful adjoint string

$$\mathbf{Fam}\cdot\mathfrak{y}\dashv\mathfrak{m}\dashv\mathfrak{y}\cdot\mathbf{Fam},$$

we note that, in particular, $\mathfrak{m}$ is a right adjoint, and therefore preserves Φ-limits. ∎

In [39], we study the pseudodistributive laws of the free product completion pseudomonad $\mathcal{L}_{\mathbf{Set}}=\mathbf{Fam}\big((-)^{\mathrm{op}}\big)^{\mathrm{op}}$ and the free finite product completion pseudomonad $\mathcal{L}_{\mathbf{finSet}}=\mathbf{FinFam}\big((-)^{\mathrm{op}}\big)^{\mathrm{op}}$ over **Fam**, taking **Set** (**finSet**) to be the class of small (finite), discrete categories. The composite pseudomonads $\mathbf{Dist}=\mathbf{Fam}\circ\mathcal{L}_{\mathbf{Set}}$ and $\mathbf{Fam}\circ\mathcal{L}_{\mathbf{finSet}}$ are the pseudomonads whose pseudoalgebras are the doubly-infinitary distributive categories and infinitary distributive categories, respectively. Under the terminology we introduced, these are the *product-coproduct distributive* categories and the *finite product-coproduct distributive* categories. In the current work, we shall see that:

- Φ-coproduct distributive categories are infinitary lextensive categories, for the class Φ of finite categories (which corresponds to distributivity of finite limits over coproducts);
- Φ-coproduct distributive categories are the infinitary extensive categories with pullbacks, for the singleton class Φ consisting of the cospan category $\cdot \to \cdot \leftarrow \cdot$ (which corresponds to distributivity of pullbacks over coproducts);
- Φ-coproduct distributive categories are the infinitary lextensive categories that are doubly-infinitary distributive as well, for the class Φ of all small categories (which corresponds to distributivity of limits over coproducts).

2.1. Infinitary lextensive categories

We recall that a category with small coproducts $\mathbb{C}$ is *infinitary extensive* if it has pullbacks along coproduct inclusions, and if the coproducts are *disjoint* and *pullback-stable*. This can be expressed in three conditions:

(a) for every pair of objects $A, B \in \mathbb{C}$, we have a pullback diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & A + B \end{array}$$

(b) for each morphism $f\colon Y \to \coprod_{i\in I} X_i$, if we take pullbacks along the coproduct inclusions $X_i \xrightarrow{\iota_i} \coprod_{i\in I} X_i$,

$$\begin{array}{ccc} Y_i & \xrightarrow{\iota_i} & Y \\ \downarrow & \lrcorner & \downarrow f \\ X_i & \xrightarrow[\iota_i]{} & \coprod_{i\in I} X_i \end{array}$$

we have that $Y_i \xrightarrow{\iota_i} Y$ form a coproduct diagram as well, and

(c) for every family $(f_i\colon Y_i \to X_i)_{i\in I}$ of morphisms, the following commutative square

$$\begin{array}{ccc} Y_i & \xrightarrow{\iota_i} & \coprod_{i\in I} Y_i \\ f_i \downarrow & \lrcorner & \downarrow \coprod_{i\in I} f_i \\ X_i & \xrightarrow[\iota_i]{} & \coprod_{i\in I} X_i \end{array}$$

is a pullback diagram.

We also make use of the following notation: if $\mathbb{C}$ is a category with coproducts and a terminal object $\mathbb{1}$, we let $- * \mathbb{1}\colon \mathbf{Set} \to \mathbb{C}$ be the functor left adjoint to $\mathbb{C}(\mathbb{1}, -)\colon \mathbb{C} \to \mathbf{Set}$.

We highlight that if $\mathbb{C}$ has a terminal object $\mathbb{1}$, then so does $\mathbf{Fam}(\mathbb{C})$, so we have a functor $- * \mathbb{1}: \mathbf{Set} \to \mathbf{Fam}(\mathbb{C})$.

The following result, appearing in [10] and [48], is an important step in the characterization of the $\mathbf{Fam}_{\mathcal{L}_{\mathsf{fin}}}$-pseudoalgebras:

Lemma 2.5. *Let $\mathbb{C}$ be a category with finite limits and coproducts. Then the following are equivalent:*

(i) *$\mathbb{C}$ is infinitary lextensive;*

(ii) *$\coprod: \mathbf{Fam}(\mathbb{C}) \to \mathbb{C}$ preserves finite limits.*

Proof. For an infinitary lextensive $\mathbb{C}$, [48, Lemma 7.1] guarantees that we have an equivalence $\mathbf{Fam}(\mathbb{C}) \simeq \big(\mathbb{C} \downarrow (- * \mathbb{1})\big)$, and that the projection $\big(\mathbb{C} \downarrow (- * \mathbb{1})\big) \to \mathbb{C}$ preserves finite limits. Moreover, we also establish that the composite

$$\mathbf{Fam}(\mathbb{C}) \xrightarrow{\simeq} \big(\mathbb{C} \downarrow (- * \mathbb{1})\big) \longrightarrow \mathbb{C}$$

corresponds to the coproduct functor $\mathbf{Fam}(\mathbb{C}) \to \mathbb{C}$. This shows that (i) $\implies$ (ii).

Now, if we assume (ii), it follows in particular that $\coprod$ preserves pullbacks. So, we consider the following pullback diagrams in $\mathbf{Fam}(\mathbb{C})$

$$\begin{array}{ccc} \varnothing & \longrightarrow & A_0 \\ \downarrow & \lrcorner & \downarrow \\ A_1 & \longrightarrow & (A_i)_{i\in\{0,1\}} \end{array} \qquad \begin{array}{ccc} (Y_i)_{i\in I} & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow f \\ (X_i)_{i\in I} & \longrightarrow & \coprod_{i\in I} X_i \end{array} \qquad \begin{array}{ccc} V_j & \longrightarrow & (V_j)_{j\in J} \\ \downarrow & \lrcorner & \downarrow \\ W_j & \longrightarrow & (W_j)_{j\in J} \end{array} \tag{2.1}$$

for objects $A_0, A_1 \in \mathbb{C}$, a morphism $f: Y \to \coprod_{i\in I} X_i$ in $\mathbb{C}$, and a family of morphisms $(g_j: V_j \to W_j)_{j\in J}$ in $\mathbb{C}$.

Since the coproduct functor preserves pullbacks, it can be composed with each diagram (2.1) to respectively obtain the pullback diagrams in (a), (b) and (c). Hence, we witness the infinitary extensivity of $\mathbb{C}$, thereby confirming that (ii) $\implies$ (i). ∎

Now, by Lemma 2.2 and the description for $\mathbf{Fam}_{\mathcal{L}_{\mathsf{fin}}}$-pseudoalgebras, we conclude, as a corollary, that:

Theorem 2.6. *The 2-category $(\mathbf{Fam} \circ \mathcal{L}_{\mathsf{fin}})$-$\mathbf{PsAlg}$ consists of infinitary lextensive categories, and functors preserving coproducts and finite limits.*

2.2. Infinitary extensive categories with pullbacks

We can still obtain results analogous to Lemma 2.5 even in the absence of terminal objects.

Lemma 2.7. *Let $\mathbb{C}$ be a category with coproducts and pullbacks. The following are equivalent:*

(i) *The coproduct functor $\coprod : \mathbf{Fam}(\mathbb{C}) \to \mathbb{C}$ preserves pullbacks.*

(ii) *$\mathbb{C}$ is infinitary extensive.*

Proof. If $\mathbb{C}$ is infinitary extensive and has pullbacks, then $\mathbb{C} \downarrow X$ is infinitary lextensive for all objects X. Thus, we may apply Lemma 2.5 to conclude that

$$\mathbf{Fam}(\mathbb{C}) \downarrow X \simeq \mathbf{Fam}(\mathbb{C} \downarrow X) \xrightarrow{\coprod} \mathbb{C} \downarrow X$$

preserves finite limits. Since $\mathbf{Fam}(\mathbb{C})$ is infinitary extensive, we have

$$\mathbf{Fam}(\mathbb{C}) \downarrow (X_i)_{i\in I} \simeq \prod_{i\in I} \mathbf{Fam}(\mathbb{C}) \downarrow X_i,$$

and a product of finite limit preserving functors preserves finite limits as well. Thus, we deduce that $\coprod : \mathbf{Fam}(\mathbb{C}) \to \mathbb{C}$ preserves pullbacks, confirming that (ii) $\implies$ (i).

Conversely, if $\coprod : \mathbf{Fam}(\mathbb{C}) \to \mathbb{C}$ preserves pullbacks, we follow the same argument used for Lemma 2.5: we compose the coproduct functor with each of the diagrams (2.1) to respectively obtain (a), (b) and (c), exhibiting infinitary extensiveness. This proves that (i) $\implies$ (ii). ■

As a consequence, by Lemma 2.2 and the description of $\mathbf{Fam}_{\mathcal{L}_{\mathsf{pb}}}$-pseudoalgebras, we conclude that:

Theorem 2.8. *The 2-category $(\mathbf{Fam} \circ \mathcal{L}_{\mathsf{pb}})$-$\mathbf{PsAlg}$ consists of infinitary extensive categories with pullbacks, and functors which preserve coproducts and pullbacks.*

2.3. Doubly infinitary lextensive categories

Inspired by the terminology of [39], we call the $(\mathbf{Fam} \circ \mathcal{L})$-pseudoalgebras *doubly-infinitary lextensive* categories.

Theorem 2.9. *Let $\mathbb{C}$ be a category with coproducts and limits. The following are equivalent:*

(i) *The coproduct functor $\coprod : \mathbf{Fam}(\mathbb{C}) \to \mathbb{C}$ preserves limits;*

(ii) *$\mathbb{C}$ is doubly infinitary extensive;*

(iii) *$\mathbb{C}$ is lextensive and doubly infinitary distributive.*

Proof. We have the equivalence (i) $\iff$ (ii) by definition.

The equivalence (iii) $\iff$ (ii) follows by Lemma 2.7 and [39, Lemma 3.1]. We use the basic facts that any limit can be obtained via pullbacks and arbitrary products, and that infinitary extensive categories with products are, in particular, infinitary distributive (see [9, Proposition 4.5]). ■

3. Exponentiability in freely generated structures

The purpose of this section is to study the exponentiable objects of the free completions $\mathbf{Fam}(\mathcal{L}_{\mathsf{fin}}(\mathbb{C}))$ and $\mathbf{Fam}(\mathcal{L}(\mathbb{C}))$, which constitute the main results of this work. Aiming for a self-contained account of exponentiability, we begin by recalling the definition of *exponentiable object*, as well as some elementary properties.

In order to fix notation, we recall that an object E in a category $\mathbb{C}$ with finite products is *exponentiable* at X if there exists an object $E \Rightarrow X$ and a natural isomorphism

$$\mathbb{C}(- \times E, X) \cong \mathbb{C}(-, E \Rightarrow X). \tag{3.1}$$

We say that E is *exponentiable* if (3.1) holds naturally for every object X in $\mathbb{C}$.

We revisit the following elementary observation about exponentiable objects used in [39, Remark 1]:

Lemma 3.1. *Let $\mathbb{C}$ be a category with finite products and $\mathbb{J}$-limits for a small category $\mathbb{J}$. If $F\colon \mathbb{J} \to \mathbb{C}$ is a diagram, and E is an object such that E is exponentiable at Fj for each j in $\mathbb{J}$, then E is exponentiable at $\lim_{j\in\mathbb{J}} F$ and*

$$E \Rightarrow \lim_{j\in\mathbb{J}} Fj \cong \lim_{j\in\mathbb{J}} (E \Rightarrow Fj)$$

Proof. For each object A, we have a natural isomorphism

$$\mathbb{C}(A \times E, \lim_{j\in\mathbb{J}} Fj) \cong \lim_{j\in\mathbb{J}} \mathbb{C}(A \times E, F_j) \cong \lim_{j\in\mathbb{J}} \mathbb{C}(A, E \Rightarrow F_j) \cong \mathbb{C}\big(A, \lim_{j\in\mathbb{J}}(E \Rightarrow F_j)\big)$$

as desired. ■

We recall from [6, Definition 6.1.3] that an object A of a category $\mathbb{C}$ is *connected* if the hom-functor $\mathbb{C}(A, -)$ preserves coproducts. It is an immediate consequence of Lemma 1.2 that the objects in the essential image of $\mathfrak{y}\colon \mathbb{C} \to \mathbf{Fam}(\mathbb{C})$ are precisely the connected objects in $\mathbf{Fam}(\mathbb{C})$. We confirm that an analogous characterization is available for the internal hom-functor:

Lemma 3.2. *If $\mathbb{C}$ is a category with finite products, and C is an exponentiable object in $\mathbf{Fam}(\mathbb{C})$, then the following are equivalent:*

(i) *C is connected.*

(ii) *$C \Rightarrow -$ preserves coproducts.*

Proof. Let $(A_i)_{i\in I}$ be a family of objects in $\mathbb{C}$, and let $(X_j)_{j\in J}$ be a family of objects in $\mathbf{Fam}(\mathbb{C})$. If $\mathbb{C}$ is connected, then we have natural isomorphisms

$$\begin{aligned}
&\mathbf{Fam}(\mathbb{C})\Big((A_i)_{i\in I}\times C, \coprod_{j\in J} X_j\Big) \\
&\qquad \cong \mathbf{Fam}(\mathbb{C})\Big((A_i\times C)_{i\in I}, \coprod_{j\in J} X_j\Big) && \text{products in } \mathbf{Fam}(\mathbb{C}), \\
&\qquad \cong \prod_{i\in I}\coprod_{j\in J}\mathbf{Fam}(\mathbb{C})(A_i\times C, X_j) && (1.6), \\
&\qquad \cong \prod_{i\in I}\coprod_{j\in J}\mathbf{Fam}(\mathbb{C})(A_i, C\Rightarrow X_j) && C \text{ exponentiable}, \\
&\qquad \cong \mathbf{Fam}(\mathbb{C})\Big((A_i)_{i\in I}, \coprod_{j\in J}(C\Rightarrow X_j)\Big) && (1.6).
\end{aligned}$$

Hence, we conclude that

$$\coprod_{j\in J}(C\Rightarrow X_j)\cong C\Rightarrow \coprod_{j\in J} X_j,$$

which confirms that (i) $\Longrightarrow$ (ii).

Conversely, if $C\Rightarrow -$ preserves coproducts, then for a family $(X_j)_{j\in J}$ of objects in $\mathbf{Fam}(\mathbb{C})$, we have

$$\begin{aligned}
\mathbf{Fam}(\mathbb{C})\Big(C, \coprod_{j\in J} X_j\Big) &\cong \mathbf{Fam}(\mathbb{C})\Big(\mathbb{1}, C\Rightarrow \coprod_{j\in J} X_j\Big) && C \text{ exponentiable}, \\
&\cong \mathbf{Fam}(\mathbb{C})\Big(\mathbb{1}, \coprod_{j\in J} C\Rightarrow X_j\Big) && \text{by hypothesis (ii)}, \\
&\cong \coprod_{j\in J}\mathbf{Fam}(\mathbb{C})(\mathbb{1}, C\Rightarrow X_j) && \text{terminal connected}, \\
&\cong \coprod_{j\in J}\mathbf{Fam}(\mathbb{C})(C, X_j),
\end{aligned}$$

hence, we conclude that (ii) $\Longrightarrow$ (i). ∎

Let Φ be a class of small categories that includes all finite, discrete categories, so that every $\mathcal{L}_\Phi$-pseudoalgebra has finite products. For the sake of succinctness, we say that an object of $\mathbf{Fam}(\mathcal{L}_\Phi(\mathbb{C}))$ is a *generator* if it is in the essential image of the inclusion $\mathbb{C}\to\mathbf{Fam}(\mathcal{L}_\Phi(\mathbb{C}))$.

We will give an inductive perspective on exponentials in $\mathbf{Fam}\big(\mathcal{L}_\Phi(\mathbb{C})\big)$, and the following result is the cornerstone for our development (see [39, Remark 1]):

Lemma 3.3. *If X is a generator and D is connected in* $\mathbf{Fam}\big(\mathcal{L}_\Phi(\mathbb{C})\big)$, *then D is exponentiable at X and we have*

$$D \Rightarrow X \cong X + \hat{\mathbb{C}}(D, X) * \mathbb{1}$$

where $\hat{\mathbb{C}} = \mathbf{Fam}\big(\mathcal{L}_\Phi(\mathbb{C})\big)$.

Proof. Let $(E_i)_{i\in I}$ be a family of objects in $\mathcal{L}_\Phi(\mathbb{C})$. We have natural isomorphisms

$$\begin{aligned}
&\hat{\mathbb{C}}\big((E_i)_{i\in I} \times D, X\big) \\
&\quad\cong \hat{\mathbb{C}}\big((E_i \times D)_{i\in I}, X\big) && \text{products in } \hat{\mathbb{C}} \\
&\quad\cong \prod_{i\in I} \hat{\mathbb{C}}(E_i \times D, X) && \hat{\mathbb{C}}(-, X) \text{ preserves products,} \\
&\quad\cong \prod_{i\in I} \mathcal{L}_{\mathsf{fin}}(\mathbb{C})(E_i \times D, X) && \text{full faithfulness,} \\
&\quad\cong \prod_{i\in I} \mathcal{L}_{\mathsf{fin}}(\mathbb{C})(E_i, X) + \mathcal{L}_{\mathsf{fin}}(\mathbb{C})(D, X) && (1.7), \\
&\quad\cong \prod_{i\in I} \hat{\mathbb{C}}(E_i, X) + \hat{\mathbb{C}}(D, X) && \text{full faithfulness,} \\
&\quad\cong \hat{\mathbb{C}}\big((E_i)_{i\in I}, X + \hat{\mathbb{C}}(D, X) * \mathbb{1}\big) && (1.6)
\end{aligned}$$

■

3.1. Exponentials for free doubly infinitary lextensive categories

Having reviewed the elementary properties of exponentiable objects, we proceed to prove our main result on exponentiability of the objects of freely generated doubly-infinitary lextensive categories:

Theorem 3.4. *The category* $\mathbf{Fam}\big(\mathcal{L}(\mathbb{C})\big)$ *is cartesian closed.*

Proof. First, we note that connected objects are exponentiable:

- By Lemma 3.3, we have that any connected object in $\mathbf{Fam}\big(\mathcal{L}(\mathbb{C})\big)$ is exponentiable at the generators.
- Any connected object is a limit of generators, so by Lemma 3.1 we conclude that connected objects are exponentiable at any connected object in $\mathbf{Fam}\big(\mathcal{L}(\mathbb{C})\big)$.
- Since any object in $\mathbf{Fam}\big(\mathcal{L}(\mathbb{C})\big)$ is a coproduct of connected objects, we simply apply Lemma 3.2 to deduce our claim.

Now, let $(E_i)_{i\in I}$ and $(D_j)_{j\in J}$ be families of objects in $\mathcal{L}(\mathbb{C})$, and X any object in $\hat{\mathbb{C}}$. We have natural isomorphisms

$$\begin{aligned}
&\hat{\mathbb{C}}\big((E_i)_{i\in I} \times (D_j)_{j\in J}, X\big) \\
&\quad\cong \hat{\mathbb{C}}\big((E_i \times D_j)_{(i,j)\in I\times J}, X\big) && \text{binary products in } \hat{\mathbb{C}}, \\
&\quad\cong \prod_{i\in I}\prod_{j\in J} \hat{\mathbb{C}}(E_i \times D_j, X) && \hat{\mathbb{C}}(-, X) \text{ preserves limits}, \\
&\quad\cong \prod_{i\in I}\prod_{j\in J} \hat{\mathbb{C}}(E_i, D_j \Rightarrow X) && D_j \text{ connected (exponentiable)}, \\
&\quad\cong \prod_{i\in I} \hat{\mathbb{C}}\Big(E_i, \prod_{j\in J}(D_j \Rightarrow X)\Big) \\
&\quad\cong \hat{\mathbb{C}}\Big((E_i)_{i\in I}, \prod_{j\in J}(D_j \Rightarrow X)\Big)
\end{aligned}$$

Thus, we obtain

$$(D_j)_{j\in J} \Rightarrow X \cong \prod_{j\in J}(D_j \Rightarrow X),$$

confirming that coproducts of connected objects are exponentiable. But every object in $\mathbf{Fam}(\mathcal{L}(\mathbb{C}))$ is a coproduct of connected objects, hence the result follows. ∎

3.2. Explicit descriptions of the exponentials

Let $(D_j)_{j\in J}$ and $(E_k \colon \mathbb{A}_k \to \mathbb{C})_{k\in K}$ be families of objects in $\mathbb{L}$, where $\mathbb{A}_k$ is a small category for each $k \in K$.

The results of the previous subsection can be used to calculate an explicit expression for the exponential $(D_j)_{j\in J} \Rightarrow (E_k)_{k\in K}$ in $\mathbf{Fam}\big(\mathcal{L}(\mathbb{C})\big)$: via Lemmas 3.1–3.3, one of Theorems 3.7 or 3.4, and the key ideas of the proof of [39, Theorem 2.3], we obtain

$$(D_j)_{j\in J} \Rightarrow (E_k)_{k\in K} \cong \Big(\prod_{j\in J} \lim_{l\in\mathbb{A}_{f_j^K}} \Delta_{f,j,l}\Big)_{f\in\Omega} \tag{3.2}$$

where

$$\Omega = \prod_{j\in J}\coprod_{k\in K} \lim_{l\in\mathbb{A}_k} \big(\mathbb{1} + \mathcal{L}(\mathbb{C})(D_j, E_{k,l})\big),$$

$$\Delta_{f,j,l} = \begin{cases} E_{f_j^K,l} & \text{if } f_j(l) \in \mathbb{1} \\ \mathbb{1} & \text{if } f_j(l) \in \mathcal{L}(\mathbb{C})(D_j, E_{f_j^K,l}) \end{cases}$$

and f_j^K is the projection of f_j onto K for each $f \in \Omega$, $j \in J$.

Remark 3.5. As long as $\mathbb{C}$ has an initial object $\mathbb{0}$, the exponentials may be given explicitly by

$$(D_j)_{j\in J} \Rightarrow (E_k)_{k\in K} \cong \left(\prod_{j\in J} \mathfrak{d}^c\Big(\pi_2\big(f(j)\big)\Big)\right)_{f\in\Omega}$$

where

$$\Omega = \prod_{j\in J}\coprod_{k\in K}\big(\mathcal{L}_{\mathsf{fin}}(\mathbb{C})\big)(D_j \times \mathbb{0}, E_k),$$

and $\mathfrak{d}^c(g)$ is defined via the following pushout in $\mathcal{L}_{\mathsf{fin}}(\mathbb{C})$, by co-extensivity:

$$\begin{array}{ccc} D_i \times \mathbb{0} & \xrightarrow{\ g\ } & E_{\pi_1(f(i))} \\ {\scriptstyle \pi_2}\downarrow & \ulcorner & \downarrow \\ \mathbb{0} & \longrightarrow & \mathfrak{d}^c(g). \end{array}$$

3.3. Exponentials for free infinitary lextensive categories

As we remarked in Section 1, we have a fully faithful, finite limit preserving functor

$$\mathfrak{u}\colon \mathcal{L}_{\mathsf{fin}}(\mathbb{C}) \to \mathcal{L}(\mathbb{C})$$

for every category $\mathbb{C}$. By studying the fully faithful functor

$$\overline{\mathfrak{u}} = \mathbf{Fam}(\mathfrak{u})\colon \mathbf{Fam}\big(\mathcal{L}_{\mathsf{fin}}(\mathbb{C})\big) \to \mathbf{Fam}\big(\mathcal{L}(\mathbb{C})\big),$$

we can deduce results about exponentiability of objects in $\mathbf{Fam}\big(\mathcal{L}_{\mathsf{fin}}(\mathbb{C})\big)$. More precisely, we have

Lemma 3.6. *The functor* $\overline{\mathfrak{u}}$ *reflects exponentials of finite coproducts of connected objects.*

Proof. Let $(D_j)_{j\in J}$ be a *finite* family of objects in $\mathcal{L}_{\mathsf{fin}}(\mathbb{C})$, and let $(E_k\colon \mathbb{A}_k \to \mathbb{C})_{k\in K}$ be a family of objects in $\mathcal{L}_{\mathsf{fin}}(\mathbb{C})$, where $\mathbb{A}_k$ is a finite category for each $k \in K$. Given any object X in $\mathbf{Fam}\big(\mathcal{L}_{\mathsf{fin}}(\mathbb{C})\big)$, we have

$$\begin{aligned} &\mathbf{Fam}\big(\mathcal{L}_{\mathsf{fin}}(\mathbb{C})\big)\big(X \times (D_j)_{j\in J}, (E_k)_{k\in K}\big) \\ &\quad\cong \hat{\mathbb{C}}\Big(\overline{\mathfrak{u}}\big(X \times (D_j)_{j\in J}\big), \overline{\mathfrak{u}}\big((E_k)_{k\in K}\big)\Big) && \overline{\mathfrak{u}} \text{ fully faithful,} \\ &\quad\cong \hat{\mathbb{C}}\Big(\overline{\mathfrak{u}}(X) \times \overline{\mathfrak{u}}\big((D_j)_{j\in J}\big), \overline{\mathfrak{u}}\big((E_k)_{k\in K}\big)\Big) && \overline{\mathfrak{u}} \text{ preserves binary products} \\ &\quad\cong \hat{\mathbb{C}}\Big(\overline{\mathfrak{u}}(X), \overline{\mathfrak{u}}\big((D_j)_{j\in J}\big) \Rightarrow \overline{\mathfrak{u}}\big((E_k)_{k\in K}\big)\Big) && \mathbf{Fam}\big(\mathcal{L}(\mathbb{C})\big) \text{ is cartesian closed,} \end{aligned}$$

where $\hat{\mathbb{C}} = \mathbf{Fam}\big(\mathcal{L}(\mathbb{C})\big)$. Moreover, we have

$$\overline{\mathfrak{u}}\big((D_j)_{j\in J}\big) \Rightarrow \overline{\mathfrak{u}}\big((E_k)_{k\in K}\big) \cong \big(\mathfrak{u}(D_j)\big)_{j\in J} \Rightarrow \big(\mathfrak{u}(E_k)\big)_{k\in K},$$

and calculating the exponential as in (3.2), we obtain

$$\big(\mathfrak{u}(D_j)\big)_{j\in J} \Rightarrow \big(\mathfrak{u}(E_k)\big)_{k\in K} \cong \Big(\prod_{j\in J} \lim_{l\in \mathbb{L}_{f_j^K}} \mathfrak{u}(\Gamma_{f,j,l})\Big)_{f\in\Xi}$$

where

$$\Xi = \prod_{j\in J} \coprod_{k\in K} \lim_{l\in\mathcal{L}_k} \big(\mathbb{1} + \mathcal{L}_{\mathsf{fin}}(\mathbb{C})(D_j, E_{k,l})\big),$$

$$\Gamma_{f,j,l} = \begin{cases} E_{f_j^K,l} & \text{if } f_j(l) \in \mathbb{1} \\ \mathbb{1} & \text{if } f_j(l) \in \mathcal{L}_{\mathsf{fin}}(\mathbb{C})(D_j, E_{f_j^K,l}), \end{cases}$$

and f_j^K is the projection of f_j onto K for each $f \in \Xi$, $j \in J$.

Now, since $\mathfrak{u}$ is fully faithful and preserves finite limits, it must reflect them as well. Since we are given that J is finite, as well as $\mathbb{A}_k$ for all $k \in K$, we have

$$\prod_{j\in J} \lim_{l\in \mathbb{A}_{f_j^K}} \mathfrak{u}(\Gamma_{f,j,l}) \cong \mathfrak{u}\Big(\prod_{j\in J} \lim_{l\in \mathbb{A}_{f_j^K}} \Gamma_{f,j,l}\Big).$$

and thus

$$\overline{\mathfrak{u}}\big((D_j)_{j\in J}\big) \Rightarrow \overline{\mathfrak{u}}\big((E_k)_{k\in K}\big) \cong \overline{\mathfrak{u}}\Big(\prod_{j\in J} \lim_{l\in \mathbb{L}_{f_j^K}} \Gamma_{f,j,l}\Big)_{f\in\Xi}$$

so, since $\overline{\mathfrak{u}}$ is fully faithful, we conclude that the exponential $(D_j)_{j\in J} \Rightarrow (E_k)_{k\in K}$ in $\mathbf{Fam}(\mathcal{L}_{\mathsf{fin}}(\mathbb{C}))$ exists and

$$(D_j)_{j\in J} \Rightarrow (E_k)_{k\in K} \cong \Big(\prod_{j\in J} \lim_{l\in \mathbb{L}_{f_j^K}} \Gamma_{f,j,l}\Big)_{f\in\Xi},$$

as desired. ■

As an immediate corollary, we obtain our second main result:

Theorem 3.7. *Finite coproducts of connected objects in* $\mathbf{Fam}(\mathcal{L}_{\mathsf{fin}}(\mathbb{C}))$ *are exponentiable.*

4. Examples

In this section, we intend to give a brief discussion on examples of the various notions of (l)extensive categories arising from the Φ-coproduct distributive categories discussed herein. More interestingly, we discuss examples of the *doubly-infinitary lextensive categories* introduced in 2.3.

Recall that we consider the notion of doubly-infinitary distributive categories introduced in [39], and the 2-functor $\mathbf{Dist} = \mathbf{Fam}\left(\mathbf{Fam}\,(-)^{\mathsf{op}}\right)^{\mathsf{op}}$. By Theorem 2.9, doubly-infinitary lextensive categories are precisely the doubly-infinitary distributive categories which are also lextensive. With this in mind, we refer the reader to the examples discussed in [39], and we make some considerations tailored to our setting.

4.1. Fundamental examples

Let $\mathbb{1}$ be the terminal category (the category with precisely one object and the identity morphism), and $\varnothing$ the initial category (the empty category).

Let Φ be a class of small categories containing $\varnothing$. The category of sets

$$\mathbf{Set} \simeq \mathbf{Fam}(\mathbb{1}) \simeq \mathbf{Fam}(\mathcal{L}_{\Phi}(\varnothing))$$

is the free Φ-coproduct distributive category on the empty category. Hence, it is the initial object in the 2-category of Φ-coproduct distributive categories.

Let Φ be a class of small categories containing all discrete categories. Then the category

$$\mathbf{Fam}(\mathbf{Set}^{\mathsf{op}}) \simeq \mathbf{Fam}\big(\mathcal{L}(\mathbb{1})\big) \simeq \mathbf{Dist}(\mathbb{1})$$

is the free Φ-coproduct distributive category on the terminal category $\mathbb{1}$. As such, $\mathbf{Fam}(\mathbf{Set}^{\mathsf{op}})$ is both the free doubly-infinitary lextensive category, and the free doubly-infinitary distributive category on $\mathbb{1}$. By Theorem 3.4 (or [39, Theorem 2.3]), we conclude that $\mathbf{Fam}(\mathbf{Set}^{\mathsf{op}})$ (also known as the category of polynomials) is cartesian closed – recovering the result of [4].

4.2. Monadicity and presheaves

Let Φ be a class of categories. The "Φ-coproduct distributivity" properties can be lifted through functors that create Φ-limits and coproducts. To be precise, we have the following elementary result:

Lemma 4.1. *Let $G\colon \mathbb{D} \to \mathbb{C}$ be a functor that creates coproducts and Φ-limits. If $\mathbb{C}$ is a Φ-coproduct distributive category, then so is $\mathbb{D}$.*

Since pseudomonadic pseudofunctors create bicategorical products (see, for instance, [31, 32] for lifting results on the pseudomonad setting, and [50, 51] and [33, 3.8] for bilimits), we find that:

Lemma 4.2. *The 2-categories Φ-**Lim** and $(\mathbf{Fam} \circ \mathcal{L}_{\Phi})$-**PsAlg** have bicategorical products, given by the product of the underlying categories.*

More specifically, if $(C_i)_{i\in I}$ is a family of Φ-limit complete (Φ-coproduct distributive) categories, then

$$\prod_{i\in I}\mathbb{C}_i$$

is Φ-limit complete (Φ-coproduct distributive).

By applying Corollary 4.1, we conclude that:

Theorem 4.3. *Let $\mathbb{J}$ be a small category. If $\mathbb{C}$ is a Φ-coproduct distributive category, then the functor category $\mathbf{CAT}(\mathbb{J},\mathbb{C})$ is Φ-coproduct distributive as well.*

Proof. The result follows from the fact that the restriction/forgetful functor

$$\mathbf{CAT}(\mathbb{J},\mathbb{C}) \to \mathbf{CAT}(\mathsf{ob}\,\mathbb{J},\mathbb{C}) \cong \prod_{j\in\mathsf{ob}\,\mathbb{J}}\mathbb{C}$$

creates limits and colimits that exist in $\mathbb{C}$, and Lemma 4.2. ■

As a consequence, if $\mathbb{A}$ is a small category, the presheaf category $\mathbf{CAT}(\mathbb{A}^{\mathsf{op}},\mathbf{Set})$ is Φ-coproduct distributive, provided that Φ contains $\varnothing$. In particular, $\mathbf{CAT}(\mathbb{A}^{\mathsf{op}},\mathbf{Set})$ is doubly-infinitary extensive.

4.3. Finite and small bicategorical biproducts

As remarked in [39, 4.3], the 2-category of categories with products is bicategorically semi-additive. This observation also extends to our setting.

Let Φ be a class of small categories containing the finite discrete categories. We note that the 2-category Φ-**Lim** of Φ-limit complete categories is naturally enriched over the 2-category of symmetric monoidal categories with the multilinear multicategorical structure. Together with Lemma 4.2, we conclude that the 2-category Φ-**Lim** has finite bicategorical coproducts, which are equivalent to the finite bicategorical products. In other words:

Lemma 4.4. *The 2-category Φ-**Lim** has finite bicategorical biproducts.*

Moreover, it is clear that the hom-categories in the 2-category Φ-**Lim** are themselves Φ-limit complete – moreover, noting that composition of Φ-limit preserving functors preserve Φ-limits componentwise, we conclude that:

Lemma 4.5. *The 2-category of Φ-limit complete categories is naturally enriched over itself, with the multilinear multicategorical structure.*

If Φ contains all (small) discrete categories, then by Lemmas 4.2 and 4.5, we conclude that the 2-category of Φ-limit complete categories has bicategorical coproducts, which are equivalent to the bicategorical products. This is given as:

Lemma 4.6. *If Φ is a class of small categories containing the discrete categories, then the* 2*-category of* Φ*-limit complete categories has infinite bicategorical biproducts.*

This allows us to understand freely generated Φ-coproduct distributive categories over coproducts of categories, as we, for instance, show in Subsection 4.4.

4.4. Freely generated categorical structures on discrete categories

Now, we assume Φ be a class of small categories that contains all the small (respectively, finite) discrete categories.

If $\mathbb{C}$ is a small (finite) discrete category, we have $\mathbb{C} \simeq \coprod_{c \in \mathrm{ob}\,\mathbb{C}} \mathbb{1}$. Since $\mathcal{L}_\Phi$ preserves small (finite) bicategorical biproducts and $\mathcal{L}_\Phi(\mathbb{1}) \simeq \mathbf{Set}^{\mathrm{op}}$, we have that

$$\mathcal{L}_\Phi(\mathbb{C}) \simeq \mathcal{L}_\Phi\Big(\coprod_{c \in \mathrm{ob}\,\mathbb{C}} \mathbb{1}\Big) \simeq \prod_{c \in \mathrm{ob}\,\mathbb{C}} \mathcal{L}_\Phi(\mathbb{1}) \simeq \prod_{c \in \mathrm{ob}\,\mathbb{C}} \mathbf{Set}^{\mathrm{op}}$$

by Lemma 4.6. Therefore:

Theorem 4.7. *If* $\mathbb{C}$ *is a small discrete category, then*

$$\mathbf{Fam}(\mathcal{L}_\Phi(\mathbb{C})) \simeq \mathbf{Fam}\Big(\prod_{c \in \mathrm{ob}\,\mathbb{C}} \mathbf{Set}^{\mathrm{op}}\Big).$$

In particular, this result describes the free doubly-infinitary distributive categories, and free doubly-infinitary lextensive categories on a small, discrete category $\mathbb{C}$.

4.5. More on doubly-infinitary lextensive categories via free coproduct completions

As we showed in Lemma 2.4, **Fam** lifts to a pseudomonad $\mathbf{Fam}_{\mathcal{L}_\Phi}$ on $\mathcal{L}_\Phi$-**PsAlg**. Thus, if a category $\mathbb{C}$ has Φ-limits, then $\mathbf{Fam}(\mathbb{C})$ has Φ-limits as well, which are preserved by the coproduct $\mathfrak{m}\colon \mathbf{Fam}(\mathbf{Fam}(\mathbb{C})) \to \mathbf{Fam}(\mathbb{C})$. In particular,

- if $\mathbb{C}$ has pullbacks, then $\mathbf{Fam}(\mathbb{C})$ is infinitary extensive with pullbacks,
- if $\mathbb{C}$ has finite limits, then $\mathbf{Fam}(\mathbb{C})$ is infinitary lextensive,
- if $\mathbb{C}$ has small limits, then $\mathbf{Fam}(\mathbb{C})$ is doubly-infinitary lextensive,
- if $\mathbb{C}$ has products, then $\mathbf{Fam}(\mathbb{C})$ is doubly-infinitary distributive by [39, Example 1].

So, even if a category $\mathbb{C}$ with products does not have small limits, we can still establish that the category $\mathbf{Fam}(\mathbb{C})$ is doubly-infinitary distributive, and it is extensive [9] by virtue of being a free coproduct completion. Hence, if $\mathbf{Fam}(\mathbb{C})$ has small limits, we conclude that it is doubly-infinitary lextensive, by Theorem 2.9.

Before discussing our examples, we let $\mathbf{Conn}(\mathbb{C})$ be the full subcategory of a category $\mathbb{C}$ with coproducts consisting of the *connected objects* [6, Definition 6.1.3].

We begin by noting that the category **Cat** $\simeq$ **Fam**(**Conn**(**Cat**)) of small categories is doubly-infinitary lextensive, as it is both doubly-infinitary distributive and extensive, and **Cat** has small limits. Likewise, we can prove that the category ω-**CPO** $\simeq$ **Fam**(**Conn**(ω-**CPO**)) of ω-complete partial orders is also a doubly-infinitary lextensive category.

Again similarly, the category **LocConTop** of locally connected topological spaces and continuous functions is doubly-infinitary lextensive. Indeed, from [39, Example 8], we learn that **LocConTop** $\simeq$ **Fam**(**Conn**(**LocConTop**)) is both doubly-infinitary distributive and extensive, as the free coproduct completion of a category with products. Moreover, **LocConTop** is a coreflective subcategory of **Top** [17], therefore, **LocConTop** has small limits, letting us conclude that **LocConTop** is doubly-infinitary lextensive.

4.6. Doubly-infinitary distributive categories that are not extensive

As observed in [39], a distributive lattice $\mathbb{D}$ (seen as a distributive, thin category) is extensive if and only if $\mathbb{D} \simeq \mathbb{1}$, so any non-trivial example of a completely distributive lattice $\mathbb{D}$ will be doubly-infinitary distributive, but not extensive.

Another example is the full subcategory $\mathbf{Set}^2_\bullet$ of **Set** $\times$ **Set** consisting of those pairs of sets that are either both empty, or both non-empty. Since coproducts and products are calculated componentwise in $\mathbf{Set}^2_\bullet$, this category is doubly-infinitary distributive as well, but it is not extensive.

4.7. Cartesian closedness vs. doubly-infinitary lextensivity

The category **Fam**(**Top**) is an example of a doubly-infinitary lextensive category that is *not* cartesian closed. We note that the category **Top** of topological spaces is infinitary distributive, but not cartesian closed. So, by [39, Theorem 4.2], we conclude that **Fam**(**Top**) is not cartesian closed as well. However, **Fam**(**Top**) is doubly-infinitary lextensive, since **Top** has small limits.

An example of a cartesian closed category with all coproducts and limits, but not doubly-infinitary lextensive, is given in [39, Counter-example 2], the category of Quasi-Borel spaces.

5. Epilogue

Motivated particularly by the insights from [37, 39], the present work explores the distributive properties of limits over coproducts through the lens of two-dimensional monad theory [5, 31].

We have demonstrated that the canonical (pseudo)distributivity of pullbacks over coproducts leads to a pseudomonad whose pseudoalgebras are precisely the infinitary extensive categories equipped with pullbacks. Similarly, the distributivity of finite limits over coproducts leads to the notion of a pseudomonad whose 2-category of pseudoalgebras is precisely the 2-category of infinitary lextensive categories. Finally, we showed that the distributivity of limits over coproducts leads to the concept of *doubly-infinitary lextensivity*, characterized as infinitary extensive categories that are also doubly-infinitary distributive as introduced in [39].

We also studied the exponentiable objects of the free completions $\mathbf{Fam}(\mathcal{L}_{\mathsf{fin}}(\mathbb{C}))$ and $\mathbf{Fam}(\mathcal{L}(\mathbb{C}))$, confirming that the latter is a cartesian closed category for any category $\mathbb{C}$. These free completions enjoy various other known properties since they end up being the free coproduct completion of a well-behaved category – we refer the reader to [1, 9, 37–39] for further results.

Free finite coproduct completion

By replacing the free coproduct pseudomonad **Fam** with its *finite* counterpart **FinFam**, we recover nearly all of our results, provided we make some adaptations to be finitary setting. Namely, we obtain a pseudodistributive law

$$\mathcal{L}_\Phi \circ \mathbf{FinFam} \to \mathbf{FinFam} \circ \mathcal{L}_\phi,$$

for any class Φ of *finite* categories, by reworking the proof of Lemma 2.4. We then obtain two more characterizations:

- the $(\mathbf{FinFam} \circ \mathcal{L}_{\mathsf{pb}})$-pseudoalgebras are precisely the extensive categories with pullbacks,
- the $(\mathbf{FinFam} \circ \mathcal{L}_{\mathsf{fin}})$-pseudoalgebras are precisely the lextensive categories.

Most consequentially, an adaptation of our exponentiability results will confirm that $\mathbf{FinFam}(\mathcal{L}_{\mathsf{fin}}(\mathbb{C}))$ is a cartesian closed category whenever $\mathbb{C}$ is *locally finite*.

Descent theory

Effective descent morphisms [16, 23] (see also [36, Sections 3 and 4]) are the backbone of Grothendieck's descent theory [22, 33], which has significant consequences in various fields [8, 44, 49]. Besides their wide range of applications, effective descent morphisms hold intrinsic interest, as their purpose is the reconstruction of data over the codomain from given data over the domain, plus some additional algebraic structure.

Of particular relevance to the present work are effective descent morphisms of freely generated categorical structures. For instance, [46, Section 4] studied categories of descent data for families of morphisms $\phi\colon (X_i)_{i\in I} \to Y$, as well as conditions under which ϕ is an effective descent morphism in $\mathbf{Fam}(\mathbb{C})$, provided that $\mathbb{C}$ has finite limits. Namely, it was shown that all such descent data is a coproduct of connected descent data, which provided simpler conditions for a morphism $\phi\colon (X_i)_{i\in I} \to Y$ to be of effective descent – this gives evidence that $\mathbf{Fam}(\mathbb{C})$ is a good proxy for the study effective descent morphisms of $\mathbb{C}$. This perspective was useful in the study of effective descent functors between enriched categories, establishing precise connections between the work of [15], [33, Theorem 9.11], [47], and the work of [11, 12, 49].

Since the free completions $\mathbf{Dist}(\mathbb{C})$ and $\mathbf{Fam}(\mathcal{L}(\mathbb{C}))$ are even better behaved categories, enjoying properties such as cartesian closedness, an inquiry on whether studying effective descent morphisms in such free completions seems to be a reasonable avenue for future work.

Non-canonical isomorphisms

In analogy with [39, Subsection 5.2], we may use the results of [35] to prove that a category $\mathbb{C}$ is Φ-coproduct distributive if it has coproducts, Φ-limits, and there exists a(ny) invertible natural isomorphism

$$\coprod_{x\in \lim\limits_{j\in\mathbb{J}} UF} \lim_{j\in\mathbb{J}} F_{j,x_j} \xrightarrow{\cong} \lim_{j\in\mathbb{J}} \coprod_{x\in UFj} F_{j,x}$$

for every functor $F\colon \mathbb{J} \to \mathbf{Fam}(\mathbb{C})$ with $\mathbb{J} \in \Phi$, where we let $U\colon \mathbf{Fam}(\mathbb{C}) \to \mathbf{Set}$ be the functor that outputs the underlying indexing set.

More generally, if we have a pseudomonad $\mathcal{T}$ on $\mathbf{CAT}$ and a pseudodistributive law $\delta\colon \mathcal{T} \circ \mathbf{Fam} \to \mathbf{Fam} \circ \mathcal{T}$, then for any category $\mathbb{C}$ with coproducts and the structure of a $\mathcal{T}$-pseudoalgebra, the coproduct functor

$$\coprod\colon \mathbf{Fam}(\mathbb{C}) \to \mathbb{C}$$

is an oplax $\mathcal{T}$-morphism by doctrinal adjunction [24, 34]. The (codual version of the) techniques of non-canonical isomorphisms from [35] can be applied just as well to this setting.

Comparison to Cockett and Lack [14]

In [14], the authors address the extensive completion $\mathbf{Bool}(\mathbb{C})$ of a distributive category $\mathbb{C}$, whereas our work concerns, among others, the free lextensive category on any (possibly non-distributive) category $\mathbb{C}$.

If $\mathbb{C}$ is already distributive, this raises the question of whether our completion coincides with Cockett and Lack's construction. The answer is "no". In our setting, the canonical inclusion

$$\mathfrak{y}\colon \mathbb{C} \to \mathbf{FinFam}(\mathcal{L}_{\mathsf{fin}}(\mathbb{C})) \tag{5.1}$$

does *not* preserve finite coproducts nor finite limits, as we are dealing with a *free* completion. In contrast, the embedding constructed in [14]

$$I\colon \mathbb{C} \to \mathbf{Bool}(\mathbb{C})$$

preserves coproducts and products, so this is not a free completion. In fact, if $\mathbb{C}$ is extensive to begin with, we obtain an equivalence $I\colon \mathbb{C} \simeq \mathbf{Bool}(\mathbb{C})$, but this is far from the case for the embedding (5.1).

This distinction between free and non-free completions is encompassed by the difference between lax idempotent monads and pseudo-idempotent monads, which is a topic we plan to discuss in future work.

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Fernando Lucatelli Nunes
Department of Information and Computing Sciences, Utrecht University, Utrecht, The Netherlands; Department of Mathematics, University of Coimbra, Coimbra, Portugal;
f.lucatellinunes@uu.nl

Rui Prezado
Department of Mathematics, University of Coimbra, Coimbra, Portugal;
ruiprezado@gmail.com

Matthijs Vákár
Department of Information and Computing Sciences, Utrecht University, Utrecht, The Netherlands; matthijsvakar@gmail.com