MONOIDAL CLOSURE OF GROTHENDIECK CONSTRUCTIONS VIA Σ-TRACTABLE MONOIDAL STRUCTURES AND DIALECTICA FORMULAS

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ABSTRACT. We study the categorical structure of the Grothendieck construction of an indexed category $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$, starting with the study of fibred limits, colimits, and monoidal structures. Next, we give sufficient conditions for the monoidal closure of the total category $\Sigma_{\mathcal{C}}\mathcal{L}$ of a Grothendieck construction of an indexed category $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$. Our analysis is a generalization of Gödel's Dialectica interpretation, and it relies on a novel notion of Σ-tractable monoidal structure. As we will see, Σ-tractable coproducts simultaneously generalize cocartesian coclosed structures, biproducts and extensive coproducts. We analyse when the closed structure is fibred – usually it is not.

1. INTRODUCTION

Similarly to how we can represent an indexed family $S_{(-)} : I \to \mathbf{Set}$ of sets equivalently as a pair of sets $\Sigma_{i\in I}S_i$ and I with a projection function $\pi_1 : \Sigma_{i\in I}S_i \to I$, we can represent an indexed category $\mathcal{L}_{(-)} : \mathcal{C}^{op} \to \mathbf{CAT}$ equivalently as a pair of categories $\Sigma_{\mathcal{C}}\mathcal{L}$ and $\mathcal{C}$ with a projection functor $\pi_1 : \Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ that is a cloven fibration[1]. This construction relating indexed categories and (cloven) fibrations is known as the Grothendieck construction or the Σ-type of categories [Grothendieck and Raynaud(1971)].

Given the fundamental role that this construction plays in category theory, it is important to understand the categorical structure (e.g., limits, colimits, closure) of such categories $\Sigma_{\mathcal{C}}\mathcal{L}$. In fact, the construction also plays a fundamental role in computer science, where various special cases have been reinvented several times under the names of "polynomials" (see e.g., [Gambino and Kock(2013)]), "containers" (see e.g., [Abbott et al.(2003)]), and "lenses" (see e.g., [Foster et al.(2007)]). The basic idea is that objects in $\Sigma_{\mathcal{C}}\mathcal{L}$ (pairs (C, L) of an object C of $\mathcal{C}$ and an object L of $\mathcal{L}(C)$) get interpreted as a pair of a type of shapes C and a corresponding dependent type L of values of shape C. This can lead to convenient representations of data types. A typical example is given by types of arrays that have a shape (rank and size, for example) and a dependent type that specifies a type for each of the values stored in each position in the shape (e.g., a Boolean or integer). Understanding the categorical structure of categories $\Sigma_{\mathcal{C}}\mathcal{L}^{op}$ is important in this case, as it gives us guidance on what are principled programming idioms when manipulating data in such a polynomial/container/lens representation.

The structure of products in $\Sigma_{\mathcal{C}}\mathcal{L}$ has long been known [Gray(1966)] and the structure of coproducts and initial algebras and terminal coalgebras in $\Sigma_{\mathcal{C}}\mathcal{L}$ was recently analysed by [Lucatelli Nunes and Vákár(2023)]. The structure of equalizers and coequalizers is entirely analogous to that of products and coproducts, and, as such, is not unexpected. Essentially, the limit of $J : \mathcal{E} \to \Sigma_{\mathcal{C}}\mathcal{L}$ can be constructed as the limit of $\pi_1 \circ J$ in $\mathcal{C}$ and the limit of $\pi_2 \circ J$ in the fibre categories of $\mathcal{L}$, as long as the change-of-base functors preserve the latter; the colimit of J in $\Sigma_{\mathcal{C}}\mathcal{L}$ arises from the colimit of $\pi_1 \circ J$ in $\mathcal{C}$ if $\mathcal{L}(\lambda)$ has a left adjoint for the coprojections λ corresponding to the colimit (so in particular if $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ preserves the limit of $(\pi_1 \circ J)^{op}$). In fact, as we show, these are precisely necessary and sufficient conditions for fibred limits and colimits in Grothendieck constructions.

However, the basic question of when $\Sigma_{\mathcal{C}}\mathcal{L}$ has exponentials has not been studied much in literature, except for some special cases [Hyland(2002), Shulman(2008), Altenkirch et al.(2010), Moss and von Glehn(2018),

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[1] The fact that we consider $\mathcal{C}^{op}$ here rather than $\mathcal{C}$ is merely to adhere to the convention in the literature to phrase results in terms of fibrations rather than opfibrations. When considering $\mathcal{C}^{op}$-indexed categories (i.e., functors $\mathcal{L} : \mathcal{C} \to \mathbf{CAT}$), we can construct an equivalent opfibration $\pi_1 : (\Sigma_{\mathcal{C}^{op}}\mathcal{L}^{op})^{op} \to \mathcal{C}$ over $\mathcal{C}$.

Lucatelli Nunes and Vákár(2023)]. This surprisingly challenging question is the focus of the present paper, as well as its generalisation to non-cartesian monoidal closed structures. We provide an answer that arises as a generalisation of Gödel's Dialectica interpretation of exponentials.

Contributions. Briefly, this paper makes the following contributions:

- a proof of necessary and sufficient conditions for the existence of fibred limits and colimits in a Grothendieck construction $\Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ (Lemmas 1 and 2); while similar results (particularly for limits) were known [Gray(1966)], we believe our particular phrasing does not appear in the literature yet;
- a proof of necessary and sufficient conditions for fibred monoidal closure of a Grothendieck construction $\Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ (Lemma 5);
- a definition of the notion of Σ-*(co)tractable* monoidal structures and demonstration that many examples of monoidal categories that arise in practice are Σ-(co)tractable (Section 3);
- a proof that Σ-cotractability combined with the existence of Π-types (fibred products) yield sufficient conditions for a Grothendieck construction over a nice[2] base category to be (non-fibred) monoidal closed, via a generalised Dialectica formula for exponentials (Theorem 1);
- a demonstration that many interesting examples of such non-fibred monoidal closed and cartesian closed structures arise in practice (Section 4).

A remark about size and set-theoretic concerns. We work with a standard Von Neumann–Bernays–Gödel set theory as a basis for our constructions. In particular, we assume the existence of a predicative hierarchy of universes of which we will only use the first three levels. We refer to sets at those three levels as small sets, large sets and very large sets. As a convention, all of our categories, unless stated otherwise, can be large, but are locally small (meaning that each of their homsets is a small set). We write **Set**, **Cat** and **2Cat** for the large, locally small (2-)categories of small sets, small categories and small 2-categories, respectively. We will only consider three very large categories: **SET**, **CAT**, **2CAT**, the very large, locally large (2-)categories of large sets, large categories and large 2-categories, respectively.

2. Grothendieck construction basics

2.1. **Basic definition.** Here, we recall some basics about the Grothendieck construction [Grothendieck and Raynaud(1971)].

Let $\mathcal{C}$ be a category and let $\mathcal{L}$ be a $\mathcal{C}$-indexed category in the sense of a pseudofunctor $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ to the 2-category **CAT** of (large) categories, functors and natural transformations.

Definition 1 (Indexed category). A (large) $\mathcal{C}$-indexed category $\mathcal{L}$ consists of the following data:

- for each object C of $\mathcal{C}$, a (large) category $\mathcal{L}(C)$;
- for each morphism $c : C' \to C$ of $\mathcal{C}$, a functor $\mathcal{L}(c) : \mathcal{L}(C) \to \mathcal{L}(C')$;
- natural isomorphisms $\eta^C : \mathrm{id}_{\mathcal{L}(C)} \to \mathcal{L}(\mathrm{id}_C)$ (for C in $\mathcal{C}$) and $\mu^{c',c} : \mathcal{L}(c') \circ \mathcal{L}(c) \to \mathcal{L}(c \circ c')$ (for $C'' \xrightarrow{c'} C' \xrightarrow{c} C$ in $\mathcal{C}$);
- coherence conditions, for all $C''' \xrightarrow{c''} C'' \xrightarrow{c'} C' \xrightarrow{c} C$ in $\mathcal{C}$,

$$
\begin{array}{ccccc}
 & \swarrow^{\eta^{C''}\mathcal{L}(c')} & \mathcal{L}(c') & \searrow^{\mathcal{L}(c')\eta^{C'}} & \\
 & & \| & & \\
\mathcal{L}(\mathrm{id}_{C''}) \circ \mathcal{L}(c') & \xrightarrow[\mu^{\mathrm{id}_{C''},c'}]{} & \mathcal{L}(c') & \xleftarrow[\mu^{c',\mathrm{id}_{C'}}]{} & \mathcal{L}(c') \circ \mathcal{L}(\mathrm{id}_{C'})
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{L}(c'') \circ \mathcal{L}(c') \circ \mathcal{L}(c) & \xrightarrow{\mathcal{L}(c'')\mu^{c',c}} & \mathcal{L}(c'') \circ \mathcal{L}(c \circ c') \\
{\scriptstyle \mu^{c'',c'}\mathcal{L}(c)}\downarrow & & \downarrow{\scriptstyle \mu^{c'',c\circ c'}} \\
\mathcal{L}(c' \circ c'') \circ \mathcal{L}(c) & \xrightarrow[\mu^{c'\circ c'',c}]{} & \mathcal{L}(c \circ c' \circ c'').
\end{array}
$$

In many but not all interesting examples, $\mathcal{L}$ will be a strict functor $\mathcal{C}^{op} \to \mathbf{CAT}$ (i.e., $\eta = \mathrm{id}$ and $\mu = \mathrm{id}$), which we call a strict indexed category[3].

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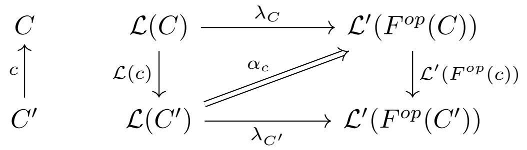
[2] A model of dependent type theory with Π-types and strong Σ-types.

[3] By the classical strictification constructions due to Giraud and Bénabou (see [Lumsdaine and Warren(2015), Section 2.2] for a good discussion), any indexed category is equivalent to a strict one (in two different ways, corresponding to left and right 2-adjoints to the inclusion of the category of strictly indexed categories into the category of indexed categories). This observation follows from the more recent general coherence theorem for pseudoalgebras, where indexed categories can be seen as pseudocoalgebras and pseudoalgebras as the forgetful 2-functor from the 2-category of indexed categories to the 2-category of families is 2-monadic and 2-comonadic [Blackwell et al.(1989), Lack(2002), Lucatelli Nunes(2016)].

Definition 2 (Morphisms of indexed categories)**.** We consider the following three notions of morphisms between indexed categories $(\mathcal{C}, \mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT})$ and $(\mathcal{C}', \mathcal{L}' : \mathcal{C}'^{op} \to \mathbf{CAT})$:

- colax morphisms: a pair of a functor $F : \mathcal{C} \to \mathcal{C}'$ and a colax natural transformations, i.e. for all objects C of $\mathcal{C}$ a functor $\lambda_C : \mathcal{L}(C) \to \mathcal{L}'(F^{op}(C))$ together with $\{\alpha_c : \lambda_{C'} \circ \mathcal{L}(c) \to \mathcal{L}'(F^{op}(c)) \circ \lambda_C\}_{C',C\in \mathrm{ob}\mathcal{C}, c\in\mathcal{C}(C',C)}$ making up a colax natural transformation:



$+$ the usual coherence conditions for α_c.

- pseudomorphisms: colax morphisms for which the 2-cell α is an isomorphism;
- strict morphisms: pseudomorphisms for which the 2-cell α is the identity.

We can further consider modifications between the morphisms to obtain a 2-categories of (strict) indexed categories colax/pseudo/strict morphisms and modifications.

We define the Grothendieck construction $\Sigma_{\mathcal{C}}\mathcal{L}$ as follows.

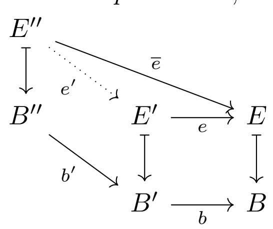
Definition 3 (Grothendieck construction)**.** For a $\mathcal{C}$-indexed category, we define the Grothendieck construction $\Sigma_{\mathcal{C}}\mathcal{L}$ to be the following category

- objects are dependent pairs (C, L) of an object C in $\mathcal{C}$ and an object L in $\mathcal{L}(C)$; i.e., $\mathrm{ob}\,\Sigma_{\mathcal{C}}\mathcal{L} \stackrel{\mathrm{def}}{=} \Sigma_{C\in \mathrm{ob}\,\mathcal{C}}\mathrm{ob}\,\mathcal{L}(C)$;
- morphisms $(C, L) \to (C', L')$ are dependent pairs (c, l) of a morphism $c : C \to C'$ in $\mathcal{C}$ and a morphism $l : L \to \mathcal{L}(c)(L')$ in $\mathcal{L}(C)$; i.e., $\Sigma_{\mathcal{C}}\mathcal{L}((C, L),(C', L')) \stackrel{\mathrm{def}}{=} \Sigma_{c\in\mathcal{C}(C,C')}\mathcal{L}(C)(L, \mathcal{L}(f)(L'))$;
- identities are defined as $\mathrm{id}_{(C,L)} \stackrel{\mathrm{def}}{=} (\mathrm{id}_C, \eta^C_L)$;
- composition of $(C'', L'') \xrightarrow{(c',l')} (C', L') \xrightarrow{(c,l)} (C, L)$ is defined as $(c, l)\circ(c', l') \stackrel{\mathrm{def}}{=} (c\circ c', \mu^{c',c}_L \circ \mathcal{L}(c')(l)\circ l')$.

Then, we have a functor $\pi_1 : \Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ that projects to the first component of the dependent pairs: $\pi_1(c, l) = c$. π_1 has the interesting property that it is a fibration. In fact, up to some choices of representatives of equivalence classes (of a cleavage, see below), fibrations are equivalent to indexed categories. In particular, this means that the fibration $\pi_1 : \Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ contains almost all the information present in the original category. We note that $\Sigma_{\mathcal{C}}\mathcal{L}$ is (locally) small (resp., large) if $\mathcal{C}$ and the fiber categories $\mathcal{L}(C)$ are (locally) small (resp., large).

We recall the basic definitions of Grothendieck fibrations.

Definition 4 (Cartesian lift)**.** Let $p : \mathcal{E} \to \mathcal{B}$ be some functor and let $b : B' \to B$ be a morphism in $\mathcal{B}$. We call a morphism $e : E' \to E$ a *cartesian lift* of b if $pe = b$ and for any other $b' : B'' \to B'$ in $\mathcal{B}$ and $\overline{e} : E'' \to E$ such that $p\overline{e} = b \circ b'$, there is a unique $e' : E'' \to E'$ such that $pe' = b'$ and $e \circ e' = \overline{e}$:



Given a functor $p : \mathcal{E} \to \mathcal{B}$, a morphism $b : B' \to B$ and an object E, we easily see that all cartesian lifts $e_{b,E}$ of b with codomain E are isomorphic. In that sense, cartesian lifts are unique up to isomorphism. They are not always guaranteed to exist, however.

Definition 5 ((Grothendieck) fibration)**.** We call a functor $p : \mathcal{E} \to \mathcal{B}$ a (Grothendieck) fibration if for every morphism $b : B' \to B$ and E in $\mathcal{E}$ such that $pE = B$, there *exists* a cartesian lift $e_{b,E}$ of b with codomain E.

We call a functor $p : \mathcal{E} \to \mathcal{B}$ for which $p^{op} : \mathcal{E}^{op} \to \mathcal{B}$ is a fibration an *opfibration.* A functor that is both a fibration and an opfibration is commonly called a *bifibration.*

Definition 6 (Cleavage)**.** Given a fibration $p : \mathcal{E} \to \mathcal{B}$, we call a choice $e_{b,E}$ of representatives of the isomorphism class of cartesian liftings of b along p with codmain E a *cleavage* of p. We call a fibration with a chosen cleavage a *cloven fibration*.

Definition 7 (Split fibration)**.** We a cloven fibration *split* if the cleavage has the property that $e_{\mathrm{id}_B,E} = \mathrm{id}_E$ and $e_{b,E} \circ e_{b',E'} = e_{b \circ b',E}$, where we write E' for the domain of $e_{b,E}$.

Definition 8 (Morphisms of Fibrations)**.** Given two fibrations $p : \mathcal{E} \to \mathcal{B}$ and $p' : \mathcal{E}' \to \mathcal{B}'$, we consider the following two notions of morphisms:

- colax morphisms: simply commutative squares of functors

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F_1} & \mathcal{E}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathcal{B} & \xrightarrow[F_0]{} & \mathcal{B}' \end{array}$$

- pseudomorphisms (sometimes called fibred functors): colax morphisms for which F_1 sends cartesian liftings along p of morphisms b in $\mathcal{B}$ to cartesian liftings along p' of $F_0(b)$.

If we have further chosen cleavages $e_{-,-}$ and $e'_{-,-}$ for both fibrations p and p', we can also consider the following:

- strict morphisms (sometimes called split functors): pseudomorphisms that respect the cleavages in the sense that $F_1(e_{b,E}) = e'_{F_0(b),F_1(E)}$.

Definition 9 (Fibred natural transformation)**.** Given two (colax) morphisms $(p : \mathcal{E} \to \mathcal{B}) \to (p' : \mathcal{E}' \to \mathcal{B}')$ of fibrations (F_1, F_0) and (G_1, G_0), then we consider fibred natural transformations as 2-cells: pairs of natural transformations $\alpha_1 : F_1 \to G_1$ and $\alpha_0 : F_0 \to G_0$ such that $p'\alpha_1 = \alpha_0 p$.

That gives us 2-categories of general/cloven/split fibrations, colax/pseudo/strict morphisms and fibred natural transformations.

The classic result relating fibrations and indexed categories is as follows. See for example [Shulman(2008), Proposition 3.8] or [Johnstone(2002), Proposition B1.3.6] for details.

Proposition 1. The Grothendieck construction defines an equivalence of 2-categories between

- indexed categories, colax morphisms of indexed categories, modifications;
- cloven fibrations, colax morphisms of fibrations, fibred natural transformations.

Further, these equivalences restrict in the following way:

- pseudomorphisms of indexed categories correspond to pseudomorphisms of fibrations;
- strict morphisms of indexed categories correspond to strict morphisms of cloven fibrations;
- strict indexed categories correspond to split fibrations;
- cloven bifibrations correspond precisely to indexed categories that factor over the category $\mathbf{Cat}_{Adj}$ of categories and adjunctions (i.e., cloven bifibrations are equivalent to indexed categories for which $\mathcal{L}(f)$ always has a left adjoint $\mathcal{L}_!(f)$) (see [Jacobs(1999), Lemma 9.1.2] for this result).

Proof sketch. We only sketch a very limited part of the proof that we will use and refer the reader to the cited references for more details. Given an colax (resp. pseudo, resp. strict) morphism $(F, \lambda, \alpha) : (\mathcal{C}, \mathcal{L}) \to (\mathcal{C}', \mathcal{L}')$ of indexed categories, we can define a functor $\Sigma_F \lambda : \Sigma_{\mathcal{C}} \mathcal{L} \to \Sigma_{\mathcal{C}'} \mathcal{L}'$ such that $(F, \Sigma_F \lambda)$ is an colax (resp. pseudo, resp. strict) morphism of fibrations. Indeed, we define $(\Sigma_F \lambda)(C, L) = (FC, \lambda_C(L))$ on objects and $(\Sigma_F \lambda)(c, l) = (Fc, ((\alpha_C)_L \circ -) \circ \lambda_{C'}(l)$ on morphisms $(c, l) : (C', L') \to (C, L)$. □

Assuming the axiom of choice, we can equip any fibration with a cleavage, showing precisely how fibrations generalize indexed categories.

From now on, we shall work with indexed categories (equivalently, cloven fibrations) and pseudomorphisms, unless explicitly stated otherwise. If we talk about indexed structures (such as indexed monoidal structures), we shall mean structures on indexed categories built out of pseudomorphisms of indexed categories. Similarly, if we talk about fibred structures (such as fibred monoidal structures), we shall mean structures on fibrations built out of pseudomorphisms of fibrations.

Example 1 (Pullback)**.** Given an indexed category $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ and a (pseudo)functor $F : \mathcal{D} \to \mathcal{C}$, we obtain a new indexed category $\mathcal{L} \circ F^{op} : \mathcal{D}^{op} \to \mathbf{CAT}$. By choosing some cleavage, this immediately shows that for fibrations $p : \mathcal{E} \to \mathcal{C}$, the pullback F^*p in $\mathbf{CAT}$ is also a fibration.

Example 2 (Composition)**.** The functor composition of two fibrations is another fibration. In the language of indexed categories, this tells us that given an indexed category $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ and an indexed category $\mathcal{L}' : (\Sigma_{\mathcal{C}}\mathcal{L})^{op} \to \mathbf{CAT}$, we can take the indexed Grothendieck construction to obtain an indexed category $(\Sigma_{\mathcal{L}}\mathcal{L}') : \mathcal{C}^{op} \to \mathbf{CAT}$ with fibres $(\Sigma_{\mathcal{L}}\mathcal{L}')(C) = \Sigma_{\mathcal{L}(C)}\mathcal{L}'(C, -)$.

Example 3 (Dual)**.** Postcomposition with the functor $op : \mathbf{CAT} \to \mathbf{CAT}$ lets us construct an indexed category $\mathcal{L}^{op} : \mathcal{C}^{op} \to \mathbf{CAT}$ from an existing indexed category $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$.

Example 4 (Domain fibration)**.** For any category $\mathcal{C}$, using function composition, the undercategories define an indexed category $\mathcal{L} : \mathcal{C}^{op} \to \mathcal{C}$: $\mathcal{L}(C) = C/\mathcal{C}$. The corresponding fibration is the domain functor from the arrow category $\mathrm{dom} : \mathcal{C}^{\to} \to \mathcal{C}$ (Note that the overcategories always define an opfibration for this reason.)

Example 5 (Codomain fibration)**.** For any category $\mathcal{C}$ with pullbacks, the codomain functor from the arrow category $\mathrm{cod} : \mathcal{C}^{\to} \to \mathcal{C}$ is a fibration. Using the axiom of choice, we can choose a cleavage and turn this into an indexed category with the overcategories of $\mathcal{C}$ as fiber categories.

Definition 10 (Locally indexed category)**.** We sometimes call a $\mathbf{CAT}(\mathcal{C}^{op}, \mathbf{Set})$-enriched category a *locally indexed category* (terminology introduced by [Levy(2012)]). The reason is that they can equivalently axiomatised as indexed categories $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ for which $\mathrm{ob}\,\mathcal{L}(C)$ does not depend on C and for which $\mathcal{L}(c)$ acts as the identity on objects, for any $c : C' \to C$.

Example 6 ($\mathcal{C}$-enriched category)**.** Any $\mathcal{C}$-enriched category $\mathcal{D}$ for a cartesian monoidal category $\mathcal{C}$ is in particular $\mathbf{CAT}(\mathcal{C}^{op}, \mathbf{Set})$-enriched, via the Yoneda embedding, so it defines a locally $\mathcal{C}$-indexed category.

Example 7 (Product self-indexing)**.** Given a category $\mathcal{C}$ with (chosen) finite products, we can define a locally indexed category $\mathrm{self}(\mathcal{C}) : \mathcal{C}^{op} \to \mathbf{CAT}$ by $\mathrm{ob}\,\mathrm{self}(\mathcal{C})(C) = \mathrm{ob}\,\mathcal{C}$ and $\mathrm{self}(\mathcal{C})(C)(C', C'') = \mathcal{C}(C \times C', C'')$. Observe that for $c : C' \to C$, we get an identity on objects functor $\mathrm{self}(\mathcal{C})(C) \to \mathrm{self}(\mathcal{C})(C')$ that acts on morphisms $f : C \times C_1 \to C_2$ as $f \circ (c \times \mathrm{id}_{C_1})$.

Example 8 (Scone)**.** Combining Examples 1 and 5, we get that for a functor $F : \mathcal{D} \to \mathcal{C}$ to a category with pullbacks $\mathcal{C}$, we get a $\mathcal{D}$-indexed category $\mathcal{L}$ with $\mathcal{L}(D) = \mathcal{C}/FD$. This indexed category is commonly known as the *Scone* or *Artin gluing* of F. It is commonly used in programming languages theory to structure logical relations arguments over a denotational semantics in $\mathcal{D}$ as a denotational semantics valued in $\Sigma_{\mathcal{D}}\mathcal{L}$ [Mitchell and Scedrov(1992)].

Example 9 (Lax comma)**.** Given a 2-category $\mathcal{C}$, we can take the $\mathbf{CAT}$-enriched Yoneda embedding to obtain a strict indexed category $\mathbf{2CAT}(-^{op}, \mathcal{C}) : \mathbf{Cat}^{op} \to \mathbf{CAT}$ of functors and colax natural transformations. The corresponding Grothendieck construction $\Sigma_{\mathbf{Cat}}\mathbf{2CAT}(-^{op}, \mathcal{C})$ is commonly known as the lax comma over $\mathcal{C}$. Similarly, we can take $\mathcal{C}$ to be the 2-category $\mathbf{Cat}$ of small categories, we get a large indexed category $\mathbf{2CAT}(-^{op}, \mathbf{Cat}) : \mathbf{Cat}^{op} \to \mathbf{CAT}$ of small strict indexed categories and colax natural transformations.

Example 10 (Families construction)**.** We can precompose the indexed category of Example 9 with the embedding $\mathbf{Set} \hookrightarrow \mathbf{Cat}$ as discrete categories, to obtain an indexed category $\mathbf{Cat}(-, \mathcal{C}) : \mathbf{Set}^{op} \to \mathbf{CAT}$. The Grothendieck construction $\Sigma_{\mathbf{Set}}\mathbf{Cat}(-, \mathcal{C})$ is commonly known as $\mathbf{Fam}(\mathcal{C})$ and is the free coproduct completion of $\mathcal{C}$.

2.2. **Limits in Grothendieck constructions.** We have the following general characterisation of fibred limits in a Grothendieck construction $\pi_1 : \Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$, in the sense of limits that are preserved by π_1. (A variation of this result appears as [Gray(1966), Theorem 4.2], but we are not aware of this precise phrasing appearing in the literature.)

Lemma 1 (Fibred limits in a Grothendieck construction)**.** A functor $J = (J_1, J_2) : \mathcal{E} \to \Sigma_{\mathcal{C}}\mathcal{L}$ has a fibred limit iff $J_1 : \mathcal{E} \to \mathcal{C}$ has a limit (L, λ) and $\mathcal{L}(\lambda)(J_2) : \mathcal{E} \to \mathcal{L}(L)$ has a limit that is preserved by $\mathcal{L}(u) : \mathcal{L}(L) \to \mathcal{L}(K)$ for any $u : K \to L$.

Proof. First, we need to explain what we mean by $\mathcal{L}(\lambda)(J_2) : \mathcal{E} \to \mathcal{L}(L)$. Observe that, for $e : E \to E'$ in $\mathcal{E}$, $J_2(e) : J_2(E) \to \mathcal{L}(J_1(e))(J_2(E'))$ is a morphism in $\mathcal{L}(J_1(E))$. Further, $\lambda_E : L \to J_1(E)$. Therefore, by naturality of λ,

$$\begin{aligned}\mathcal{L}(\lambda_E)(J_2(e)) \in &\mathcal{L}(L)(\mathcal{L}(\lambda_E)(J_2(E)), \mathcal{L}(\lambda_E)(\mathcal{L}(J_1(e))(J_2(E')))) \cong \\ &\mathcal{L}(L)(\mathcal{L}(\lambda_E)(J_2(E)), \mathcal{L}(J_1(e) \circ \lambda_E)(J_2(E'))) = \\ &\mathcal{L}(L)(\mathcal{L}(\lambda_E)(J_2(E)), \mathcal{L}(\lambda_{E'})(J_2(E')))).\end{aligned}$$

As a consequence of the functoriality of J, this gives us a functor $\mathcal{L}(\lambda)(J_2) : \mathcal{E} \to \mathcal{L}(L)$.

Then, observe that we have natural isomorphisms of homsets

$$\begin{aligned}
&\mathbf{CAT}(\mathcal{E}, \Sigma_{\mathcal{C}}\mathcal{L})(E \mapsto (C_1, C_2), (J_1, J_2)) \\
&= \Sigma_{f \in \mathbf{CAT}(\mathcal{E},\mathcal{C})(E \mapsto C_1, J_1)} \mathbf{CAT}(\mathcal{E}, \mathcal{L}(C_1))(E \mapsto C_2, E \mapsto \mathcal{L}(f_E)(J_2(E))) \\
&\cong \Sigma_{u \in \mathcal{C}(C_1, \lim_E J_1(E))} \mathbf{CAT}(\mathcal{E}, \mathcal{L}(C_1))(E \mapsto C_2, E \mapsto \mathcal{L}(\lambda_E \circ u)(J_2(E))) && \{ \text{ limit in } \mathcal{C} \ \} \\
&\cong \Sigma_{u \in \mathcal{C}(C_1, \lim_E J_1(E))} \mathcal{L}(C_1)(C_2, \lim_E \mathcal{L}(\lambda \circ u)(J_2)(E)) && \{ \text{ limit in } \mathcal{L}(C_1) \ \} \\
&\cong \Sigma_{u \in \mathcal{C}(C_1, \lim_E J_1(E))} \mathcal{L}(C_1)(C_2, \lim_E \mathcal{L}(u)(\mathcal{L}(\lambda)(J_2)(E))) && \{ \text{ pseudofunctoriality } \mathcal{L} \ \} \\
&\cong \Sigma_{u \in \mathcal{C}(C_1, \lim_E J_1(E))} \mathcal{L}(C_1)(C_2, \mathcal{L}(u)(\lim_E \mathcal{L}(\lambda)(J_2)(E))) && \{ \ \mathcal{L}(u) \text{ preserves limit } \} \\
&= \Sigma_{\mathcal{C}}\mathcal{L}((C_1, C_2), (\lim_E J_1(E), \lim_E \mathcal{L}(\lambda)(J_2)(E))).
\end{aligned}$$

□

[Lucatelli Nunes and Vákár(2023), Theorem 52] shows that a similar construction relates fibred terminal coalgebras of fibred endofunctors on $\Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ to pairs of a terminal coalgebra L in $\mathcal{C}$ and a terminal coalgebra in $\mathcal{L}(L)$ that is preserved by change of base.

2.3. **Colimits in Grothendieck constructions.** By duality (and as noted in [Gray(1966), Theorem 4.2]), Lemma 1 also tells us how to compute fibred colimits in an op-fibration. In particular, it applies to colimits in bifibrations $\pi_1 : \Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ in the sense of a fibration such that all $\mathcal{L}(f)$ have a left adjoint $\mathcal{L}_!(f)$. Indeed, in that case, $(\Sigma_{\mathcal{C}}\mathcal{L})^{op} \cong \Sigma_{\mathcal{C}^{op}}\mathcal{L}_!^{op} \to \mathcal{C}^{op}$, which lets us construct fibred colimits in $\Sigma_{\mathcal{C}}\mathcal{L}$ out of colimits in $\mathcal{C}$ and colimits in $\mathcal{L}$ that are preserved by change of base in $\mathcal{L}_!$. However, there are also other cases of colimits in $\Sigma_{\mathcal{C}}\mathcal{L}$ that we are interested in. In general, we have the following result. (We imagine that it is known, but have not found a reference to it in the literature.)

Lemma 2 (Fibred colimits in a Grothendieck construction)**.** A functor $J = (J_1, J_2) : \mathcal{E} \to \Sigma_{\mathcal{C}}\mathcal{L}$ has a fibred colimit iff $J_1 : \mathcal{E} \to \mathcal{C}$ has a colimit (L, λ) and the functor $\mathcal{L}(\lambda) : \mathcal{L}(L) \to \mathbf{CAT}(\mathcal{E}, \mathcal{L} \circ J_1^{op})$ given by $C_2 \mapsto E \mapsto \mathcal{L}(\lambda_E)(C_2)$ has a left adjoint $\mathcal{L}(\lambda)_!$. The colimit of J is then given by $(L, \mathcal{L}(\lambda)_!(J_2))$.

Proof. Indeed, we have the following natural isomorphisms of homsets

$$\begin{aligned}
&\mathbf{CAT}(\mathcal{E}, \Sigma_{\mathcal{C}}\mathcal{L})(J_1, J_2), E \mapsto (C_1, C_2)) \\
&= \Sigma_{f \in \mathbf{CAT}(\mathcal{E},\mathcal{C})(J_1, E \mapsto C_1)} \mathbf{CAT}(\mathcal{E}, \mathcal{L} \circ J_1^{op})(J_2, E \mapsto \mathcal{L}(f_E)(C_2)) \\
&\cong \Sigma_{g \in \mathcal{C}(\operatorname{colim}_E J_1(E), C_1)} \mathbf{CAT}(\mathcal{E}, \mathcal{L} \circ J_1^{op})(J_2, E \mapsto \mathcal{L}(g \circ \lambda_E)(C_2))) && \{ \text{ colimit in } \mathcal{C} \ \} \\
&\cong \Sigma_{g \in \mathcal{C}(\operatorname{colim}_E J_1(E), C_1)} \mathbf{CAT}(\mathcal{E}, \mathcal{L} \circ J_1^{op})(J_2, E \mapsto \mathcal{L}(\lambda_E)(\mathcal{L}(g)(C_2))) && \{ \text{ pseudofunctoriality } \mathcal{L} \ \} \\
&\cong \Sigma_{g \in \mathcal{C}(\operatorname{colim}_E J_1(E), C_1)} \mathcal{L}(\operatorname{colim}_E J_1(E))(\mathcal{L}(\lambda)_!(J_2), \mathcal{L}(g)(C_2)) && \{ \ \mathcal{L}(\lambda)_! \dashv \mathcal{L}(\lambda) \ \} \\
&= \Sigma_{\mathcal{C}}\mathcal{L}((\operatorname{colim}_E J_1(E), \mathcal{L}(\lambda)_!(J_2)), (C_1, C_2)).
\end{aligned}$$

□

We call an indexed category $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ *$\mathcal{E}$-extensive* if $\mathcal{L}$ preserves all limits of shape $\mathcal{E}$. In particular, in case $\mathcal{E}$ ranges over small discrete categories (sets), we obtain the notion of an extensive indexed category of [Lucatelli Nunes and Vákár(2023), Section 6.5]: a pseudofunctor $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ that preserves products. If further, $\mathcal{L}$ corresponds to the codomain fibration of $\mathcal{C}$ (with a chosen cleavage), then we retrieve the classical notion of (infinitary) extensive category.

Corollary 1 (Colimits in extensive fibrations)**.** For $\mathcal{E}$-extensive indexed categories $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$, $\Sigma_{\mathcal{C}}\mathcal{L}$ has colimits of shape $\mathcal{E}$.

Proof. In that case, we have an equivalence $\mathcal{L}(\mathrm{colim}^{\mathcal{C}}_{E} J_1(E)) \simeq \lim^{\mathbf{Cat}}_{E} \mathcal{L}(J_1(E)) \simeq \mathbf{CAT}(\mathcal{E}, \mathcal{L} \circ J_1^{op})$, so in particular have the left adjoint required by Lemma 2. □

In particular, we recover the results of [Lucatelli Nunes and Vákár(2023), Section 6.5] that show that extensive indexed categories have coproducts in their Grothendieck construction.

Further, [Lucatelli Nunes and Vákár(2023), Corollary 49] shows that a similar construction relates fibred initial algebras of fibred endofunctors on $\Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ to pairs of an initial algebra L in $\mathcal{C}$ and initial an algebra in $\mathcal{L}(L)$ that is preserved by change-of-base.

2.4. **Monoidal structures on Grothendieck constructions.** Lemma 1 tells us, in particular, that given a cartesian monoidal category $\mathcal{C}$, we have an equivalence between indexed cartesian monoidal structures on $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ (in the sense of monoidal structures on all fibres $\mathcal{L}(C)$ that are preserved by change-of-base functors $\mathcal{L}(f)$) and fibred cartesian monoidal structures (in the sense of being a fibred functor/pseudomorphism of fibrations) on $\Sigma_{\mathcal{C}}\mathcal{L}$ by taking $\mathbb{1}_{\Sigma_{\mathcal{C}}\mathcal{L}} = (\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{L}(\mathbb{1}_{\mathcal{C}})})$ and $(C, L) \times_{\Sigma_{\mathcal{C}}\mathcal{L}} (C', L') = (C \times_{\mathcal{C}} C', \mathcal{L}(\pi_1)(L) \times_{\mathcal{L}(C \times_{\mathcal{C}} C')} \mathcal{L}(\pi_2)(L'))$, where we write π_1 and π_2 for the product projections $C \xleftarrow{\pi_1} C \times C' \xrightarrow{\pi_2} C'$ in $\mathcal{C}$. As shown by Shulman, this correspondence works more generally for monoidal structures, braided monoidal structures, and symmetric monoidal structures on $\Sigma_{\mathcal{C}}\mathcal{L}$, as long as we make sure the monoidal structure on $\mathcal{C}$ is cartesian.

Lemma 3 (Monoidal structures on a Grothendieck construction, [Shulman(2008), Theorem 12.7])**.** *As long as $\mathcal{C}$ is a cartesian monoidal category*, the following definitions for the monoidal unit I and product $\otimes$ define an equivalence between fibred monoidal structures on $\Sigma_{\mathcal{C}}\mathcal{L}$ and indexed monoidal structures on $\mathcal{L}$:

$$I_{\Sigma_{\mathcal{C}}\mathcal{L}} = (\mathbb{1}_{\mathcal{C}}, I_{\mathcal{L}(\mathbb{1}_{\mathcal{C}})}) \quad \text{and} \quad (C, L) \otimes_{\Sigma_{\mathcal{C}}\mathcal{L}} (C', L') = (C \times_{\mathcal{C}} C', \mathcal{L}(\pi_1)(L) \otimes_{\mathcal{L}(C \times_{\mathcal{C}} C')} \mathcal{L}(\pi_2)(L'))$$
$$I_{\mathcal{L}(C)} = \mathcal{L}(!_C)(\pi_2(I_{\Sigma_{\mathcal{C}}\mathcal{L}})) \quad \text{and} \quad L \otimes_{\mathcal{L}(C)} L' = \mathcal{L}(\langle \mathrm{id}_C, \mathrm{id}_C \rangle)(\pi_2((C, L) \otimes_{\Sigma_{\mathcal{C}}\mathcal{L}} (C, L')))$$

Further, fibred braidings for $\otimes_{\Sigma_{\mathcal{C}}\mathcal{L}}$ are in 1-1 correspondence with indexed braidings for $\otimes_{\mathcal{L}}$, and a braiding for $\otimes_{\Sigma_{\mathcal{C}}\mathcal{L}}$ is symmetric precisely if the corresponding one for $\otimes_{\mathcal{L}}$ is.

(For the corresponding definitions for the associators, unitors and braidings, see [Shulman(2008)].)

2.5. **Monoidal closed structures on Grothendieck constructions.** A natural question is whether monoidal closure of $\pi_1 : \Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ is related to monoidal closure of (the fibre categories of) $\mathcal{L}$, and how. We dedicate the rest of this paper primarily to that question.

The closest existing result in this direction that we are aware of is the following result.

Lemma 4 ([Shulman(2008), Proposition 13.25])**.** Suppose that $\mathcal{L}$ is a indexed monoidal category over a cartesian monoidal category $\mathcal{C}$, such that $\mathcal{L}(f) : \mathcal{L}(C') \to (C)$ has a right adjoint $\mathcal{L}_*(f)$ for any $f : C \to C'$ in $\mathcal{C}$, satisfying the right Beck-Chevalley condition. Then,

- the functors $(-) \otimes_{\Sigma_{\mathcal{C}}\mathcal{L}} (C_2, L_2) : \mathcal{L}(C_1) \to \mathcal{L}(C_1 \times C_2)$ have right adjoints if and only if the functors $(-) \otimes_{\mathcal{L}(C)} L : \mathcal{L}(C) \to \mathcal{L}(C)$ have a right adjoint;
- the functors $(C_1, L_1) \otimes_{\Sigma_{\mathcal{C}}\mathcal{L}} (-) : \mathcal{L}(C_2) \to \mathcal{L}(C_1 \times C_2)$ have right adjoints if and only if the functors $L \otimes_{\mathcal{L}(C)} (-) : \mathcal{L}(C) \to \mathcal{L}(C)$ have a right adjoint.

As observed in [Shulman(2008), Remark 13.12], this result does not imply monoidal closure of $\Sigma_{\mathcal{C}}\mathcal{L}$. Interestingly, many naturally occurring monoidal closed structures on total categories $\Sigma_{\mathcal{C}}\mathcal{L}$ of fibrations are not fibred, and sufficient conditions for their construction are involved. In particular, the sufficient conditions we present in this paper will involve a technical condition on the monoidal structure that we call $\Sigma, \mathcal{C}$-cotractability.

This condition is, in particular, implied by indexed monoidal closure of $\mathcal{L}$, but is far more general. For the particular case of fibred monoidal closed structures, we can easily prove a variation on Shulman's result.

Lemma 5 (Fibred monoidal closure)**.** Suppose that $\mathcal{L}$ is a indexed monoidal category over a cartesian monoidal category $\mathcal{C}$.

- Assume that $\mathcal{L}(\pi_2) : \mathcal{L}(C') \to (C \times C')$ has a right adjoint[4] $\mathcal{L}_*(\pi_2)$ for any product projection $\pi_2 : C \times C' \to C'$ in $\mathcal{C}$, satisfying the right Beck-Chevalley condition in the sense that the canonical

[4]In terms of type theory, we can think of such a right adjoint as a kind of dependent function type Π_C.

map $\mathcal{L}(g) \circ \mathcal{L}_*(\pi_2) \to \mathcal{L}_*(\pi_2) \circ \mathcal{L}(\mathrm{id} \times g)$ is an isomorphism. Then, $\Sigma_{\mathcal{C}}\mathcal{L}$ is fibred monoidal left-closed (resp., right-closed) if $\mathcal{C}$ is cartesian closed and $\mathcal{L}$ is indexed monoidal left-closed (resp., right-closed):

$$(C, L) \multimap_{\Sigma_{\mathcal{C}}\mathcal{L}} (C', L') = (C \Rightarrow_{\mathcal{C}} C', \mathcal{L}_*(\pi_2)(\mathcal{L}(\pi_1)(L) \multimap_{\mathcal{L}(C \times (C \Rightarrow_{\mathcal{C}} C'))} \mathcal{L}(\mathrm{ev})(L')))$$

where π_1 and π_2 are the first and second product projection and $\mathrm{ev} : C \times_{\mathcal{C}} (C \Rightarrow_{\mathcal{C}} C') \to C'$ is the co-unit of the exponential adjunction in $\mathcal{C}$, and

- Assume that $\mathcal{L}(\langle \mathrm{id}_C, \mathrm{id}_C \rangle) : \mathcal{L}(C \times C) \to (C)$ has a right adjoint[5] $\mathcal{L}_*(\langle \mathrm{id}_C, \mathrm{id}_C \rangle)$ for any object C in $\mathcal{C}$, satisfying the right Beck-Chevalley condition in the sense that the canonical map[6] $\mathcal{L}(f) \circ \mathcal{L}_*(\langle \mathrm{id}, \mathrm{id} \rangle) \to \mathcal{L}_*(f^* \langle \mathrm{id}, \mathrm{id} \rangle) \circ \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle^* f)$ is an isomorphism (for any $f : C' \to C \times C$). Then, $\mathcal{C}$ is cartesian closed and $\mathcal{L}$ is indexed monoidal left-closed (resp., right-closed) if $\Sigma_{\mathcal{C}}\mathcal{L}$ is fibred monoidal left-closed (resp., right-closed):

$$L \multimap_{\mathcal{L}(C)} L' = \mathcal{L}(\Lambda(\mathrm{id}_{C \times C}))(\pi_2((C, L) \multimap_{\Sigma_{\mathcal{C}}\mathcal{L}} (C \times_{\mathcal{C}} C, \mathcal{L}_*(\langle \mathrm{id}_C, \mathrm{id}_C \rangle)(L''))))),$$

  where $\Lambda(\mathrm{id}_{C \times C}) : C \to C \Rightarrow_{\mathcal{C}} (C \times_{\mathcal{C}} C)$ corresponds to $\mathrm{id}_{C \times C} : C \times C \to C \times C$ under the exponential adjunction in $\mathcal{C}$.

In case $\mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)$ and $\mathcal{L}(\pi_2)$ both have right adjoints satisfying the right Beck-Chevalley condition, we have an equivalence between indexed monoidal left-closed (resp. right-closed) structures on $\mathcal{L}$ and fibred monoidal left-closed (resp. right-closed) structures on $\Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$.

Proof. Suppose that $\mathcal{C}$ is cartesian closed and that $\mathcal{L}$ is indexed monoidal left-closed. Then, we have the following natural isomorphisms

[5] As far as we are aware, right adjoints to $\mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)$ are not so commonly considered in type theory. More popular are left adjoints $\mathcal{L}_!(\langle \mathrm{id}, \mathrm{id} \rangle)$, which correspond to identity types. In fact, we can define this right adjoint in terms of the left adjoints to change of base and a biclosed monoidal structure. That is, a common type theoretic interpretation of $\mathcal{L}_*(\langle \mathrm{id}_C, \mathrm{id}_C \rangle)(L)$ is a type depending on two copies of C that is equal to $\mathcal{L}(\pi_1)(L)$ (or equivalently $\mathcal{L}(\pi_2)(L)$) if both copies of C are equal (i.e., above the diagonal) and equal to the terminal type $\mathbb{1}$ otherwise. Indeed, then we have the following natural isomorphisms:

$$\begin{array}{ll}
\mathcal{L}(C)(\mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)(L), L') \cong & \{ \text{ monoidal structure } \} \\
\mathcal{L}(C)(I \otimes \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)(L), L') \cong & \{ \text{ monoidal right-closure } \} \\
\mathcal{L}(C)(I, \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)(L) \multimap^r L') \cong & \{ \text{ pseudofunctoriality } \mathcal{L} \} \\
\mathcal{L}(C)(I, \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)(L) \multimap^r \mathcal{L}(\mathrm{id})(L')) \cong & \{ \text{ Cartesian monoidal } \mathcal{C} \} \\
\mathcal{L}(C)(I, \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)(L) \multimap^r \mathcal{L}(\pi_2 \circ \langle \mathrm{id}, \mathrm{id} \rangle)(L')) \cong & \{ \text{ pseudofunctor } \mathcal{L} \} \\
\mathcal{L}(C)(I, \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)(L) \multimap^r \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)(\mathcal{L}(\pi_2)(L'))) \cong & \{ \multimap^r \text{ indexed } \} \\
\mathcal{L}(C)(I, \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)(L \multimap^r \mathcal{L}(\pi_2)(L'))) \cong & \{ \mathcal{L}_!(\langle \mathrm{id}, \mathrm{id} \rangle) \dashv \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle) \} \\
\mathcal{L}(C)(\mathcal{L}_!(\langle \mathrm{id}, \mathrm{id} \rangle)(I), L \multimap^r \mathcal{L}(\pi_2)(L')) \cong & \{ \text{ monoidal right-closure } \} \\
\mathcal{L}(C)(\mathcal{L}_!(\langle \mathrm{id}, \mathrm{id} \rangle)(I) \otimes L, \mathcal{L}(\pi_2)(L')) \cong & \{ \text{ monoidal left-closure } \} \\
\mathcal{L}(C)(L, \mathcal{L}_!(\langle \mathrm{id}, \mathrm{id} \rangle)(I) \multimap^l \mathcal{L}(\pi_2)(L')) &
\end{array}$$

[6] Here, we use the following notational convention for pullback squares:

$$\begin{array}{ccc}
A & \xrightarrow{g^* f} & B \\
{\scriptstyle f^* g}\downarrow & & \downarrow{\scriptstyle g} \\
C & \xrightarrow[f]{} & D
\end{array}$$

$$
\begin{aligned}
&\Sigma_{\mathcal{C}}\mathcal{L}((C_1, L_1) \otimes (C_2, L_2), (C_3, L_3)) = \\
&\Sigma_{f \in \mathcal{C}(C_1 \times C_2, C_3)} \mathcal{L}(C_1 \times C_2)(\mathcal{L}(\pi_1)(L_1) \otimes \mathcal{L}(\pi_2)(L_2), \mathcal{L}(f)(L_3)) \cong && \{ \text{ monoidal left-closure of } \mathcal{L} \ \} \\
&\Sigma_{f \in \mathcal{C}(C_1 \times C_2, C_3)} \mathcal{L}(C_1 \times C_2)(\mathcal{L}(\pi_2)(L_2), \mathcal{L}(\pi_1)(L_1) \multimap \mathcal{L}(f)(L_3)) \cong && \{ \ \mathcal{L}(\pi_2) \dashv \mathcal{L}_*(\pi_2) \ \} \\
&\Sigma_{f \in \mathcal{C}(C_1 \times C_2, C_3)} \mathcal{L}(C_2)(L_2, \mathcal{L}_*(\pi_2)(\mathcal{L}(\pi_1)(L_1) \multimap \mathcal{L}(f)(L_3))) = && \{ \text{ cartesian closure of } \mathcal{C} \ \} \\
&\Sigma_{g \in \mathcal{C}(C_2, C_1 \Rightarrow C_3)} \mathcal{L}(C_2)(L_2, \mathcal{L}_*(\pi_2)(\mathcal{L}(\pi_1)(L_1) \multimap \mathcal{L}(\mathrm{ev} \circ (C_1 \times g))(L_3))) = && \{ \text{ cartesian monoidal structure } \mathcal{C} \ \} \\
&\Sigma_{g \in \mathcal{C}(C_2, C_1 \Rightarrow C_3)} \mathcal{L}(C_2)(L_2, \mathcal{L}_*(\pi_2)(\mathcal{L}(\pi_1 \circ (C_1 \times g))(L_1) \multimap \mathcal{L}(\mathrm{ev} \circ (C_1 \times g))(L_3))) \cong && \{ \text{ pseudofunctoriality of } \mathcal{L} \ \} \\
&\Sigma_{g \in \mathcal{C}(C_2, C_1 \Rightarrow C_3)} \mathcal{L}(C_2)(L_2, \mathcal{L}_*(\pi_2)(\mathcal{L}(C_1 \times g)(\mathcal{L}(\pi_1))(L_1) \multimap \mathcal{L}(C_1 \times g)(\mathcal{L}(\mathrm{ev})(L_3)))) \cong && \{ \ \multimap \text{ indexed } \} \\
&\Sigma_{g \in \mathcal{C}(C_2, C_1 \Rightarrow C_3)} \mathcal{L}(C_2)(L_2, \mathcal{L}_*(\pi_2)(\mathcal{L}(C_1 \times g)(\mathcal{L}(\pi_1)(L_1) \multimap \mathcal{L}(\mathrm{ev})(L_3)))) \cong && \{ \text{ Beck-Chevalley } \} \\
&\Sigma_{g \in \mathcal{C}(C_2, C_1 \Rightarrow C_3)} \mathcal{L}(C_2)(L_2, \mathcal{L}(g)(\mathcal{L}_*(\pi_2)(\mathcal{L}(\pi_1)(L_1) \multimap \mathcal{L}(\mathrm{ev})(L_3)))) = \\
&\Sigma_{\mathcal{C}}\mathcal{L}((C_2, L_2), (C_1 \Rightarrow C_3, \mathcal{L}_*(\pi_2)(\mathcal{L}(\pi_1)(L_1) \multimap \mathcal{L}(\mathrm{ev})(L_3)))) = \\
&\Sigma_{\mathcal{C}}\mathcal{L}((C_2, L_2), (C_1, L_1) \multimap (C_3, L_3)).
\end{aligned}
$$

So, our formula defines a left-exponential for $\Sigma_{\mathcal{C}}\mathcal{L}$.

Conversely, suppose that $\Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C}$ is fibred monoidal left-closed. Observe that by definition of $\Sigma_{\mathcal{C}}\mathcal{L}$, for any $c : C \to C'$, we have $(*)$:

$$\mathcal{L}(C)(L, \mathcal{L}(c')(L')) \cong \{(c, l) \in (\Sigma_{\mathcal{C}}\mathcal{L})((C, L), (C', L')) \mid c = c'\}.$$

Therefore, we have the following string of isomorphisms (natural in L, L', L''):

$$
\begin{aligned}
&\mathcal{L}(C)(L \otimes L', L'') = && \{ \text{ Lemma 3 } \} \\
&\mathcal{L}(C)(\mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle)(\pi_2((C, L) \otimes (C, L''))), L'') = && \{ \ \mathcal{L}(\langle \mathrm{id}, \mathrm{id} \rangle) \dashv \mathcal{L}_*(\langle \mathrm{id}, \mathrm{id} \rangle) \ \} \\
&\mathcal{L}(C \times C)(\pi_2((C, L) \otimes (C, L'')), \mathcal{L}_*(\langle \mathrm{id}, \mathrm{id} \rangle) L'') \cong && \{ \ \mathcal{L} \text{ pseudofunctor } \} \\
&\mathcal{L}(C \times C)(\pi_2((C, L) \otimes (C, L'')), \mathcal{L}(\mathrm{id}_{C \times C})(\mathcal{L}_*(\langle \mathrm{id}, \mathrm{id} \rangle) L'')) \cong && \{ \ (*) \ \} \\
&\{(c, l) \in (\Sigma_{\mathcal{C}}\mathcal{L})((C \times C, \pi_2((C, L) \otimes (C, L''))), (C \times C, \mathcal{L}_*(\langle \mathrm{id}, \mathrm{id} \rangle) L'')) \mid c = \mathrm{id}_{C \times C}\} = && \{ \ \Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C} \text{ fibred monoidal } \} \\
&\{(c, l) \in (\Sigma_{\mathcal{C}}\mathcal{L})((C, L) \otimes (C, L'), (C \times C, \mathcal{L}_*(\langle \mathrm{id}, \mathrm{id} \rangle) L'')) \mid c = \mathrm{id}_{C \times C}\} \cong && \{ \ \Sigma_{\mathcal{C}}\mathcal{L} \to \mathcal{C} \text{ fibred monoidal left-closed } \} \\
&\{(c', l') \in (\Sigma_{\mathcal{C}}\mathcal{L})((C, L'), (C, L) \multimap (C \times C, \mathcal{L}_*(\langle \mathrm{id}, \mathrm{id} \rangle L'')) \mid c' = \Lambda(\mathrm{id}_{C \times C})\} \cong && \{ \ (*) \ \} \\
&\mathcal{L}(C)(L', L(\Lambda(\mathrm{id}_{C \times C}))(\pi_2((C, L) \multimap (C \times C, \mathcal{L}_*(\langle \mathrm{id}, \mathrm{id} \rangle L''))))).
\end{aligned}
$$

We get naturality in C (hence indexedness of $\multimap$) from the Beck-Chevalley condition.

Finally, our formulas are easily seen to be pseudoinverse to each other, hence define an equivalence between indexed monoidal closed and fibred monoidal closed structures. □

Next, we turn to the more general case of closed structures on $\Sigma_{\mathcal{C}}\mathcal{L}$ that might not be fibred.

3. Σ-(co)tractable monoidal structures

3.1. Σ-(co)tractability. Next, we want to give more general sufficient conditions for fibred monoidal structures on $\Sigma_{\mathcal{C}}\mathcal{L}$ to be closed. To do so, we restrict our attention to a particularly well-behaved sort of monoidal structure on $\mathcal{L}$, which we call Σ-*tractable*. Essentially, they are monoidal structures for which we can decompose morphisms $A \to B \otimes C$ into a component that does not involve C and a residual morphism into C.

Definition 11 ($\Sigma, \mathcal{C}$-tractable monoidal structure)**.** Given a category $\mathcal{C}$ with a terminal object, we call a monoidal category $(\mathcal{L}, I, \otimes, a, l, r)$ $\Sigma, \mathcal{C}$-*tractable* if we have

- a functor $(-) \multimap T(-) : \mathcal{L}^{op} \times \mathcal{L} \to \mathcal{C}$;
- a functor $\mathfrak{d}^c : \mathbb{1} \downarrow ((-) \multimap T(-)) \to \mathcal{L}$, from the comma category of $\mathbb{1} : \mathbb{1} \to \mathcal{C}$ and $(-) \multimap T(-) : \mathcal{L}^{op} \times \mathcal{L} \to \mathcal{C}$;
- a natural isomorphism between functors $\mathcal{L}^{op} \times \mathcal{L} \times \mathcal{L} \to \mathbf{Set}$:

$$\mathcal{L}(A, B \otimes C) \cong \Sigma f \in \mathcal{C}(\mathbb{1}, A \multimap TB).\mathcal{L}(\mathfrak{d}^c(A, B, f), C)$$

Dually, we say that $(\mathcal{L}, I, \otimes, a, l, r)$ is $\Sigma, \mathcal{C}$-*cotractable* if $(\mathcal{L}^{op}, I, \otimes^{op}, a^{-1}, l^{-1}, r^{-1})$ is $\Sigma, \mathcal{C}$-tractable.

Most of the time, $\Sigma, \mathcal{C}$-tractable monoidal categories $\mathcal{L}$ arise from Σ-tractable monoidal categories in the following sense.

Definition 12 (Σ-tractable monoidal structure)**.** We call a monoidal category $(\mathcal{L}, I, \otimes, a, l, r)$ Σ-*tractable* if we have

- a functor $T : \mathcal{L} \to \mathcal{L}$;
- a functor $\mathfrak{d}^c : \mathcal{L} \downarrow T \to \mathcal{L}$, from the comma category $\mathcal{L} \downarrow T$ of the functors $\mathrm{id}_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ and $T : \mathcal{L} \to \mathcal{L}$;
- a natural isomorphism
$$\mathcal{L}(A, B \otimes C) \cong \Sigma f \in \mathcal{L}(A, TB).\mathcal{L}(\mathfrak{d}^c(A, B, f), C)$$
between functors
$$\mathcal{L}^{op} \times \mathcal{L} \times \mathcal{L} \to \mathbf{Set}.$$

Dually, we say that $(\mathcal{L}, I, \otimes, a, l, r)$ is Σ-*cotractable* if $(\mathcal{L}^{op}, I, \otimes^{op}, a^{-1}, l^{-1}, r^{-1})$ is Σ-tractable.

Lemma 6 (Σ-tractable, $\mathcal{C}$-enriched $\Rightarrow \Sigma, \mathcal{C}$-tractable)**.** Suppose that

- $\mathcal{L}$ is a Σ-tractable monoidal category;
- $\mathcal{L}(-,-) : \mathcal{L}^{op} \times \mathcal{L} \to \mathbf{Set}$ factors over $\mathcal{C}(\mathbb{1}, -) : \mathcal{C} \to \mathbf{Set}$ for some category $\mathcal{C}$ with a terminal object $\mathbb{1}$ (for example because $\mathcal{L}$ is enriched over $\mathcal{C}$).

Then, $\mathcal{L}$ is $\Sigma, \mathcal{C}$-tractable.

Proof. Write $(-) \multimap (-)$ for the functor $\mathcal{L}^{op} \times \mathcal{L} \to \mathcal{C}$, such that $\mathcal{C}(\mathbb{1}, A \multimap B) \cong \mathcal{L}(A, B)$. Then, using $T : \mathcal{L} \to \mathcal{L}$ and $(-) \multimap (-)$, we have a composite functor $(-) \multimap T(-) : \mathcal{L}^{op} \times \mathcal{L} \to \mathcal{C}$. Observe that we have an equivalence of comma categories $\mathbb{1} \downarrow ((-) \multimap T(-)) \simeq \mathcal{L} \downarrow T$ to get our desired $\mathfrak{d}^c$. We get the desired natural isomorphism by definition of $(-) \multimap (-)$. □

Corollary 2. Any Σ-tractable monoidal category $\mathcal{L}$ is, in particular, $\Sigma, \mathbf{Set}$-tractable.

We will be most interested in such Σ-tractable monoidal structures as they give us the vast majority of our examples of $\Sigma, \mathcal{C}$-tractable monoidal structures.

Given a Σ-tractable monoidal structure, observe that we have a natural transformation $\mathcal{L}(A, B \otimes C) \to \mathcal{L}(A, TB)$, hence, by the Yoneda lemma, a natural transformation $B \otimes C \to TB$.

Lemma 7. In case $\mathcal{L}$ has a terminal object $\mathbb{1}$ and a Σ-tractable monoidal structure, then $T \cong (-) \otimes \mathbb{1}$.

Proof. Take $C = \mathbb{1}$ in the definition of Σ-tractable monoidal structure:
$$\mathcal{L}(A, B \otimes \mathbb{1}) \cong \Sigma f \in \mathcal{L}(A, TB).\mathcal{L}(\mathfrak{d}^c(A, B, f), \mathbb{1}) \cong \Sigma f \in \mathcal{L}(A, TB).\mathbb{1} \cong \mathcal{L}(A, TB).$$
Then use that the Yoneda embedding is fully faithful. □

The basic example of Σ-(co)tractable monoidal structures are given by products and coproducts.

Example 11 (Products and coproducts)**.** A cartesian (resp., cocartesian) monoidal structure on $\mathcal{L}$ is always Σ-tractable (resp., Σ-cotractable) with $T = \mathrm{id}_{\mathcal{L}}$ and $\mathfrak{d}^c(A, B, f) = A$.

In Section 3.2, we study examples of categories for which the coproducts are Σ-tractable or the products are Σ-cotractable, which is not always guaranteed.

Example 12 (Monoidal closure)**.** Suppose that $\mathcal{L}$ has a left-closed monoidal structure. Then, $\mathcal{L}$ is $\Sigma, \mathcal{C}$-cotractable for any category $\mathcal{C}$ with a terminal object:

- we take $T(-) \multimap (-) : \mathcal{L}^{op} \times \mathcal{L} \to \mathcal{C}$ to be the constantly $\mathbb{1}$ functor $\Delta_{\mathbb{1}}$;
- we take $\mathfrak{d}^c = (\pi_1 \multimap \pi_2) : \mathbb{1} \downarrow \Delta_{\mathbb{1}} \cong (\mathcal{L}^{op} \times \mathcal{L}) \to \mathcal{L}$ to be the functor $(B, A, \mathrm{id} : \mathbb{1} \to \mathbb{1}) \mapsto B \multimap A$);
- we have the natural isomorphism
$$\mathcal{L}(B \otimes C, A) \cong \mathcal{L}(C, B \multimap A) \cong \Sigma f \in \mathcal{C}(\mathbb{1}, \mathbb{1}).\mathcal{L}(C, (\pi_1 \multimap \pi_2)(B, A, f))$$
because $B \multimap (-)$ is right adjoint to $B \otimes (-)$.

In case $\mathcal{L}$ further has an initial object $\mathbb{0}$, we can define $T : \mathcal{L} \to \mathcal{L}$ to be the constantly $\mathbb{0}$ functor $\Delta_{\mathbb{0}}$, to obtain the functor $T(-) \multimap (-) = \Delta_{\mathbb{1}}$. We see that $\mathcal{L}$ is Σ-cotractable in that case.

Concrete examples of such $\mathcal{L}$ are given by any cartesian closed category such as **Set**, **Pos**, or **Cat**. Another large class of examples of such $\mathcal{L}$ arises from Eilenberg-Moore categories S-$\mathbf{Alg}(\mathcal{C})$ for a commutative monad S on a symmetric monoidal closed category $\mathcal{C}$ with equalizers and coequalizers. We believe that [Keigher(1978)] is the original reference for the induced symmetric monoidal structure on S-**Alg**$(\mathcal{C})$ and [Kock(1971)] is the original reference for the construction of the closed structure; a concise account is given in [Vákár(2017), Theorem 2.3.3]. Other typical sources of monoidal closed categories (that are therefore Σ-tractable monoidal structures) are given by categories $\mathcal{V}$-$\mathbf{CAT}(\mathcal{C}^{op}, \mathcal{V})$ of $\mathcal{V}$-enriched presheaves on $\mathcal{C}$, equipped with the Day convolution [Day(1970)].

Example 13 (Product categories)**.** Observe that any product $\prod_{i\in I} \mathcal{V}_i$ of categories $\mathcal{V}_i$ with Σ-tractable (resp., Σ-cotractable) monoidal structures has a Σ-tractable (resp., Σ-cotractable) monoidal structure. Similarly, if $\mathcal{C}$ has products, then any product $\prod_{i\in I} \mathcal{V}_i$ of categories $\mathcal{V}_i$ with $\Sigma, \mathcal{C}$-tractable (resp., $\Sigma, \mathcal{C}$-cotractable) monoidal structures has a $\Sigma, \mathcal{C}$-tractable (resp., $\Sigma, \mathcal{C}$-cotractable) monoidal structure.

3.2. **Σ-tractable coproducts and Σ-cotractable products.** Recall that coproducts $B \sqcup C$ in a category $\mathcal{L}$ (like all colimits and left adjoints) are defined via a mapping-out property: morphisms $B \sqcup C \xrightarrow{a} A$ *out of* the coproduct are easy to analyse. Such a always correspond precisely to pairs of $B \xrightarrow{a_B} A$ and $C \xrightarrow{a_C} A$. Put differently, we have a natural isomorphism

$$\mathcal{L}(B \sqcup C, A) \cong \mathcal{L}(B, A) \times \mathcal{L}(C, A).$$

We can convert between both representations using coprojections and copairing. Indeed, that is precisely the universal property of coproducts. In particular, coproducts are always Σ-cotractable.

Problematically, however, we might not have any tools for analysing morphisms $A \to B \sqcup C$ *into* a coproduct. To be able to say anything about such morphisms, we need to impose extra axioms. The same goes for analysing morphisms $B \times C \to A$ out of a product.

Interestingly, large classes of coproducts (resp., products) we encounter in practice are Σ-tractable (resp., Σ-cotractable). We give some important classes of examples.

Example 14 (Biproducts)**.** Suppose that $\mathcal{L}$ has binary products and binary coproducts that coincide (for example, because $\mathcal{L}$ has biproducts/has finite products and is **CMon**-enriched). Then, these are Σ-tractable coproducts, and, by duality, Σ-cotractable products, by Example 11.

Concrete examples are the categories **CMon** of commutative monoids and homomorphisms and **Vect** of vector spaces and linear functions.

Example 15 (Cartesian closure)**.** By Example 12, products are $\Sigma, \mathcal{C}$-cotractable in a cartesian closed category $\mathcal{L}$ (for any $\mathcal{C}$ with a terminal object). Further, they are Σ-cotractable if $\mathcal{L}$ additionally has an initial object.

Example 16 (Extensive category)**.** Recall that a (finitely) extensive category $\mathcal{L}$ is a category with finite coproducts such that

$$(-) \sqcup (-) : \mathcal{L}/B \times \mathcal{L}/C \to \mathcal{L}/(B \sqcup C)$$

defines an equivalence. (That is, if the codomain fibration is extensive.) As a consequence, the equivalence inverse is given by the pullbacks $g \mapsto (\iota_1^* g, \iota_2^* g)$.

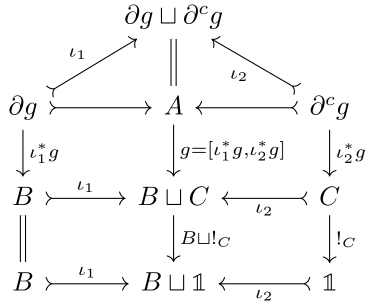
Assuming that $\mathcal{L}$ is extensive, let us write $\partial g \rightarrowtail A \leftarrowtail \partial^c g$ for the coproduct diagram that is obtained as the pullback along $g : A \to B \sqcup C$ of the coproduct diagram $B \rightarrowtail B \sqcup C \leftarrowtail C$.

Then, if $\mathcal{L}$ further has a terminal object, its coproducts are Σ-tractable:

- we have coproducts by assumption;
- we take $T = (-) \sqcup \mathbb{1} : \mathcal{L} \to \mathcal{L}$ to be the functor that takes the coproduct with the terminal object;
- we take $\mathfrak{d}^c : \mathcal{L} \downarrow (-) \sqcup \mathbb{1} \to \mathcal{L}$ to be the functor that takes $(A, B, f : A \to B \sqcup \mathbb{1})$ to the pullback $\partial^c f$;
- we have the natural isomorphism

$$\begin{array}{rcr} \mathcal{L}(A, B \sqcup C) & \cong & \Sigma f \in \mathcal{L}(A, B \sqcup \mathbb{1}).\mathcal{L}(\partial^c f, C) \\ g & \mapsto & ((B \sqcup !_C) \circ g, \iota_2^* g) \\ [\iota_1 \circ (\iota_1^* f), \iota_2 \circ f'] & \leftarrow\!\shortmid & (f, f'). \end{array}$$

because in the following diagram all commutative rectangles are pullbacks and all horizontal and diagonal arrows are coproduct inclusions



Some concrete examples are the categories **Set** of sets and functions and **Top** of topological spaces and continuous functions.

Example 17 (Free coproduct completions)**.** Consider the free coproduct completion $\mathbf{Fam}(\mathcal{C})$ of $\mathcal{C}$. Recall that $\mathbf{Fam}(\mathcal{C})$ has objects that are a pair of a set I and an I-indexed family $[C_i \mid i \in I]$ of $\mathcal{C}$-objects C_i. The homset $\mathbf{Fam}(\mathcal{C})([C_i \mid i \in I], [C'_{i'} \mid i' \in I'])$ is $\Pi_{i\in I}\Sigma_{i'\in I'}\mathcal{C}(C_i, C'_{i'})$.

In case $\mathcal{C}$ has a terminal object, then $\mathbf{Fam}(\mathcal{C})$ is an extensive category with a terminal object, so by Example 16, it is has Σ-tractable coproducts. However, even if $\mathcal{C}$ does not have a terminal object, $\mathbf{Fam}(\mathcal{C})$ always has $\Sigma, \mathbf{Set}$-tractable coproducts. Indeed, while we cannot define the monad $(-) \sqcup \mathbb{1}$ on $\mathbf{Fam}(\mathcal{C})$, unless $\mathcal{C}$ has a terminal object, we can always define the functor

$$(-) \multimap (-) \sqcup \mathbb{1} : \mathbf{Fam}(\mathcal{C})^{op} \times \mathbf{Fam}(\mathcal{C}) \to \mathbf{Set}$$

by

$$([C'_{i'} \mid i' \in I'], [C_i \mid i \in I]) \mapsto \Pi_{i'\in I'}\Sigma_{i\in I\sqcup\{\perp\}}\mathcal{C}(C_i, C'_{i'}) \text{ if } i \neq \perp \text{ else } \{\perp\}.$$

Further, we can define

$$\mathfrak{d}^c([C'_{i'} \mid i' \in I'], [C_i \mid i \in I], f) = [C'_{i'} \mid i' \in I', f(i') = \langle \perp, \perp \rangle].$$

Then,

$$\begin{aligned}
&\mathbf{Fam}(\mathcal{C})([C''_{i''} \mid i'' \in I''], [C'_{i'} \mid i' \in I'] \sqcup [C_i \mid i \in I]) \cong \\
&\mathbf{Fam}(\mathcal{C})([C''_{i''} \mid i'' \in I''], [\begin{array}{ll} C'_k & \text{if } k \in I' \\ C_k & \text{if } k \in I \end{array} \mid k \in I' \sqcup I]) = \\
&\Pi_{i''\in I''}\Sigma_{k\in I'\sqcup I}\mathcal{C}\left(C''_{i''}, \begin{array}{ll} C'_k & \text{if } k \in I' \\ C_k & \text{if } k \in I \end{array}\right) \cong \\
&\Sigma_{f\in\Pi_{i''\in I''}\Sigma_{k\in I'\sqcup\{\perp\}}\mathcal{C}(C''_{i''}, C'_{i'}) \text{ if } k\neq\perp \text{ else } \{\perp\}}\Pi_{i''\in I'', f(i'')=\langle\perp,\perp\rangle}\Sigma_{i\in I}\mathcal{C}(C''_{i''}, C_i) = \\
&\Sigma f \in \mathbf{Set}(\mathbb{1}, [C''_{i''} \mid i'' \in I''] \multimap [C'_{i'} \mid i' \in I'] \sqcup \mathbb{1}).\mathbf{Fam}(\mathcal{C})(\mathfrak{d}^c([C''_{i''} \mid i'' \in I''], [C'_{i'} \mid i' \in I'], f), [C_i \mid i \in I]).
\end{aligned}$$

By duality, products are always $\Sigma, \mathbf{Set}$-cotractable in a free product completion $\mathbf{Fam}(\mathcal{C}^{op})^{op}$ of $\mathcal{C}$.

Example 18 (Product categories)**.** Specialising Example 13, observe that any product of categories with Σ-tractable coproducts has Σ-tractable coproducts. This gives us examples of Σ-tractable coproducts that do not arise from our Examples 14, 15, and 16, like $\mathbf{Set}^{op} \times \mathbf{Set}$, which has Σ-tractable coproducts, but does not have biproducts (as **Set** does not have biproducts), is not (co)-cartesian (co)-closed (as $\mathbf{Set}^{op}$ is not cartesian closed), and is not extensive (as coproducts in $\mathbf{Set}^{op}$ are not disjoint). By a similar argument (as a self-dual category) $\mathbf{Set}^{op} \times \mathbf{Set}$ has Σ-cotractable products.

Example 19 (Partial functions)**.** Consider the category **pSet** of sets and partial functions. The coproduct $S \sqcup S'$ is the usual disjoint union of sets while the product $S \times_p S'$ is given by $S \times S' \sqcup S \sqcup S'$, where we write $\times_p$ for the product in **pSet** and $\times$ for the usual product in **Set**. Obviously, **pSet** does not have biproducts. Clearly, **pSet** is not distributive hence not extensive:

$$\begin{aligned}
X \times_p (Y \sqcup Z) &\cong X \times (Y \sqcup Z) \sqcup X \sqcup (Y \sqcup Z) \\
&\cong X \times Y \sqcup X \times Z \sqcup X \sqcup Y \sqcup Z
\end{aligned}$$

$$\begin{aligned} &\ncong X \times Y \sqcup X \times Z \sqcup X \sqcup Y \sqcup Z \sqcup X \\ &\cong X \times Y \sqcup X \sqcup Y \sqcup X \times Z \sqcup X \sqcup Z \cong X \times_p Y \sqcup X \times_p Z. \end{aligned}$$

Moreover, **pSet** with the cocartesian monoidal structure is not monoidal coclosed as $X \sqcup (-)$ does not preserve products:

$$\begin{aligned} X \sqcup (Y \times_p Z) &\cong X \sqcup Y \times Z \sqcup Y \sqcup Z \\ &\ncong X \sqcup Y \times Z \sqcup Y \sqcup Z \sqcup X \times X \sqcup X \times Z \sqcup Y \times X \sqcup X \\ &\cong X \times X \sqcup X \times Z \sqcup Y \times X \sqcup Y \times Z \sqcup X \sqcup Y \sqcup X \sqcup Z \\ &\cong (X \sqcup Y) \times (X \sqcup Z) \sqcup X \sqcup Y \sqcup X \sqcup Z \\ &\cong (X \sqcup Y) \times_p (X \sqcup Z) \end{aligned}$$

However, **pSet** does have Σ-tractable coproducts, for $T = \mathrm{id}$ and $\mathfrak{d}^c(A, B, f) = A \setminus f^{-1}(B)$. Indeed,

$$\mathbf{pSet}(A, B \sqcup C) \cong \Sigma f \in \mathbf{pSet}(A, B).\mathbf{pSet}(A \setminus f^{-1}(B), C).$$

This shows that the cocartesian structure on **pSet** is a Σ-tractable coproduct that does not arise from our Examples 14, 15, and 16.

Example 20 (Σ-tractable posets)**.** Let X be a poset with Σ-tractable coproducts. Observe that we have a natural transformation $X(x, y \vee z) \to X(x, Ty)$. Therefore, by the full and faithfulness of the Yoneda embedding, we get a morphism $y \vee z \leq Ty$ and, in particular, a morphism $z \leq Ty$. We see that Ty is the terminal object $\top$ of X. Therefore, the condition for Σ-tractability is that

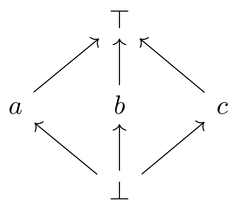
$$\begin{aligned} X(x, y \vee z) &\cong \Sigma f \in X(x, Ty).X(\mathfrak{d}^c(x, y, f), z) \cong X(x, Ty) \times X(\mathfrak{d}^c(x, y), z) \\ &\cong X(x, \top) \times X(\mathfrak{d}^c(x, y), z) \cong 1 \times X(\mathfrak{d}^c(x, y), z) \cong X(\mathfrak{d}^c(x, y), z). \end{aligned}$$

That is, X having finite coproducts that are Σ-tractable is equivalent to X^{op} being cartesian closed with an initial object (Example 15).

As an aside, note that posets with biproducts are trivial ($a \wedge b \leq a, b \leq a \vee b$ implies that $a \wedge b = a \vee b$ iff $a = b$) and extensive posets are trivial (extensive posets are distributive lattices, by definition, and disjointness of coproducts implies that $a \wedge b = \bot$ if $a \neq b$; in particular if $a < b$, $a = a \wedge b = \bot$).

The following is a counter example.

Counter example 1 (Non-distributive lattices)**.** From Example 20, we see that for any non-distributive lattice X (the typical examples being M_3 and N_5), X^{op} has coproducts that are not Σ-tractable. To make this counter example very concrete, consider the lattice M_3:



Then, $X \cong X^{op}$ is not distributive as $a \wedge (b \vee c) = a \wedge \top = a$ while $(a \wedge b) \vee (a \wedge c) = \bot \vee \bot = \bot$. In particular, $X \cong X^{op}$ is not cartesian closed (not a Heyting algebra), as $a \wedge (-)$ does not preserve coproducts. Therefore, the coproducts in X are not all Σ-tractable.

4. A Dialectica-like formula for the monoidal closure of Grothendieck constructions, based on Σ-cotractability

Now we turn to consider our novel sufficient conditions for monoidal closure of the Grothendieck construction. Our formula for the closed structure is a generalization of Gödel's Dialectica construction [Gödel(1958)], and it requires certain dependent types (Σ- and Π-types) to phrase. While the intuitions behind these dependent types are quite natural from a type-theoretic or proof-theoretic point of view, they are a bit verbose to phrase in terms of category theory. We formulate these precise conditions now.

4.1. Sufficient conditions for the monoidal closure of $\Sigma_{\mathcal{C}}\mathcal{L}$.

A model of (cartesian) dependent type theory $\mathcal{C}' : \mathcal{C}^{op} \to \mathbf{CAT}$. Let $\mathcal{C}' : \mathcal{C}^{op} \to \mathbf{CAT}$ be a model of (cartesian) dependent type theory with Π-types and strong Σ-types. That is, $\mathcal{C}'$ is an indexed category satisfying full and faithful democratic comprehension in the sense of [Vákár(2017), Definition 2.1.4]) or, equivalently, a full, cloven, democratic comprehension category (with unit) in the sense of [Jacobs(1999), Definitions 10.4.2, 10.4.7]) with Π-types, $\mathbb{1}$-types, and strong Σ-types ([Vákár(2017), Theorem 2.1.7] or, equivalently, ([Jacobs(1999), Definitions 10.5.1, 10.5.2(i)]). We spell out all the details here to be self-contained.

Definition 13 (Model of dependent type theory with Π-types and strong Σ-types)**.** We consider the following data when we speak of a model of dependent type theory:

- an indexed category $\mathcal{C}' : \mathcal{C}^{op} \to \mathbf{CAT}$ over a category $\mathcal{C}$ with a terminal object $\mathbb{1}$;
- indexed terminal objects in the sense of a right adjoint $\mathbb{1} : \mathcal{C} \to \Sigma_{\mathcal{C}}\mathcal{C}'$ to the projection $\pi_1 : \Sigma_{\mathcal{C}}\mathcal{C}' \to \mathcal{C}$;
- a further right adjoint $(\_.\_) : \Sigma_{\mathcal{C}}\mathcal{C}' \to \mathcal{C}$ to the indexed terminal object functor, such that the induced *comprehension functor* $\mathbf{p}_{\_} : \Sigma_{\mathcal{C}}\mathcal{C}' \to \mathcal{C}^{\to}$; $\mathbf{p}_{W,w} \stackrel{\text{def}}{=} \pi_1(\epsilon_{(W,w)})$ is full and faithful and restricts to an equivalence $\mathcal{C}'(\mathbb{1}) \simeq \mathcal{C}$, where we write ϵ for the co-unit of the adjunction $\mathbb{1} \dashv (-.-)$; given $W \in \mathrm{ob}\,\mathcal{C}$ and $w \in \mathrm{ob}\,\mathcal{C}'(W)$, we think of $W.w \in \mathrm{ob}\,\mathcal{C}$ as a Σ-type and $\mathbf{p}_{W,w} \in \mathcal{C}(W.w, W)$ as a projection that sends dependent pairs to their first component;
- Σ-types in the sense of left adjoint functors $\Sigma_w : \mathcal{C}'(W.w) \to \mathcal{C}'(W)$ to $\mathcal{C}'(\mathbf{p}_{W,w})$ that satisfy the left Beck-Chevalley condition, i.e., the canonical natural transformations $\Sigma_{\mathcal{C}'(f)(w)} \circ \mathcal{C}'(\mathbf{q}_{f,w}) \to \mathcal{C}'(f) \circ \Sigma_w$ are an isomorphism, where $\mathbf{q}_{f,w}$ is the unique morphism making the following square a pullback
$$\begin{array}{ccc} W'.\mathcal{C}'(f)(w) & \xrightarrow{\mathbf{q}_{f,w}} & W.w \\ \downarrow{\scriptstyle \mathbf{p}_{W',\mathcal{C}'(f)(w)}} & & \downarrow{\scriptstyle \mathbf{p}_{W,w}} \\ W' & \xrightarrow{\quad f \quad} & W; \end{array}$$
in particular, $\mathcal{C}'$ has indexed binary products given by $w \times w' = \Sigma_w \mathcal{C}'(\mathbf{p}_{W,w})(w')$ if $w, w' \in \mathrm{ob}\,\mathcal{C}'(W)$;
- Π-types in the sense of Σ-types in $\mathcal{C}'^{op}$; that is, right adjoint functors $\Pi_w : \mathcal{C}'(W.w) \to \mathcal{C}'(W)$ to $\mathcal{C}'(\mathbf{p}_{W,w})$ that satisfy the right Beck-Chevalley condition, i.e., the canonical natural transformations $\mathcal{C}'(f) \circ \Pi_w \to \Pi_{\mathcal{C}'(f)(w)} \circ \mathcal{C}'(\mathbf{q}_{f,w})$ are an isomorphism;
in particular, $\mathcal{C}'$ has indexed exponentials given by $w \Rightarrow w' = \Pi_w \mathcal{C}'(\mathbf{p}_{W,w})(w')$ if $w, w' \in \mathrm{ob}\,\mathcal{C}'(W)$; in fact, for our purposes, the weaker assumption of *non-indexed* Π-types in the sense of right adjoint functors $\Pi_w : \mathcal{C}'(\mathbb{1}.w) \to \mathcal{C}'(\mathbb{1})$ to $\mathcal{C}'(\mathbf{p}_{\mathbb{1},w})$ are enough (i.e., the case of $W = \mathbb{1}$);
- a *strong (or dependent)* elimination rule for the Σ-types in the sense that the canonical maps $\mathbf{p}_{W,w} \circ \mathbf{p}_{W.w,w'} \to \mathbf{p}_{W,\Sigma_w w'}$ are isomorphisms;

Example 21 (Families of sets)**.** For example, $\mathcal{C}$ could be **Set** and $\mathcal{C}'(S) \stackrel{\text{def}}{=} \mathbf{CAT}(S, \mathbf{Set})$, in which case $\Sigma_{\mathcal{C}}\mathcal{C}' = \mathbf{Fam}(\mathbf{Set})$, the category of families of sets (the free coproduct completion of **Set**). Then, the comprehension $(-.-) : \mathbf{Fam}(\mathbf{Set}) \to \mathbf{Set}$ is given by the disjoint union. The Σ-types are given by disjoint unions and Π-types are given by products. See, for example, [Hofmann and Hofmann(1997)] for more details.

Example 22 (Continuous families of ω-cpos)**.** Another typical example is to take $\mathcal{C}$ to be the category $\boldsymbol{\omega}\mathbf{CPO}$ of ω-cocomplete partial orders and ω-cocontinuous functors and to take $\mathcal{C}'(X) = \omega\mathbf{ContFunc}(X, \boldsymbol{\omega}\mathbf{CPO}_{ep})$ to be the category of ω-cocontinuous functors from X into the cateogry of ω-cpos and embedding-projection pairs and lax natural transformations. For more details on this model of ω-continuous families of ω-cpos, see, for example, [Palmgren and Stoltenberg-Hansen(1990), Ahman et al.(2016)].

Example 23 (Locally cartesian closed categories)**.** Another large source of examples is given by the codomain fibrations $\mathrm{cod} : \mathcal{C}^{\to} \to \mathcal{C}$ (with some choice of pullbacks to obtain a cleavage) of locally cartesian closed categories [Seely(1984), Clairambault and Dybjer(2014)]. In that case, we define $\mathcal{C}'(C) = \mathcal{C}/C$. Comprehension $(-.-) : \mathcal{C}^{\to} \to \mathcal{C}$ is given by the domain functor $\mathrm{dom} : \mathcal{C}^{\to} \to \mathcal{C}$. Σ-types are then given by composition and Π-types are given by the right adjoints to pullback functors.

Example 24 (Product self-indexing)**.** Given a cartesian closed category $\mathcal{C}$, we can form the locally indexed category $\mathrm{self}(\mathcal{C}) : \mathcal{C}^{op} \to \mathbf{CAT}$ (see Example 7). Then, $\mathrm{self}(\mathcal{C})$ is a model of dependent type theory. Indeed, the comprehension $(.-.) : \Sigma_{\mathcal{C}}\mathrm{self}(\mathcal{C}) \to \mathcal{C}$ is defined as $(C, C') \mapsto C \times C'$ on objects and $(f : C_1 \to C_2, g : C_1 \times C_1' \to C_2') \mapsto (\langle f \circ \pi_1, g\rangle : C_1 \times C_1' \to C_2 \times C_2')$ on morphisms. Strong Σ-types are defined as $\Sigma_C C' = C \times C$ and Π-types are defined as $\Pi_C C' = C \Rightarrow C'$.

Example 25 (Indexed category of indexed categories)**.** This example categorifies the families construction of Example 21 and replaces **Set** with **Cat** and **CAT** with **2CAT**. We have a model of dependent type theory: $\mathcal{C} = \mathbf{Cat}$ and $\mathcal{C}'(\mathcal{C}) = \mathbf{2CAT}(\mathcal{C}^{op}, \mathbf{Cat})_{colax}$ is the category of (strict) $\mathcal{C}$-indexed categories and colax natural transformations. This indexed category satisfies the comprehension axiom with $(-.-) : \Sigma_{\mathbf{Cat}}\mathbf{2CAT}(\mathcal{C}^{op}, \mathbf{Cat})_{colax} \to \mathbf{Cat}$ given by the Grothendieck construction [North(2019)]. It has (strong) Σ-types $\Sigma_{\mathcal{C}}\mathcal{L}$ given by the colax colimit, which exists as a functor $\Sigma_{\mathcal{D}} : \mathbf{2CAT}((\mathcal{C}.\mathcal{D})^{op}, \mathbf{Cat})_{colax} \to \mathbf{2CAT}(\mathcal{C}^{op}, \mathbf{Cat})_{colax}$ and is precisely the Grothendieck construction (as should be clear from Proposition 1; see [Gray(1974)] for the original reference and details – note that Gray calls these colax (co)limits quasi-(co)limits). It also has non-parameterised Π-types $\Pi_{\mathcal{C}}\mathcal{L}$ given by the colax limit, which exists as a functor $\Pi_{\mathcal{D}} : \mathbf{2CAT}(\mathcal{D}^{op}, \mathbf{Cat})_{colax} \cong \mathbf{2CAT}((\mathbb{1}.\mathcal{D})^{op}, \mathbf{Cat})_{colax} \to \mathbf{2CAT}(\mathbb{1}^{op}, \mathbf{Cat})_{colax} \cong \mathbf{Cat}$ and is given by the category of sections of the Grothendieck construction (i.e., functors $F : \mathcal{C} \to \Sigma_{\mathcal{C}}\mathcal{L}$ such that $\pi_1 F = \mathrm{id}_{\mathcal{C}}$ and natural transformations $\alpha : F \to G$ such that $\pi_1\alpha = \mathrm{id}_{\mathrm{id}_{\mathcal{C}}}$) [Gray(1974)].

Example 26 (Indexed groupoids)**.** [Hofmann and Streicher(1998)] restricts Example 25 to indexed groupoids indexed by another groupoid. That gives us another model of dependent type theory and it is the starting point for homotopy type theory, where people consider variants of this model based on ∞-groupoids rather than 1-groupoids [Kapulkin and Lumsdaine(2021)].

Needless to say, many other models exist, such as ones based on polynomials [Moss and von Glehn(2018)]. A Σ-cotractable indexed monoidal category $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ with Π-types. Further, assume that we have a model $\mathcal{L}$ of linear dependent type theory [Vákár(2017), Chapter 2] over the same base category $\mathcal{C}$, with a Σ-tractable monoidal structure, in the following sense:

- an indexed category $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$;
- $\mathcal{L}$ has Π-types in the sense of right adjoints $\Pi_w : \mathcal{L}(W.w) \to \mathcal{L}(W)$ to $\mathcal{L}(\mathbf{p}_{W,w})$ that satisfy the right Beck-Chevalley condition, i.e., the canonical natural transformations $\mathcal{L}(f) \circ \Pi_w \to \Pi_{\mathcal{C}'(f)(w)} \circ \mathcal{L}(\mathbf{q}_{f,w})$ are an isomorphism; in fact, for our purposes, the weaker assumption of *non-dependent* Π-types in the sense of right adjoint functors to $\mathcal{L}(\pi_1)$ for (non-dependent) product projections $\pi_1 : W \times W' \to W$ suffice;
- $\mathcal{L}$ has an indexed $\Sigma, \mathcal{C}'$-cotractable monoidal structure[7] in the sense of an indexed monoidal structure on $\mathcal{L}$ such that on each fibre $\mathcal{L}(C)$ the monoidal structure is $\Sigma, \mathcal{C}'(C)$-cotractable and $T(-) \multimap (-)$ and $\mathfrak{d}^c$ are $\mathcal{C}$-indexed functors.

For example, the fibre categories of $\mathcal{L}$ could have Σ-cotractable monoidal structure because they are monoidal closed with an initial object, because they have biproducts, or because they are co-extensive with an initial object).

Example 27 ($\mathcal{L} = \mathcal{C}'$)**.** Take $\mathcal{L} = \mathcal{C}' : \mathcal{C}^{op} \to \mathbf{CAT}$ to be any model of dependent type theory with Π-types and strong Σ-types. Then, $\mathcal{L}$ is an indexed cartesian closed category and $\mathcal{C}$ has a terminal object. By Example 12, products in $\mathcal{L}$ are $\Sigma, \mathcal{C}'$-cotractable. Observe that $\Sigma_{\mathcal{C}}\mathcal{L} = \Sigma_{\mathcal{C}}\mathcal{C}'$.

Example 28 ($\mathcal{L} = \mathcal{C}'^{op}$, extensive)**.** Take $\mathcal{C}' : \mathcal{C}^{op} \to \mathbf{CAT}$ to be any model of dependent type theory with Π-types and strong Σ-types. Further, assume that $\mathcal{C}'$ has indexed coproducts and that the categories $\mathcal{C}'(C)$ are extensive. Then, by Example 16, $\mathcal{L} = \mathcal{C}'^{op}$ has indexed Σ-cotractable products. Further, it has Π-types, given by the Σ-types of $\mathcal{C}'$. Observe that $\Sigma_{\mathcal{C}}\mathcal{L} = \Sigma_{\mathcal{C}}\mathcal{C}'^{op}$.

Example 29 (Coproducts/biproducts)**.** Let $\mathcal{C}' : \mathcal{C}^{op} \to \mathbf{CAT}$ be a model of dependent type theory with Π-types and strong Σ-types. Let $\mathcal{L} : \mathcal{C}^{op} \to \mathbf{CAT}$ be any indexed category with finite indexed coproducts, such that the hom-functor of $\mathcal{L}$ factors over $\mathcal{C}'$. (For example, we can take $\mathcal{L} = \mathcal{C}'^{op}$.) Seeing that coproducts always form a Σ-cotractable monoidal structure, it follows that they are a $\Sigma, \mathcal{C}'$-cotractable monoidal structure.

[7] Observe that this last condition is, in particular, implied by the following pair of conditions that often holds in practice:

– $\mathcal{L}$ is enriched over $\mathcal{C}'$, or more weakly, we have $\mathcal{L}-\ \multimap$-types in $\mathcal{C}'$ in the sense of that we have an indexed functor $(-) \multimap (-) : \mathcal{L}^{op} \times \mathcal{L} \to \mathcal{C}'$ and a natural isomorphism
$$\mathcal{L}(W)(A, B) \cong \mathcal{C}'(W)(\mathbb{1}, A \multimap B);$$
– $\mathcal{L}$ has an indexed Σ-cotractable monoidal structure in the sense of an indexed monoidal structure on $\mathcal{L}$ such that on each fibre $\mathcal{L}(C)$ the monoidal structure is Σ-cotractable and T and $\mathfrak{d}^c$ are $\mathcal{C}$-indexed functors.

Observe that the case where $\mathcal{L}$ has indexed biproducts is of particular interest as, in that case, $\mathcal{L}$ has $\Sigma, \mathcal{C}'$-cotractable products.

Example 30 (Locally indexed categories)**.** This Example builds on the choice $\mathcal{C}' = \mathrm{self}(\mathcal{C})$ of Example 24. Suppose that $\mathcal{D}$ is a $\mathcal{C}$-enriched category. Then, it, in particular, defines a locally $\mathcal{C}$-indexed category $\mathcal{L}(C)(D, D') = \mathcal{C}(C, \mathcal{D}(D, D'))$. If $\mathcal{D}$ is $\mathcal{L}$ is $\mathcal{C}$-powered in the sense that $\mathcal{C}(C, \mathcal{D}(D, D')) \cong \mathcal{D}(D, C \Rightarrow D')$, then $\mathcal{L}$ has Π-types: $\Pi_C D = C \Rightarrow D$. If $\mathcal{D}$ has a Σ-cotractable monoidal structure, then it meets our conditions.

Example 31 (Dual product self-indexed)**.** This Example builds on the choice $\mathcal{C}' = \mathrm{self}(\mathcal{C})$ for a cartesian closed category $\mathcal{C}$ of Example 24, and it specialises Example 29. Observe that $\mathrm{self}(\mathcal{C})^{op}$ is a (locally) $\mathcal{C}$-indexed category with indexed coproducts (products in $\mathcal{C}$). Further, it has Π-types given by $\Pi_C C' = C \times C'$ products in $\mathcal{C}$. Seeing that coproducts are always Σ-tractable and seeing that $\mathrm{self}(\mathcal{C})^{op}$ is $\mathrm{self}(\mathcal{C})$ enriched, it follows that $\mathrm{self}(\mathcal{C})^{op}$ has a $\Sigma, \mathrm{self}(\mathcal{C})$-cotractable monoidal structure.

Example 32 (Families)**.** Building on the choice of $\mathcal{C}'$ of Example 21, for any category $\mathcal{D}$ with a $\Sigma, \mathbf{Set}$-cotractable monoidal structure (for example, $\mathcal{D}$ monoidal closed, a free product completion, co-extensive with an initial object, or a category with biproducts) and small products, $\mathcal{L} : \mathbf{Set}^{op} \to \mathbf{CAT}$ with $\mathcal{L}(S) = \mathbf{CAT}(S, \mathcal{D})$ meets our conditions. The Π-types are given by products in $\mathcal{D}$ (see [Vákár(2015)]).

For example, we may take $\mathcal{D}$ to be a product-complete monoidal closed category such as a category of algebras for a commutative algebraic theory on **Set**. Observe that $\Sigma_{\mathcal{C}}\mathcal{L} = \mathbf{Fam}(\mathcal{D})$.

Example 33 (ω-Continuous families)**.** This Example builds on the choice of $\mathcal{C}'$ of Example 22. Given an $\boldsymbol{\omega}\mathbf{CPO}$-enriched Lawvere theory, we may take $\mathcal{D}$ to be its category of algebras in $\boldsymbol{\omega}\mathbf{CPO}$ and $\mathcal{L}(X) = \omega\mathbf{ContFunc}(X, \mathcal{D}_{ep})$ to be the $\boldsymbol{\omega}\mathbf{CPO}$-indexed category of ω-cocontinuous functors into the category of $\mathcal{D}$-objects and embedding-projection pairs. Then, $\mathcal{L}$ is an indexed monoidal closed category, hence an indexed $\Sigma, \mathcal{C}'$-cotractable monoidal category. Details are discussed in [Ahman et al.(2016), Section 6]. Observe that $\Sigma_{\mathcal{C}}\mathcal{L} = \omega\mathbf{ContFam}(\mathcal{D})$ is the category ω-continuous families of $\mathcal{D}$-objects.

Example 34 (Lextensive locally cartesian closed categories)**.** This example specializes Examples 23 and 28. Assume that $\mathcal{C}$ is a lextensive locally cartesian closed category category (for example, $\mathcal{C}$ an elementary topos). Consider the codomain fibration $\mathcal{C}'(C) = \mathcal{C}/C$, which we can turn into an indexed category by making use of the axiom of choice to choose pullbacks. Observe that $\mathcal{C}/C$ is also lextensive (lextensive categories are locally lextensive [Carboni et al.(1993), Proposition 4.8]) with a terminal object hence has Σ-tractable coproducts. Define $\mathcal{L} = \mathcal{C}'^{op}$. Then, $\mathcal{L}$ has Σ-cotractable products and Π-types (Σ-types in $\mathcal{C}'$). Observe that $\Sigma_{\mathcal{C}}\mathcal{L} = \Sigma_{\mathcal{C}}(\mathcal{C}/-)^{op}$ is a kind of generalised category of polynomials (or containers).

Example 35 (Lax comma)**.** This Example takes $\mathcal{C}'$ to be defined as in Example 25. Let $\mathcal{D}$ be some 2-category with colax limits. (For example, we already obtain many interesting examples for $\mathcal{D}$ a 1-category with limits.) We have a **Cat**-indexed category $\mathcal{L}(\mathcal{C}) = \mathbf{2CAT}(\mathcal{C}^{op}, \mathcal{D})_{colax}$. Its non-dependent Π-types are simply given by colax limits (ordinary limits, if $\mathcal{D}$ is a 1-category). If $\mathcal{D}$ has a $\Sigma, \mathbf{Cat}$-cotractable monoidal structure (such as a $\Sigma, \mathbf{Set}$-cotractable one), then $\mathcal{L}$ meets our conditions. Observe that $\Sigma_{\mathcal{C}}\mathcal{L} = \mathbf{Cat}//\mathcal{D}$ is the lax comma category of $\mathcal{D}$ in **Cat**. Some important subcases of this example are worked out in more detail in [Clementino et al.(2024)], where the structure of exponentials is presented in terms of ends.

4.2. Monoidal closure of $\Sigma_{\mathcal{C}}\mathcal{L}$.

We can now phrase our main theorem.

Theorem 1 (Monoidal closure of $\Sigma_{\mathcal{C}}\mathcal{L}$ via a Dialectica formula)**.** Assuming the conditions of Section 4.1, $\Sigma_{\mathcal{C}}\mathcal{L}$ is monoidal left-closed with

$$(X, x) \multimap (Y, y) \stackrel{\text{def}}{=} (\Pi_X \Sigma_Y T x \multimap y, \Pi_X \mathcal{L}(\zeta)(\mathfrak{d}^c \tilde{v}))$$

for two morphisms $\tilde{v}$ and ζ that we define below. By co-duality, we obtain monoidal right-closure if $\mathcal{L}^{co}$ is $\Sigma, \mathcal{C}'$-cotractable instead.

Proof. By Lemma 3, $(\mathbb{1}, I)$ is the monoidal unit of $\Sigma_{\mathcal{C}}\mathcal{L}$ and $(X \times Y, \mathcal{L}(\pi_1)(x) \otimes \mathcal{L}(\pi_2)(y))$ is the monoidal product of (X, x) and (Y, y) in $\Sigma_{\mathcal{C}}\mathcal{L}$.

The novel part is the existence of exponentials, which we turn to next. We have (natural) bijections (where, to aid legibility, we abuse notations a bit by leaving implicit: (1) some weakening functors $\mathcal{L}(\mathbf{p}_{W,w})$

and $\mathcal{C}'(\mathbf{p}_{W,w})$, (2) the equivalence $\mathcal{C}'(\mathbb{1}) \simeq \mathcal{C}$, and (3) isomorphisms $X.\Sigma_Y Z \cong X.Y.Z$ where they are obvious from the context):

$$
\begin{aligned}
&\Sigma_{\mathcal{C}}\mathcal{L}((X,x)\otimes(W,w),(Y,y)) = \\
&= \Sigma_{\mathcal{C}}\mathcal{L}((X\times W,\mathcal{L}(\pi_1)(x)\otimes\mathcal{L}(\pi_2)(w)),(Y,y)) \\
&= \Sigma_{f\in\mathcal{C}(X\times W,Y)}\mathcal{L}(X\times W)(\mathcal{L}(\pi_1)(x)\otimes\mathcal{L}(\pi_2)(w),\mathcal{L}(f)(y)) \\
&\cong \Sigma_{f\in\mathcal{C}(X\times W,Y)}\Sigma_{g\in\mathcal{C}'(X\times W)(\mathbb{1},T\mathcal{L}(\pi_1)(x)\multimap\mathcal{L}(f)(y))}\mathcal{L}(X\times W)(\mathcal{L}(\pi_2)(w),\mathfrak{d}^c g) && \{\ \otimes\,\Sigma,\mathcal{C}'\text{-cotractable}\ \} \\
&\cong \Sigma_{f\in\mathcal{C}'(X\times W)(\mathbb{1},Y)}\Sigma_{g\in\mathcal{C}'(X\times W)(\mathbb{1},T\mathcal{L}(\pi_1)(x)\multimap\mathcal{L}(f)(y))}\mathcal{L}(X\times W)(\mathcal{L}(\pi_2)(w),\mathfrak{d}^c g) && \{\ \text{comprehension } \mathcal{C}'\ \} \\
&\cong \Sigma_{(f,g)\in\Sigma_{f\in\mathcal{C}'(X\times W)(\mathbb{1},Y)}\mathcal{C}'(X\times W)(\mathbb{1},T\mathcal{L}(\pi_1)(x)\multimap\mathcal{L}(f)(y))}\mathcal{L}(X\times W)(\mathcal{L}(\pi_2)(w),\mathfrak{d}^c g) && \{\ \text{strong } \Sigma\text{-types in } \mathbf{Set}\ \} \\
&\cong \Sigma_{(f,g)\in\Sigma_{f\in\mathcal{C}'(X\times W)(\mathbb{1},Y)}\mathcal{C}'(X\times W)(\mathbb{1},T\mathcal{L}(\pi_1)(x)\multimap\mathcal{L}(f)(y))}\mathcal{L}(W)(w,\Pi_X\mathfrak{d}^c g) && \{\ \Pi\text{-types in } \mathcal{L}\ \} \\
&= \Sigma_{(f,g)\in\Sigma_{f\in\mathcal{C}'(X\times W)(\mathbb{1},Y)}\mathcal{C}'(X\times W)(\mathbb{1},Tx\multimap\mathcal{L}(f)(y))}\mathcal{L}(W)(w,\Pi_X\mathfrak{d}^c g) && \{\ \text{implicit } \mathcal{L}(\pi_1) \text{ for legibility}\ \} \\
&= \Sigma_{(f,g)\in\Sigma_{f\in\mathcal{C}'(X\times W)(\mathbb{1},Y)}\mathcal{C}'(X\times W)(\mathbb{1},Tx\multimap\mathcal{L}(f)(y))}\mathcal{L}(W)(w,\Pi_X\mathfrak{d}^c\mathcal{L}((\pi_1,f,g))(v)) && \{\ \text{definition } v\ \} \\
&\cong \Sigma_{(f,g)\in\Sigma_{f\in\mathcal{C}'(X\times W)(\mathbb{1},Y)}\mathcal{C}'(X\times W)(\mathbb{1},Tx\multimap\mathcal{L}(f)(y))}\mathcal{L}(W)(w,\Pi_X\mathcal{L}((\pi_1,f,g))(\mathfrak{d}^c v)) && \{\ \mathfrak{d}^c \text{ indexed functor}\ \} \\
&= \Sigma_{(f,g)\in\Sigma_{f\in\mathcal{C}'(X\times W)(\mathbb{1},Y)}\mathcal{C}'(X\times W)(\mathbb{1},Tx\multimap\mathcal{L}(f)(y))}\mathcal{L}(W)(w,\Pi_X\mathcal{L}(\zeta\circ(\Lambda(f,g),\pi_1))(\mathfrak{d}^c v))) && \{\ \text{definition } \zeta\ \} \\
&\cong \Sigma_{(f,g)\in\Sigma_{f\in\mathcal{C}'(X\times W)(\mathbb{1},Y)}\mathcal{C}'(X\times W)(\mathbb{1},Tx\multimap\mathcal{L}(f)(y))}\mathcal{L}(W)(w,\Pi_X\mathcal{L}((\Lambda(f,g),\pi_1))(\mathcal{L}(\zeta)(\mathfrak{d}^c v))) && \{\ \mathcal{L} \text{ pseudofunctor}\ \} \\
&\cong \Sigma_{(f,g)\in\Sigma_{f\in\mathcal{C}'(X\times W)(\mathbb{1},Y)}\mathcal{C}'(X\times W)(\mathbb{1},Tx\multimap\mathcal{L}(f)(y))}\mathcal{L}(W)(w,\mathcal{L}(\Lambda(f,g))(\Pi_X\mathcal{L}(\zeta)(\mathfrak{d}^c v))) && \{\ \text{Beck-Chevalley for } \Pi\ \} \\
&\cong \Sigma_{h\in\mathcal{C}'(X\times W)(\mathbb{1},\Sigma_Y Tx\multimap y))}\mathcal{L}(W)(w,\mathcal{L}(\Lambda(h))(\Pi_X\mathcal{L}(\zeta)(\mathfrak{d}^c v))) && \{\ \text{strong } \Sigma\text{-types in } \mathcal{C}'\ \} \\
&\cong \Sigma_{h\in\mathcal{C}'(X\times W)(\mathcal{L}(\pi_1)(\mathbb{1}),\Sigma_Y Tx\multimap y))}\mathcal{L}(W)(w,\mathcal{L}(\Lambda(h))(\Pi_X\mathcal{L}(\zeta)(\mathfrak{d}^c v))) && \{\ \text{indexed } \mathbb{1} \text{ in } \mathcal{C}'\ \} \\
&\cong \Sigma_{k\in\mathcal{C}'(W)(\mathbb{1},\Pi_X\Sigma_Y Tx\multimap y))}\mathcal{L}(W)(w,\mathcal{L}(k)(\Pi_X\mathcal{L}(\zeta)(\mathfrak{d}^c v))) && \{\ \Pi\text{-types in } \mathcal{C}'\ \} \\
&\cong \Sigma_{k\in\mathcal{C}(W,\Pi_X\Sigma_Y Tx\multimap y)}\mathcal{L}(W)(w,\mathcal{L}(k)(\Pi_X\mathcal{L}(\zeta)(\mathfrak{d}^c v))) && \{\ \text{comprehension } \mathcal{C}'\ \} \\
&= \Sigma_{\mathcal{C}}\mathcal{L}((W,w),(\Pi_X\Sigma_Y Tx\multimap y,\Pi_X\mathcal{L}(\zeta)(\mathfrak{d}^c v))).
\end{aligned}
$$

Here, we have used the obvious morphisms (again leaving weakening / change of base along projections implicit, for legibility):

$$
\begin{aligned}
&v\in\mathcal{C}'(X.Y.Tx\multimap y)(\mathbb{1},Tx\multimap y) && \{\ \text{representing element of the comprehension}\ \} \\
&\mapsto \mathfrak{d}^c v\in\mathcal{L}(X.Y.Tx\multimap y) && \{\ \mathfrak{d}^c:\mathbb{1}\downarrow T(-)\multimap(-)\to\mathcal{L}\ \} \\
&\mapsto \mathcal{L}(\zeta)(\mathfrak{d}^c v)\in\mathcal{L}(\Pi_X\Sigma_Y Tx\multimap y.X) && \{\ \text{change of base along } \zeta\ \} \\
&\mapsto \Pi_X\mathcal{L}(\zeta)(\mathfrak{d}^c v)\in\mathcal{L}(\Pi_X\Sigma_Y Tx\multimap y) && \{\ \Pi\text{-types in } \mathcal{L}\ \}
\end{aligned}
$$

and

$$\zeta = (\pi_2,\pi_1\circ\mathrm{ev},\pi_2\circ\mathrm{ev}) : \Pi_X\Sigma_Y Z.X\to X.Y.Z.$$

□

Observe that any of the Examples from Section 4.1 now give us monoidal closed Grothendieck constructions. We would like to highlight just a few concrete Examples, because they show up a lot in practice.

Example 36 (Monoidal closure of $\mathbf{Fam}(-)$-constructions)**.** By Examples 12, 14, 16, 17, and 19, we have that

- $\mathbf{Fam}(\mathcal{D})$ is monoidal left-closed (resp., right-closed) for any monoidal left-closed (resp., right-closed) category $\mathcal{D}$; in this case, the monoidal-closed structure on $\mathbf{Fam}(\mathcal{D})$ is fibred over the cartesian closed structure on **Set**:

$$[D_i\mid i\in I]\Rightarrow[D'_{i'}\mid i'\in I'] = \left[\prod_{i\in I}D_i\Rightarrow D_{f(i)}\mid f\in I\Rightarrow I'\right];$$

- $\mathbf{Fam}(\mathcal{D})$ is (non-fibred) cartesian closed for a category $\mathcal{D}$ with biproducts and small products (such as $\mathcal{D}=\mathbf{CMon}$ or $\mathcal{D}=\mathbf{CMon}^{op}$):

$$[D_i\mid i\in I]\Rightarrow[D'_{i'}\mid i'\in I'] = \left[\prod_{i\in I}D_{\pi_1(f(i))}\mid f\in\Pi_{i\in I}\Sigma_{i'\in I'}\mathcal{D}(D_i,D'_{i'})\right];$$

- $\mathbf{Fam}(\mathcal{D}^{op})$ is (non-fibred) cartesian closed for an extensive category $\mathcal{D}$ with a small coproducts and a terminal object (such as $\mathcal{D} = \mathbf{Set}$ or $\mathcal{D} = \mathbf{Top}$):

$$[D_i \mid i \in I] \Rightarrow [D'_{i'} \mid i' \in I'] = \left[\bigsqcup_{i \in I} \mathfrak{d}^c(\pi_2(f(i))) \mid f \in \Pi_{i\in I}\Sigma_{i'\in I'}\mathcal{D}(D'_{i'}, D_i \sqcup \mathbb{1})\right]$$

  here, $\mathfrak{d}^c(g)$ should be thought of as the complement of the domain of g; in particular, for $\mathcal{D} = \mathbf{Set}$ that is precisely what it is;
- free doubly-infinitary distributive categories $\mathbf{Dist}(\mathcal{C}) = \mathbf{Fam}(\mathbf{Fam}(\mathcal{C}^{op})^{op})$ are always (non-fibred) cartesian closed (see also [Lucatelli Nunes and Vákár(2024)]):

$$\begin{aligned}&[\langle C_{ji} \mid i \in I_j\rangle \mid j \in J] \Rightarrow [\langle C'_{j'i'} \mid i' \in I'_j\rangle \mid j' \in J'] = \\ &\quad [\langle C'_{j'i'} \mid j \in J, \langle j', g\rangle = f(j), i' \in I'_{j'}, g(i') = \langle \bot, \bot\rangle\rangle \mid \\ &\qquad\qquad f \in \Pi_{j\in J}\Sigma_{j'\in J'}\Pi_{i'\in I'_{j'}}\Sigma_{i\in I_j\sqcup\{\bot\}}\mathcal{C}(C_{ji}, C'_{j'i'}) \text{ if } i \neq \bot \text{ else } \{\bot\}];\end{aligned}$$

  using the same formula for the exponentials, we see that finite coproducts of products of $\mathcal{C}$-objects are exponentiable in the free infinitary distributive category on $\mathcal{C}$; further, [Nunes et al.(2024)] shows that a similar formula for exponentials also exists in free lextensive categories;
- $\mathbf{Fam}(\mathbf{pSet}^{op})$ is (non-fibred) cartesian closed:

$$[D_i \mid i \in I] \Rightarrow [D'_{i'} \mid i' \in I'] = \left[\bigsqcup_{i \in I} D'_{\pi_1(f(i))} \setminus (\pi_2(f(i)))^{-1}(D_i) \mid f \in \Pi_{i\in I}\Sigma_{i'\in I'}\mathbf{pSet}(D'_{i'}, D_i)\right];$$

  this example is reminiscent of the variant of the Dialectica interpretation discussed by [Biering(2008)].

Example 37 (Monoidal closure of lax comma categories)**.** We can categorify Example 36 by building on Example 35. Theorem 1 tells us that the lax comma category $\mathbf{Cat}//\mathcal{D}$ is monoidal closed for any small complete category $\mathcal{D}$ with a $\Sigma, \mathbf{Cat}$-cotractable monoidal structure (including any $\Sigma, \mathbf{Set}$-cotractable one). In particular, this is true for any small complete category $\mathcal{D}$ that is monoidal closed, has finite biproducts, or is co-extensive with an initial object. In the last two cases, $\mathbf{Cat}//\mathcal{D}$ is cartesian closed. Similarly, $\mathbf{Cat}//\mathbf{pSet}^{op}$ is cartesian closed.

Example 38 (Predicate-free Dialectica)**.** Building on Example 31, we have a symmetric monoidal structure $(U, X) \otimes (V, Y) = (U \times V, X \times Y)$ on $\mathbf{Dial}_{pf} = \Sigma_{\mathcal{C}}\mathrm{self}(\mathcal{C})^{op}$. This has a corresponding closed structure: $(U, X) \multimap (V, Y) = (U \Rightarrow V \times (Y \Rightarrow X), U \times Y)$. This is a predicate-free version of the Dialectica interpretation [Gödel(1958)]. The original Dialectica interpretation has a further fibration of predicates over this category, which we omit as it would distract from the main point of this paper. See Section 5 and [Hyland(2002)] for details.

Example 39 (Predicate-free Diller-Nahm)**.** Building on Example 30, assume that $\mathcal{D}$ is a $\mathcal{C}$-enriched category with biproducts and $\mathcal{C}$-copowers $C \otimes D$. Then, $\mathcal{L}(C)(D, D') = \mathcal{C}(C, \mathcal{D}^{op}(D, D'))$ defines a locally $\mathcal{C}$-indexed category with Π-types given by $\Pi_C D = C \otimes D$. It then follows that $\mathbf{Dill}_{pf} = \Sigma_{\mathcal{C}}\mathcal{L}$ is cartesian closed with products given by $(U, X) \times (V, Y) = (U \times V, X \times Y)$ and exponentials given by $(U \Rightarrow V \times \mathcal{D}(Y, X), U \otimes Y)$. This is a predicate-free version of the Diller-Nahm interpretation, where one classically considers the case where $\mathcal{D}$ is the Kleisli category for an additive monad on $\mathcal{C}$. Like the Dialectica interpretation, the Diller-Nahm variant can also be extended with a further fibration of predicates over this category. See Section 5 and [Hyland(2002)] for details.

Example 40 (Fibred closed structures)**.** From the formula given in Theorem 1, it is immediately clear that the monoidal left-closed structure on $\Sigma_{\mathcal{C}}\mathcal{L}$ will be fibred, if $\mathcal{L}$ is an $\mathcal{C}$-indexed left-closed monoidal category (Example 12), as we can then choose $Tx \multimap y = \mathbb{1} \in \mathcal{C}'(W)$ for all $x, y \in \mathcal{L}(W)$, resulting in the formula

$$(X, x) \multimap (Y, y) \cong (X \Rightarrow Y, \Pi_X x \multimap \mathcal{L}(\mathrm{ev})(y)),$$

for the left-exponential in $\Sigma_{\mathcal{C}}\mathcal{L}$. The converse also holds: if the monoidal left-closed structure resulting from Theorem 1 is fibred, then $Tx \multimap y \cong \mathbb{1} \in \mathcal{C}'(W)$. Then, $\Sigma, \mathcal{C}$-cotractability of $\otimes$ tells us that

$$\mathcal{L}(W)(y \otimes z, x) \cong \Sigma f \in \mathcal{C}'(W)(\mathbb{1}, Tx \multimap y).\mathcal{L}(W)(z, \mathfrak{d}^c(x, y, f)) \cong \mathcal{L}(W)(z, \mathfrak{d}^c(x, y, !_{\mathbb{1}})).$$

We see that $\mathcal{L}(W)$ is monoidal left-closed with exponential $y \multimap x = \mathfrak{d}^c(x, y, !_{\mathbb{1}})$, which is an indexed functor, as $\mathfrak{d}^c$ and $\mathbb{1}$ are. Co-dually, we get fibred right-exponentials in $\Sigma_{\mathcal{C}}\mathcal{L}$ from our Theorem 1 if and only if $\mathcal{L}$ is an indexed monoidal right-closed category.

These results are mostly a special case of those of Lemma 5.

Example 41 (Cartesian closure for indexed co-extensive categories)**.** We build on Example 28. In the special case that $\mathcal{L}$ is an indexed extensive category with an indexed terminal object $\mathbb{1}$, $\Sigma_{\mathcal{C}}\mathcal{L}^{op}$ is cartesian closed and we have the following formula for exponentials:

$$(X, x) \Rightarrow (Y, y) = (\Pi_X \Sigma_Y \mathcal{L}(\pi_2)(y) \multimap (\mathcal{L}(\pi_1)(x) \sqcup \mathbb{1}), \Sigma_X \mathcal{L}(\zeta)(\partial^c \tilde{v})),$$

i.e. the second component is the Σ-type (sum, in $\mathcal{L}$, so product in $\mathcal{L}^{op}$) of all complements of the domains $\partial^c(g)$ of definition of the morphisms $g : \mathcal{L}(\pi_2)(y) \multimap \mathcal{L}(\pi_1)(x) \sqcup \mathbb{1}$ (which we think of as partial functions) in the first component. This special case can be seen as a generalisation of the results of [Altenkirch et al.(2010)] on higher-order containers.

Example 42 (Cartesian closure for indexed coproduct/biproduct categories)**.** We build on Example 29. In the special case that $\mathcal{L}$ is a $\mathcal{C}'$-enriched indexed category with indexed coproducts and Π-types, $\Sigma_{\mathcal{C}}\mathcal{L}$ is symmetric monoidal closed and we have the following formula for exponentials:

$$(X, x) \multimap (Y, y) = (\Pi_X \Sigma_Y \mathcal{L}(\pi_1)(x) \multimap \mathcal{L}(\pi_2)(y), \Pi_X \mathcal{L}(\mathrm{ev1})(y)),$$

where we use the obvious morphism

$$\mathrm{ev1} : \Pi_X \Sigma_Y Z.X \to Y,$$

that is, the morphism obtained as the composition (where we write π_1 for the projection $\Sigma_Y Z \to Y$)

$$\Pi_X \Sigma_Y Z.X \cong (\Pi_X \Sigma_Y Z) \times X \xrightarrow{(\Pi_X \pi_1) \times X} (\Pi_X Y) \times X \cong (X \Rightarrow Y) \times X \xrightarrow{\mathrm{ev}} Y.$$

Of particular interest are the cases that

- $\mathcal{L} = \mathcal{C}'^{op}$: in this case, required the Π-types and coproducts always exist (as $\mathcal{C}'$ has Σ-types) and the $\mathcal{C}'$-enrichment exists as $\mathcal{C}'$ has Π-types so its fibres are cartesian closed; we conclude that for any model $\mathcal{C}' : \mathcal{C}^{op} \to \mathbf{CAT}$ of dependent type theory with Π-types and strong Σ-types, $\Sigma_{\mathcal{C}}\mathcal{C}'^{op}$ is symmetric monoidal closed;
- $\mathcal{L}$ has biproducts: in this case, the monoidal structure, if it exists is a cartesian one, giving us a cartesian closed structure on $\Sigma_{\mathcal{C}}\mathcal{L}$, assuming that the required $\mathcal{C}'$-enrichment and Π-types exist; further, observe that $\Sigma_{\mathcal{C}}\mathcal{L}^{op}$ is then also cartesian closed as long as the required Σ-types exist in $\mathcal{L}$:

$$(X, x) \Rightarrow (Y, y) = (\Pi_X \Sigma_Y \mathcal{L}(\pi_2)(y) \multimap \mathcal{L}(\pi_1)(x), \Sigma_X \mathcal{L}(\mathrm{ev1})(y))$$

This shows that we reproduce the results of [Moss(2018), Proposition 4.6.1] and [Lucatelli Nunes and Vákár(2023), Section 6.4], as a special case of Theorem 1.

Finally, we can also use our result as a tool to show that a monoidal structure is not Σ-cotractable.

Counter example 2. Let $\mathcal{D}$ be a infinitary distributive category, i.e. a category with small coproducts and finite products such that $\bigsqcup : \mathbf{Fam}(\mathcal{D}) \to \mathcal{D}$ preserves finite products. Suppose further that $\mathcal{D}$ is not cartesian closed. For example, $\mathcal{D}$ could be the category of locally connected topological spaces and continuous functions [Lucatelli Nunes and Vákár(2024), Example 8] or the category of finite dimensional smooth manifolds of varying dimension and smooth functions [Huot et al.(2020), Appendix A]. Then, by [Lucatelli Nunes and Vákár(2024), Theorem 4.2], $\mathbf{Fam}(\mathcal{D})$ is not cartesian closed. As a consequence, by Example 36, it follows that the products in $\mathcal{D}$ are not Σ-cotractable.

5. Related work and outlook

Dialectica and Diller-Nahm interpretations (with predicates). The earliest examples of similar techniques for constructing exponentials on Grothendieck constructions that we are aware of arose in proof theory when demonstrating the relative consistency of Heyting arithmetic: Gödel's Dialectica interpretation [Gödel(1958)] and Diller and Nahm's **CMon**-enriched variant of that interpretation [Diller(1974)]. In Examples 38 and 40, we give simplified (predicate-free) presentations $\mathbf{Dial}_{pf}$ and $\mathbf{Dill}_{pf}$ of these constructions. Here, we briefly point out how to extend them with predicates, following [Hyland(2002)]'s categorical presentation of these interpretations. We first quote the presentation in [Hyland(2002)] for definitions and next briefly explain how the closed structures are obtained from Theorem 1 by building on the closed structures described in Examples 38 and 40.

> [[Hyland(2002)], Dialectica] Suppose that we have a category T which we can think of as interpreting some type theory; and suppose that over the category T we have a pre-ordered fibration $p : P \to T$, which we can regard as providing for each $I \in T$ a pre-ordered collection of (possibly non-standard) predicates $P(I) = (P(I), \vdash)$. Starting with this data we construct a new category **Dial** = **Dial**(p) which we regard as a category of propositions and proofs. We do this as follows.
>
> - The objects A of **Dial** are $U, X \in T$ together with $\alpha \in P(U \times X)$. $(\cdots)$ Our understanding of the predicate α is not symmetric as regards U and X : we read α as $\exists u \in U.\forall x \in X.\alpha(u, x)$, in accord with the form of propositions in the image of the Dialectica interpretation.
> - Maps of **Dial** from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ are $(\cdots)$ of the form $f : U \to V$, $F : U \times Y \to X$ with $\alpha(u, F(u, y)) \vdash \beta(f(u), y)$ in $P(U \times Y)$.

We can observe that[8] **Dial** is precisely the category

$$\Sigma_{U\in T}\mathbf{Dial}(U) \qquad \text{for} \qquad \mathbf{Dial}(U) = (\Sigma_{X\in \mathrm{self}(T)(U)} P(U \times X)^{op})^{op}.$$

It is a more involved version of the category discussed in Example 38, where we additionally endow all objects with predicates.

If we assume that $P \to T$ is fibred cartesian closed, it follows from our Theorem 1 that **Dial** is monoidal closed for the (symmetric) monoidal structure $(U, X, \alpha) \otimes (V, Y, \beta) = (U \times V, X \times Y, \alpha \wedge \beta)$. Then, $(V, Y, \beta) \multimap (W, Z, \gamma) = ((V \Rightarrow W) \times (V \times Z \Rightarrow Y), V \times Z, \rho)$, where $\rho((g, G),(v, z)) = \beta(v, G(v, z)) \Rightarrow \gamma(g(v), z)$. Indeed, the indexed monoidal structure on **Dial**(U) with unit $(\mathbb{1}, \top)$ and product $(X, \alpha) \otimes (X', \alpha) = (X \times X', P(\mathrm{id} \times \pi_1)(\alpha) \wedge P(\mathrm{id} \times \pi_2)(\alpha')$ is $\Sigma, \mathrm{self}(T)(U)$-cotractable because

$$\begin{aligned}
&\mathbf{Dial}(U)((X, \alpha) \otimes (X', \alpha'),(X'', \alpha'')) = \\
&\{F : U \times X'' \to X \times X' \mid \alpha(u, \pi_1(F(u, x''))) \wedge \alpha'(u, \pi_2(F(u, x''))) \vdash \alpha''(u, x'')\} = \\
&\Sigma F_1 \in \mathrm{self}(T)(U)(\mathbb{1}, X'' \Rightarrow X).\mathbf{Dial}(U)((X', \alpha'),(X'', \rho)),
\end{aligned}$$

where $\rho(u, x'') = \alpha(u, F_1(u)(*)(x'')) \Rightarrow \alpha''(u, x'')$. Therefore,

$$\begin{aligned}
&\mathbf{Dial}(U)((X', \alpha'),(X'', \rho)) = \\
&\quad \{F_2 : U \times X'' \to X' \mid \alpha'(u, F_2(u, x'')) \vdash \alpha(u, F_1(u)(*)(x'')) \Rightarrow \alpha''(u, x'')\},
\end{aligned}$$

showing that we can choose $T(X'', \alpha'') \multimap (X, \alpha) = X'' \Rightarrow X$ and $\mathfrak{d}^c((X'', \alpha''),(X, \alpha), F_1) = (X'', \rho)$ where $\rho(u, x'') = \alpha(u, F_1(u)(*)(x'')) \Rightarrow \alpha''(u, x'')$. Further, the indexed category $U \mapsto \mathbf{Dial}(U)$ has Π-types, given by $\Pi_V(X, \alpha) = (V \times X, P(\pi_2)(\alpha))$, meaning that the assumptions of Theorem 1 are met.

> [[Hyland(2002)], Diller-Nahm] Suppose again that we have a pre-ordered set fibration $p : P \to T$, providing for each type $I \in T$ a collection of (possibly non-standard) predicates $P(I)$ over I. We need some additional structure. We suppose that $p : P \to T$ is equipped with a commutative monoid $(-)^\bullet$ in the following sense.
>
> - Firstly, T is a category with products and $(-)^\bullet$ is a strong monad on T such that each algebra is equipped naturally with the structure of a commutative monoid.
> - Secondly, we suppose that we have an indexed extension of $(-)^\bullet$ to P. For $\phi \in P(I \times A)$ we have $\phi^\bullet \in P(I \times X^\bullet)$. For each $I \in T$, the strength gives an action of $(-)^\bullet$ on the (simple slice) category T/I . And the operation $\phi \to \phi^\bullet$ just described is an extension of this to the global category $P/I \to T/I$.
>
> The example to have in mind here is the finite multiset monad on the category of sets; of course, that is exactly the monad whose algebras are commutative monoids. This monad extends naturally to the subset lattices: if $\phi \subseteq I \times X$ then $\phi^\bullet \subseteq I \times X^\bullet$ is defined by $\phi^\bullet(i, \xi)$

[8] We use the locally indexed category self(T) for the category with products T here. See Example 7.

if and only if $\forall x \in \xi.\phi(i,x)$. From the data just described we construct a new category $\mathbf{Dill} = \mathbf{Dill}(p)$ which we regard again as a category of propositions and proofs.

- The objects of **Dill** are still pairs $U, X \in T$ together with $\alpha \in P(U \times X)$. $(\cdots)$
- Maps of **Dill** from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ are $(\cdots)$ of the form $f : U \to V$, $F : U \times Y \to X^\bullet$ with $\alpha^\bullet(u, F(u, y)) \vdash \beta(f(u), y)$ in $P(U \times Y)$.

That is, **Dill** is precisely the category

$$\Sigma_{U\in T}\mathbf{Dill}(U) \qquad \text{for} \qquad \mathbf{Dill}(U) = \Sigma_{U\in T}(\text{Kleisli}((-)^\bullet)(\Sigma_{X\in \text{self}(T)(U)}P(U\times X)^{op}))^{op},$$

for the lifted monad $(-)^\bullet$ on $\Sigma_{X\in\text{self}(T)(U)}P(U\times X)^{op}$. It is a more involved version of the category discussed in Example 40, where we additionally endow all objects with predicates.

If we assume that $P \to T$ is a fibred cartesian closed category over a bicartesian closed category T and that P is an extensive indexed category in the sense that we have a natural isomorphism $[-] : \prod_{i=1}^N P(X_i) \cong P(\bigsqcup_{i=1}^N X_i)$ and that $(-)^\bullet$ is an additive monad in the sense that $(\bigsqcup_{i=1}^N X_i)^\bullet \cong \prod_{i=1}^N X_i^\bullet$ (we will abuse notation slightly and leave these two isomorphisms implicit), then it follows from our Theorem 1 that **Dial** has the cartesian closed structure structure $(U, X, \alpha) \times (V, Y, \beta) = (U \times V, X \sqcup Y, [\alpha, \beta])$ and $(V, Y, \beta) \Rightarrow (W, Z, \gamma) = \big((V \Rightarrow W) \times (V \times Z \Rightarrow Y^\bullet),\, V \times Z, \rho\big)$, where $\rho((g, G), (v, z)) = \beta^\bullet(v, G(v, z)) \Rightarrow \gamma(g(v), z)$.

Indeed, the indexed product structure on $\mathbf{Dill}(U)$ with unit $(\mathbb{0}, [])$ and product $(X, \alpha) \times (X', \alpha) = (X \sqcup X', [\alpha, \alpha'])$ is $\Sigma, \text{self}(T)(U)$-cotractable because

$$\begin{aligned}
&\mathbf{Dill}(U)((X, \alpha) \times (X', \alpha'), (X'', \alpha'')) = \\
&\{F : U \times X'' \to X^\bullet \times X'^\bullet \cong (X \sqcup X')^\bullet \mid \alpha^\bullet(u, \pi_1(F(u, x''))) \wedge \alpha'^\bullet(u, \pi_2(F(u, x''))) \vdash \alpha''(u, x'')\} = \\
&\Sigma F_1 \in \text{self}(T)(U)(\mathbb{1}, X'' \Rightarrow X).\mathbf{Dill}(U)((X', \alpha'), (X'', \rho)),
\end{aligned}$$

where $\rho(u, x'') = \alpha^\bullet(u, F_1(u)(*)(x'')) \Rightarrow \alpha''(u, x'')$. Therefore,

$$\begin{aligned}
&\mathbf{Dill}(U)((X', \alpha'), (X'', \rho)) = \\
&\quad \{F_2 : U \times X'' \to X'^\bullet \mid \alpha'^\bullet(u, F_2(u, x'')) \vdash \alpha^\bullet(u, F_1(u)(*)(x'')) \Rightarrow \alpha''(u, x'')\},
\end{aligned}$$

showing that we can choose $T(X'', \alpha'') \multimap (X, \alpha) = X'' \Rightarrow X^\bullet$ and $\mathfrak{d}^c((X'', \alpha''), (X, \alpha), F_1) = (X'', \rho)$ where $\rho(u, x'') = \alpha^\bullet(u, F_1(u)(*)(x'')) \Rightarrow \alpha''(u, x'')$. Further, the indexed category $U \mapsto \mathbf{Dill}(U)$ has Π-types, given by $\Pi_V(X, \alpha) = (V \times X, P(\pi_2)(\alpha))$, meaning that the assumptions of Theorem 1 are met.

These Examples raise the more general question under what circumstances, given two fibrations $p : P \to Q$ and $q : Q \to R$, the fibration $(q \circ p^{op})^{op}$ (using the fibrewise opposite fibration and composition of fibrations) has a monoidal closed total space. Assuming that q is a model of dependent type theory with Π-types and strong Σ-types, that amounts, in the light of our Theorem 1, to characterising when an indexed monoidal structure on the fibres of $(q \circ p^{op})^{op}$ is $\Sigma, Q(-)$-cotractable and when $(q \circ p^{op})^{op}$ has Π-types.

Higher order containers. [Altenkirch et al.(2010)] previously gave the special case of our formula for exponentials in $\mathbf{Fam}(\mathbf{Set}^{op}) = \Sigma_{\mathbf{Set}}\mathbf{CAT}(-, \mathbf{Set}^{op})$ which they interpret as a category of containers (or polynomial endofunctors). Such containers are useful in programming as they give a certain, concrete representation of datatypes. As such, the authors use it to give a notion of "higher-order container". Our construction shows that the same construction can be carried out for more general notions of containers valued in a category with a Σ-tractable monoidal structure, like an extensive category with its coproduct structure or a category with biproducts. Some examples of such containers (such as additive containers, as in CHAD, see below) have already found useful programming applications. However, we believe there might be potential for many more notions of container and lens to find use in programming. We hope that the formulas given in his work can contribute to principled programming idioms for such data representations.

CHAD. Recent work [Vákár(2021), Vákár and Smeding(2021), Lucatelli Nunes and Vákár(2023)] has analysed the special case of our formula for exponentials in the case that the fibre categories $\mathcal{L}$ have biproducts. They show that this case can be used to prove correct (see loc. cit.) and give an efficient implementation of (see [Smeding and Vákár(2024)]) a programming technique called Automatic Differentiation (AD), typically the method of choice these days for efficiently computing derivatives of numerical programs. It is tempting to give a similar analysis, based on Grothendieck constructions, for reverse-mode AD methods for calculating higher derivatives [Betancourt(2018), Huot et al.(2022)].

Freely generated categorical structures. The case of our formula for exponentials in Grothendieck constructions $\Sigma_{\mathcal{C}}\mathcal{L}$ indexed cartesian closed categories indexed by a cartesian closed category seems to be well-known. It is used, in particular, for the case of families $\mathbf{Fam}(\mathcal{D}) = \Sigma_{\mathbf{Set}}\mathbf{CAT}(-, \mathcal{D})$ valued in a cartesian closed category $\mathcal{D}$ (i.e., the freely generated category with small coproducts on $\mathcal{D}$) by [Adámek and Rosický(2020)]. Recently, [Lucatelli Nunes and Vákár(2024)] and [Nunes et al.(2024)] analysed exponentiability in freely generated distributive and lextensive categories generated from an arbitrary locally small category $\mathcal{D}$ (which need not be cartesian closed), respectively. The formula used for the exponentials arises as a special case of the present work. These works raise the question whether our method is suitable for a study of exponentiability in further kinds of freely generated categorical structures.

Dependently typed Dialectica. In a tour de force, [Von Glehn(2015), Moss and von Glehn(2018), Moss(2018)] show that the Dialectica and Diller-Nahm interpretations can be extended to dependently typed languages. In particular, they show the following two results, which are in a sense dependently typed variants of two of our examples:

- starting from a model of dependent type theory with strong Σ-types, Π-types and identity types that is *extensive* in a suitable sense, they construct another model of dependent type theory with Σ-types, Π-types and identity types, generalizing our Example 28, in a sense;
- starting from a model of dependent type theory with strong Σ-types, Π-types and identity types with an *additive monad, using a Kleisli construction*, they construct another model of dependent type theory with Σ-types, Π-types and identity types; this is closely related but not quite a generalisation of our Example 29;

Compared to their work, on the one hand, we do not consider the considerable amount of structure needed to interpret dependent types in a Grothendieck construction, so, in this sense, our work is more limited. On the other hand, we generalise from two examples of products in extensive categories and Kleisli categories of additive monads to Σ-cotractable monoidal structures. The latter also give rise to various new examples of cartesian and non-cartesian (even non-symmetric) monoidal closed structures on Grothendieck constructions. In that sense, our work is more general.

Dependently typed (Σ-type) equivalents of non-cartesian monoidal structures are surprisingly subtle [Vákár(2017)], so it is not clear if a common generalisation of both approaches is possible. The most promising avenue might be to limit oneself to cartesian type theories and to pursue a notion of Σ-cotractable Σ-type to generalise Σ-cotractable binary products as well as [Moss and von Glehn(2018)]'s examples.

Other Σ-(co)tractable monoidal structures. So far, we have shown that typical examples of Σ-cotractable monoidal structures are:

- coproducts in any category;
- products in a co-extensive category with an initial object;
- a monoidal left-closed structure on a category with an initial object;
- products in $\mathbf{pSet}^{op}$.

In fact, we have seen that for posets (and, more generally, preorders) Σ-cotractability of the product is equivalent to cartesian closure plus an initial object. For non-thin categories, we have no such characterisation of Σ-cotractability, however. This raises the question if there are other interesting, naturally occurring examples of Σ-cotractable products and non-cartesian monoidal structures for non-thin categories.

Efficient implementation. [Smeding and Vákár(2024)] shows that when interpreted as a recipe for generating code in a functional programming language, programs that make use of the Dialectica-like monoidal closed structure presented in this paper can be inefficient. Interestingly, the non-fibred nature of the exponentials can result in recomputation. In the particular example of CHAD-style automatic differentiation, a workaround is possible by closure converting the code (essentially, by using a representation for the exponential as a co-end, using the co-Yoneda lemma).

Containers and lenses are a more and more important data-representation, particularly in machine learning applications where data needs to flow in both directions [Cruttwell et al.(2022)]. Therefore, it would be interesting to have a better understanding of the precise nature of these efficiency pathologies arising for higher-order containers, as well as generally applicable solutions.

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