On semidirect products of quantale enriched monoids

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Abstract

We consider monoids equipped with a compatible quantale valued relation, to which we call quantale enriched monoids, and study semidirect products of such structures. It is well-known that semidirect products of monoids are closely related to Schreier split extensions which, in the setting of monoids, play the role of split extensions of groups. We will thus introduce certain split extensions of quantale enriched monoids, which generalize the classical Schreier split extensions of monoids, and investigate their connections with semidirect products. We then restrict our study to a class of quantale enriched monoids whose behavior mimics the fact that the preorder on a preordered group is completely determined by its cone of positive elements. Finally, we instantiate our results for preordered monoids and compare them with existing literature.

Keywords: quantale enriched monoid, preordered monoid, semidirect product, split extension

1 Introduction

In the theory of groups, the study of split extensions plays an important role as it allows for the decomposition of each group as a semidirect product of each of its normal subgroups and corresponding quotients. In recent years, there had been generalizations of this result in two different directions.

On the one hand, there were considered in [1, 3] groups equipped with a preorder compatible with the multiplication (but not with the inversion!) and split extensions of preordered groups were characterized. Later, the theory was further generalized to groups enriched in a quantale $\mathcal{V}$ [2], the so-called $\mathcal{V}$-groups, thereby obtaining results that apply not only to preordered groups but also to various structures such as generalized (ultra)metric groups and probabilistic (ultra)metric groups.

On the other hand, there were considered in [6] split extensions of monoids. In *loc. cit.*, it was shown that not every split extension of monoids would give rise to a semidirect product but one should instead restrict to the so-called *Schreier split extensions*. In the case where the structures at play are groups, the concepts of Schreier split extension and of split extension coincide. Once again, Schreier split extensions of monoids were later explored in an enriched setting by considering monoids equipped with a preorder compatible with the multiplication [7]. We note however that the results of [7] are not a generalization of those of [1, 3, 2]. For one, the definition of *Schreier split*

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extension of [7] is not a generalization of the definition of *split extension* of [1, 3] for preordered groups (and thus, neither of [2] for $\mathcal{V}$-groups). Moreover, the focus of [1, 3] and [2] is different of that of [7]. While in the former the authors characterize the preorders (respectively, $\mathcal{V}$-group structures) on a given semidirect product of groups that determine a split extension of preodered groups (respectively, of $\mathcal{V}$-groups), in the latter the authors abstractly provide a characterization of (certain) Schreier split extensions of preordered monoids without paying attention to the preorders that may possibly occur.

In this paper we study split extensions of quantale-enriched monoids, thereby generalizing both approaches at once. After briefly recalling, in Section 2, the most relevant concepts and results used in the remaining paper, we introduce, in Section 3, the category of $\mathcal{V}$-monoids, that is, of *monoids enriched in a quantale* $\mathcal{V}$. Split extensions of $\mathcal{V}$-monoids are considered in Section 4 where we introduce the notion of *U-Schreier split extension*, for a suitable functor U, and characterize the $\mathcal{V}$-monoid structures that may possibly appear in the semidirect product component of a U-Schreier split extension. Section 5 is then devoted to studying those $\mathcal{V}$-monoids whose quantale enrichment is determined by a suitable analogue of the positive cone for preordered monoids, as it happens for $\mathcal{V}$-groups. Finally, in Section 6, we instantiate the results of Section 5 for preordered groups and compare them with those of [7].

2 Preliminaries

The reader is assumed to have some acquaintance with basic category theory and semidirect products of monoids. With the aim of setting up the notation, we recall the most relevant concepts that will be used in the paper. For more on general category theory, including the missing definitions, we refer to [5], for $\mathcal{V}$-categories to [4], and for Schreier split extensions and semidirect products of monoids to [6].

2.1 $\mathcal{V}$-categories

A *quantale* is a tuple $\mathcal{V} = (V, \leq, \otimes, k)$ such that $(V, \leq)$ is a complete lattice, $(V, \otimes, k)$ is a monoid, and the following equalities hold:

$$a \otimes (\bigvee_i b_i) = \bigvee_i (a \otimes b_i) \qquad \text{and} \qquad (\bigvee_i a_i) \otimes b = \bigvee_i (a_i \otimes b). \tag{1}$$

The top and bottom elements of $(V, \leq)$ will be denoted by $\top$ and $\bot$, respectively. Notice that (1) implies that $\otimes$ is monotone with respect to the partial order on V, that is, if $v_1 \leq v_2$ and $w_1 \leq w_2$, then $v_1 \otimes w_1 \leq v_2 \otimes w_2$. We say that $\mathcal{V}$ is *commutative* provided the monoid $(V, \otimes, k)$ is commutative. The quantale $\mathbf{2} = (\{0, 1\}, \leq, \wedge, 1)$, where $0 \leq 1$, will be important in this work, namely when considering preordered monoids on Section 6.

Let $\mathcal{V}$ be a quantale, and let X, Y be two sets. A *$\mathcal{V}$-relation* from X to Y is a function $a : X \times Y \to V$. A *$\mathcal{V}$-category* is a pair (X, a), where X is a set and a is a reflexive and transitive $\mathcal{V}$-relation from X to X, that is, for every $x, y, z \in X$, we have

- $k \leq a(x, x)$ (reflexivity);
- $a(x, y) \otimes a(y, z) \leq a(x, z)$ (transitivity).

A *$\mathcal{V}$-functor* from (X, a) to (Y, b) is a set function $f : X \to Y$ such that, for all $x_1, x_2 \in X$,

$$a(x_1, x_2) \leq b(f(x_1), f(x_2)).$$

We denote by $\mathcal{V}$-**Cat** the category whose objects are $\mathcal{V}$-categories and morphisms are $\mathcal{V}$-functors. Note that a **2**-category may be simply seen as a preordered set, while a morphism of **2**-categories is a monotone function.

When $\mathcal{V}$ is a commutative quantale, for $\mathcal{V}$-categories (X, a) and (Y, b) we may define a new $\mathcal{V}$-category $(X \times Y, a \otimes b)$, where

$$(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y'),$$

for all $x, x' \in X$ and $y, y' \in Y$ [4, Proposition III.1.3.3]. As this is an essential construction in this paper, *all* the quantales we consider will be commutative, even if we do not mention it explicitly.

2.2 Semidirect products of monoids

In the theory of groups, it is a well-known result that, for groups H and N, semidirect products of the form $N \rtimes H$ are in a bijective correspondence with split exact sequences $N \to G \to H$. This result has later been extended to preordered groups [1] and further generalized to $\mathcal{V}$-groups [2]. On the other hand, in the context of monoids, if one aims at describing semidirect products of monoids via suitable short exact sequences, then one should restrict to the so-called *Schreier split extensions* [6].

In what follows, we fix three monoids $(X, +)$, $(Y, \cdot)$, and $(Z, \star)$. For the sake of readability we denote the operation on X additively, though X is not assumed to be commutative.

Definition 2.1. A *Schreier point of monoids* is a split epimorphism $p : Z \to Y$, together with a section $s : Y \to Z$, for which there exists a unique set map $q : Z \to X$ satisfying

$$z = kq(z) \star sp(z), \tag{2}$$

where $k : X \to Z$ is the kernel of p.

A *Schreier split extension of monoids* is a diagram of the form

$$X \overset{k}{\hookrightarrow} Z \underset{s}{\overset{p}{\rightleftarrows}} Y, \tag{3}$$

where p is a Schreier point of monoids with section s, and k the kernel of p.

We have that every Schreir split extension of monoids is a split exact sequence [6, Proposition 2.7]. For a Schreier split extension as in (3), we will often denote by q the unique set map satisfying (2).

Given a function $\alpha : Y \times X \to X$, we define a binary operation on $X \times Y$ by

$$(x_1, y_1) \star (x_2, y_2) = (x_1 + \alpha(y_1, x_2), y_1 \cdot y_2).$$

The algebra thus obtained is denoted by $X \rtimes_\alpha Y$. Note that α may be recovered from the binary operation on $X \rtimes_\alpha Y$. Indeed, for every $x \in X$ and $y \in Y$, we have

$$(0, y) \star (x, 1) = (\alpha(y, x), y).$$

It is well-known that α is a monoid action if, and only if, $(X \rtimes_\alpha Y, \star)$ is a monoid.

Theorem 2.2 ([6, Theorem 2.9]). *There is a one-to-one correspondence between the Schreier split extensions* $X \overset{k}{\hookrightarrow} Z \underset{s}{\overset{p}{\rightleftarrows}} Y$ *and the monoid actions of* Y *on* X *(and thus, with the semidirect products* $(X \rtimes_\alpha Y, \star)$*). More precisely,*

(a) *if* $X \overset{k}{\hookrightarrow} Z \underset{s}{\overset{p}{\rightleftarrows}} Y$ *is a Schreir split extension of monoids and* $q : Z \to X$ *is the unique set map satisfying* (2)*, then* $\alpha : Y \times X \to X$ *defined by* $\alpha(y, x) = q(s(b) \star k(x))$ *is a monoid action,*

(b) *if* $(X \rtimes_\alpha Y, \star)$ *is a semidirect product of monoids, then* $X \overset{\iota_1}{\hookrightarrow} X \rtimes_\alpha Y \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} Y$ *is a Schreir split extension of monoids whose unique set map satisfying* (2) *is the projection* $\pi_1 : X \rtimes_\alpha Y \to X$*.*

Moreover, these two assignments are mutually inverse and, if $X \overset{k}{\hookrightarrow} Z \underset{s}{\overset{p}{\rightleftarrows}} Y$ *is a Schreir split extension of monoids, then the maps*

$$\varphi : X \rtimes_\alpha Y \to Z, \qquad (x, y) \mapsto k(x) \star s(y) \tag{4}$$

and

$$\psi : Z \to X \rtimes_\alpha Y, \qquad z \mapsto (q(z), p(z)) \tag{5}$$

are mutually inverse monoid homomorphisms.

The preordered version of Theorem 2.2 was considered in [7], but only for a special class of preordered monoids that retain a certain group-like behavior. Further details on this will be provided in Section 6.

3 The category of quantale enriched monoids

Let $\mathcal{V}$ be a (commutative) quantale. A *$\mathcal{V}$-monoid* is a triple $(X, a, +)$ such that (X, a) is a $\mathcal{V}$-category, $(X, +)$ is a monoid and the monoid operation induces a $\mathcal{V}$-functor $(\_+\_) : (X, a) \otimes (X, a) \to (X, a)$. We recall that, although we are denoting the monoid operation additively, we do not assume that X is commutative and this will be the usual practice in the remaining paper. When the monoid operation is clear from the context (or irrelevant), we may simply say that (X, a) is a $\mathcal{V}$-monoid. A *morphism of $\mathcal{V}$-monoids* $h : (X, a, +) \to (Y, b, \cdot)$ is a set map $h : X \to Y$ such that $h : (X, a) \to (Y, b)$ is a $\mathcal{V}$-functor and $h : (X, +) \to (Y, \cdot)$ is a monoid homomorphism. We denote by $\mathcal{V}$-**Mon** the category of $\mathcal{V}$-monoids and corresponding homomorphisms.

Starting from the category of $\mathcal{V}$-monoids, we may either forget the $\mathcal{V}$-category or the monoid structure, thereby obtaining two forgetful functors $U : \mathcal{V}\text{-}\mathbf{Mon} \to \mathbf{Mon}$ and $V : \mathcal{V}\text{-}\mathbf{Mon} \to \mathcal{V}\text{-}\mathbf{Cat}$, respectively. Similarly to what happens for $\mathcal{V}$-groups [2] and for preordered monoids [7], U is a topological functor and V a monadic one. We do not include the proofs of these two facts, as they are simple adaptations of the proof of [2, Theorem 4.1]. In particular, it follows that the category of $\mathcal{V}$-monoids is both complete and cocomplete. Moreover, limits are preserved by both forgetful functors, while colimits are preserved by U. We make a few observations that will be relevant in the sequel. First note that the initial object of $\mathcal{V}$-**Mon** is $(\{*\}, \kappa)$, where $\kappa(*, *) = k$, while its terminal object is $(\{*\}, \tau)$, where $\tau(*, *) = \top$. In particular, $\mathcal{V}$-**Mon** is a pointed category if, and only if, $k = \top$. When that is the case, the kernel of a morphism $h : (X, a) \to (Y, b)$ of $\mathcal{V}$-monoids is the $\mathcal{V}$-monoid (Z, c), where

$$Z = \{x \in X \mid h(x) = 1\}$$

is a submonoid of X and c is the suitable restriction of a. Finally, an epimorphism of $\mathcal{V}$-monoids is simply a morphism whose underlying monoid homomorphism is an epimorphism.

The following result is a simple observation, but we will occasionally use it in the remaining paper.

Lemma 3.1. *Let (X,a) be a $\mathcal{V}$-category and $(X,+)$ be a monoid. Then, $(\_+\_) : (X,a)\otimes(X,a)\to(X,a)$ is a $\mathcal{V}$-functor if, and only if, for every $x,y,z\in X$,*

$$a(x,y)\le a(x+z,y+z) \quad \textit{and} \quad a(x,y)\le a(z+x,z+y). \tag{6}$$

Proof. By definition, $+$ is a $\mathcal{V}$-functor if, and only if, for all $x_1,x_2,y_1,y_2\in X$, the following equality holds

$$a(x_1,y_1)\otimes a(x_2,y_2)\le a(x_1+x_2,y_1+y_2).$$

In particular, using the fact that a is reflexive, we have

$$a(x,y)=a(x,y)\otimes k\le a(x,y)\otimes a(z,z)\le a(x+z,y+z)$$

and

$$a(x,y)=k\otimes a(x,y)\le a(z,z)\otimes a(x,y)\le a(z+x,z+y).$$

Conversely, using (6) and transitivity of a, we may derive that

$$a(x_1,y_1)\otimes a(x_2,y_2)\le a(x_1+x_2,y_1+x_2)\otimes a(y_1+x_2,y_1+y_2)\le a(x_1+x_2,y_1+y_2). \qquad \square$$

4 Semidirect products of quantale enriched monoids

In this section we will assume that the quantale $\mathcal{V}$ is such that $\mathcal{V}$-**Mon** is a pointed category, that is, $k=\top$ in $\mathcal{V}$. We will also fix $\mathcal{V}$-monoids $(X,a,+)$, $(Y,b,\cdot)$, and $(Z,c,\star)$.

As already mentioned, semidirect products of monoids are closely related to the so-called Schreier split extensions of monoids. In the context of quantale enriched monoids, we shall consider the following definitions. Recall that we have a forgetful functor $U:\mathcal{V}\text{-}\mathbf{Mon}\to\mathbf{Mon}$.

Definition 4.1. We call *U-Schreier point of $\mathcal{V}$-monoids* to a split epimorphism of $\mathcal{V}$-monoids $p:(Z,c,\star)\to(Y,b,\cdot)$, together with a section s, such that $Up:UZ\to UY$, together with the section Us, is a Schreier point of monoids.

A *U-Schreier split extension of $\mathcal{V}$-monoids* is a diagram of the form

$$(X,a)\overset{k}{\hookrightarrow}(Z,c)\underset{s}{\overset{p}{\rightleftarrows}}(Y,b)$$

in $\mathcal{V}$-**Mon**, where p is a U-Schreier point of $\mathcal{V}$-monoids with section s, and k is the kernel of p.

Our goal is to present a characterization of the U-Schreier points of $\mathcal{V}$-monoids with codomain (Y,b) and kernel (X,a) or, in other words, the U-Schreier split extensions of $\mathcal{V}$-monoids of the form

$$(X,a)\overset{k}{\hookrightarrow}(Z,c)\underset{s}{\overset{p}{\rightleftarrows}}(Y,b). \tag{7}$$

We remark that, since the diagram formed by the underlying monoid homomorphisms of (7) is a Schreier split extension of monoids, by Theorem 2.2, the monoid Z is isomorphic to a semidirect product of the form $X\rtimes_\alpha Y$, the maps k and s are isomorphic to the inclusions ι_1 and ι_2, respectively, and p is isomorphic to the projection π_2. Moreover, the unique set map q satisfying (2) is isomorphic to the projection $\pi_1:X\rtimes_\alpha Y\to X$. In particular, the diagram (7) is isomorphic to

$$(X,a)\overset{\iota_1}{\hookrightarrow}(X\rtimes_\alpha Y,c)\underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}}(Y,b) \tag{8}$$

and the latter is a U-Schreier split extension of $\mathcal{V}$-monoids if, and only if, for all $x,x'\in X$ and $y,y'\in Y$, the following conditions hold:

(S.1) π_2 is a $\mathcal{V}$-functor, that is,
$$c((x,y),(x',y')) \leq b(y,y');$$

(S.2) ι_1 is the kernel of π_2, that is,
$$a(x,x') = c((x,1),(x',1));$$

(S.3) ι_2 is a $\mathcal{V}$-functor, that is,
$$b(y,y') \leq c((0,y),(0,y')).$$

In [2], for $\mathcal{V}$-monoids (X,a) and (Y,b), the (reverse) lexicographic $\mathcal{V}$-relation lex : $(X \times Y) \times (X \times Y) \to \mathcal{V}$ was defined by

$$\mathrm{lex}((x,y),(x',y')) = \begin{cases} a(x,x'), & \text{if } y = y'; \\ b(y,y'), & \text{else.} \end{cases}$$

Here, we consider its weaken version wlex : $(X \times Y) \times (X \times Y) \to V$ given by

$$\mathrm{wlex}((x,y),(x',y')) = \begin{cases} a(x,x'), & \text{if } y = y' = 1; \\ b(y,y'), & \text{else.} \end{cases}$$

We note that the $\mathcal{V}$-relations lex and wlex coincide on every tuple $((x,y),(x',y'))$ unless $y = y' \neq 1$ and, in that case, we have

$$\mathrm{lex}((x,y),(x',y)) = a(x,x') \leq k = b(y,y) = \mathrm{wlex}((x,y),(x',y)).$$

This in particular shows that lex $\leq$ wlex. In fact, the two relations coincide only in very particular cases.

Lemma 4.2. *Let (X,a) and (Y,b) be $\mathcal{V}$-monoids. Then, the following are equivalent:*

(a) lex = wlex*,*

(b) *Y is trivial or $a(x,x') = k$ for all $x,x' \in X$.*

Proof. We first observe that lex = wlex if, and only if, for every $x,x' \in X$ and $y \in Y \setminus \{1\}$, the equality
$$\mathrm{lex}((x,y),(x',y)) = \mathrm{wlex}((x,y),(x',y))$$
holds. By definition of lex and wlex this is equivalent to having

$$a(x,x') = b(y,y) = k,$$

from where we may conclude that (a) and (b) are indeed equivalent. □

We also note that the $\mathcal{V}$-relations lex and wlex are always reflexive but they may not be transitive. Indeed, we have the following:

Lemma 4.3. *The $\mathcal{V}$-relation* wlex *is transitive if, and only if, for every $x,x' \in X$ and every $y \in Y \setminus \{1\}$, we have $b(1,y) \otimes b(y,1) \leq a(x,x')$.*

Proof. If the $\mathcal{V}$-relation wlex is transitive then, for $y \neq 1$, we have

$$\begin{aligned} b(1,y) \otimes b(y,1) &= \mathrm{wlex}((x,1),(0,y)) \otimes \mathrm{wlex}((0,y),(x',1)) \qquad \text{(by definition of wlex)} \\ &\leq \mathrm{wlex}((x,1),(x',1)) \qquad \text{(because wlex is transitive)} \\ &= a(x,x') \qquad \text{(by definition of wlex).} \end{aligned}$$

Conversely, let (x,y), (x',y'), and (x'',y'') belong to $X \times Y$. If $y = y' = y'' = 1$, then

$$\begin{aligned} \mathrm{wlex}((x,y),(x',y')) \otimes \mathrm{wlex}((x',y'),(x'',y'')) &= a(x,x') \otimes a(x',x'') \qquad \text{(by definition of wlex)} \\ &\leq a(x,x'') \qquad \text{(because } a \text{ is transitive)} \\ &= \mathrm{wlex}((x,y),(x'',y'')) \qquad \text{(by definition of wlex).} \end{aligned}$$

If, on the other hand, we have $y = y'' = 1$ but $y' \neq 1$, then

$$\begin{aligned} \mathrm{wlex}((x,y),(x',y')) \otimes \mathrm{wlex}((x',y'),(x'',y'')) &= b(1,y') \otimes b(y',1) \qquad \text{(by definition of wlex)} \\ &\leq a(x,x'') \qquad \text{(by hypothesis)} \\ &= \mathrm{wlex}((x,y),(x'',y'')) \qquad \text{(by definition of wlex).} \end{aligned}$$

Finally, suppose that we do not have $y = y'' = 1$. Then, we have

$$\mathrm{wlex}((x,y),(x'',y'')) = b(y,y'')$$

and the expression

$$\mathrm{wlex}((x,y),(x',y')) \otimes \mathrm{wlex}((x',y'),(x'',y'')) \tag{9}$$

is equal to one of the following:

(*i*) $a(x,x') \otimes b(y',y'')$,

(*ii*) $b(y,y') \otimes a(x',x'')$, or

(*iii*) $b(y,y') \otimes b(y',y'')$.

Then, we have that (9) equals the expression in (*i*) if, and only if, $y = y' = 1$ and $y'' \neq 1$. And, in that case,

$$\begin{aligned} \mathrm{wlex}((x,y),(x',y')) \otimes \mathrm{wlex}((x',y'),(x'',y'')) &= a(x,x') \otimes b(y',y'') \\ &\leq b(1,y'') \qquad \text{(because } a(x,x') \leq k\text{)} \\ &= \mathrm{wlex}((x,y),(x'',y'')) \qquad \text{(because } y'' \neq 1\text{).} \end{aligned}$$

Similarly, if (9) equals the expression in (*ii*), then we must have $y \neq 1$ and $y' = y'' = 1$, and thus,

$$\begin{aligned} \mathrm{wlex}((x,y),(x',y')) \otimes \mathrm{wlex}((x',y'),(x'',y'')) &= b(y,y') \otimes a(x',x'') \\ &\leq b(y,1) \qquad \text{(because } a(x',x'') \leq k\text{)} \\ &= \mathrm{wlex}((x,y),(x'',y'')) \qquad \text{(because } y \neq 1\text{).} \end{aligned}$$

Finally, if (9) equals the expression in (*iii*) then,

$$\begin{aligned} \mathrm{wlex}((x,y),(x',y')) \otimes \mathrm{wlex}((x',y'),(x'',y'')) &= b(y,y') \otimes b(y',y'') \\ &\leq b(y,y'') \qquad \text{(because } b \text{ is transitive)} \\ &= \mathrm{wlex}((x,y),(x'',y'')) \\ &\qquad \text{(because we do not have } y = y'' = 1\text{).} \qquad \square \end{aligned}$$

Lemma 4.4. *The $\mathcal{V}$-relation* lex *is transitive if, and only if, for every $x, x' \in X$ and every $y, y' \in Y$ with $y \neq y'$, we have $b(y, y') \otimes b(y', y) \leq a(x, x')$.*

We omit the proof of this result as it is analogous to that of Lemma 4.3 with the obvious adaptations.

While in the context of $\mathcal{V}$-groups, the relation lex is the biggest possible $\mathcal{V}$-enrichment of a semidirect product [2, Proposition 7.6], in the setting of $\mathcal{V}$-monoids we have the following:

Proposition 4.5. *Let $\alpha : Y \times X \to X$ be a monoid action and $c : (X \times Y) \times (X \times Y) \to V$ be a $\mathcal{V}$-relation on $X \times Y$ that turns $X \rtimes_\alpha Y$ into a $\mathcal{V}$-monoid. Then, the following are equivalent:*

(*a*) $(X, a) \overset{\iota_1}{\hookrightarrow} (X \rtimes_\alpha Y, c) \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} (Y, b)$ *is a U-Schreier split extension of $\mathcal{V}$-monoids,*

(*b*) $a \otimes b \leq c \leq \mathrm{wlex}$.

Proof. Suppose that (*a*) holds. Using (S.2) and (S.3), and the fact that $(X \rtimes_\alpha Y, c)$ is a $\mathcal{V}$-monoid, we have

$$\begin{aligned} a(x, x') \otimes b(y, y') &\leq c((x, 1), (x', 1)) \otimes c((0, y), (0, y')) \\ &\leq c((x, 1)(0, y), (x', 1)(0, y')) = c((x, y), (x', y')), \end{aligned}$$

which proves that $a \otimes b \leq c$. Now, using (S.2) again, we have

$$c((x, 1), (x', 1)) \leq a(x, x') = \mathrm{wlex}((x, 1), (x', 1));$$

and, by (S.1),

$$c((x, y), (x', y')) \leq b(y, y') = \mathrm{wlex}((x, y), (x', y')),$$

where the last equality holds if at least one of y and y' is different from 1. This shows that $c \leq \mathrm{wlex}$.

Conversely, let us suppose that $a \otimes b \leq c \leq \mathrm{wlex}$. By definition of wlex, we have

$$\mathrm{wlex}((x, y), (x', y')) \leq b(y, y')$$

and thus (S.1) holds because we are assuming that $c \leq \mathrm{wlex}$. This inequality also yields

$$c((x, 1), (x', 1)) \leq \mathrm{wlex}((x, 1), (x', 1)) = a(x, x'),$$

which is half of the equality (S.2). Finally, using the assumption $a \otimes b \leq c$, we have

$$a(x, x') = (a \otimes b)((x, 1), (x', 1)) \leq c((x, 1), (x', 1))$$

and

$$b(y, y') = (a \otimes b)((0, y), (0, y')) \leq c((0, y), (0, y')),$$

which show the other half of (S.2) and (S.3), respectively. □

As an immediate consequence, we have the following:

Corollary 4.6. *Let $\alpha : Y \times X \to X$ be a monoid action and $c : (X \times Y) \times (X \times Y) \to V$ be a $\mathcal{V}$-relation satisfying $a \otimes b \leq c \leq \mathrm{wlex}$. Then, the following are equivalent:*

(*a*) $(X, a) \overset{\iota_1}{\hookrightarrow} (X \rtimes_\alpha Y, c) \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} (Y, b)$ *is a U-Schreier split extension of $\mathcal{V}$-monoids,* (*b*) $(X \rtimes_\alpha Y, c)$ *is a* $\mathcal{V}$*-monoid.*

We will now characterize under which conditions the pair $(X \rtimes_\alpha Y, c)$ is a $\mathcal{V}$-monoid, when c is each of the bounds identified in Proposition 4.5, as well as when $c = \text{lex}$.

Given a monoid action $\alpha : Y \times X \to X$, we consider the function

$$\overline{\alpha} : Y \times X \to X \times Y, \qquad (y, x) \mapsto (\alpha(y, x), y).$$

The proof of the following result is included for the sake of completeness, but we note that it is similar to that of [2, Proposition 7.2].

Proposition 4.7. *Let* $\alpha : Y \times X \to X$ *be a monoid action. Then, the following are equivalent:*

(*a*) $(X \rtimes_\alpha Y, a \otimes b)$ *is a* $\mathcal{V}$*-monoid,*

(*b*) $\overline{\alpha}$ *is a* $\mathcal{V}$*-functor.*

Proof. (*a*) $\implies$ (*b*): Suppose that $(X \rtimes_\alpha Y, a \otimes b)$ is a $\mathcal{V}$-monoid. Then, the following computations show that $\overline{\alpha}$ is a $\mathcal{V}$-functor:

$$\begin{aligned}(b \otimes a)((y, x), (y', x')) &= a(0, 0) \otimes (b \otimes a)((y, x), (y', x')) \otimes b(1, 1) \\ &= (a \otimes b)((0, y), (0, y')) \otimes (a \otimes b)((x, 1), (x', 1)) \\ &\leq (a \otimes b)(\overline{\alpha}(y, x), \overline{\alpha}(y', x')) \qquad \text{(because } (X \rtimes_\alpha Y, a \otimes b) \text{ is a } \mathcal{V}\text{-monoid).}\end{aligned}$$

(*b*) $\implies$ (*a*): We need to show that the operation on $X \rtimes_\alpha Y$ induces a $\mathcal{V}$-functor, that is, that for all $x_1, x_1', x_2, x_2' \in X$ and $y_1, y_1', y_2, y_2' \in Y$, the following inequality holds:

$$(a \otimes b)((x_1, y_1), (x_1', y_1')) \otimes (a \otimes b)((x_2, y_2), (x_2', y_2')) \leq (a \otimes b)((x_1, y_1)(x_2, y_2), (x_1', y_1')(x_2', y_2')). \quad (10)$$

Indeed, we have:

$$\begin{aligned}&(a \otimes b)((x_1, y_1), (x_1', y_1')) \otimes (a \otimes b)((x_2, y_2), (x_2', y_2')) \\ &= a(x_1, x_1') \otimes (b \otimes a)((y_1, x_2), (y_1', x_2')) \otimes b(y_2, y_2') \\ &\leq a(x_1, x_1') \otimes (a \otimes b)(\overline{\alpha}(y_1, x_2), \overline{\alpha}(y_1', x_2')) \otimes b(y_2, y_2') \qquad \text{(because } \overline{\alpha} \text{ is a } \mathcal{V}\text{-functor)} \\ &\leq a(x_1 + \alpha(y_1, x_2), x_1' + \alpha(y_1', x_2')) \otimes b(y_1 y_1', y_2 y_2') \qquad \text{(because } (X, a) \text{ and } (Y, b) \text{ are } \mathcal{V}\text{-monoids)} \\ &= (a \otimes b)((x_1, y_1)(x_2, y_2), (x_1', y_1')(x_2', y_2')). \quad \square\end{aligned}$$

Proposition 4.8. *Let* $\alpha : Y \times X \to X$ *be a monoid action. Then, the following are equivalent:*

(*a*) $(X \rtimes_\alpha Y, \text{wlex})$ *is a* $\mathcal{V}$*-monoid,*

(*b*) *the* $\mathcal{V}$*-relation* wlex *is transitive and, for all* $y_1, y_2, y_1', y_2' \in Y \setminus \{1\}$ *satisfying* $y_1 y_2 = y_1' y_2' = y_1' y_2 = 1$*, the following inequality holds:*

$$b(y_1, y_1') \otimes b(y_2, y_2') \leq \bigwedge_{x, x' \in X} a(x, x'). \quad (11)$$

Proof. (*a*) $\implies$ (*b*): If $(X \rtimes_\alpha Y, \text{wlex})$ is a $\mathcal{V}$-monoid then wlex is transitive. Let us show that (11) holds. Let $y_1, y_2, y_1', y_2' \in Y \setminus \{1\}$ be such that $y_1 y_2 = y_1' y_2' = 1$ and $x, x' \in X$. Then, since $(X \rtimes_\alpha Y, \text{wlex})$ is a $\mathcal{V}$-monoid, we have

$$\begin{aligned}b(y_1, y_1') \otimes b(y_2, y_2') &= \text{wlex}((x, y_1), (x', y_1')) \otimes \text{wlex}((0, y_2), (0, y_2')) \\ &\leq \text{wlex}((x, y_1 y_2), (x', y_1' y_2')) = a(x, x').\end{aligned}$$

Since $x, x' \in X$ are arbitrary, we have (11).

$(b) \implies (a)$: We have that $(X \rtimes_\alpha Y, \text{wlex})$ is a $\mathcal{V}$-monoid if, and only if, the following inequality holds:

$$\text{wlex}((x_1, y_1), (x_1', y_1')) \otimes \text{wlex}((x_2, y_2), (x_2', y_2')) \leq \text{wlex}((x_1 + \alpha(y_1, x_2), y_1 y_2), (x_1' + \alpha(y_1', x_2'), y_1' y_2')). \tag{12}$$

We consider the following cases, according to the value v of the left-hand side of (12):

- If $v = a(x_1, x_1') \otimes a(x_2, x_2')$ then, we must have $y_1 = y_1' = y_2 = y_2' = 1$ and inequality (12) follows from (X, a) being a $\mathcal{V}$-monoid.
- If $v = a(x_1, x_1') \otimes b(y_2, y_2')$ then it is because we have $y_1 = y_1' = 1$ but we do not have $y_2 = y_2' = 1$. Thus, we cannot either have $y_1 y_2 = y_1' y_2' = 1$ and thus, the right-hand side of (12) is $b(y_1 y_2, y_1' y_2') = b(y_2, y_2')$ which is greater than or equal to v.
- If $v = b(y_1, y_1') \otimes a(x_2, x_2')$ then the argument is similar to the one of the previous case.
- If $v = b(y_1, y_1') \otimes b(y_2, y_2')$ then it is because neither $y_1 = y_1' = 1$ nor $y_2 = y_2' = 1$. We consider the following three further cases:
  - If $y_1 y_2 \neq 1$ or $y_1' y_2' \neq 1$, then the right-hand side of (9) is $b(y_1 y_2, y_1' y_2')$ which, since (Y, b) is a $\mathcal{V}$-monoid, is greater than or equal to v.
  - If $y_1 y_2 = y_1' y_2' = 1$, but $y_1' y_2 \neq 1$, then
  $$\begin{aligned} v &= b(y_1, y_1') \otimes b(y_2, y_2') \\ &\leq b(y_1 y_2, y_1' y_2) \otimes b(y_1' y_2, y_1' y_2') && \text{(because } (X, b) \text{ is a } \mathcal{V}\text{-monoid)} \\ &= b(1, y_1' y_2) \otimes b(y_1' y_2, 1) \\ &\leq a(x_1 + \alpha(y_1, x_2), x_1' + \alpha(y_1', x_2')) && \text{(by Lemma 4.3).} \end{aligned}$$
  - If $y_1 y_2 = y_1' y_2' = y_1' y_2 = 1$, then we use (11). □

Given a function $\alpha : Y \times X \to X$ and $y \in Y$, we let $\alpha_y : X \to X$ be defined by $\alpha_y(x) = \alpha(y, x)$.

Proposition 4.9. *Let $\alpha : Y \times X \to X$ be a monoid action. Then, the following are equivalent:*

(a) $(X \rtimes_\alpha Y, \text{lex})$ is a $\mathcal{V}$-monoid,

(b) lex is transitive, for every $y \in Y$, α_y is a $\mathcal{V}$-functor and, for every $y_0, y, y' \in Y$ with $y \neq y'$, if $y_0 y = y_0 y'$ then
$$b(y, y') \leq \bigwedge_{x, x' \in X} a(\alpha(y_0, x), \alpha(y_0, x')) \tag{13}$$
and if $y y_0 = y' y_0$ then
$$b(y, y') \leq \bigwedge_{x, x' \in X} a(x, x'). \tag{14}$$

Proof. $(a) \implies (b)$: If $(X \rtimes_\alpha Y, \text{lex})$ is a $\mathcal{V}$-monoid, then lex is, by definition, transitive. Let $y \in Y$. Then, for all $x, x' \in X$, we have

$$\begin{aligned} a(x, x') &= \text{lex}((0, y), (0, y)) \otimes \text{lex}((x, 1), (x', 1)) \\ &\leq \text{lex}((\alpha(y, x), y), (\alpha(y, x'), y)) && \text{(because } (X \rtimes_\alpha Y, \text{lex}) \text{ is a } \mathcal{V}\text{-monoid)} \\ &= a(\alpha_y(x), \alpha_y(x')), \end{aligned}$$

and therefore, α_y is a $\mathcal{V}$-functor. Now, we let $y_0, y, y' \in Y$ be such that $y \neq y'$, and pick any $x, x' \in X$. If $y_0 y = y_0 y'$ then

$$\begin{aligned} b(y,y') &= \mathrm{lex}((0,y_0),(0,y_0)) \otimes \mathrm{lex}((x,y),(x',y')) \\ &\leq \mathrm{lex}((\alpha(y_0,x),y_0y),(\alpha(y_0,x'),y_0y')) \qquad \text{(because } (X \rtimes_\alpha Y, \mathrm{lex}) \text{ is a } \mathcal{V}\text{-monoid)} \\ &= a(\alpha(y_0,x),\alpha(y_0,x')) \qquad \text{(because } y_0y = y_0y'\text{)} \end{aligned}$$

and this proves (13). If, on the other hand, we have $yy_0 = y'y_0$, then

$$\begin{aligned} b(y,y') &= \mathrm{lex}((x,y),(x',y')) \otimes \mathrm{lex}((0,y_0),(0,y_0)) \\ &\leq \mathrm{lex}((x,yy_0),(x',y'y_0)) \qquad \text{(because } (X \rtimes_\alpha Y, \mathrm{lex}) \text{ is a } \mathcal{V}\text{-monoid)} \\ &= a(x,x') \qquad \text{(because } yy_0 = y'y_0\text{)} \end{aligned}$$

which shows (14).

$(b) \implies (a)$: Since lex is always reflexive, we have that $(X \rtimes_\alpha Y, \mathrm{lex})$ is a $\mathcal{V}$-category. Thus, it suffices to show that, for all $x_1, x_1', x_2, x_2' \in X$ and $y_1, y_1', y_2, y_2' \in Y$, the following inequality holds:

$$\mathrm{lex}((x_1,y_1),(x_1',y_1'))\otimes\mathrm{lex}((x_2,y_2),(x_2',y_2')) \leq \mathrm{lex}((x_1+\alpha(y_1,x_2),y_1y_2),(x_1'+\alpha(y_1',x_2'),y_1'y_2')). \quad (15)$$

For that, we consider the following cases:

- If $y_1 = y_1'$ and $y_2 = y_2'$, then (15) is equivalent to

  $$a(x_1,x_1') \otimes a(x_2,x_2') \leq a(x_1+\alpha(y_1,x_2), x_1'+\alpha(y_1',x_2')).$$

  Using the fact that $\alpha_{y_1} = \alpha_{y_1'}$ is a $\mathcal{V}$-functor and that (X,a) is a $\mathcal{V}$-monoid, we have

  $$a(x_1,x_1') \otimes a(x_2,x_2') \leq a(x_1,x_1') \otimes a(\alpha_{y_1}(x_2),\alpha_{y_1'}(x_2')) \leq a(x_1+\alpha(y_1,x_2), x_1'+\alpha(y_1',x_2')),$$

  as required.

- If $y_1 = y_1'$, $y_2 \neq y_2'$ and $y_1y_2 = y_1'y_2'$ then (15) is equivalent to

  $$a(x_1,x_1') \otimes b(y_2,y_2') \leq a(x_1+\alpha(y_1,x_2), x_1'+\alpha(y_1',x_2')),$$

  which holds because, by (13), the inequality

  $$a(x_1,x_1') \otimes b(y_2,y_2') \leq a(x_1,x_1') \otimes a(\alpha(y_1,x),\alpha(y_1',x'))$$

  holds and $(X \rtimes_\alpha, \mathrm{lex})$ is a $\mathcal{V}$-monoid.

- If $y_1 \neq y_1'$, $y_2 = y_2'$ and $y_1y_2 = y_1'y_2'$ then (15) is equivalent to

  $$b(y_1,y_1') \otimes a(x_2,x_2') \leq a(x_1+\alpha(y_1,x_2), x_1'+\alpha(y_1',x_2')),$$

  which holds thanks to (14).

- If $y_1y_2 \neq y_1'y_2'$ then at least one of the equalities $y_1 = y_1'$ and $y_2 = y_2'$ must fail. If $y_i \neq y_i'$ for exactly one $i \in \{1,2\}$, then we have

  $$\begin{aligned} \mathrm{lex}((x_1,y_1),(x_1',y_1')) \otimes \mathrm{lex}((x_2,y_2),(x_2',y_2')) &\leq b(y_i,y_i') \\ &\leq b(y_1y_2,y_1'y_2') \qquad \text{(by Lemma 3.1)} \\ &= \mathrm{lex}((x_1+\alpha(y_1,x_2),y_1y_2),(x_1'+\alpha(y_1',x_2'),y_1'y_2')). \end{aligned}$$

If we have both $y_1 \neq y_1'$ and $y_2 \neq y_2'$ then (15) is equivalent to

$$b(y_1, y_1') \otimes b(y_2, y_2') \leq b(y_1 y_2, y_1' y_2'),$$

which holds because (Y, b) is a $\mathcal{V}$-monoid. $\square$

We have thus provided characterizations of the $\mathcal{V}$-monoids $(X \rtimes_\alpha Y, c)$ for three particular instances of c. Now, since the underlying diagram of monoids of a U-Schreier split extension of $\mathcal{V}$-monoids forms a Schreier split extension of monoids, in the diagram (7), there is a unique set map $q : Z \to X$ satisfying (2). We recall that, in (8), such a map is the first projection $\pi_1 : X \rtimes_\alpha Y \to X$. It is then natural to ask for necessary and sufficient conditions for having that this map is also a $\mathcal{V}$-functor. We provide such in the next result.

Lemma 4.10. *If* $(X, a) \overset{\iota_1}{\hookrightarrow} (X \rtimes_\alpha Y, c) \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} (Y, b)$ *is a U-Schreier split extension of $\mathcal{V}$-monoids then, π_1 is a $\mathcal{V}$-functor if, and only if, $c \leq a \wedge b$.*

Proof. By definition, π_1 is a $\mathcal{V}$-functor if, and only if, for all $x, x' \in X$ and $y, y' \in Y$, the following inequality holds:

$$c((x, y), (x', y')) \leq a(x, x').$$

Noting that, by definition of wlex, we have $\mathrm{wlex}((x, y), (x', y')) \leq b(y, y')$ and, by Proposition 4.5, we have $c \leq \mathrm{wlex}$, the forward implication follows. The backwards implication is trivial. $\square$

In the remaining of the section, we will focus on the case where Y is a group. When that is the case, Propositions 4.8 and 4.9 may be considerably simplified as follows.

Corollary 4.11 (of Proposition 4.8)**.** *Let $\alpha : Y \times X \to X$ be a monoid action and suppose that Y is a group. Then, the following are equivalent:*

(a) $(X \rtimes_\alpha Y, \mathrm{wlex})$ *is a $\mathcal{V}$-monoid,*

(b) $\mathrm{lex} = \mathrm{wlex}$.

Proof. We recall that, by Lemma 4.2, $\mathrm{lex} = \mathrm{wlex}$ if, and only if, either Y is trivial or $a(x, x') = k$ for all $x, x' \in X$. Now, if $(X \rtimes_\alpha Y, \mathrm{wlex})$ is a $\mathcal{V}$-monoid and Y is non-trivial then we may pick $y \in Y \setminus \{1\}$ and, by Proposition 4.8, (11) holds for $y_1 = y_1' = y$ and $y_2 = y_2' = y^{-1}$, which yields $a(x, x') = k$ for all $x, x' \in X$. This shows that (a) implies (b). Conversely, if $\mathrm{lex} = \mathrm{wlex}$ then either Y is trivial and $\mathrm{wlex} = a$, or $a(x, x') = k$ for all $x, x' \in X$. In either case, we have that wlex is transitive. Furthermore, it is clear that (11) holds. Thus, by Proposition 4.8, we may conclude that $(X \rtimes_\alpha Y, \mathrm{wlex})$ is a $\mathcal{V}$-monoid, as required. $\square$

Corollary 4.12 (of Proposition 4.9)**.** *Let $\alpha : Y \times X \to X$ be a monoid action and suppose that Y is a group. Then, the following are equivalent:*

(a) $(X \rtimes_\alpha Y, \mathrm{lex})$ *is a $\mathcal{V}$-monoid,*

(b) lex *is transitive and α_y is a $\mathcal{V}$-functor for all $y \in Y$.*

Proof. This is a trivial consequence of Proposition 4.9 as, when Y is a group, there are no $y_0, y, y' \in Y$ for which $y \neq y'$ and $y_0 y = y_0 y'$ or $y y_0 = y' y_0$. $\square$

We have already identified necessary and sufficient conditions for having that lex and wlex are transitive $\mathcal{V}$-relations. Along the same lines, we may also show that, in the setting of $\mathcal{V}$-groups, the condition identified in [2, Theorem 7.4] is necessary and sufficient for lex being transitive. More precisely, if X and Y are groups, then lex is transitive if, and only if, the inequality

$$b(y,1) \otimes b(1,y) \leq a(x,0)$$

holds for every $x \in X$ and $y \in Y \setminus \{1\}$. Thus, Corollary 4.12 is a generalization of [2, Theorem 7.4]. On the other hand, Corollaries 4.11 and 4.12 also imply that, in the setting of $\mathcal{V}$-groups, if $(X \rtimes_\alpha Y, \text{wlex})$ is a $\mathcal{V}$-monoid, then the relations lex and wlex coincide. That is no surprise as in [2, Proposition 7.6] it is shown that lex is an upper bound of all relations c turning $(X \rtimes_\alpha Y, c)$ into a $\mathcal{V}$-group. The next example shows that, unlike what happens for $\mathcal{V}$-groups, considering the relation wlex is not redundant for $\mathcal{V}$-monoids in general.

Example 4.13. Let $\mathcal{V} = \mathbf{2}$, so that $\mathcal{V}$-**Mon** is the category of preordered monoids. We consider the following preordered monoids:

- $\mathbb{N}$ is the monoid of natural numbers equipped with the usual order relation,
- $\dot{\mathbb{N}}$ is the monoid of natural numbers equipped with the preorder $\dot{\leq}$ defined by

$$0 \,\dot{\leq}\, n, \text{ for all } n \in \mathbb{N}, \qquad \text{and} \qquad n \,\dot{\leq}\, m, \text{ for all } n, m \in \mathbb{N} \setminus \{0\}.$$

It is easy to verify that both $\mathbb{N}$ and $\dot{\mathbb{N}}$ are indeed preordered monoids and, using Lemma 4.3 we may also check that wlex is transitive and thus, $(\mathbb{N} \times \dot{\mathbb{N}}, \leq_{\text{wlex}})$ is a **2**-category (or preordered set). Moreover, the lexicographic and weak lexicographic relations do not coincide in this case: for instance, $(2,2)$ is below $(1,2)$ is the weak lexicographic relation, but not in the lexicographic one. Finally, we check that $(\mathbb{N} \times \dot{\mathbb{N}}, \leq_{\text{wlex}})$ is a preordered monoid. First observe that

$$(m,n) \leq_{\text{wlex}} (m',n') \iff (n = n' = 0 \text{ and } m \leq m') \text{ or } (n' \neq 0).$$

It is then clear that $\leq_{\text{wlex}}$ is invariant by shifting, as required.

5 Group-like behaved quantale enriched monoids

A crucial and useful property in the study of preordered groups is the fact that the preorder relation of a preordered group is completely determined by its cone of positive elements, in the following sense: If $(G, \leq, +)$ is a preodered group and $P_G = \{x \in G \mid x \geq 0\}$ is the cone of positive elements of G, then $x \leq y$ if, and only if, $y \in P_G + x$. That is no longer the case for preordered monoids as witnessed by [7, Example 1]. Indeed, if $(M, +, \leq)$ is a preordered monoid, $P_M = \{x \in M \mid x \geq 0\}$ is its cone of positive elements, and $\leq_{P_M}$ is the preorder on M defined by

$$x \leq_{P_M} y \iff y \in P_M + x, \tag{16}$$

then $(M, +, \leq_{P_M})$ is a preordered monoid if, and only if, P_M is a right normal submonoid of M [7, Proposition 2] and this condition does not even guarantee that $\leq_{P_M}$ is the preorder on M [7, page 5]. Given the importance, in the context of preordered groups, of having that the relations $\leq_{P_M}$ and $\leq$ coincide, in [7] the authors restrict their study of split extensions to those preordered monoids for which that is the case.

In this section we will start by investigating which property can play the role of right normality in the more general context of $\mathcal{V}$-monoids and, in the spirit of [7], we will restrict our study to the subclass of $\mathcal{V}$-monoids that, in a sense that we will make precise soon, behave like $\mathcal{V}$-groups.

Let $(M, \leq, +)$ be a preordered monoid and let $P_M = \{x \in M \mid x \geq 0\}$ be its cone of positive elements. Then, seeing M as a $\mathbf{2}$-monoid $(M, a, +)$, for

$$a(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise,} \end{cases}$$

we have that P_M is the preimage of $1 = \top$ under the projection $a(0, \_) : M \to \mathbf{2}$. This very simple observation opens the door to a generalization of the results of [7] to the setting of $\mathcal{V}$-monoids. Indeed, we will now focus on those $\mathcal{V}$-monoids (X, a) whose $\mathcal{V}$-relation a is determined by the projection $a(0, \_) : X \to V$.

Given a $\mathcal{V}$-monoid (X, a), we will denote by P_a the map $P_a : X \to V$ defined by

$$P_a(x) = a(0, x), \tag{17}$$

for all $x \in X$. We note that, in the case where (X, a) is a $\mathcal{V}$-group, the $\mathcal{V}$-relation a is completely determined by its projection P_a. Indeed, by Lemma 3.1, we have

$$a(x, y) \leq a(0, y - x) \leq a(x, y)$$

and, therefore, the equality

$$a(x, y) = P_a(y - x)$$

holds. More generally, we will consider those $\mathcal{V}$-monoids (X, a) satisfying

$$a(x, y) = \bigvee \{P_a(w) \mid y = w + x\}, \tag{18}$$

for all $x, y \in X$. We observe that, in the case where X is a group, the right-hand side of (18) is simply $P_a(y - x)$. We further observe that, for a $\mathcal{V}$-category (X, a), the function $P = P_a$ satisfies the following two properties:

(M.1) $k \leq P(0)$,

(M.2) $P(x) \otimes P(y) \leq P(x + y)$, for all $x, y \in X$.

These two properties turn out to be crucial when defining a $\mathcal{V}$-monoid structure on a given monoid X out of a function $X \to V$. Indeed, such a function $P : X \to V$, we consider the $\mathcal{V}$-relation $a_P : X \times X \to V$ defined by

$$a_P(x, y) = \bigvee \{P(w) \mid y = w + x\}. \tag{19}$$

Proposition 5.1. *Let X be a monoid, and let $P : X \to V$ be a function. Then,*

(a) a_P is reflexive if, and only if, P satisfies (M.1);

(b) a_P is transitive if, and only if, P satisfies (M.2);

(c) if (X, a_P) is a $\mathcal{V}$-category, then the monoid operation on X is a $\mathcal{V}$-functor if, and only if, for all $x, z \in X$, P satisfies

$$P(x) \leq \bigvee \{P(w) \mid z + x = w + z\}. \tag{M.3}$$

In particular, (X, a_P) is a $\mathcal{V}$-monoid if, and only if, P satisfies properties (M.1), (M.2), and (M.3).

Proof. Noticing that $P_{a_P} = P$, if a_P is reflexive and transitive then P satisfies (M.1) and (M.2), respectively. Thus, the forward implications of (a) and (b) hold. Suppose that (M.1) holds. Then, a_P is reflexive because

$$a_P(x,x) = \bigvee\{P(w) \mid x = w + x\} \geq P(0) \overset{\text{(M.1)}}{\geq} k.$$

If (M.2) holds, then a_P is transitive because

$$\begin{aligned} a_P(x,y) \otimes a_P(y,z) &= (\bigvee\{P(w) \mid y = w + x\}) \otimes (\bigvee\{P(w) \mid z = w + y\}) \\ &= \bigvee\{P(w) \otimes P(w') \mid y = w + x \text{ and } z = w' + y\} \\ &= \bigvee\{P(w') \otimes P(w) \mid y = w + x \text{ and } z = w' + y\} \\ &\overset{\text{(M.2)}}{\leq} \bigvee\{P(w' + w) \mid y = w + x \text{ and } z = w' + y\} \\ &\leq \bigvee\{P(w' + w) \mid z = w' + w + x\} = a_P(x,z), \end{aligned}$$

and this finishes the proofs of (a) and (b).

Let us now prove (c). Suppose that (X, a_P) is a $\mathcal{V}$-category. If the monoid operation on X is a $\mathcal{V}$-functor then, using the second inequality stated in Lemma 3.1, for all $x, z \in X$, we have

$$P(x) = a_P(0,x) \leq a_P(z, z + x) = \bigvee\{P(w) \mid z + x = w + z\}.$$

Conversely, let us verify that the inequalities (6) of Lemma 3.1 hold. First note that we always have $a_P(x,y) \leq a_P(x + z, y + z)$. Indeed,

$$a_P(x,y) = \bigvee\{P(w) \mid y = w + x\} \leq \bigvee\{P(w) \mid y + z = w + x + z\} = a_P(x + z, y + z).$$

Using (M.3), we may deduce that

$$\begin{aligned} a_P(x,y) &= \bigvee\{P(w) \mid y = w + x\} \\ &\leq \bigvee\{\bigvee\{P(w') \mid z + w = w' + z\} \mid y = w + x\} \\ &= \bigvee\{P(w') \mid z + w = w' + z \text{ and } y = w + x\} \\ &\leq \bigvee\{P(w') \mid z + y = w' + z + x\} = a_P(z + x, z + y). \quad \square \end{aligned}$$

We remark that, in the case of preordered monoids, having $P(x) \leq \bigvee\{P(w) \mid z + x = w + z\}$ for all $x, z \in X$ means that, if $0 \leq x$ and $z \in X$, then there exists w such that $0 \leq w$ and $z + x = w + z$. In other words, this is to say that $z + P \subseteq P + z$, that is, P is right normal.

We will denote by $\mathcal{V}$-**Mon*** the full subcategory of $\mathcal{V}$-**Mon** determined by the $\mathcal{V}$-monoids (X, a) satisfying $a = a_{P_a}$ (recall (17) and (19)). In particular, by Proposition 5.1, when that is the case, P_a must satisfy (M.1)–(M.3).

The remaining of this section will be devoted to the characterization of the U-Schreier split extensions of $\mathcal{V}$-monoids of the form

$$(X,a) \overset{k}{\hookrightarrow} (Z,c) \underset{s}{\overset{p}{\rightleftarrows}} (Y,b),$$

where (X,a), (Y,b), and (Z,c) belong to $\mathcal{V}$-**Mon***.

Definition 5.2. Let (X, a) and (Y, b) be objects of $\mathcal{V}$-**Mon***. A $\mathcal{V}$-*enriched action* of (Y, b) on (X, a) is a pair (α, P), where $\alpha : Y \times X \to X$ is a monoid action and $P : X \times Y \to V$ is a function satisfying the following axioms:

(E.0) $P(x, y) \leq P_b(y)$ for all $(x, y) \in X \times Y$,

(E.1) $P_b(y) \leq P(0, y)$ for all $y \in Y$,

(E.2) $P_a(x) = P(x, 1)$ for all $x \in X$,

(E.3) $P(x, y) \otimes P(x', y') \leq P(x + \alpha(y, x'), yy')$, for all $(x, y), (x', y') \in X \times Y$,

(E.4) $P(x, y) \leq \bigvee\{P(x', y') \mid x_0 + \alpha(y_0, x) = x' + \alpha(y', x_0) \text{ and } y_0 y = y' y_0\}$, for all $(x, y), (x_0, y_0) \in X \times Y$.

We note that properties (E.3) and (E.4) are nothing but properties (M.2) and (M.3), respectively, stated for the function $P : X \times Y \to V$ and for the monoid $X \rtimes_\alpha Y$.

Theorem 5.3. *Let (X, a) and (Y, b) be objects of $\mathcal{V}$-**Mon***. Then, up to isomorphism, there is a one-to-one correspondence between U-Schreier split extensions $(X, a) \overset{k}{\hookrightarrow} (Z, c) \underset{s}{\overset{p}{\rightleftarrows}} (Y, b)$ of $\mathcal{V}$-monoids, with (Z, c) lying in $\mathcal{V}$-**Mon***, and $\mathcal{V}$-enriched actions of (Y, b) on (X, a).*

Proof. Let $(X, a) \overset{k}{\hookrightarrow} (Z, c) \underset{s}{\overset{p}{\rightleftarrows}} (Y, b)$ be a U-Schreier split extension of $\mathcal{V}$-monoids in the category $\mathcal{V}$-**Mon***, let s be the section of p, and let $q : Z \to X$ be the unique set map satisfying (2). We consider the monoid action α defined by $\alpha(y, x) = q(s(y) \star k(x))$. By Theorem 2.2, we know that $\varphi : X \rtimes_\alpha Y \to Z$ and $\psi : Z \to X \rtimes_\alpha Y$ defined by $\varphi(x, y) = k(x) \star s(y)$ and by $\psi(z) = (q(z), p(z))$, respectively, are mutually inverse monoid isomorphisms. We let $P : X \times Y \to V$ be defined by

$$P(x, y) = P_c(k(x) \star s(y)) = P_c(\varphi(x, y))$$

and we claim that (α, P) is a $\mathcal{V}$-enriched action of (Y, b) on (X, a).

(E.0): Since p is a $\mathcal{V}$-functor, we have

$$P(x, y) = P_c(k(x) \star s(y)) \leq P_b(p(k(x) \star s(y))) = P_b(y),$$

where the last equality uses that $\psi \circ \varphi$ is the identity map on $X \times Y$.

(E.1): Since s is a $\mathcal{V}$-functor, we have

$$P_b(y) \leq P_c(s(y)) = P(0, y).$$

(E.2): Since k is the kernel of p, we have

$$P_a(x) = P_c(k(x)) = P(x, 1).$$

For proving (E.3) and (E.4), we use the fact that (Z, c) belongs to $\mathcal{V}$-**Mon*** and thus, the function P_c satisfies (M.2) and (M.3).

(E.3): By (M.2), we have

$$\begin{aligned} P(x, y) \otimes P(x', y') &= P_c(k(x) \star s(y)) \otimes P_c(k(x') \star s(y')) \leq P_c(k(x) \star s(y) \star k(x') \star s(y')) \\ &= P_c(\varphi(x, y) \star \varphi(x', y')) = P_c(\varphi(x + \alpha(y, x'), yy')) \\ &= P(x + \alpha(y, x'), yy'). \end{aligned}$$

(E.4): By (M.3), for all $z, z_0 \in Z$, we have

$$P_c(z) \leq \bigvee\{P_c(z') \mid z_0 \star z = z' \star z_0\}.$$

Since φ is a monoid isomorphism, this is equivalent to having

$$P_c(\varphi(x,y)) \leq \bigvee\{P_c(\varphi(x',y')) \mid \varphi(x_0+\alpha(y_0,x), y_0 y) = \varphi(x'+\alpha(y',x_0), y'y_0)\}$$

for all $(x,y),(x_0,y_0) \in X \times Y$, and having

$$\varphi(x_0+\alpha(y_0,x), y_0 y) = \varphi(x'+\alpha(y',x_0), y'y_0)$$

is equivalent to having

$$x_0+\alpha(y_0,x) = x'+\alpha(y',x_0) \quad \text{and} \quad y_0 y = y' y_0.$$

Thus, we have (E.4).

Conversely, let (α, P) be a $\mathcal{V}$-enriched action of (Y,b) on (X,a). By Proposition 5.1, taking

$$a_P((x,y),(x',y')) = \bigvee\{P(x'',y'') \mid (x',y') = (x'',y'') \star (x,y)\},$$

yields a $\mathcal{V}$-monoid $(X \rtimes_\alpha Y, a_P)$ provided P satisfies properties (M.1), (M.2), and (M.3). As already observed, (M.2) and (M.3) hold because so do (E.3) and (E.4), respectively. To show (M.1), we may use (E.2) and the fact that P_a satisfies (M.1). Now, by Corollary 4.6, to conclude that

$$(X,a) \overset{\iota_1}{\hookrightarrow} (X \rtimes_\alpha Y, a_P) \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} (Y,b)$$

is a U-Schreier split extension of $\mathcal{V}$-monoids, it suffices to show that $a \otimes b \leq a_P \leq \text{wlex}$. Since (X,a) and (Y,b) belong to $\mathcal{V}$-$\mathbf{Mon}^*$, for all $x,x' \in X$ and $y,y' \in Y$, we have

$$a(x,x') = \bigvee\{P_a(x'') \mid x' = x''+x\} \overset{(E.2)}{=} \bigvee\{P(x'',1) \mid x' = x''+x\} = a_P((x,1),(x',1)) \tag{20}$$

and

$$b(y,y') = \bigvee\{P_b(y'') \mid y' = y''y\} \overset{(E.1)}{\leq} \bigvee\{P(0,y'') \mid y' = y''y\} = a_P((0,y),(0,y')).$$

Thus,

$$a(x,x') \otimes b(y,y') \leq a_P((x,1),(x',1)) \otimes a_P((0,y),(0,y')) \leq a_P((x,y),(x',y')),$$

where the last inequality holds because $(X \rtimes_\alpha Y, a_P)$ is a $\mathcal{V}$-monoid. This shows that $a \otimes b \leq a_P$. Now, by (20), $a_P((x,1),(x',1)) = a(x,x')$ and, for arbitrary y, y',

$$a_P((x,y),(x',y')) = \bigvee\{P(x'',y'') \mid (x',y') = (x'',y'') \star (x,y)\} \overset{(E.0)}{\leq} \bigvee\{P_b(y'') \mid y' = y''y\} = b(y,y'),$$

which finishes showing that $a_P \leq \text{wlex}$.

It remains to check that the two correspondences just described are mutually inverse. It is easily seen that $P = P_{a_P}$. Thus, taking Theorem 2.2 into account, it remains to show that, for a U-Schreier split extension $(X,a) \overset{k}{\hookrightarrow} (Z,c) \underset{s}{\overset{p}{\rightleftarrows}} (Y,b)$, if (α, P) is the corresponding $\mathcal{V}$-enriched action, then the monoid isomorphisms φ and ψ define morphisms of $\mathcal{V}$-monoids when $X \rtimes_\alpha Y$ is equipped with the $\mathcal{V}$-relation a_P. Indeed, we have

$$\begin{aligned} a_P((x,y),(x',y')) &= \bigvee\{P(x'',y'') \mid (x',y') = (x'',y'') \star (x,y)\} \\ &= \bigvee\{P_c(\varphi(x'',y'')) \mid (x',y') = (x'',y'') \star (x,y)\} \quad \text{(by definition of } P) \\ &= \bigvee\{P_c(\varphi(x'',y'')) \mid \varphi(x',y') = \varphi(x'',y'') \star \varphi(x,y)\} \quad \text{(because } \varphi \text{ is a} \\ &\hspace{20em} \text{monoid isomorphism)} \\ &= c(\varphi(x,y),\varphi(x',y')) \quad \text{(because } (Z,c) \text{ belongs to } \mathcal{V}\text{-}\mathbf{Mon}^*) \quad \square \end{aligned}$$

We finish this section by noting that, as in [7], we could also have defined morphisms of U-Schreier split extensions of $\mathcal{V}$-monoids and of $\mathcal{V}$-enriched actions in the obvious way, thereby forming two categories that could be proved to be equivalent by following the ideas used in the proof of Theorem 5.3. We do not include details on this as we believe no further meaningful mathematical knowledge of the structures involved would be added.

6 The case of preordered monoids

In this section, we analyze the results of Section 5 in the case of preordered monoids and compare them with those of [7].

Let $\mathcal{V} = \mathbf{2}$, so that $\mathcal{V}$-**Mon** can be identified with the category of preordered monoids, and let (X, a) be a $\mathbf{2}$-monoid. The ensuing preorder on X will be denoted by $\leq_X$. A function $P : X \to \mathbf{2}$ is uniquely determined by the subset $P^{-1}(\{\top\}) \subseteq X$ and, conversely, each subset $P \subseteq X$ uniquely determines a function $X \to \mathbf{2}$. We will often abuse notation and identify a subset $P \subseteq X$ with the function $P : X \to \mathbf{2}$ it defines. As already mentioned, under this identification, P_a as defined in the previous section is the cone P_X of positive elements of X. Now, given a subset $P \subseteq X$, the $\mathbf{2}$-relation a_P defined in (19) induces the preorder $\leq_P$ on X given by

$$x \leq_P y \iff y \in P + x,$$

for all $x, y \in P$. Indeed, we have $x \leq_P y$ if, and only if, $a_P(x,y) = \top$, which holds if, and only if, there exists some $w \in X$ satisfying $P(w) = \top$ and $y = w + x$. We further observe that $P \subseteq X$ satisfies properties (M.1) and (M.2) if, and only if, $0 \in P$ and $x + y \in P$ whenever $x, y \in P$, respectively. That is, if, and only if, P is a submonoid of X. In turn, as it was already explained, requiring (M.3) is equivalent to requiring that P is right normal. Thus, our Proposition 5.1 is a generalization of [7, Proposition 2] to the setting of $\mathcal{V}$-monoids. Moreover, the category **OrdMon*** studied in [7] is our category $\mathbf{2}$-**Mon***. Let us state Definition 5.2 for $\mathcal{V} = \mathbf{2}$.

Proposition 6.1. *Let X and Y be objects of* $\mathbf{2}$-**Mon****. A* $\mathbf{2}$-enriched action *of Y on X is a pair (α, P), where $\alpha : Y \times X \to X$ is a monoid action and $P \subseteq X \times P_Y$ satisfies the following axioms:*

(B.0) $P \cap (X \times \{1\}) \subseteq P_X \times \{1\}$,

(B.1) $\{0\} \times P_Y \subseteq P$,

(B.2) $P_X \times \{1\} \subseteq P$,

(B.3) if (x, y) and (x', y') belong to P then so does $(x + \alpha(y, x'), yy')$,

(B.4) *if* $(x, y) \in P$ *and* $(x_0, y_0) \in X \times Y$*, then there exists* $(x', y') \in P$ *such that* $x_0 + \alpha(y_0, x) = x' + \alpha(y', x_0)$ *and* $y_0 y = y' y_0$*.*

Proof. This is a straightforward translation of Definition 5.2, with (E.0) corresponding to the condition $P \subseteq X \times P_Y$, (B.0) and (B.2) corresponding to (E.1) and, for $i = 1, 3, 4$, (B.i) corresponding to (E.i). □

Let us now recall the main result of [7]. *In loc. cit.*, preordered actions are defined as follows:

Definition 6.2 ([7, Definition 4]). Let X and Y belong to $\mathbf{2}$-$\mathbf{Mon}^*$. A *preordered action* of Y on X is a pair (α, ξ), where $\alpha : Y \times X \to X$ is a monoid action and $\xi : X \times P_Y \to X$ is a function satisfying

(A.1) $\xi(0, y) = 0$, for all $y \in P_Y$,

(A.2) $\xi(x, 1) = x$, for all $x \in P_X$,

(A.3) if $\xi(x, y) = x$ and $\xi(x', y') = x'$, then $\xi(x + \alpha(y, x'), yy') = x + \alpha(y, x')$,

(A.4) if $(x, y) \in X \times P_Y$ and $(x_0, y_0) \in X \times Y$, then there exists $(x', y') \in X \times P_Y$ such that $x_0 + \alpha(y_0, x) = x' + \alpha(y', x_0)$, $\xi(x', y') = x'$, and $y_0 y = y' y_0$.

The intuitive idea behind this definition is that a preordered action (α, ξ) of X on Y determines a preorder on $X \times Y$ that turns $X \rtimes_\alpha Y$ into a preordered monoid by specifying its positive cone. More precisely, the function ξ specifies the cone of positive elements of $X \rtimes_\alpha Y$ by asserting that (x, y) is positive if, and only if, $\xi(x, y) = x$. The fact that the domain of ξ is $X \times P_Y$ yields that all positive elements of $X \rtimes_\alpha Y$ belong to this set. Moreover, any value taken by ξ on a point (x, y) that is different from x is somehow irrelevant. Indeed, the authors define morphisms of preordered actions as follows: if X, X' and Y, Y' are preordered monoids, and (α, ξ) and (α', ξ') are preordered actions of Y on X and of Y' on X', respectively, then a morphism from (α, ξ) to (α', ξ') is pair (f, g) such that $f : X \to X'$ and $g : Y \to Y'$ are monoid homomorphisms restricting and co-restricting to the suitable positive cones, and that satisfy

$$f(\alpha(y, x)) = \alpha'(g(y), f(x)) \text{ and } \xi'(f(u), g(v)) = f(u),$$

for all $(x, y) \in X \times Y$, and $(u, v) \in X \times P_Y$ such that $\xi(u, v) = u$. Therefore, if (α, ξ) and (α, ξ') are preordered actions of Y on X such that, for all $x \in X$ and $y \in P_Y$

$$\xi(x, y) = x \iff \xi'(x, y) = x, \tag{21}$$

then the pair $(\mathrm{id}_X, \mathrm{id}_Y)$ consisting of the suitable identity maps defines an isomorphism between (α, ξ) and (α, ξ'). We may thus identify two preordered actions (α, ξ) and (α, ξ') whenever they satisfy (21).

We now compare the notions of $\mathbf{2}$-enriched action as in Proposition 6.1 and preordered action as in Definition 6.2.

Proposition 6.3. *Let* X *and* Y *belong to* $\mathbf{2}$-$\mathbf{Mon}^*$*. Then, there is a one-to-one correspondence between* $\mathbf{2}$*-enriched actions of* Y *on* X *and preordered actions of* Y *on* X *determined by those* (α, ξ) *that further satisfy*

(A.0) *if* $\xi(x, 1) = x$*, then* $x \in P_X$*, for all* $x \in X$*.*

Proof. The correspondence is as follows. If (α, P) is a **2**-enriched action then (α, ξ) is a preordered action satisfying (A.0), where

$$\xi(x,y) = \begin{cases} x, & \text{if } (x,y) \in P, \\ 0, & \text{else.} \end{cases}$$

Conversely, if (α, ξ) is a preordered action that satisfies (A.0) then, taking

$$P = \{(x,y) \in X \times P_Y \mid \xi(x,y) = x\}$$

defines a **2**-enriched action (α, P). It is a routine computation to check that these two assignments are indeed well-defined and are mutually inverse. We highlight that, if we start with a preordered action (α, ξ) and (α, P) it the **2**-enriched action it defines, then the preordered action (α, ξ') defined by (α, P) may not coincide with (α, ξ), but (α, ξ) and (α, ξ') do satisfy (21). □

The main result of [7] states that, up to isomorphism, there is a one-to-one correspondence between Schreier split extensions of preordered monoids in $\mathcal{V}$-**Mon*** and preordered actions. As we have just seen that the notions of *preordered action* and of **2**-*enriched action* are slightly different, the reader may now realize an apparent contradiction between this result and our Theorem 5.3. To understand what is happening, we have to analyze the definition of Schreier split extension of [7]. It is then the moment to introduce yet a new category, which turns out to be isomorphic to **OrdMon*** [7, Theorem 1].

Definition 6.4. [7, Definition 2] A monomorphism of monoids $m : P \rightarrowtail M$ is *right normal* if its image is a right normal submonoid of M. The full subcategory of the category of monomorphisms of monoids is denoted by **RNMono**(**Mon**).

The authors of [7] mostly work with the category **RNMono**(**Mon**) rather than with **OrdMon***, including for defining Schreier split extensions.

Definition 6.5. [7, Definition 3] A *Schreier split epimorphism* in **RNMono**(**Mon**) is a diagram

$$\begin{array}{ccccc} P_X & \xrightarrow{\overline{k}} & P_Z & \underset{\overline{s}}{\overset{\overline{p}}{\rightleftarrows}} & P_Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{k} & Z & \underset{s}{\overset{p}{\rightleftarrows}} & Y \end{array} \tag{22}$$

in which the lower row is a Schreier split epimorphism of monoids and the upper row consists if right normal submonoids, the positive cones P_X, P_Z, and P_Y, that turn X, Z, and Y objects of **OrdMon***. The morphisms $\overline{k}$, $\overline{p}$, and $\overline{s}$ are the corresponding restrictions.

This definition obfuscates the real nature of the morphisms in **OrdMon***. Indeed, while a monoid homomorphism between preordered monoids in **OrdMon*** is monotone if, and only if, it preserves the positive cone, the morphism k in diagram (22), although monotone, is not the kernel of p in the category of preordered monoids (nor in **OrdMon***). If we want to ensure that the Schreier split extension defined by a preordered action (α, ξ) is such that k is a kernel in the category of preordered monoids, then we must require that (α, ξ) satisfies (A.0). Moreover, even in the case of $\mathcal{V}$-groups, unlike ours, the results of [7] are not comparable with those of [1, 3, 2], as shown by the next example.

Example 6.6. Let X and Y be preordered groups and $\alpha : Y \times X \to X$ be a group action. We consider the preorder $\preceq$ on $X \times Y$ defined by

$$(x, y) \preceq (x', y') \iff y \leq_Y y'.$$

Then, $(X \rtimes_\alpha Y, \preceq)$ is a preordered group and

$$\begin{array}{ccccc}
P_X & \xrightarrow{\overline{\iota_1}} & P_{X \rtimes_\alpha Y} & \underset{\overline{\iota_2}}{\overset{\overline{\pi}_2}{\rightleftarrows}} & P_Y \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\iota_1} & X \rtimes_\alpha Y & \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} & Y
\end{array}$$

is a Schreier split extension in **RNMono**(**Mon**). However, unless the preorder on X is trivial,

$$X \overset{\iota_1}{\hookrightarrow} X \rtimes_\alpha Y \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} Y$$

is not a Schreier split extension of preordered groups in the sense of [1, 2]. Indeed, for all $x, x' \in X$ we have $(x, 1) \preceq (x', 1)$. Thus, if there are some $x \not\leq_X x'$, then $\iota_1[X]$ is not a submonoid of $X \rtimes_\alpha Y$, and therefore k is not the kernel of π_2.

References

[1] M. M. Clementino, N. Martins-Ferreira, and A. Montoli. On the categorical behaviour of preordered groups. *J. Pure Appl. Algebra*, 223(10):4226–4245, 2019.

[2] M. M. Clementino and A. Montoli. On the categorical behaviour of V-groups. *J. Pure Appl. Algebra*, 225(4):Paper No. 106550, 24, 2021.

[3] M. M. Clementino and C. Ruivo. On split extensions of preordered groups. *Port. Math.*, 80(3-4):327–341, 2023.

[4] D. Hofmann, G. J. Seal, and W. Tholen. Lax algebras. In *Monoidal topology*, volume 153 of *Encyclopedia Math. Appl.*, pages 145–283. Cambridge Univ. Press, Cambridge, 2014.

[5] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[6] N. Martins-Ferreira, A. Montoli, and M. Sobral. Semidirect products and crossed modules in monoids with operations. *J. Pure Appl. Algebra*, 217(2):334–347, 2013.

[7] N. Martins-Ferreira and M. Sobral. Schreier split extensions of preordered monoids. *J. Log. Algebr. Methods Program.*, 120:Paper No. 100643, 14, 2021.