DEGENERATE FREE BOUNDARY PROBLEMS WITH OSCILLATORY SINGULARITIES

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ABSTRACT. We extend the findings of [4] concerning free boundary problems involving varying singularities to the degenerate scenario. We establish fine geometric properties for minimizers under minimal assumptions on the oscillatory exponent.

CONTENTS

1. INTRODUCTION

In this paper, our focus is directed towards fine regularity properties of local minimizers of the p-energy functional

$$v \longmapsto \int_{\Omega} \frac{1}{p}\,|Dv|^p + \mathcal{Q}(x,v)\,dx, \tag{1.1}$$

where $p>2$ and $\mathcal{Q}\colon \Omega\times\mathbb{R}\to\mathbb{R}^+$ is a measurable function with respect to x and non-differentiable on the v variable. Functionals of this type emerge in certain chemistry phenomena, namely when the heterogeneity of external factors and the media influence the reaction rate of a porous catalyst

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region where gas is distributed. Simplified mathematical models lead to nonlinearities of the form

$$\mathcal{Q}(x,v) = v_+^\gamma \quad \text{for} \quad \gamma \in (0,1). \tag{1.2}$$

The non-differentiability of the corresponding energy functional (1.1) introduces several mathematical challenges as the system is best understood as a free boundary problem. The case described in (1.2) is referred to in the literature as the Alt-Phillips or quenching problem. Note that the corresponding Euler-Lagrange equation

$$\Delta_p u = \gamma\, u^{\gamma-1}\chi_{\{u>0\}}$$

only holds within the positivity set — a region self-dependent on the solution. The analysis of such free boundary problems boasts a rich history. The linear case, $p = 2$, is addressed in [1, 18, 19], with a modern approach detailed in the book [17]. Recent advances for the case $p > 2$ are elucidated in [7, 6]. Additionally, further attempts to tackle free boundary issues for different, though related, classes of problems can be found in [2, 10, 9, 13, 11, 12, 16, 14, 20], to cite a few.

In this paper, we analyze a broader class of degenerate free boundary problems featuring Alt-Phillips potentials with oscillatory blow-up rates, departing from a constant exponent γ and considering a measurable function $\gamma\colon \Omega \to (0,1)$ which determines the singularity of the model point-by-point in the domain. This oscillatory feature of the model leads to the appearance of multiple free boundary geometries. Such considerations lead to the analysis of minimizers of p-energy functionals of the form

$$J_\gamma^\delta(v) := \int \frac{1}{p}\,|Dv|^p + \delta(x) v_+^{\gamma(x)} dx, \tag{1.3}$$

where the functions $\gamma(x)$, $\delta(x)$ satisfy certain assumptions that will be detailed in due course. Our ultimate goal is to understand how oscillations of the parameter $\gamma(x)$ affect the regularity of the solutions near the free boundary and how it relates to the degeneracy feature of the diffusion.

In this paper, we extend the recent work [4] (see also [3, 5]) to the degenerate case. While this study does not focus on analyzing the regularity of the free boundary, as conducted in [4], we thoroughly investigate the fine geometric properties of solutions. Compared to the tools used in [4] to achieve the corresponding results, the most significant differences are: (i) the method of achieving non-degeneracy, with primary challenges arising from the nonlinear nature of the p-Laplacian, and (ii) the sharp regularity estimates along the free boundary, where the key obstacle is the lack of *explicit* $C^{1,\beta}$ regularity estimates for p-harmonic functions. We overcome this obstacle by leveraging geometric tangential analysis techniques.

The paper is organized as follows. In Section 2, we discuss the mathematical framework we will consider, particularly the notion of solution and the scaling features of the problem. We also develop an existence theory and obtain local regularity estimates and non-degeneracy. We dedicate Section 3 to gradient estimates near the free boundary. Finally, in Section 4, we refine the previously obtained estimates by considering a special oscillation regime of the parameter $\gamma(x)$ and obtain porosity estimates for the free boundary.

2. Problem formulation and preliminary findings

We first outline the mathematical framework for our problem by providing the concept of solution. For a bounded domain $\Omega \subset \mathbb{R}^n$ and a non-negative boundary datum $0 \leq \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we are interested in minimizers of p-energy functionals of the type

$$\mathcal{J}_\gamma^\delta(v, \Omega) := \int_\Omega \frac{1}{p} |Dv|^p + \delta(x) v_+^{\gamma(x)} dx, \tag{2.1}$$

among competing functions $W^{1,p}(\Omega)$ whose trace value agrees with φ on $\partial\Omega$, i.e. $W_\varphi^{1,p}(\Omega)$. More precisely, we say that $u \in W_\varphi^{1,p}(\Omega)$ is a minimizer of (2.1) if

$$\mathcal{J}_\gamma^\delta(u, \Omega) \leq \mathcal{J}_\gamma^\delta(v, \Omega), \quad \forall\, v \in W_\varphi^{1,p}(\Omega).$$

Note that minimizers as above are, in particular, local minimizers in the sense that, for any open subset $\Omega' \subset \Omega$,

$$\mathcal{J}_\gamma^\delta(u, \Omega') \leq \mathcal{J}_\gamma^\delta(v, \Omega'), \quad \forall\, v \in W_u^{1,p}(\Omega').$$

We emphasize that the minimal assumption to assure the well-posedness of the functional is that the functions $\delta, \gamma : \Omega \to \mathbb{R}_0^+$ are non-negative bounded mensurable functions. Furthermore, we shall assume

$$0 < \gamma_\star(\Omega) \leq \gamma(x) \leq \gamma^\star(\Omega) \leq 1, \quad \text{a.e. } x \in \Omega, \tag{2.2}$$

where

$$\gamma_\star(\Omega) := \operatorname*{ess\,inf}_{y \in \Omega} \gamma(y) \quad \text{and} \quad \gamma^\star(\Omega) := \operatorname*{ess\,sup}_{y \in \Omega} \gamma(y). \tag{2.3}$$

Key to the arguments in the sequel is the following scaling feature of the functional (2.1). Let $x_0 \in \Omega$ and consider parameters $A, B \in (0, 1]$. Then,

$$u \quad \text{minimizes} \quad \mathcal{J}_\gamma^\delta(v, \Omega) \iff w \quad \text{minimizes} \quad \mathcal{J}_{\tilde{\gamma}}^{\tilde{\delta}}(v, \tilde{\Omega}),$$

where $\tilde{\Omega} = x_0 - A^{-1}\,\Omega$,

$$w(x) := \frac{u(x_0 + Ax)}{B}, \quad x \in B_1, \tag{2.4}$$

and

$$\tilde{\delta}(x) := B^{\gamma(x_0 + Ax)} \left(\frac{A}{B}\right)^p \delta(x_0 + Ax) \quad \text{and} \quad \tilde{\gamma}(x) := \gamma(x_0 + Ax).$$

Indeed, by performing a change of variables, it is possible to show that

$$\mathcal{J}_{\gamma}^{\delta}(v,\Omega) = A^{n-p}B^{p}\,\mathcal{J}_{\tilde{\gamma}}^{\tilde{\delta}}(\tilde{v},\tilde{\Omega}) \quad \text{where} \quad \tilde{v}(x) = \frac{v(x_0+Ax)}{B}.$$

Observe that since $0<B\le 1$, $\tilde{\delta}$ satisfies

$$\|\tilde{\delta}\|_{L^\infty(B_1)} \le B^{\gamma_\star(\Omega)-p}A^{p}\|\delta\|_{L^\infty(B_A(x_0))}.$$

In particular, choosing $A=r$ and $B=r^{\beta}$, with $0<r\le 1$ and

$$\beta = \frac{p}{p-\gamma_\star(\Omega)},$$

we obtain $\|\tilde{\delta}\|_{L^\infty(\tilde{\Omega})} \le \|\delta\|_{L^\infty(\Omega)}$.

We remark that since the estimates are local, we can exchange Ω in the previous computations by the ball B_1 with radius one and center at the origin. To simplify the notation in the subsequent sections, we also define

$$\gamma_\star(x,r) \coloneqq \gamma_\star(B_r(x)) \qquad \text{and} \qquad \gamma^\star(x,r) \coloneqq \gamma^\star(B_r(x)).$$

2.1. **Existence of minimizers.** We begin by addressing the existence of a minimizer for the functional (2.1). Furthermore, we obtain global L^∞-bounds for minimizers. The argument follows a somewhat classical approach, but we have chosen to provide a detailed explanation here as a courtesy to the readers.

Proposition 2.1. *Assume* (2.2) *is in force. There exists a minimizer* $u\in W^{1,p}_{\varphi}(\Omega)$ *of the energy-functional* (2.1)*. Furthermore,* u *is non-negative in* Ω *and* $\|u\|_{L^\infty(\Omega)} \le \|\varphi\|_{L^\infty(\Omega)}$.

Proof. First, since J_{γ}^{δ} is non-negative, it follows that

$$m \coloneqq \inf\left\{\mathcal{J}_{\gamma}^{\delta}(v,\Omega) : v\in W^{1,p}_{\varphi}(\Omega)\right\} > -\infty.$$

This grants the existence of a minimizing sequence $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,p}_{\varphi}(\Omega)$, that is,

$$\mathcal{J}_{\gamma}^{\delta}(u_k,\Omega) \longrightarrow m \quad \text{as} \quad k\to\infty.$$

Then, for $k\gg 1$, we have

$$\begin{aligned} \|Du_k\|^p_{L^p(\Omega)} &= p\mathcal{J}_{\gamma}^{\delta}(u_k,\Omega) - p\int_{\Omega}\delta(x)(u_k)_+^{\gamma(x)}dx \\ &\le p(m+1). \end{aligned}$$

From Poincaré's inequality, we also have

$$\begin{aligned} \|u_k\|_{L^p(\Omega)} &\le \|u_k-\varphi\|_{L^p(\Omega)} + \|\varphi\|_{L^p(\Omega)} \\ &\le C\,\|Du_k - D\varphi\|_{L^p(\Omega)} + \|\varphi\|_{L^p(\Omega)} \\ &\le C\,\|Du_k\|_{L^p(\Omega)} + C\,\|D\varphi\|_{L^p(\Omega)} + \|\varphi\|_{L^p(\Omega)} \\ &\le C(p(m+1))^{\frac{1}{p}} + C\|\varphi\|_{W^{1,p}(\Omega)}, \end{aligned}$$

and so $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(\Omega)$. Consequently, for a subsequence (relabelled for convenience) and a function $u \in W^{1,p}(\Omega)$, we have

$$u_k \longrightarrow u,$$

weakly in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and pointwise for a.e. $x \in \Omega$.

The weak lower semi-continuity of the norm gives

$$\int_\Omega \frac{1}{p}\,|Du|^p\,dx \leq \liminf_{k\to\infty} \int_\Omega \frac{1}{p}\,|Du_k|^p\,dx$$

and the pointwise convergence and Lebesgue's dominated convergence give

$$\int_\Omega \delta(x)(u_k)_+^{\gamma(x)} dx \longrightarrow \int_\Omega \delta(x) u_+^{\gamma(x)} dx.$$

We conclude that

$$\mathcal{J}_\gamma^\delta(u,\Omega) \leq \liminf_{k\to\infty} \mathcal{J}_\gamma^\delta(u_k,\Omega) = m$$

and so u is a minimizer.

We now turn to the bounds on the minimizer. That u is non-negative for a non-negative boundary datum is trivial since $(u_+)_+ = u_+$, and testing the functional against $u_+ \in W^{1,p}_\varphi(\Omega)$ immediately gives the result. For the upper bound, test the functional with $v = \min\left\{u, \|\varphi\|_{L^\infty(\Omega)}\right\} \in W^{1,p}_\varphi(\Omega)$ to get, by the minimality of u,

$$\begin{aligned} 0 \leq \int_\Omega |D(u-v)|^p\,dx &= \int_{\Omega\cap\{u>\|\varphi\|_{L^\infty(\Omega)}\}} |Du|^p\,dx \\ &= \int_\Omega |Du|^p - |Dv|^p\,dx \\ &\leq 2\int_\Omega \delta(x)\left[v_+^{\gamma(x)} - u_+^{\gamma(x)}\right] dx \\ &\leq 0. \end{aligned}$$

We conclude that $v = u$ in Ω and thus $\|u\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)}$. $\square$

2.2. **Local $C^{1,\alpha}$−regularity estimates.** Next, we examine local regularity estimates of local minimizers of the p-energy functional (2.1). We emphasize that no further assumption on $\gamma(x)$, other than (2.2), is in force.

Theorem 2.1. *Let u be a minimizer of the p-energy functional* (2.1) *and assume* (2.2) *is in force. For each subdomain $\Omega' \Subset \Omega$, there exists a constant $C > 0$, depending only on n, p, $\|\delta\|_\infty$, $\gamma_\star(\Omega')$,* $\mathrm{dist}\,(\Omega', \partial\Omega)$ *and $\|u\|_\infty$, such that*

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C \quad \textit{for} \quad \alpha = \min\left\{\alpha_p^-, \frac{\gamma_\star(\Omega')}{p-\gamma_\star(\Omega')}\right\},$$

where α_p is the regularity exponent for p-harmonic functions.

Before the proof of the previously stated theorem, we comment that, without loss of generality, minimizers can be assumed normalized, that is,

$$\|u\|_{L^\infty(\Omega)} \leq 1. \tag{2.5}$$

Indeed, if u minimizes (2.1), then, the auxiliary function

$$\overline{u}(x) := \frac{u(x)}{M},$$

minimizes the functional

$$v \mapsto \int_\Omega \frac{1}{p}\,|Dv|^p + \overline{\delta}(x) v_+^{\gamma(x)}\,dx,$$

where

$$\overline{\delta}(x) := M^{\gamma(x)-p}\delta(x).$$

Taking $M = \max\{1, \|u\|_{L^\infty(\Omega)}\}$, places the new function $\overline{u}$ under condition (2.5); any regularity estimate proven for $\overline{u}$ automatically translates to u.

Furthermore, the proof uses classical estimates concerning the p-harmonic replacement (lifting), which we state here for completeness. Given a ball $B_R(x_0) \Subset \Omega$, we denote the p-harmonic replacement of u in $B_R(x_0)$ by h, *i.e.*, h is the solution of the boundary value problem

$$\Delta_p h = 0 \;\text{ in }\; B_R(x_0) \qquad \text{and} \qquad h - u \in W_0^{1,p}(B_R(x_0)).$$

By the maximum principle, we have $h \geq 0$ and

$$\|h\|_{L^\infty(B_R(x_0))} \leq \|u\|_{L^\infty(B_R(x_0))}. \tag{2.6}$$

The following two lemmata are classical results, and we refer to [15, Lemma 4.1] for a proof.

Lemma 2.1. *Let $\psi \in W^{1,p}(B_R)$ and h be the p-harmonic replacement of ψ in B_R. There exists c, depending only on n and p, such that*

$$c \int_{B_R} |D\psi - Dh|^p\,dx \leq \int_{B_R} |D\psi|^p - |Dh|^p\,dx. \tag{2.7}$$

Lemma 2.2. *Let $\psi \in W^{1,p}(B_R)$ and h be the p-harmonic replacement of ψ in B_R. There exists $C(n,p) > 0$ such that*

$$\begin{aligned}\int_{B_r} |D\psi - (D\psi)_r|^p\,dx \quad &\leq \quad C\left(\frac{r}{R}\right)^{n+p\,\alpha_p} \int_{B_R} |D\psi - (D\psi)_R|^p\,dx \\ &\qquad + C\int_{B_R} |D\psi - Dh|^p\,dx,\end{aligned}$$

for each $0 < r \leq R$. Here α_p is the regularity exponent for p-harmonic functions.

We are ready to prove the local regularity result.

Proof of Theorem 2.1. We prove the result for the case of balls $B_R(x_0) \Subset \Omega$. Without loss of generality, assume $x_0 = 0$ and denote $B_R := B_R(0)$. Since u is a local minimizer, by testing (2.1) against its p-harmonic replacement, we obtain the inequality

$$\int_{B_R} |Du|^p - |Dh|^p \, dx \leq p \int_{B_R} \delta(x) \left(h(x)^{\gamma(x)} - u(x)^{\gamma(x)} \right) dx. \tag{2.8}$$

Next, with the aid of [15, Lemma 2.5], one obtains

$$h(x)^{\gamma(x)} - u(x)^{\gamma(x)} \leq |u(x) - h(x)|^{\gamma(x)},$$

and, using (2.2), together with (2.5) and (2.6), we get

$$|u(x) - h(x)|^{\gamma(x)} \leq |u(x) - h(x)|^{\gamma_\star(0,R)}, \quad \text{a.e. in } \; B_R. \tag{2.9}$$

This readily leads to

$$\int_{B_R} \delta(x) \left(h(x)^{\gamma(x)} - u(x)^{\gamma(x)} \right) dx \leq \|\delta\|_{L^\infty(\Omega)} \int_{B_R} |u(x) - h(x)|^{\gamma_\star(0,R)} \, dx.$$

In addition, by combining Hölder and Sobolev inequalities, we obtain

$$\begin{aligned} \int_{B_R} |u - h|^{\gamma_\star(0,R)} \, dx \quad &\leq \quad C|B_R|^{1 - \frac{\gamma_\star(0,R)}{p^*}} \left(\int_{B_R} |u - h|^{p^*} \, dx \right)^{\frac{\gamma_\star(0,R)}{p^*}} \\ &\leq \quad C|B_R|^{1 - \frac{\gamma_\star(0,R)}{p^*}} \left(\int_{B_R} |Du - Dh|^p \, dx \right)^{\frac{\gamma_\star(0,R)}{p}} \end{aligned} \tag{2.10}$$

for $p^* = \dfrac{pn}{n-p}$.

Therefore, using Lemma 2.1, together with (2.8), (2.9) and (2.10), we get

$$\int_{B_R} |Du - Dh|^p \, dx \leq C|B_R|^{\frac{p(p^* - \gamma_\star(0,R))}{p^*(p - \gamma_\star(0,R))}} = CR^{n + p\frac{\gamma_\star(0,R)}{p - \gamma_\star(0,R)}}. \tag{2.11}$$

Therefore, Lemma 2.2 allows us to conclude

$$\begin{aligned} &\int_{B_r} |Du - (Du)_r|^p \, dx \\ &\leq C \left(\frac{r}{R} \right)^{n + p\,\alpha_p} \int_{B_R} |Du - (Du)_R|^p \, dx + CR^{n + p\,\frac{\gamma_\star(0,R)}{p - \gamma_\star(0,R)}}, \end{aligned}$$

for each $0 < r \leq R$. If we define the non-negative and non-decreasing function

$$\phi(t) := \int_{B_t} |Du - (Du)_t|^p \, dx,$$

then the previous inequality can be rewritten as

$$\phi(r) \leq C\left(\frac{r}{R}\right)^{n+p\,\alpha_p}\phi(R) + C\,R^{n+p\,\frac{\gamma_\star(0,R)}{p-\gamma_\star(0,R)}}.$$

A classical analysis result (see [15, Lemma 2.7]), allow us to conclude

$$r^{-n}\phi(r) \leq Cr^{p\,\beta},$$

for r small enough and

$$\beta = \min\left\{\alpha_p^-, \frac{\gamma_\star(0,R)}{p-\gamma_\star(0,R)}\right\}.$$

By Campanato's characterization of Hölder continuous functions, the proof is complete. □

Hereafter, in this paper, we assume $\Omega = B_1 \subset \mathbb{R}^n$ and, according to what was argued around (2.5), fix a normalized, non-negative minimizer, $0 \leq u \leq 1$, of the p-energy functional (2.1).

2.3. **Non-degeneracy.** Seeking now for non-degeneracy estimates, we need to assume further that the coefficient $\delta(x)$ is bounded below away from zero. More precisely, it satisfies

$$\operatorname*{ess\,inf}_{x\in B_1}\delta(x) =: \delta_0 > 0. \tag{2.12}$$

This assumption is crucial to create barriers touching the distorted minimizer from below.

Theorem 2.2. *Assume* (2.2) *and* (2.12) *are in force. For any* $y \in \overline{\{u>0\}}$ *and* $0 < r \ll 1$*, we have*

$$\sup_{\partial B_r(y)} u \geq c\,r^{\frac{p}{p-\gamma^\star(y,r)}}, \tag{2.13}$$

where $c > 0$ *depends only on* n*,* p*,* δ_0 *and* $\gamma_\star(0,1)$*.*

Proof. We only need to consider the case when $y \in \{u > 0\}$ and $0 < r \ll 1$. The general case follows by continuity. Define the auxiliary function φ by

$$\varphi(x) := u(x)^{\frac{p-\gamma^\star(y,r)}{p-1}}.$$

Letting $\lambda = \frac{p-\gamma^*(y,r)}{p-1}$, we have

$$D\varphi = \lambda u^{\lambda-1}Du \qquad \text{and} \qquad D^2\varphi = \lambda\Big((\lambda-1)u^{\lambda-2}Du\otimes Du + u^{\lambda-1}D^2u\Big).$$

Thus,

$$\begin{aligned}
\Delta_p\varphi &= |D\varphi|^{p-2}\Delta\varphi + (p-2)|D\varphi|^{p-4}\Delta_\infty\varphi \\
&= \Big|\lambda u^{\lambda-1}Du\Big|^{p-2}\lambda\Big((\lambda-1)u^{\lambda-2}|Du|^2 + u^{\lambda-1}\Delta u\Big) \\
&+ (p-2)\Big|\lambda u^{\lambda-1}Du\Big|^{p-4}\lambda^3\Big((\lambda-1)u^{2\lambda-3}|Du|^4 + u^{2(\lambda-1)}\Delta_\infty u\Big).
\end{aligned}$$

From assumption (2.2), it follows that $\lambda \geq 1$, and so

$$\begin{aligned}\Delta_p \varphi &\geq \lambda^{p-1} u^{(\lambda-1)(p-1)} \left(|Du|^{p-2}\Delta u + (p-2)|Du|^{p-4}\Delta_\infty u\right)\\ &= \delta(x)\gamma(x)\lambda^{p-1}u^{(\lambda-1)(p-1)}u^{\gamma(x)-1}\\ &\geq \delta_0\gamma_\star(0,1)u^{\gamma(x)-\gamma^\star(y,r)}.\end{aligned}$$

Since we can assume u to be normalized, it follows that $\Delta_p\varphi \geq \delta_0\,\gamma_*(0,1)$ in $\{u>0\}\cap B_r(y)$.

Now, for a positive constant $c>0$, let $\psi = c|x-y|^{\frac{p}{p-1}}$. By direct computations, it follows that

$$\Delta_p \psi = c^{p-1}\left(\frac{p}{p-1}\right)^{p-1} n,$$

and so, we can pick c in a way that $\Delta_p\psi < \delta_0\gamma_\star(0,1)$. So far, we have shown

$$\Delta_p\varphi > \Delta_p\psi \quad \text{within} \quad \{u>0\}\cap B_r(y),$$

and, as a consequence, there should hold

$$\partial\left(\{u>0\}\cap B_r(y)\right)\cap\{\varphi>\psi\} \neq \emptyset.$$

In fact, if $\varphi \leq \psi$ in $\partial\left(\{u>0\}\cap B_r(y)\right)$, then, by the comparison principle, we would have $\varphi\leq\psi$ in $\{u>0\}\cap B_r(y)$. But this is a contradiction, since

$$\varphi(y) > 0 = \psi(y).$$

Thus, there exists $x_0 \in \partial\left(\{u>0\}\cap B_r(y)\right)$ such that

$$u(x_0)^{\frac{p-\gamma^\star(y,r)}{p-1}} > c|y-x_0|^{\frac{p}{p-1}}.$$

This readily implies that $x_0 \notin \partial\{u>0\}$, and so $x_0\in\partial B_r(y)$, from which we conclude the proof. □

3. Gradient estimates near the free boundary

In regions relatively close to the free boundary, a special behaviour of the minimizers of (2.1) is expected. In this section, we are concerned with oscillation estimates for the gradients near free boundary points. We first show a result that acts as a compensatory measure for the absence of Harnack-type estimates.

Lemma 3.1. *Let u be a minimizer of the p-energy functional* (2.1) *in B_1 and assume* (2.2) *is in force. There exists a constant $C>1$, depending only on $\gamma_\star(0,1)$ and universal parameters, such that, if*

$$u(x) \leq \frac{1}{C}\, r^{\frac{p}{p-\gamma_\star(x,r)}}, \tag{3.1}$$

for $x \in B_{1/2}$ and $r \le 1/4$, then

$$\sup_{B_r(x)} u \le C r^{\frac{p}{p-\gamma_\star(x,r)}}.$$

Proof. Assume, seeking for a contradiction, the lemma fails. Therefore, given an integer $k > 0$, there should be a triple

$$(u_k, x_k, r_k) \in \mathcal{A}_p(B_1) \times B_{1/2} \times (0, 1/4),$$

where u_k is a minimizer of (2.1), such that

$$u_k(x_k) \le \frac{1}{k} r_k^{\frac{p}{p-\gamma^k}},$$

but

$$k\, r_k^{\frac{p}{p-\gamma^k}} < \sup_{B_{r_k}(x_k)} u_k =: s_k \le 1,$$

where $\gamma^k := \gamma_\star(x_k, r_k)$. Note that from the last two estimates,

$$u_k(x_k) \le \frac{1}{k} r_k^{\frac{p}{p-\gamma^k}} < \frac{1}{k^2} s_k,$$

and

$$\frac{r_k^{\frac{p}{p-\gamma^k}}}{s_k} < \frac{1}{k}. \tag{3.2}$$

Now, we define

$$\varphi_k(x) := \frac{u_k(x_k + r_k x)}{s_k} \quad \text{in } B_1,$$

and observe that

$$\sup_{B_1} \varphi_k = 1, \quad \text{and} \quad \varphi_k(0) < \frac{1}{k^2}. \tag{3.3}$$

It is clear that φ_k minimizes $\mathcal{J}_{\gamma_k}^{\delta_k}(v, B_1)$, for

$$\delta_k(x) := \delta(x_k + r_k x) \frac{r_k^p}{s_k^{p-\gamma(x_k+r_k x)}} \quad \text{and} \quad \gamma_k(x) := \gamma(x_k + r_k x).$$

From (3.2), we obtain

$$s_k^{\gamma(x_k+r_k x)-p} r_k^p \le s_k^{\gamma(x_k+r_k x)-p} \left(\frac{s_k}{k}\right)^{p-\gamma^k} = s_k^{\gamma(x_k+r_k x)-\gamma^k} \left(\frac{1}{k}\right)^{p-\gamma^k} \le \frac{1}{k},$$

for each $x \in B_1$. The last estimate is guaranteed since, for each k,

$$\gamma^k = \inf_{y \in B_{r_k}(x_k)} \gamma(y) = \inf_{x \in B_1} \gamma(x_k + r_k x) \le \gamma(x_k + r_k x).$$

Hence,

$$\|\delta_k\|_{L^\infty(B_1)} \le \|\delta\|_{L^\infty(B_1)} k^{-1}.$$

Recalling the lower bound

$$\inf_{y\in B_1} \gamma_k(y) = \inf_{y \in B_1} \gamma(x_k + r_k y) = \inf_{x \in B_{r_k}(x_k)} \gamma(x) = \gamma_\star(x_k, r_k) \ge \gamma_\star(0,1) =: \theta,$$

we apply Theorem 2.1 and get that the sequence $\{\varphi_k\}_k$ is uniformly bounded in $C^{1,\alpha}$, for α as in Theorem 2.1. Therefore, up to a subsequence, φ_k converges strongly to φ_∞ in $W^{1,\infty}(B_{1/2})$, as $k \to \infty$. Taking into account the estimates above, we conclude that φ_∞ minimizes the functional

$$v \longmapsto \int_{B_1} \frac{1}{p} |Dv|^p \, dx.$$

In particular, φ_∞ is p-harmonic in B_1, and $\varphi_\infty(0) = 0$. Therefore, by the strong maximum principle, one has $\varphi_\infty \equiv 0$ in B_1. But this contradicts

$$\sup_{B_1} \varphi_\infty = 1,$$

and the proof of the lemma is complete. □

Next, we prove a pointwise gradient estimate.

Lemma 3.2. *Let u be a minimizer of the p-energy functional* (2.1) *in B_1 and assume* (2.2) *is in force. Assume further that γ is lower semi-continuous in B_1. There exists a universal constant $\overline{C}$, such that*

$$|Du(x)|^p \leq \overline{C}\,[u(x)]^{\gamma_\star(0,1)}, \tag{3.4}$$

for each $x \in B_{1/2}$.

Proof. Due to the regularity estimates from Theorem 2.1, the lemma holds true at free boundary points since $|Du| = u = 0$ along $\partial\{u > 0\}$. Thus, with no loss of generality, we may assume $x \in \{u > 0\}$. Moreover, we may also assume $0 < u(x) < \tau$, where τ is universally small enough. Otherwise, if $u(x) \geq \tau$, then

$$|Du(x)|^p \leq L^2 = L^2 \left(\frac{\tau}{\tau}\right)^{\gamma_\star(0,1)} \leq \frac{L^2}{\tau^{\gamma_\star(0,1)}} [u(x)]^{\gamma_\star(0,1)},$$

and the lemma is proved. Assuming $0 < u < \tau$, for

$$\tau := \frac{1}{C}\left(\frac{1}{4}\right)^{\frac{p}{p-\gamma^\star(0,1)}},$$

and C as in Lemma 3.1, we note that

$$\lim_{s \to 0^+} s^{\frac{p}{p-\gamma_\star(x,s)}} = 0,$$

for each $x \in B_{1/2}$. From this and the fact that $\gamma_\star(x,\cdot)$ is lower semi-continuous, we can select $r > 0$ such that

$$r^{\frac{p}{p-\gamma_\star(x,r)}} = Cu(x) \leq \left(\frac{1}{4}\right)^{\frac{p}{p-\gamma^\star(0,1)}}.$$

The rest of the proof follows the same lines as in [4, Lemma 3.2], with the obvious modifications. □

Remark 3.1. *It is worthwhile mentioning that the lower semi-continuity assumption on $\gamma(x)$ in Lemma 3.2 can be removed. To do so, one has to prove a weaker version of Lemma 3.1, with $p/(p-\gamma_*(0,1))$ replacing $p/(p-\gamma_*(x,r))$. The reasoning follows seamlessly.*

4. Optimal regularity estimates

A key challenge in our problem is understanding how the oscillation of $\gamma(x)$ impacts the regularity of minimizers along the free boundary. Observe that the local regularity result in Theorem 2.1 yields a $(1+\alpha)-$growth control for a minimizer u near its free boundary. More precisely, if z_0 is a free boundary point, an inspection of the proof of Theorem 2.1 yields

$$u(y) \le C|y-z_0|^{1+\min\left\{\alpha_p^-, \frac{\gamma_\star(\Omega')}{p-\gamma_\star(\Omega')}\right\}}.$$

While this estimate has its merits, it also has notable drawbacks. Initially, it is (naturally) bounded by the regularity theory for p-harmonic functions, which remains largely unknown in dimensions $n \ge 3$ (*cf.* [5]). Secondly, it does not consider how γ oscillates, as it is tailored for problems without assumptions on the oscillation of γ.

In this section, we assume γ is continuous at a free boundary point z_0, with a modulus of continuity ω satisfying

$$\omega(1) + \lim_{t\to 0} \omega(t) \ln\left(\frac{1}{t}\right) \le \tilde{C}, \tag{4.1}$$

for a constant $\tilde{C} > 0$. Such a condition often appears in models involving variable exponent PDEs. We refer to [4, Section 4] for an explanation of this condition and how Dini-continuity is related.

We are ready to state a sharp pointwise regularity estimate for local minimizers of (2.1) under (4.1). We define the subsets

$$\Omega(u) := \{x \in B_1 \mid u(x) > 0\} \quad \text{and} \quad F(u) := \partial\Omega(u),$$

corresponding to the non-coincidence set and the free boundary of the problem, respectively.

Theorem 4.1. *Let u be a local minimizer of (2.1) in B_1. Assume $z_0 \in F(u) \cap B_{1/2}$ and γ satisfies (4.1) at z_0. There exist universal constants $r_0 > 0$ and $C' > 1$ such that*

$$\sup_{y\in B_r(z_0)} u(y) \le C' r^{\frac{p}{p-\gamma(z_0)}}, \tag{4.2}$$

for all $0 < r \le r_0$.

Proof. It is enough to show that

$$u(y) \le C' \, |y - z_0|^{\frac{p}{p-\gamma(z_0)}} \quad \text{for all} \quad y \in B_{r_0}(z_0).$$

First, as (4.1) is in force, we can pick $r_0 \ll 1$ such that, for $r < r_0$,

$$\omega(r) \ln\left(\frac{1}{r}\right) \le 2\left[\tilde{C} - \omega(1)\right] =: C^*. \tag{4.3}$$

Fix $y \in B_{r_0}(z_0)$ and let $r := |y - z_0| < r_0$. Since $z_0 \in F(u)$, it follows that $u(z_0) = 0$. Trivially, Lemma 3.1 holds and then

$$\sup_{x \in B_r(z_0)} u(x) \le C \, r^{\frac{p}{p-\gamma_\star(z_0,r)}}.$$

By continuity, it follows as a consequence that

$$u(y) \le C \, r^{\frac{p}{p-\gamma_*(z_0,r)}}. \tag{4.4}$$

Now, taking into account that the function $g\colon [0,1] \to [0,1]$ given by

$$g(t) := \frac{p}{p-t}$$

satisfies $\frac{1}{p} \le g'(t) \le \frac{p}{(p-1)^2}$, for all $t \in [0,1]$, we have

$$\begin{aligned} g\left(\gamma(z_0)\right) - g\left(\gamma_\star(z_0,r)\right) &\le \frac{p}{(p-1)^2}\left(\gamma(z_0) - \gamma_\star(z_0,r)\right) \\ &\le \frac{p}{(p-1)^2}\,\omega(r), \end{aligned}$$

where we used that $\gamma(z_0) - \gamma_\star(z_0,r) \le \omega(r)$. By (4.4), it follows that

$$\begin{aligned} u(y) &\le C\, r^{g(\gamma_*(z_0,r))} \\ &= C\, r^{-[g(\gamma(z_0)) - g(\gamma_*(z_0,r))]} r^{g(\gamma(z_0))}. \end{aligned}$$

Since $g\left(\gamma(z_0)\right) - g\left(\gamma_\star(z_0,r)\right) \le \frac{p}{(p-1)^2}\,\omega(r)$ and taking (4.3) into account, we reach

$$\begin{aligned} u(y) &\le C\, r^{-[g(\gamma(z_0)) - g(\gamma_*(z_0,r))]} r^{g(\gamma(z_0))} \\ &\le C\, r^{-\frac{p}{(p-1)^2}\omega(r)}\, r^{g(\gamma(z_0))} \\ &\le C\, e^{\frac{p}{(p-1)^2}C^*}\, r^{g(\gamma(z_0))} \\ &= C' \, |y - z_0|^{\frac{p}{p-\gamma(z_0)}}, \end{aligned}$$

as desired. □

We also obtain a sharp, strong non-degeneracy result.

Theorem 4.2. *Let u be a local minimizer of (2.1) in B_1. Assume $z_0 \in F(u) \cap B_{1/2}$, γ satisfies (4.1) at z_0 and (2.12) is in force. There exists a universal constant $c^* > 0$ such that*

$$\sup_{\partial B_r(z_0)} u \ge c^* \, r^{\frac{p}{p-\gamma(z_0)}},$$

for every $0 < r < 1$.

Proof. First, we note that

$$\frac{p}{p-\gamma^\star(z_0,r)} = \frac{p}{p-\gamma(z_0)} + \frac{p}{p-\gamma^\star(z_0,r)} - \frac{p}{p-\gamma(z_0)},$$

and

$$\begin{aligned}\frac{p}{p-\gamma^\star(z_0,r)} - \frac{p}{p-\gamma(z_0)} &= \frac{p(\gamma^\star(z_0,r)-\gamma(z_0))}{(p-\gamma^\star(z_0,r))\,(p-\gamma(z_0))} \\ &\leq p(\gamma^\star(z_0,r)-\gamma(z_0)) \\ &\leq p\,\omega(r).\end{aligned}$$

Also, since

$$\frac{p}{p-\gamma^\star(z_0,r)} - \frac{p}{p-\gamma(z_0)} \leq \frac{p}{p-1} - 1 = \frac{1}{p-1},$$

we have

$$\frac{p}{p-\gamma^\star(z_0,r)} - \frac{p}{p-\gamma(z_0)} \leq \min\left\{p\,\omega(r), \frac{1}{p-1}\right\}.$$

Now, define $r_0 > 0$ to be the largest number in $(0,1)$ such that (4.3) holds. If $r < r_0$, then

$$\begin{aligned} r^{\frac{p}{p-\gamma^\star(z_0,r)}} &\geq r^{p\,\omega(r)} r^{\frac{p}{p-\gamma(z_0)}} \\ &= e^{p\,\omega(r)\ln r}\, r^{\frac{p}{p-\gamma(z_0)}} \\ &\geq e^{-pC^*}\, r^{\frac{p}{p-\gamma(z_0)}}.\end{aligned}$$

If $r \geq r_0$, then

$$r^{\frac{p}{p-\gamma^\star(z_0,r)}} \geq r^{\frac{1}{p-1}} r^{\frac{p}{p-\gamma(z_0)}} \geq r_0^{\frac{1}{p-1}} r^{\frac{p}{p-\gamma(z_0)}}.$$

Either way, it follows that

$$r^{\frac{p}{p-\gamma^\star(z_0,r)}} \geq \overline{c} r^{\frac{p}{p-\gamma(z_0)}} \quad \text{where} \quad \overline{c} = \min\left\{r_0^{\frac{1}{p-1}}, e^{-pC^*}\right\}.$$

The result follows with $c^* = c\,\overline{c}$ once we apply Theorem 2.2. ☐

With sharp regularity and non-degeneracy estimates, we can now prove the positive density of the non-coincidence set, leading to the porosity of the free boundary and Hausdorff measure estimates.

Theorem 4.3. *Let u be a local minimizer of* (2.1) *in B_1. Assume $z_0 \in F(u) \cap B_{1/2}$, γ satisfies* (4.1) *at z_0 and* (2.12) *is in force. There exists a constant $\mu_0 > 0$, depending on n, p, δ_0, $\gamma_\star(0,1)$ and the constant from* (4.1)*, such that*

$$\frac{|B_r(z_0) \cap \Omega(u)|}{|B_r(z_0)|} \geq \mu_0,$$

for every $0 < r < r_0$. In particular, $F(u)$ is porous and there exists an $\epsilon > 0$ such that $\mathcal{H}^{n-\epsilon}(F(u) \cap B_{1/2}) = 0$.

Proof. The proof follows the same lines as in [4, Theorem 4.3] with the obvious modifications. □

Finally, we establish an optimized version of Lemma 3.2. Key to the argument is the fact that the oscillating parameter $\gamma(x)$ now satisfies (4.1), which allows us to obtain

$$1/2 \leq r^{\gamma_\star(x,r)-\gamma(x)} \leq 2 \quad \text{as} \quad r \to 0. \tag{4.5}$$

If $x \in \Omega(u) \cap B_{1/2}$ is such that

$$u(x) \leq \frac{1}{C} r^{\frac{p}{p-\gamma(x)}},$$

for $r \leq 1/4$, then, since $\gamma_\star(x,r) \leq \gamma(x)$, we have that (3.1) also holds at x. By Lemma 3.1, we obtain

$$\sup_{B_r(x)} u \leq C r^{\frac{p}{p-\gamma_*(x,r)}}.$$

Condition (4.1) comes into play, and proceeding as in the proof of Theorem 4.2, for a larger constant C_1, we have

$$\sup_{B_r(x)} u \leq C_1 r^{\frac{p}{p-\gamma(x)}}, \tag{4.6}$$

for r universally small. This remark leads to the following result.

Lemma 4.1. *Let u be a local minimizer of the energy-functional* (2.1) *in B_1. Assume* (2.12) *and* (4.1) *are in force. There exists a constant C, depending on $\gamma_\star(0,1)$ and universal parameters, such that*

$$|Du(x)|^p \leq C\,[u(x)]^{\gamma(x)},$$

for each $x \in B_{1/2}$.

Proof. The proof is essentially the same as the proof of Lemma 3.2, except for the steps we highlight below. Pick $r > 0$ so that

$$r^{\frac{p}{p-\gamma(x)}} = Cu(x).$$

As a consequence, (4.6) implies that the function, defined in B_1 by

$$v(y) := u(x+ry) r^{-\frac{p}{p-\gamma(x)}},$$

is uniformly bounded. What remains to be shown is that the parameters in the functional that v minimizes are also controlled. Due to the scaling properties from section 2, we have v minimizes a scaled functional with $\tilde{\delta}$ satisfying

$$\|\tilde{\delta}\|_{L^\infty(B_1)} \leq r^{\frac{p}{p-\gamma(x)}\gamma_*(x,r)-p} r^p \|\delta\|_{L^\infty(B_1)} \leq r^{\gamma_*(x,r)-\gamma(x)} \|\delta\|_{L^\infty(B_1)},$$

which is uniformly bounded by (4.5). The Lemma is proved once we apply Lipschitz estimates available for v. □

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References

[1] H.W. Alt and D. Phillips, *A free boundary problem for semilinear elliptic equations*, J. Reine Angew. Math. 368 (1986), 63–107.
[2] J. Andersson, E. Lindgren and H. Shahgholian, *Optimal regularity for the obstacle problem for the $p-$Laplacian*, J. Differential Equations 259 (2015), 2167–2179.
[3] D.J. Araújo, G. Sá, E. Teixeira and J.M. Urbano, *Oscillatory free boundary problems in stochastic materials*, arXiv:2404.03060.
[4] D.J. Araújo, A. Sobral, E. Teixeira and J.M. Urbano, *On free boundary problems shaped by oscillatory singularities*, arXiv:2401.08071.
[5] D.J. Araújo, E. Teixeira and J.M. Urbano, *A proof of the $C^{p'}-$regularity conjecture in the plane*, Adv. Math. 316 (2017), 541–553.
[6] D.J. Araújo, R. Teymurazyan and J.M. Urbano, *Hausdorff measure estimates for the degenerate quenching problem*, arXiv:2402.11536.
[7] D.J. Araújo, R. Teymurazyan and V. Voskanyan, *Sharp regularity for singular obstacle problems*, Math. Ann. 387 (2023), 1367–1401.
[8] L. Caffarelli, *Compactness methods in free boundary problems*, Comm. Partial Differential Equations 5 (1980), 427–448.
[9] D. Danielli and A. Petrosyan, *A minimum problem with free boundary for a degenerate quasilinear operator*, Calc. Var. Partial Differential Equations 23 (2005), 97–124.
[10] J. Fernandez Bonder, S. Martínez and N. Wolanski, *A free boundary problem for the $p(x)-$Laplacian*, Nonlinear Anal. 72 (2010), 1078–1103.
[11] F. Ferrari and C. Lederman, *Regularity of flat free boundaries for a $p(x)-$Laplacian problem with right hand side*, Nonlinear Anal. 212 (2021), 112444, 25 pp.
[12] F. Ferrari and C.Lederman, *Regularity of Lipschitz free boundaries for a $p(x)$-Laplacian problem with right hand side*, J. Math. Pures Appl. 171 (2023), 26–74.
[13] A. Figalli, B. Krummel and X. Ros-Oton, *On the regularity of the free boundary in the $p-$Laplacian obstacle problem*, J. Differential Equations 263 (2017), 1931–1945.
[14] K. Lee and H. Shahgholian, *Hausdorff measure and stability for the $p-$obstacle problem ($2 < p < \infty$)*, J. Differential Equations 195 (2003), 14–24.
[15] R. Leitão, O. de Queiroz and E.V. Teixeira, *Regularity for degenerate two-phase free boundary problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015), 741–762.
[16] R. Leitão and G. Ricarte, *Free boundary regularity for a degenerate problem with right-hand side*, Interfaces Free Bound. 20 (2018), 577–595.
[17] A. Petrosyan, H. Shahgholian and N. Uraltseva, *Regularity of free boundaries in obstacle-type problems*, Grad. Stud. Math. 136, American Mathematical Society, Providence, RI, 2012. x+221 pp.
[18] D. Phillips, *A minimization problem and the regularity of solutions in the presence of a free boundary*, Indiana Univ. Math. J. 32 (1983), 1–17.
[19] D. Phillips, *Hausdorff measure estimates of a free boundary for a minimum problem*, Comm. Partial Differential Equations 8 (1983), 1409–1454.

[20] G. Ricarte, R. Teymurazyan and J.M. Urbano, *Singularly perturbed fully nonlinear parabolic problems and their asymptotic free boundaries*, Rev. Mat. Iberoam. 35 (2019), 1535–1558.

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