KEYS AND EVACUATION VIA VIRTUALIZATION

OLGA AZENHAS, NICOLLE GONZÁLEZ, DAOJI HUANG, AND JACINTA TORRES

Abstract. We prove that the key map on crystals can be reduced to the simply-laced types by using virtualization of crystals. For this we use the original definition of virtualization coming from the dilation of crystals by Kashiwara. As a direct application we obtain algorithms to compute orthogonal evacuation, keys and Demazure atoms in type B_n in terms of Kasiwara–Nakashima tableaux. In particular, we are able to use type A_n and C_n methods. For type C_n, we apply the results obtained by Azenhas–Tarighat Feller–Torres, Azenhas-Santos and Santos.

Contents

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and λ a dominant integral weight. To the corresponding irreducible finite-dimensional complex representation of $\mathfrak{g}$ of highest weight λ is associated a crystal $\mathcal{B}(\lambda)$, which can be thought of as a combinatorial skeleton of the representation $V(\lambda)$. To each crystal $\mathcal{B}(\lambda)$ is associated a so-called crystal graph, namely a weighted, directed coloured graph with as many vertices as the dimension of $V(\lambda)$.

The classical Borel–Weil theorem establishes a connection between representation theory and the geometry of flag varieties. It is well-known that the theory of crystals is closely related to standard monomial theory on flag varieties. Demazure modules were originally described as the spaces of global sections of a suitable line bundle on a Schubert variety embedded in a flag variety [Dem74, Paragraphe 5, Théorème 1]. This description exhibits the natural correspondence between Schubert varieties and Demazure modules. Let W be the Weyl group together with its strong Bruhat order $\leq$. For any $\tau \leq w$ in W, let X_τ and $V_\tau(\lambda)$ denote the Schubert variety, respectively the Demazure module corresponding to τ: $X_\tau \subseteq X_w$ means $V_\tau(\lambda) \subseteq V_w(\lambda) \subseteq V(\lambda)$. To a Demazure module $V_w(\lambda)$ is associated a Demazure crystal $B_w(\lambda)$. It seems natural to ask whether a given vertex $b \in B(\lambda)$ belongs to a certain Demazure crystal $B_w(\lambda)$.

The answer to this question is provided by the right, respectively left *key map*, in the case of an opposite Demazure crystal, which associates to each vertex $b \in B(\lambda)$ an element in the W-orbit of the highest weight element in $B(\lambda)$ called its right key respectively left key. This so-called *extremal* element indicates the smallest possible Demazure crystal, respectively opposite, containing the vertex b.These so-called *extremal* elements indicate the smallest possible Demazure crystal, respectively opposite Demazure crystal, containing the vertex b. A Demazure atom consists of the elements of a crystal with the same right key (cf. [LS90b, Theorem 3.8] for the type A_n description). The left key map detects *opposite* Demazure atoms. The study of various properties of Demazure crystals and atoms is an extremely active topic of research today [ADG23, AA24, AGL24, AS24, Ava08, BBBG21, CK17, FK13, FL94, JL20b, Kou20, KRSKV24, KV16, LL03, Lam19, ?, PR99, San21b, San21a].

The effective computation of the key map has captured the interest of many mathematicians over the last few decades. In type A_n, that is, when $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, there is a well-known algorithm for semi-standard Young tableaux due to Lascoux–Schützenberger [LS90a]. Now there are various algorithms for computing the key map in type A_n [Ava08, BBBG21, KRV23, Mas09, Wil13], as well as a recursive procedure for Kac–Moody Lie algebras using combinatorial R-matrices [JL20a], which generalizes the original procedure by Lascoux–Schützenberger. In the case of type C_n Kashiwara–Nakashima tableaux, there are two known procedures due to Santos [San21b, San21a].

The Lusztig–Schützenberger involution is defined as the unique set involution $\xi : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda)$ such that for all $i \in I$, where I is the Dynkin diagram associated to $\mathfrak{g}$. and $b \in \mathcal{B}(\lambda)$:

$$e_i\xi(b) = \xi f_{\theta(i)}(b),$$
$$f_i\xi(b) = \xi e_{\theta(i)}(b),$$
$$\mathrm{wt}(\xi(b)) = w_0\mathrm{wt}(b),$$

where θ is the automorphism of I defined by applying the longest element $w_0 \in W$ to the simple roots. (see Section 2.4). In type A_n, the Lusztig–Schütznberger involution is easily computed on semi-standard Young tableaux by the well-known procedure evacuation. For type C_n Kashiwara–Nakashima tableaux, a similar procedure was obtained by Santos [San21b]. The relevance of keys stems from standard monomial theory. In the crystal model of Lakshmibai–Seshadri paths for $\mathcal{B}(\lambda)$ the initial and final directions of a path exhibit the right key respectively left key. The left and the right keys of a vertex of $\mathcal{B}(\lambda)$ are interchangeable by Lusztig –Schütznberger involution.

1.1. **Main results.** In this paper, we introduce a new technique for computing the key map and the Schützenberger–Lusztig involution using *virtualization* of crystals. In particular, it reduces the computation of these maps for non-simply laced complex finite-dimensional simple Lie algebras to the simply-laced cases, where the former is obtained from the latter via folding of the corresponding Dynkin diagrams. Let $X \subset Y$ be an embedding of Dynkin diagrams as in Figure 1, and let $\tilde{\mathcal{B}}$ be a $\mathfrak{g}_Y$-crystal, where $\mathfrak{g}_D$ is the complex simple Lie algebra associated to the Dynkin diagram D. A virtual $\mathfrak{g}_X$-crystal is a subset $\mathcal{V} \subset \tilde{\mathcal{B}}$ such that $\mathcal{V}$ has a normal $\mathfrak{g}_X$-crystal structure is given as in Section 2.2. Additionally, if a $\mathfrak{g}_X$-crystal $\mathcal{B}$ is isomorphic to a virtual $\mathfrak{g}_X$-crystal $\mathcal{V} \subset \tilde{\mathcal{B}}$, we call the associated isomorphism $\mathcal{B} \to \mathcal{V}$ the virtualization map. In Proposition 2.9 we show that the Schützenberger–Lusztig involution commutes with virtualization. In Section 3.4, we show that vitualization commutes with dilation and consequently with the right and left key maps (cf. Corollary 3.21). Our results constitute a type-independent generalization of the results obtained in [AS24, AFT22] for the embedding $C_n \hookrightarrow A_{2n-1}$. A more general version of Proposition 2.9 in the context of the cactus group was obtained in [Tor24] for Littelmann paths and the virtualization map from [PS18].

1.2. **Applications.** In general, if the virtualization map at hand has a well-defined left inverse which can be computed effectively, then our results indeed reduce the computation of keys, atoms and the Schützenberger–Lusztig involution to the simply-laced types. In Section 4 we pursue these applications for Kashiwara–Nakashima tableaux in type B_n using different virtualization maps, for instance due to Baker [Bak00a] and Pappe–Pfannerer–Simone–Schilling [PPSS23]. In particular we obtain an algorithmic description of the key map and an evacuation algorithm for Kashiwara–Nakashima tableaux of type B_n.

1.3. **Organization of the paper.** In Section 2 we introduce normal crystals and their tensor products, the Lusztig–Schützenberger involution, and virtual crystals. In Section 3 we first introduce Demazure crystals and atoms, followed by the dilation of crystals. We then state and prove Theorem 3.9, which allows us to recall the definition of the key map (cf. 3.10). In Proposition 3.15 we show that the Lusztig–Schützenberger involution exchanges the left and right key maps. In Theorem 3.19 we show that virtualization and dilation of crystals commute. We conclude that virtualization commutes with key maps (cf. Corollary 3.21). In Section 4 we present applications of our results to Kashiwara–Nakashima tableaux. In the appendix A we collect larger examples for the comfort of the reader.

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2. Crystals

2.1. **Normal crystals.** Let $\mathfrak{g}$ be a finite complex semisimple Lie algebra with weight lattice P, simple roots $\alpha_i \in P$ with $i \in I$, simple coroots $\alpha_i^\vee \in P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$, and canonical pairing $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$. The fundamental weights ω_i and coweights $\omega_i^\vee$ are defined via $\langle \alpha_i^\vee, \omega_j \rangle = \langle \omega_j^\vee, \alpha_i \rangle = \delta_{ij}$. We denote the corresponding Weyl group as W; it is generated as a Coxeter group by simple reflections $s_i, i \in I$ and relations defined by the associated Dynkin diagram. For $u, v \in W$, we say that $u \leq v$ in the *strong Bruhat order* if, for every reduced expression for v, there exists a subexpression that is a reduced expression for u [Hum97, BB05].

We review Kashiwara's theory of $\mathfrak{g}$-crystals but refer the reader to [Kas91, Kas95, Kas02] for details. A (normal) ***$\mathfrak{g}$-crystal*** is a nonempty finite set $\mathcal{B}$ with maps:

$$\mathrm{wt} : \mathcal{B} \to P, \qquad \varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}, \qquad e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\},$$

where $0 \notin \mathcal{B}$ is an auxiliary symbol, subject to the following conditions for all $i \in I$ and $b, b' \in \mathcal{B}$:

- (C1) $\varphi_i(b) - \varepsilon_i(b) = \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$;
- (C2) $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$ if $e_i(b) \in \mathcal{B}$ and $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$ if $f_i(b) \in \mathcal{B}$;
- (C3) $b' = e_i(b)$ if and only if $b = f_i(b')$;
- (C4) $\varepsilon_i(b) = \max\{k \geq 0 | e_i^k(b) \in \mathcal{B}\}$ and $\varphi_i(b) = \max\{k \geq 0 | f_i^k(b) \in \mathcal{B}\}$.

We call the maps ε_i and φ_i the *string operators*, wt the *weight map*, and e_i and f_i the *crystal operators*.

For any $i \in I$, an *i-string* of length k is any subset of the form $\{f_i^n(b) \neq 0 \mid n \geq 0\} \subset \mathcal{B}(\lambda)$ for some $b \in \mathcal{B}(\lambda)$ satisfying $e_i(b) = 0$ and $\varphi_i(b) = k$.

Definition 2.1. Given any $\mathcal{B}, \mathcal{C} \in \mathfrak{g}$-crystals, a map $\phi : \mathcal{B} \to \mathcal{C} \cup \{0\}$ is a ***crystal morphism*** if for all $b \in \mathcal{B}$ and $\phi(b) \in \mathcal{C}$ and any $i \in I$ the following conditions hold:

- (*a*) $\mathrm{wt}(b) = \mathrm{wt}(\phi(b))$,
- (*b*) $\varepsilon_i(b) = \varepsilon_i(\phi(b))$
- (*c*) $\varphi_i(b) = \varphi_i(\phi(b))$,
- (*d*) if $e_i(b) \in \mathcal{B}$ and $\phi(e_i(b)) \in \mathcal{C}$ then $e_i(\phi(b)) = \phi(e_i(b))$, and
- (*e*) if $f_i(b) \in \mathcal{B}$ and $\phi(f_i(b)) \in \mathcal{C}$ then $f_i(\phi(b)) = \phi(f_i(b))$.

Moreover, we say ϕ an *isomomorphism* (resp. epimorphism, monomorphism) if the underlying set map $\phi : \mathcal{B} \to \mathcal{C}$ is a bijection (resp. injective, surjective).

The category of $\mathfrak{g}$-crystals is endowed with a monoidal structure that is compatible with the tensor product structure of $U_q(\mathfrak{g})$. Thus, given $\mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{g}$-crystals the *tensor product* $\mathcal{B}_1 \otimes \mathcal{B}_2$ is defined as the set

$$\mathcal{B}_1 \otimes \mathcal{B}_2 = \{b_1 \otimes b_2 \mid b_1 \in \mathcal{B}_1 \text{ and } b_2 \in \mathcal{B}_2\}$$

with wt and string operators given by

$$\begin{aligned} \mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}_i(b_1)), \\ \varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varphi_i(b_1) + \mathrm{wt}_i(b_2)), \end{aligned} \tag{1}$$

where $\mathrm{wt}_i(b) = \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle$, and with crystal operators defined by [1]

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2); \end{cases} \tag{2}$$

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{3}$$

Definition 2.2. Let $\mathcal{E}$ and $\mathcal{F}$ be the semigroups generated by $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$, respectively. We say a element $b \in \mathcal{B}$ is a ***highest weight vector*** (resp. ***lowest weight vector***) if $\mathcal{E}\{b\} = 0$ (resp. $\mathcal{F}\{b\} = 0$).

It is a classical theorem that the irreducible finite-dimensional integrable highest weight modules $V(\lambda)$ of $U_q(\mathfrak{g})$ are indexed by the set of dominant weights P^+. Thus, given $\lambda \in P^+$ we denote by $\mathcal{B}(\lambda)$ the normal crystal associated to $V(\lambda)$, whose highest weight vector b_λ is the unique element in $\mathcal{B}(\lambda)$ satisfying the property that $\mathrm{wt}(b_\lambda) = \lambda$ and

$$\mathcal{E}\{b_\lambda\} = 0 \qquad \text{and} \qquad \mathcal{F}\{b_\lambda\} = \mathcal{B}(\lambda).$$

More generally, we have that $b_1 \otimes b_2$ is a highest weight vector of $\mathcal{B}_1 \otimes \mathcal{B}_2$ if and only if

$$e_i(b_1) = 0 \text{ and } \varepsilon_i(b_2) \leq \varphi_i(b_1) \text{ for all } i \in I. \tag{4}$$

Theorem 2.3 ([Kas95],[Kas02])**.** *For $\lambda, \mu \in P^+$, $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is the crystal for $V(\lambda) \otimes V(\mu)$.*

The following decomposition formula is well known.

Theorem 2.4 ([Kas95],[Kas02])**.** *For $\lambda, \mu \in P^+$,*

$$\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \simeq \bigoplus_{\substack{b \in \mathcal{B}(\mu) \\ \varepsilon_i(b) \leq \varphi_i(b_\lambda) \ \ i \in I}} \mathcal{B}(\lambda + \mathrm{wt}(b)).$$

The action of crystal operators on $\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_r$ can be computed using *signature rules*. As noted in [BS17, p.23], these rules are obtained by assigning to each factor b_j in $b_1 \otimes \cdots \otimes b_r$ the sign $(-)^{\varepsilon_i(b_j)}(+)^{\varphi_i(b_j)}$ and then successively bracketing any pair of the form $(+-)$ until all unbracketed symbols are of the form $(-)^a(+)^b$. It then follows that

$$\varepsilon_i(b_1 \otimes \cdots \otimes b_r) = a \text{ and } \varphi_i(b_1 \otimes \cdots \otimes b_r) = b, \tag{5}$$

so that e_i will act on the factor associated to the rightmost unbracketed $(-)$ and f_i will act on the factor associated to the leftmost unbracketed $(+)$.

2.2. **Virtual crystals.** For any Dynkin diagram D, denote by P_D the corresponding integral weight lattice and by ω_i^D the corresponding fundamental weights. Let X and Y be two Dynkin diagrams and let *aut* be an automorphism of Y such that distinct nodes of Y in the same *aut*-orbit are not connected by an edge. We say there is an embedding $\psi : X \hookrightarrow Y$ if there exists a bijection $\Psi : X \to Y/aut$ inducing a map

$$P_X \to P_Y$$

given by the assignment

$$\omega_i^X \mapsto \sum_{j \in \Psi(i)} \gamma_i(\omega^Y)_j,$$

with γ_i given as in the table in Figure 1.

Consequently, we have a natural embedding of the Weyl groups W^X into W^Y, identifying W^X with the set of elements $\tilde{W}^X$ in W^Y that are fixed under the Dynkin symmetry:

$$W^X \cong \tilde{W}^X := \langle \Pi_{j \in \psi(i)} \tilde{s}_j \mid i \in I^X \rangle \subset W^Y = \langle \tilde{s}_j \mid j \in I^Y \rangle,$$

via the group isomorphism $s_i \mapsto \Pi_{j\in\psi(i)}\tilde{s}_j$. We abuse notation and use ψ to also denote the induced maps on weight lattices, Weyl groups, and indices $\psi : I^X \to I^Y$. In particular, ψ preserves strong Bruhat order and reduced expressions for elements.

[1]This article follows the Kashiwara convention for crystal tensor products which differs from the Bump-Schilling convention [BS17] by exchanging the order of the factors.

X	Y	γ_i
C_n	A_{2n-1}	$\gamma_i = 1, 1 \le i < n, \gamma_n = 2$
B_n	D_{n+1}	$\gamma_i = 2, 1 \le i < n, \gamma_n = 1$
F_4	E_6	$\gamma_1 = \gamma_2 = 2, \gamma_3 = \gamma_4 = 1$
G_2	D_4	$\gamma_1 = 1, \gamma_2 = 3$
B_n	C_n	$\gamma_i = 2, 1 \le i < n, \gamma_n = 1$
B_n	A_{2n-1}	$\gamma_i = 1, 1 \le i \le n$

FIGURE 1. The cases when $X = B_n, Y = C_n$, and $X = B_n, C_n, Y = A_{2n-1}$ were considered respectively in [PPSS23] and [Kas96, Bak00c]. The rest appear in [BS17].

Definition 2.5. Suppose X and Y are Dynkin diagrams with an embedding $\psi : X \hookrightarrow Y$ as above. Let $(\tilde{\mathcal{B}}; \tilde{e}_j, \tilde{f}_j, \tilde{\varphi}_j, \tilde{\varepsilon}_j)_{j \in I^Y}$ be a normal $\mathfrak{g}_Y$-crystal. A ***virtual*** $\mathfrak{g}_X$***-crystal*** is a subset $\mathcal{V} \subset \tilde{\mathcal{B}}$ such that $\mathcal{V}$ has a normal $\mathfrak{g}_X$-crystal structure where for any $i \in I^X$ the crystal operators are given by:

$$e_i^{\mathbf{v}} := \prod_{j \in \psi(i)} \tilde{e}_j^{\gamma_i}, \qquad f_i^{\mathbf{v}} := \prod_{j \in \psi(i)} \tilde{f}_j^{\gamma_i}, \tag{6}$$

and for any choice of $j \in \psi(i)$, the string operators defined as:

$$\varepsilon_i := \gamma_i^{-1} \tilde{\varepsilon}_j \quad \varphi_i := \gamma_i^{-1} \tilde{\varphi}_j. \tag{7}$$

Additionally, if a $\mathfrak{g}_X$-crystal $\mathcal{B}$ is isomorphic to a virtual $\mathfrak{g}_X$-crystal $\mathcal{V} \subset \tilde{\mathcal{B}}$, we call the associated isomorphism $\Upsilon_\psi : \mathcal{B} \to \mathcal{V}$ the ***virtualization*** map.

A priori, it is not clear that a virtual crystal is well-defined; hence a few important observations are in order.

(1) As noted in [BS17, Rem. 5.2] the elements $j, j' \in \psi(i)$ are not connected in Y, hence the associated operators $\tilde{f}_j, \tilde{f}_{j'}$ commute, so that their order in (6) does not matter. Thus, $e_i^{\mathbf{v}}$ and $f_i^{\mathbf{v}}$ are well-defined.
(2) For any $b \in \tilde{\mathcal{B}}$ and $i \in I^X$, the string operators $\tilde{\varepsilon}_j(b)$ are constant with value a multiple of γ_i over all $j \in \psi(i)$. Thus, the string operators in (7) are independent of the choice of $j \in \psi(i)$ and thus well-defined.
(3) A proof that the operators $(e_i^{\mathbf{v}}, f_i^{\mathbf{v}}, \varepsilon_i, \varphi_i)$ in (6) and (7) endow $\mathcal{V}$ with the structure of a normal $\mathfrak{g}_X$-crystal is given in [BS17, Prop. 5.4].
(4) For any $b \in \mathcal{V}$, the weight map for $\mathcal{V}$ as a virtual $\mathfrak{g}_X$-crystal is given by:

$$\mathrm{wt}(b) = \sum_{i \in I^X} (\varphi_i(b) - \varepsilon_i(b)) \omega_i^X$$

and satisfies the property that $\psi(\mathrm{wt}(b)) = \widetilde{\mathrm{wt}}(b)$ where $\widetilde{\mathrm{wt}}(b) = \sum_{i \in I^X} (\tilde{\varphi}_j(b) - \tilde{\varepsilon}_j(b)) \omega_i^Y$ where $\widetilde{\mathrm{wt}}$ is the weight map of $\tilde{\mathcal{B}}$ [BS17, Rem. 5.3].

Remark 2.6. In the literature various virtualization maps have been studied. In [Bak00c], Baker introduced virtualization maps corresponding to the embeddings $B_n, C_n \hookrightarrow A_{2n-1}$ directly on Kashiwara-Nakashima tableaux, and in [PPSS23] Pappe–Pfannerer–Schilling have defined a virtualization map $B_n \hookrightarrow C_n$. In [PS18], Pan–Scrimshaw have defined a virtualization map for the Littelmann path model in arbitrary type. The first appearance of virtualization known to the authors is [Kas96].

Remark 2.7. Notice that for any virtualization map, by (7) the highest weight vector of $\mathcal{B}$ gets mapped to the highest weight vector of $\mathcal{V}$, which in turn must coincide with the highest weight vector of $\tilde{\mathcal{B}}$ [BS17, Proposition 5.7]. Consequently, for every choice of embedding $\psi : X \hookrightarrow Y$ there is a unique virtualization associated to it.

2.3. **Weyl Group Orbits.** Suppose $\mathcal{B}(\lambda) \in \mathfrak{g}$-crystals with $\lambda \in P^+$. There is a natural action of the Weyl group W on the set of weights P, determined by the action of the simple reflections s_i on any $\mu \in P$:

$$s_i(\mu) := \mu - \langle \alpha_i^\vee, \mu \rangle \alpha_i.$$

Given $\lambda \in P^+$, we call the weights $\sigma\lambda$ in the W-orbit of λ the ***extremal weights*** of $\mathcal{B}(\lambda)$.

The W-action on P, in turn, induces an action of W on $\mathcal{B}(\lambda)$ given by flipping an element $b \in \mathcal{B}(\lambda)$ across the associated i-string; more precisely, for each $i \in I$:

$$s_i(b) := \begin{cases} f_i^{\varphi_i(b)-\varepsilon_i(b)}(b) & \text{if } \varphi_i(b) - \varepsilon_i(b) \geq 0 \\ e_i^{\varepsilon_i(b)-\varphi_i(b)}(b) & \text{if } \varphi_i(b) - \varepsilon_i(b) < 0. \end{cases} \tag{8}$$

Setting $b_{\sigma\lambda} := \sigma(b_\lambda)$ for any $\sigma \in W$, the ***W-orbit*** of b_λ is the set

$$\mathcal{O}(\lambda) := \{b_{\sigma\lambda} \in \mathcal{B}(\lambda) \mid \sigma \in W\}. \tag{9}$$

Let $f_j^{\max}(b) := f_i^{\varphi_i(b)}(b)$ for any $b \in \mathcal{B}$, and for any $w \in W$ with reduced expression $\mathrm{rex}(w) = s_{i_1} \dots s_{i_k}$, let

$$\mathcal{F}_w^* := f_{i_1}^{\max} \dots f_{i_k}^{\max} \in \mathcal{F}, \tag{10}$$

then

$$\mathcal{O}(\lambda) = \{f_{i_1}^{\max} f_{i_2}^{\max} \cdots f_{i_k}^{\max}(b_\lambda) \mid s_{i_1} s_{i_1} \dots s_{i_k} = \mathrm{rex}(w) \text{ with } w \leq w_0\} = \bigcup_{w \leq w_0} \mathcal{F}_w^* \{b_\lambda\}, \tag{11}$$

where the union is taken over all $w \in W$, with w_0 the longest element in W. As before, we call the elements $b_{\sigma\lambda} \in \mathcal{O}(\lambda)$ the ***extremal weight vectors*** or ***keys*** of $\mathcal{B}(\lambda)$.

Evidently, for any $\lambda \in P^+$ and $w \in W$, if $v \in wW_\lambda$, with W_λ the stabilizer subgroup of λ in W, then $v\lambda = w\lambda$, so that $b_{v\lambda} = b_{w\lambda}$. Hence, there is a natural correspondence from $\mathcal{O}(\lambda)$ and the set of minimal coset representatives of W/W_λ.

2.4. **The Lusztig–Schützenberger involution.** Given any $\lambda \in P^+$, the ***Lusztig–Schützenberger involution*** $\xi = \xi_{\mathcal{B}(\lambda)} : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda)$ is defined as the unique set involution such that for all $i \in I$ and $b \in \mathcal{B}(\lambda)$:

- $e_i \xi(b) = \xi f_{\theta(i)}(b)$,
- $f_i \xi(b) = \xi e_{\theta(i)}(b)$,
- $\mathrm{wt}(\xi(b)) = w_0 \mathrm{wt}(b)$,

where θ is the automorphism of I defined by applying the longest element $w_0 \in W$ to the simple roots:

$$w_0 \alpha_i = -\alpha_{\theta(i)}.$$

For example, in types B_n and C_n we have $\theta = Id$, whereas in type A_n $\theta(i) = n - i$ (see [AFT22]).

Moreover, the Lusztig–Schützenberger involution exchanges the string length operators as follows:

$$\varepsilon_{\theta(j)}(b) = \varphi_j(\xi(b)), \text{ for } j \in I,\, b \in \mathcal{B}(\lambda). \tag{12}$$

Thus, given any normal $\mathfrak{g}$-crystal $\mathcal{B}$ we can define $\xi_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ by applying the appropriate $\xi_{\mathcal{B}(\lambda)}$ to each connected component $\mathcal{B}(\lambda)$ of $\mathcal{B}$.

Remark 2.8. The Lusztig–Schützenberger involution ξ acts on $\mathcal{O}(\lambda)$ as $\xi(b_{\sigma\lambda}) = b_{w_0\sigma\lambda}$ because

$$\mathrm{wt}(\xi(b_{\sigma\lambda})) = w_0 \mathrm{wt}(b_{\sigma\lambda}) = w_0 \sigma\lambda.$$

In type A_n all fundamental weights are minuscule. In type B_n, respectively C_n, the minuscule fundamental weights are the spin weight $\omega_n^{B_n}$ and respectively $\omega_1^{C_n}$ [BS17, Section 5.4]. For the crystals corresponding to those fundamental weights, the weights form a single W-orbit i.e. they are all extremal weights. For types B_n and C_n, $w_0 = -I$, and $\xi b_{\sigma\lambda} = b_{-\sigma\lambda}$. For the spin case in type B_n, we get, as sets, $\mathcal{B}(\omega_n^B) = \mathcal{O}(\omega_n^B) = W.b_{\omega_n^B}$ with max = 0 or 1 in (11).

Proposition 2.9. *Given a $\mathfrak{g}_X$-crystal $\mathcal{B}$ and virtualization map $\Upsilon : \mathcal{B} \to \mathcal{V} \subset \tilde{\mathcal{B}}$, with $\tilde{\mathcal{B}}$ a $\mathfrak{g}_Y$-crystal, we have that:*

$$\Upsilon(\xi_{\mathcal{B}}(\mathcal{B})) = \xi_{\tilde{\mathcal{B}}}(\Upsilon(\mathcal{B})),$$

Thus, virtualization commutes with the Lusztig–Schützenberger involution.

Proof. Since any virtualization maps highest weights to highest weights and lowest weights to lowest weights (see Remark 2.7), the result follows. □

Given any $\mathcal{B}, \mathcal{C} \in \mathfrak{g}$-crystals, it is well known that although $\mathcal{B} \otimes \mathcal{C}$ is isomorphic to $\mathcal{C} \otimes \mathcal{B}$, the isomorphism is nontrivial. In [HK06] Henriques and Kamnitzer defined this isomorphism $\sigma_{\mathcal{B},\mathcal{C}} : \mathcal{B} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{B}$ in terms of the Lusztig–Schützenberger involution as follows:

$$\sigma_{\mathcal{B},\mathcal{C}}(b \otimes c) := \xi_{\mathcal{C}\otimes\mathcal{B}}(\xi_{\mathcal{C}}(c) \otimes \xi_{\mathcal{B}}(b)). \tag{13}$$

Proposition 2.10. *Let $\lambda \in P^+$ and m be a positive integer. Consider any connected component $\mathcal{B}(\mu) \subset \mathcal{B}(\lambda)^{\otimes m}$ with $\mu \in P^+$. If $\mathcal{B}(\mu)$ appears with multiplicity one in $\mathcal{B}(\lambda)^{\otimes m}$, then for any $b_1 \otimes \cdots \otimes b_m \in \mathcal{B}(\mu)$ we have that*

$$\xi_{\mathcal{B}(\mu)}(b_1 \otimes \cdots \otimes b_m) = \xi_{\mathcal{B}(\lambda)}(b_m) \otimes \cdots \otimes \xi_{\mathcal{B}(\lambda)}(b_1).$$

Proof. We begin by noting that $\mathcal{B}(\lambda)^{\otimes k} \otimes \mathcal{B}(\lambda)^{\otimes(m-k)}$ is equal (not just isomorphic) to $\mathcal{B}(\lambda)^{\otimes m}$ for any $k \leq m$. Since $\sigma = \sigma_{\mathcal{B}(\lambda)^{\otimes k},\mathcal{B}(\lambda)^{\otimes(m-k)}} : \mathcal{B}(\lambda)^{\otimes m} \to \mathcal{B}(\lambda)^{\otimes m}$ is a crystal-morphism, it follows from Schur's lemma that σ preserves isotypic components. Thus, if $\mathcal{B}(\mu)$ has multiplicity one in $\mathcal{B}(\lambda)^{\otimes m}$, it follows $\mathcal{B}(\mu)$ is mapped to itself under σ. Thus, for any $b_1 \otimes \cdots \otimes b_m \in \mathcal{B}(\mu)$ we have:

$$b_1 \otimes \cdots \otimes b_m = \sigma(b_1 \otimes \cdots \otimes b_m) = \xi(\xi(b_{k+1} \otimes \cdots \otimes b_m) \otimes \xi(b_1 \otimes \cdots \otimes b_k)) \tag{14}$$

Hence, $\xi(b_1 \otimes \cdots \otimes b_m) = \xi(b_{k+1} \otimes \cdots \otimes b_m) \otimes \xi(b_1 \otimes \cdots \otimes b_k)$ for any $0 \leq k \leq m$. If we then consider the connected components $\mathcal{B}(\nu_1) \ni b_{k+1} \otimes \cdots \otimes b_m$ and $\mathcal{B}(\nu_2) \ni b_1 \otimes \cdots \otimes b_k$, notice that if either were in isotypic components with multiplicities higher than one in $\mathcal{B}(\lambda)^{\otimes k}$ and $\mathcal{B}(\lambda)^{\otimes(m-k)}$ respectively, this would in turn imply that $\mathcal{B}(\mu)$ had multiplicity higher than one in $\mathcal{B}(\lambda)^{\otimes m}$, a contradiction. Thus, recursively applying (14) we get the desired result. □

3. Demazure keys, Demazure atoms and virtualization

In this section we give a type-independent proof that left and right keys are preserved under virtualization. We refer the reader to [Kas96], [Kas02, Chapitre 8] and [BS17, Section 5] for additional background information.

3.1. **Demazure crystals.** Let $V(\lambda)$ be an integrable highest weight module with highest weight $\lambda \in P^+$ and $w \in W$. The ***Demazure module*** $V_w(\lambda)$ is a $\mathfrak{b}$-module generated by the one dimensional weight space $V(\lambda)_{w\lambda}$ of weight $w\lambda$ under action of any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ [Dem74, Paragraphe 5, Théorème 1].

Littelmann [Lit95] proved in all classical types and Kashiwara [Kas93] generalized to any symmetrizable Kac-Moody Lie algebra that $V_w(\lambda)$ admits a crystal basis that arises as a certain subset of $\mathcal{B}(\lambda)$, which we now describe.

For any $w \in W$ with reduced expression $s_{i_1} \dots s_{i_k}$ define,

$$\mathcal{F}_w := \bigcup_{m_i \in \mathbb{Z}_{\geq 0}} \{f_{i_1}^{m_1} \dots f_{i_k}^{m_k}\} \subset \mathcal{F} \qquad \text{and} \qquad \mathcal{E}_w := \bigcup_{m_i \in \mathbb{Z}_{\geq 0}} \{e_{i_1}^{m_1} \dots e_{i_k}^{m_k}\} \subset \mathcal{E}.$$

Definition 3.1. Given $\lambda \in P^+$ and $w \in W$, the ***Demazure crystal*** $\mathcal{B}_w(\lambda)$ is given by,

$$\mathcal{B}_w(\lambda) := \mathcal{F}_w\{b_\lambda\} \subset \mathcal{B}(\lambda).$$

Similarly, we define the ***opposite Demazure crystal*** $\mathcal{B}^w(\lambda)$ as

$$\mathcal{B}^w(\lambda) := \mathcal{E}_w\{b_{w_0\lambda}\} \subset \mathcal{B}(\lambda).$$

In particular, we have $\mathcal{F}_w^*(b_\lambda) = b_{w\lambda}$ and $\bigcup_{v \leq w} \mathcal{F}_v^*\{b_\lambda\} \subseteq \mathcal{B}_w(\lambda)$ with $\mathcal{B}_{w_0}(\lambda) = \mathcal{B}(\lambda)$ and $\mathcal{B}_e(\lambda) = \{b_\lambda\}$. Hence, Demazure crystals can be seen as certain subsets of $\mathcal{B}(\lambda)$ with lowest weight vector $b_{w\lambda}$ that interpolate between the highest weight b_λ and the complete irreducible crystal $\mathcal{B}(\lambda)$.

Moreover, by definition $\mathcal{B}_w(\lambda)$ admits a filtration by Demazure crystals, so that for any chain in W, $w_1 \leq w_2 \leq \cdots \leq w_k$, we have

$$\mathcal{B}_{w_1}(\lambda) \subseteq \mathcal{B}_{w_2}(\lambda) \subseteq \cdots \subseteq \mathcal{B}_{w_k}(\lambda). \tag{15}$$

Thus, any $\mathfrak{g}$-crystal has filtration by Demazure crystals.

In a similar fashion, Demazure crystals can be decomposed into smaller constituents called ***atoms***,

$$\mathring{\mathcal{B}}_w(\lambda) := \mathcal{B}_w(\lambda) \setminus \bigcup_{\substack{\nu \in W \\ \nu < w}} \mathcal{B}_\nu(\lambda).$$

In particular, each atom $\mathring{\mathcal{B}}_w(\lambda)$ uniquely contains the extremal weight vector $b_{w\lambda}$.

Remark 3.2. Demazure modules were originally described as the space of global sections of a suitable line bundle on a Schubert variety [Dem74, Paragraphe 5, Théorème 1]. This description exhibits the natural correspondence between Schubert varieties and Demazure modules so that for any $\tau \leq w$ in W, $X_\tau \subseteq X_w$ means $V_\tau(\lambda) \subseteq V_w(\lambda) \subseteq V(\lambda)$.

Remark 3.3. The term Demazure atom has been used in the literature to mean both the *crystal* subset $\mathring{\mathcal{B}}_w(\lambda) \subset \mathcal{B}(\lambda)$ and its corresponding *polynomial* character $\mathcal{A}_{w\lambda} \in \mathbb{Z}[x]$, with the notation $\overline{\mathcal{B}}_w(\lambda)$ used by Kashiwara to denote this subset instead [Kas02, Ch. 9.1], [Mas09]. In this article, we will only refer to the crystal subset by this name and never discuss its character.

3.2. **Dilation of crystals.**

Definition 3.4. [Kas02, Chp. 8],[Kas96] Given a positive integer m and $\lambda \in P^+$, the m-***dilation*** of $\mathcal{B}(\lambda)$ is the unique embedding:

$$\mathbb{D}_m : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(m\lambda)$$

mapping the highest weights to each other, $b_\lambda \mapsto b_{m\lambda}$, and extending to any $b = f_{i_1} \cdots f_{i_l}(b_\lambda) \in \mathcal{B}(\lambda)$ as follows:

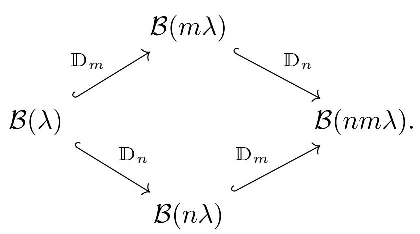
$$f_{i_1} \cdots f_{i_l}(b_\lambda) \mapsto f_{i_1}^m \cdots f_{i_l}^m(b_{m\lambda}).$$

It follows directly from the definition[2] that for any vertex $b \in \mathcal{B}(\lambda)$ and $i \in I$,

$$\mathbb{D}_m(f_i b) = f_i^m \mathbb{D}_m(b), \;\; \mathbb{D}_m(e_i b) = e_i^m \mathbb{D}_m(b) \text{ and} \tag{16}$$

$$\varphi_i(\mathbb{D}_m(b)) = m\varphi_i(b), \; \varepsilon_i(\mathbb{D}_m(b)) = m\varepsilon_i(b), \; \mathrm{wt}(\mathbb{D}_m(b)) = m\mathrm{wt}(b). \tag{17}$$

Moreover, for m, n positive integers $\mathbb{D}_{mn}$ factors through $\mathbb{D}_n$ and $\mathbb{D}_m$,



Recall (4) and Theorem 2.4. Thus, more generally, for any crystal $\mathcal{B}$, the m-dilation $\mathbb{D}_m$ acts by dilating each connected component individually.

Proposition 3.5. [Kas02, Lem. 8.1.2], [Lec03, Cor. 2.1.3] *For any $\lambda, \mu \in P^+$ and positive integer m, the dilation map $\mathbb{D}_m$ preserves highest weight vectors, so that for any $u \in \mathcal{B}(\mu)$ with $\varphi_i(b_\lambda) \geq \varepsilon_i(u)$,*

$$b_{\lambda+\mathrm{wt}(u)} \mapsto b_{m\lambda+m\mathrm{wt}(u)}.$$

Thus $\mathbb{D}_m : \mathcal{B}(\lambda + \mathrm{wt}(u)) \hookrightarrow \mathcal{B}(m\lambda + m\mathrm{wt}(u))$ and consequently,

$$\mathbb{D}_m(\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)) \subset \mathbb{D}_m(\mathcal{B}(\lambda)) \otimes \mathbb{D}_m(\mathcal{B}(\mu)).$$

Thus, $\mathbb{D}_m \otimes \mathbb{D}_m : \mathcal{B}(\mu) \otimes \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(m\mu) \otimes \mathcal{B}(m\lambda)$ is an m-dilation map. More generally, $\mathbb{D}_m(\bigotimes_i \mathcal{B}(\lambda_i)) \subset \bigotimes_i \mathbb{D}_m(\mathcal{B}(\lambda_i))$ for any family $\lambda_i \in P^+$.

[2]The original definition introduced by Kashiwara used (16) and (17) since it was given in the more general context of Kac-Moody algebras.

Now, let $\mathcal{F}\{b_\lambda^{\otimes m}\} \subset \mathcal{B}(\lambda)^{\otimes m}$ denote the unique connected component with highest weight $m\lambda$ in $\mathcal{B}(\lambda)^{\otimes m}$ and let

$$G_m : \mathcal{B}(m\lambda) \longrightarrow \mathcal{F}\{b_\lambda^{\otimes m}\}$$

be the induced crystal isomorphism mapping $b_{m\lambda} \to b_\lambda^{\otimes m}$. Thus, we have a canonical embedding:

$$\Theta_m := G_m \circ \mathbb{D}_m : \mathcal{B}(\lambda) \hookrightarrow \mathcal{F}\{b_\lambda^{\otimes m}\} \subset \mathcal{B}(\lambda)^{\otimes m}. \tag{18}$$

Thus, for any positive integers m, n we have:

$$\mathcal{B}(\lambda) \overset{\Theta_m}{\hookrightarrow} \mathcal{F}(b_\lambda^{\otimes m}) \cong \mathcal{B}(m\lambda) \overset{\Theta_n}{\hookrightarrow} \mathcal{F}(b_\lambda^{\otimes mn}) \cong \mathcal{B}(mn\lambda).$$

Proposition 3.6. [Kas02, Prop. 8.3.2] *Given positive integers m, n, $\lambda \in P^+$, and $w \in W$, then*

(1) for any extremal weight vector $b_{w\lambda} \in \mathcal{O}(\lambda)$, we have $\Theta_m(b_{w\lambda}) = b_{w\lambda}^{\otimes m}$. In particular, $\Theta_m(\mathcal{O}(\lambda)) = \mathcal{O}(m\lambda)$, and

(2) $\Theta_{mn} = \Theta_m \circ \Theta_n = \Theta_n \circ \Theta_m$.

Corollary 3.7. *For $b \in \mathcal{B}(\lambda)$ and $w_1, \ldots, w_m \in W$, if $\Theta_m(b) = b_{w_1\lambda} \otimes \cdots \otimes b_{w_m\lambda} \in \mathcal{O}(\lambda)^{\otimes m}$ then, up to repetitions, $\Theta_{nm}(b) = \Theta_n\Theta_m(b) = b_{w_1\lambda}^{\otimes n} \otimes \cdots \otimes b_{w_m\lambda}^{\otimes n}$ produces the same sequence $w_1, \ldots, w_m \in W$.*

Remark 3.8. Let us consider the $q\ell$-dilation of $\mathcal{B}(\lambda)$. By Proposition 3.5 and Proposition 3.6 (2), Corollary 3.7 implies that

$$\Theta_{q\ell}(b) = \Theta_\ell\Theta_q(b) = \Theta_\ell(b_{w_1\lambda} \otimes \cdots \otimes b_{w_q\lambda}) = b_{w_1\lambda}^{\otimes \ell} \otimes \cdots \otimes b_{w_q\lambda}^{\otimes \ell}$$

and up to repetition the sequence $w_1, \ldots, w_q$ is the same in W as the one produced by $\Theta_\ell(b)$.

It was also shown in [Kas02, Prop. 8.3.2] that for any $b \in \mathcal{B}(\lambda)$ if m is adequate then there exist $w_i \in W$ satisfying $w_1 \geq \cdots \geq w_m$ in W such that

$$\Theta_m(b) = b_{w_1\lambda} \otimes b_{w_2\lambda} \otimes \cdots \otimes b_{w_m\lambda} \in \mathcal{O}(\lambda)^{\otimes m}. \tag{19}$$

This proof, however, is inductive and does not provide explicit bounds for how large such an m must be. In the following theorem, we generalize Proposition 3.6(1) for any $b \in \mathcal{B}(\lambda)$ by providing sufficient and necessary conditions for when the decomposition $\Theta_m(b) = b_{w\lambda} \otimes b' \otimes b_{w'\lambda}$ exhibiting the extremal end weights occurs for some $b' \in \mathcal{B}(\lambda)^{\otimes(m-2)}$.

Theorem 3.9. *Let $m \in \mathbb{N}$. For all $b \in \mathcal{B}(\lambda)$, there exist $b' \in \mathcal{B}(\lambda)^{\otimes(m-2)}$ and fixed $w \geq w' \in W$ such that*

$$\Theta_m(b) = b_{w\lambda} \otimes b' \otimes b_{w'\lambda} \quad \text{if and only if} \quad m \geq \ell = \max\{\text{length}(\rho) \mid \rho \text{ is an } i\text{-string for } i \in I\}.$$

Proof. Fix $i \in I$. Let m be given and suppose $\rho = (b_0 \overset{i}{\to} b_1 \overset{i}{\to} \ldots \overset{i}{\to} b_k) \subset \mathcal{B}(\lambda)$ is any i-string of length k. Now, for any i-string there exists a filtration by Demazure crystals of $\mathcal{B}(\lambda)$ (see (15)) such that $\{b_\lambda\} \subset \mathcal{B}_w(\lambda) \subset \mathcal{B}_{s_iw}(\lambda) \subset \mathcal{B}(\lambda)$ for some $s_iw > w \in W$. In particular, w may be chosen such that the i-string connecting the Demazure lowest weight $b_{s_iw} \in \mathcal{B}_{s_iw}(\lambda)$ to $\mathcal{B}_w(\lambda)$ is the i-string of maximal length in $\mathcal{B}(\lambda)$. Since b_{s_iw} is extremal, it follows that $e_i^{\varepsilon_i(b_{s_iw})}(b_{s_iw}) = b_w$ is also extremal. Hence, if b_0 in $\rho = (b_0 \overset{i}{\to} b_1 \overset{i}{\to} \ldots \overset{i}{\to} b_k)$ is not extremal then ρ can be replaced with another i-string ρ' of length $k' \geq k$ such that $b'_0 \in \mathcal{O}(\lambda)$. Thus, without loss of generality we can assume that b_0 is an extremal weight vector.

Now, since $b_0 \in \mathcal{O}(\lambda)$, then so is b_k. Hence by Proposition 3.6 $\Theta_m(\mathcal{O}(\lambda)) \subset \mathcal{O}(m\lambda)$, implies that $\Theta_m(\rho) \subseteq \rho \otimes \Theta_{m-1}(\rho) \subseteq \mathcal{B}(\lambda)^{\otimes m}$. Thus, by (20) below

$$f_i^n(b_1 \otimes b_2) = \begin{cases} f_i^n(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) + n, \\ f_i^{\varphi_i(b_1)-\varepsilon_i(b_2)}(b_1) \otimes f_i^{n-\varphi_i(b_1)+\varepsilon_i(b_2)}(b_2) & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1) \leq \varepsilon_i(b_2) + n, \\ b_1 \otimes f_i^n(b_2) & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1). \end{cases} \tag{20}$$

one has

$$f_i^r\Theta_m(b_0) = f_i^r(b_0 \otimes b_0^{\otimes m-1}) = \begin{cases} f_i^r(b_0) \otimes b_0^{\otimes m-1} & ; r < k, \\ b_k \otimes f_i^{r-k}(b_0^{\otimes m-1}) & ; r \geq k. \end{cases} \tag{21}$$

Thus, if $m < k \leq \ell$ then $\Theta_m(b_1) = f_i^m(b_0) \otimes b_0^{\otimes m-1}$ with $f_i^m(b_0) \notin \mathcal{O}(\lambda)$.

Conversely, if $m \geq \ell \geq k$ and $b \in \mathcal{B}(\lambda)$ lies in an i-string with b_0 extremal, then it follows that $\Theta_m(b_i) = b_k \otimes f_i^{r-k}(b_1^{\otimes m-1})$ for all $1 \leq i \leq k$ with $b_k \in \mathcal{O}(\lambda)$. If however, b does not lie on an i-string with b_0 extremal, let $b_0' \in \mathcal{O}(\lambda)$ be such that $b = f_{i_1}^{n_1} \dots f_{i_s}^{n_s}(b_0')$. Moreover, since $m \geq \ell$ then by (21) there exists some subset $\{j_\alpha\} \subset \{i_\beta\}$ such that

$$\Theta_m(b) = f_{i_1}^{m(n_1)} \dots f_{i_s}^{m(n_s)}({b_0'}^{\otimes m}) = f_{j_1}^{\max} \dots f_{j_t}^{\max}(b_0') \otimes b'' \in \mathcal{O}(\lambda) \otimes \mathcal{B}(\lambda)^{\otimes(m-1)}.$$

Setting $b_{w_1\lambda} = f_{j_1}^{\max} \dots f_{j_t}^{\max}(b_0')$ the first claim follows.

The proof that any $b \in \mathcal{B}(\lambda)$ satisfies $b = b' \otimes b_{w_2\lambda}$ if and only $m \geq \ell$ follows similarly to the first part by considering $\Theta_m(b) \in \mathcal{B}(\lambda)^{\otimes(m-1)} \otimes \mathcal{B}(\lambda)$ and noting that for any i-string with b_k extremal we have:

$$e_i^r \Theta_m(b_k) = e_i^r(b_k^{\otimes m-1} \otimes b_k) = \begin{cases} b_k^{\otimes m-1} \otimes e_i^r(b_k) & ; r < k, \\ e_i^{r-k}(b_k^{\otimes m-1}) \otimes b_0 & ; r \geq k. \end{cases} \tag{22}$$

□

Definition 3.10. For any $b \in \mathcal{B}(\lambda)$ let m be large enough so that $b_{w_1\lambda}, b_{w_2\lambda} \in \mathcal{O}(\lambda)$ exist as in Theorem 3.9. The ***right key*** $K^+(b)$ and ***left key*** $K^-(b)$ of b are defined to be the unique vectors,

$$K^+(b) = b_{w_1\lambda} \text{ and } K^-(b) = b_{w_2\lambda}.$$

Evidently, it immediately follows from Proposition 3.6 that $K^+(b) = K^-(b)$ if and only if $b \in \mathcal{O}(\lambda)$, with $K^+(b_{w\lambda}) = K^-(b_{w\lambda}) = b_{w\lambda}$ for any $w \in W$. Observe that one always has $K^+(b) \geq K^-(b)$ for any $b \in \mathcal{B}(\lambda)$.

Remark 3.11. Conceptually, if we view $\mathcal{B}(\lambda)$ as a ranked poset, then the right (resp. left) key of b is the closest extremal weight vector to b that lies below (resp. above) it and is connected to b via some sequence of lowering (resp. raising) operators and these vectors are unique.

Remark 3.12. Lascoux's original definition [LS90b], based on the tableau model, does not fit with the positions of $b_{w_1\lambda}$ and $b_{w_m\lambda}$ in $\Theta_m(b)$ ($b_{w_1\lambda}$ is left-most while $b_{w_m\lambda}$ is right-most). Nonetheless, we choose to keep Lascoux's terminology in order to stay consistent with the literature.

Remark 3.13. Originally, Lascoux-Schützenberger [LS90b] termed an atom *standard basis*. Keys in type A_{n-1} have its origin in the $GL(n, \mathbb{C})$ *standard* bases to detect the semistandard tableaux which are *standard* (in the sense of standard monomial theory) on a Schubert variety. In each Demazure crystal atom there exists exactly one key tableau and the right key map detects the Demazure crystal atom that contains a given semistandard Young tableau [LS90b, Theorem 3.8].

3.3. **Demazure atoms and keys.** The right (left) key assigns to each vertex of a crystal an extremal weight to indicate the smallest (opposite) Demazure crystal containing the given vertex.

Definition 3.14. [Kas02, LS90b] Given $\lambda \in P^+$ and $w \in W$, define the ***Demazure atom*** $\mathring{\mathcal{B}}_w(\lambda)$ as the set,

$$\mathring{\mathcal{B}}_w(\lambda) := \{b \in \mathcal{B}(\lambda) : K^+(b) = b_{w\lambda}\}$$

and the ***opposite Demazure atom*** $\mathring{\mathcal{B}}^w(\lambda)$ as

$$\mathring{\mathcal{B}}^w(\lambda) := \{b \in \mathcal{B}(\lambda) : K^-(b) = b_{w\lambda}\}.$$

It is classical fact [Kas02, LS90b, San21b], and evident from Definition 3.10, that (opposite) Demazure crystals can be built from (opposite) atoms,

$$\mathcal{B}_w(\lambda) = \bigsqcup_{\nu \leq w} \mathring{\mathcal{B}}_\nu(\lambda) = \{b \in B(\lambda) : K^+(b) = b_{\nu\lambda},\ \nu \leq w\}.$$

We now turn our attention to how the Lusztig–Schützenberger involution interacts with dilation and consequently with right (left) keys, and (opposite) Demazure crystals. (See also [San21a, San21b].)

Proposition 3.15. *For any $\lambda \in P^+$ and positive integer m, the Lusztig–Schützenberger involution commutes with m-dilation; namely, for any $b \in \mathcal{B}(\lambda)$,*

$$\xi_{\mathcal{B}(m\lambda)} \Theta_m(b) = \Theta_m \xi_{\mathcal{B}(\lambda)}(b).$$

Proof. Suppose $b = f_{i_1} \cdots f_{i_l}(b_\lambda) \in \mathcal{B}(\lambda)$. Then, by Proposition 3.6(1) since $\Theta_m(b_{w_0\lambda}) = b_{w_0\lambda}^{\otimes m}$, it follows that

$$\begin{aligned}\xi_{\mathcal{B}(m\lambda)}\Theta_m(b) &= \xi_{\mathcal{B}(m\lambda)}(f_{i_1}^m \cdots f_{i_l}^m(b_\lambda^{\otimes m})) = e_{\theta(i_1)}^m \cdots e_{\theta(i_l)}^m(b_{w_0\lambda}^{\otimes m}) \\ &= \Theta_m(e_{\theta(i_1)} \cdots e_{\theta(i_l)}(b_{w_0\lambda})) = \Theta_m\xi(b).\end{aligned}$$

□

Theorem 3.16. *For any $\lambda \in P^+$ and $b \in \mathcal{B}(\lambda)$,*

$$K^+(\xi(b)) = \xi K^-(b).$$

Proof. Suppose $b \in \mathcal{B}(\lambda)$, then by Proposition 3.6 for m adequate we can write $\Theta_m(b) = b_{w_1\lambda} \otimes \cdots \otimes b_{w_m\lambda}$ for some $w_i \in W$, so that $K^+(b) = b_{w_1\lambda}$. Now, by Proposition 3.15 and Proposition 2.10 applied to the connected component $\mathcal{F}\{b_\lambda^{\otimes m}\} = \mathcal{B}(m\lambda)$, as well as Remark 2.8, it follows that

$$\begin{aligned}\Theta_m\xi_{\mathcal{B}(\lambda)}(b) =& \xi_{\mathcal{B}(m\lambda)}\Theta_m(b) = \xi_{\mathcal{B}(m\lambda)}(b_{w_1\lambda} \otimes \cdots \otimes b_{w_m\lambda}) = \xi_{\mathcal{B}(\lambda)}(b_{w_m\lambda}) \otimes \cdots \otimes \xi_{\mathcal{B}(\lambda)}(b_{w_1\lambda}) \\ =& b_{w_0w_m\lambda} \otimes \cdots \otimes b_{w_0w_1\lambda}. \end{aligned} \tag{23}$$

Thus, $K^+(\xi(b)) = \xi b_{w_m\lambda} = b_{w_0w_m\lambda} = \xi K^-(b)$. □

As an immediate consequence we also obtain the equalities,

$$\xi\mathring{\mathcal{B}}_w(\lambda) = \mathring{\mathcal{B}}^{w_0w}(\lambda) \qquad \text{and} \qquad \mathcal{B}^{w_0w}(\lambda) = \xi\mathcal{B}_w(\lambda). \tag{24}$$

Remark 3.17. Left and right keys in a $\mathfrak{g}$-crystal are shown in Theorem 3.16 to be mapped to each other through the Lusztig-Schützenberger involution. For the next $\mathfrak{g}$-crystal models it translates as follows. The sequences in W (23) produced by Theorem 3.16 are Lakshmibai-Seshadri (L-S) paths (see also [Lit94, Lemma 3.1 (b)]). Indeed Theorem 3.16 induces an action of the Lusztig-Schützenberger involution on the crystal of L-S paths $\mathbf{B}(\lambda)$. More precisely, $\xi(\tau; \mathbf{a}) = \xi(\tau_0 > \cdots > \tau_r; 0 < a_1 < \cdots < a_r < 1) = (\omega_0\tau_r > \cdots > \omega_0\tau_0; 0 < 1 - a_r < \cdots < 1 - a_1 < 1)$ where $(\tau; \mathbf{a})$ is an L-S path of $\mathbf{B}(\lambda)$. Within the Lenart–Postnikov alcove path model [LP08], [Len07, Definition 5.1, Remark 5.3, Corollary 6.2] the initial key and the final key are interchanged by the Lusztig-Schützenberger involution.

3.4. **Commutation of virtualization and dilation.** We are now ready to prove that virtualization preserves left and right. To do this, we first need to show that dilation and virtualization commute.

Lemma 3.18. [BS17, Thm. 5.8] *Given a Dynkin diagram embedding $\psi : X \hookrightarrow Y$ and associated virtualization maps $\Upsilon_\psi : \mathcal{B} \to \mathcal{V} \subset \tilde{\mathcal{B}}$ and $\Upsilon'_\psi : \mathcal{C} \to \mathcal{W} \subset \tilde{\mathcal{C}}$, the map $\Upsilon \otimes \Upsilon' : \mathcal{B} \otimes \mathcal{C} \to \mathcal{V} \otimes \mathcal{W}$ is a virtualization map.*

The following results were proven by Azenhas-Santos [AS24] for $X = C_n$ and $Y = A_{2n-1}$ using Baker virtualization [Bak00c]. We now generalize these results to any virtualization map associated to a Dynkin diagram embedding $\psi : X \hookrightarrow Y$.

Theorem 3.19. *For any any virtualization map $\Upsilon_\psi : \mathcal{B}(\lambda) \to \mathcal{V} \subset \tilde{\mathcal{B}}(\psi(\lambda))$ and any positive integer m, the induced map $\Upsilon_\psi^{\otimes m} : \Theta_m(\mathcal{B}(\lambda)) \to \Theta_m(\mathcal{V})$ is a virtualization map. Consequently, the following diagram commutes,*

$$\begin{array}{ccccc} \mathcal{B}(\lambda) & \overset{\mathbb{D}_m}{\hookrightarrow} & \mathcal{B}(m\lambda) & \xrightarrow{G_m} & \mathcal{F}\{b_\lambda^{\otimes m}\} \subseteq \mathcal{B}(\lambda)^{\otimes m} \\ \big\downarrow \Upsilon_\psi & & \big\downarrow \Upsilon_\psi & & \big\downarrow \Upsilon_\psi^{\otimes m} \\ \tilde{\mathcal{B}}(\psi(\lambda)) & \overset{\mathbb{D}_m}{\hookrightarrow} & \tilde{\mathcal{B}}(m\psi(\lambda)) & \xrightarrow{G_m} & \mathcal{F}\{b_{\psi(\lambda)}^{\otimes m}\} \subseteq \tilde{\mathcal{B}}(\psi(\lambda))^{\otimes m} \end{array}$$

and thus $\Theta_m\Upsilon_\psi = \Upsilon_\psi^{\otimes m}\Theta_m$.

Proof. We begin by noting that the underlying Dynkin diagram embedding ψ also induces a virtualization map $\Upsilon_\psi : \mathcal{B}(m\lambda) \to \tilde{\mathcal{B}}(m\psi(\lambda))$ sending $b = f_{i_1} \ldots f_{i_k}(b_{m\lambda}) \mapsto f_{i_1}^{\mathbf{v}} \ldots f_{i_k}^{\mathbf{v}}(b_{m\psi(\lambda)})$. Hence, letting $\Upsilon = \Upsilon_\psi$, for any $b = f_{i_1} \ldots f_{i_k}(b_\lambda) \in \mathcal{B}(\lambda)$, we see that

$$\Upsilon\mathbb{D}_m(b) = \Upsilon(f_{i_1}^m \ldots f_{i_k}^m(b_{m\lambda})) = (f_{i_1}^{\mathbf{v}})^m \ldots (f_{i_k}^{\mathbf{v}})^m(b_{m\psi(\lambda)}) = \mathbb{D}_m(f_{i_1}^{\mathbf{v}} \ldots f_{i_k}^{\mathbf{v}}(b_{\psi(\lambda)})) = \mathbb{D}_m\Upsilon(b).$$

Thus, $\mathbb{D}_m \Upsilon = \Upsilon \mathbb{D}_m$.

Consider the map

$$\Upsilon^{\otimes m} : \mathcal{F}\{b_\lambda^{\otimes m}\} \to \mathcal{F}^{\mathbf{v}}\{b_{\psi(\lambda)}^{\otimes m}\} \subset \mathcal{F}\{b_{\psi(\lambda)}^{\otimes m}\}$$

where $\mathcal{F}^{\mathbf{v}}\{b_{\psi(\lambda)}^{\otimes m}\}$ is the induced subset $\mathcal{F}\{b_{\psi(\lambda)}^{\otimes m}\} \cap \mathcal{V}^{\otimes m} \subset \tilde{\mathcal{B}}(\psi(\lambda))^{\otimes m}$. By Lemma 3.18 we know that

$$\Upsilon^{\otimes m} : \mathcal{B}(\lambda)^{\otimes m} \to \tilde{\mathcal{B}}(\psi(\lambda))^{\otimes m}$$

is a virtualization map. Since a virtualization map descends to the corresponding subsets, it follows that $\Upsilon^{\otimes m} : \mathcal{F}\{b_\lambda^{\otimes m}\} \to \mathcal{F}^{\mathbf{v}}\{b_{\psi(\lambda)}^{\otimes m}\}$ is a virtualization.

Now, recall that any virtualization is a crystal morphism and thus commutes with the crystal operators. Hence for any $f_{i_1} \dots f_{i_k}(b_{m\lambda}) \in \mathcal{B}(m\lambda)$,

$$\begin{aligned}\Upsilon^{\otimes m} G_m(f_{i_1} \dots f_{i_k}(b_{m\lambda})) &= \Upsilon^{\otimes m}(f_{i_1} \dots f_{i_k}(b_\lambda^{\otimes m})) \\ &= f_{i_1}^{\mathbf{v}} \dots f_{i_k}^{\mathbf{v}}(b_{m\psi(\lambda)}^{\otimes m}) \\ &= G_m(f_{i_1}^{\mathbf{v}} \dots f_{i_k}^{\mathbf{v}}(b_{m\psi(\lambda)})) = G_m \Upsilon(f_{i_1} \dots f_{i_k}(b_{m\lambda})).\end{aligned}$$

Thus, $\Theta_m \circ \Upsilon = \Upsilon^{\otimes m} \circ \Theta_m$. □

Recall that for any Dynkin diagram embedding $\psi : X \to Y$ we have an induced Weyl group embedding

$$\psi : W^X \to \widetilde{W}^X := \langle s_i := \prod_{j \in \psi(i)} \tilde{s}_i \mid i \in I^X \rangle \subset W^Y = \langle \tilde{s}_j \mid j \in I^Y \rangle.$$

Denote by $\mathcal{O}^X(\lambda)$ the W^X-orbit of b_λ and by $\widetilde{\mathcal{O}}^X(\psi(\lambda))$ the $\widetilde{W}^X$-orbit of $b_{\psi(\lambda)}$.

Lemma 3.20. *For any Dynkin diagram embedding $\psi : X \to Y$ with virtualization map $\Upsilon_\psi : \mathcal{B}(\lambda) \to \mathcal{V} \subset \tilde{\mathcal{B}}(\psi(\lambda))$, we have that*

$$\Upsilon_\psi \mathcal{O}^X(\lambda) = \widetilde{\mathcal{O}}^X(\psi(\lambda)) := \{b_{w\psi(\lambda)} \mid w \in \widetilde{W}^X\} \subset \mathcal{O}^Y(\psi(\lambda)).$$

Proof. Suppose $b_{w\lambda} \in \mathcal{O}^X(\lambda)$ for some $w \in W^X$. Then for any reduced expression $s_{i_1} \dots s_{i_k}$ of w, we have $f_{i_1}^{\max} \dots f_{i_k}^{\max}(b_\lambda) = b_{w\lambda}$. Hence,

$$\Upsilon_\psi(b_{w\lambda}) = (f_{i_1}^{\mathbf{v}})^{\max} \dots (f_{i_k}^{\mathbf{v}})^{\max}(b_{\psi(\lambda)}) = \prod_{j \in \psi(i_1)} \tilde{f}_j^{\max} \cdots \prod_{j \in \psi(i_k)} \tilde{f}_j^{\max}(b_{\psi(\lambda)}) = b_{\psi(w)\psi(\lambda)},$$

where the last equality holds since, by definition, $s_i = \prod_{j \in \psi(i)} \tilde{s}_i \in \tilde{W}^X \subset W^Y$, in which case $s_{i_1} \dots s_{i_k}$ is a reduced expression for $\psi(w)$ in $\tilde{\mathcal{O}}^X(\psi(\lambda))$.

□

We now conclude that a virtualization map preserves left and right keys and thereby embeds type X Demazure crystals and atoms into those of type Y.

Corollary 3.21. *Let $\psi : X \to Y$ be a Dynkin diagram embedding with virtualization map $\Upsilon = \Upsilon_\psi : \mathcal{B}(\lambda) \to \tilde{\mathcal{B}}(\psi(\lambda))$. Then, for any $b \in \mathcal{B}$:*

$$\Upsilon(K^+(b)) = K^+(\Upsilon(b)) \text{ and } \Upsilon(K^-(b)) = K^-(\Upsilon(b)).$$

Thus, virtualization embeds Demazure crystals and atoms correspondingly, so that for any $w \in W^X$ we have

$$\mathcal{B}_w(\lambda) \overset{\Upsilon}{\hookrightarrow} \tilde{\mathcal{B}}_{\psi(w)}(\psi(\lambda)) \qquad \text{and} \qquad \mathring{\mathcal{B}}_w(\lambda) \overset{\Upsilon}{\hookrightarrow} \mathring{\tilde{\mathcal{B}}}_{\psi(w)}(\psi(\lambda))$$

and similarly for their opposites.

Proof. Let $b \in \mathcal{B}(\lambda)$ and m be such that $\Theta_m(b) = b_{w_1\lambda} \otimes \cdots \otimes b_{w_m\lambda} \in \mathcal{O}(\lambda)^{\otimes m}$ with $w_1 \geq \cdots \geq w_m$. Then by Theorem 3.19 and Lemma 3.20,

$$\Theta_m \Upsilon(b) = \Upsilon^{\otimes m} \Theta_m(b) = \Upsilon(b_{w_1\lambda}) \otimes \cdots \otimes \Upsilon(b_{w_m\lambda}) \in \tilde{\mathcal{O}}^X(\psi(\lambda))^{\otimes m}$$

where $\psi(w_1) \geq \cdots \geq \psi(w_m)$. Thus, $K^+(\Upsilon(b)) = \Upsilon(b_{w_1\lambda}) = \Upsilon K^+(b)$ and $K^-(\Upsilon(b)) = \Upsilon(b_{w_m\lambda}) = \Upsilon K^-(b)$ as desired.

□

3.5. **Computational application to standard monomial theory.** Lakshmibai–Littelmann [LL03] identifies the standard monomials of Richardson varieties X_v^w with certain L–S paths. This allows an identification of a standard monomial of degree m of a Richardson variety with an ordered sequence of m elements in the corresponding crystal, satisfying certain conditions on their left and right keys. This condition, from a crystal-theoretic point of view, can be obtained by characterizing when the tensor of m elements in $\mathcal{B}_v^w(\lambda) := \mathcal{B}_v(\lambda) \cap \mathcal{B}^w(\lambda)$, $w \leq v$, is an element in $\mathcal{B}_v^w(m\lambda)$. Furthermore, since keys are not manifest in every model of crystals and dilation is not an effective way to compute keys, one can use virtualization to reduce the computation to simply-laced root systems.

3.6. **Computational consequences of our results.** Let $\psi : X \to Y$ be a Dynkin diagram embedding with virtualization map $\Upsilon = \Upsilon_\psi : \mathcal{B}(\lambda) \to \tilde{\mathcal{B}}(\psi(\lambda))$. Assume that one has a combinatorial model for $\mathcal{B}(\lambda), \tilde{\mathcal{B}}(\psi(\lambda))$ and that we have an algorithm for computing Υ and its inverse

$$\Upsilon^{-1} : \Upsilon(\mathcal{B}(\lambda)) \to \mathcal{B}(\lambda).$$

Additionally, assume that the model for $\tilde{\mathcal{B}}(\psi(\lambda))$ includes algorithms for computing $\xi(b)$ as well as $K^+(b)$ and $K^-(b)$ for any $b \in \tilde{\mathcal{B}}(\psi(\lambda))$. Then Corollary 3.21 and Proposition 2.9 imply that, to compute $\xi(b)$ as well as $K^+(b)$ and $K^-(b)$ for any $b \in \mathcal{B}(\lambda)$, it suffices to compute the virtualization map Υ, apply the corresponding existing algorithm, and apply the inverse map Υ^{-1}.

4. Applications to Kashiwara–Nakashima Tableaux of type B

In this section we follow [Bak00a, KN94, Lec03] and [HK02]. Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and consider $\lambda \in P^+$. Then, we may write $\lambda = \sum_{i=1}^n a_i \omega_i^{B_n}$ where

$$\omega_i^{B_n} = \begin{cases} \epsilon_1 + \cdots + \epsilon_i & i \neq n \\ \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n) & i = n \end{cases}$$

denote the fundamental weights of $\mathfrak{g}$. Hence, we have a decomposition for each part λ_j of λ,

$$\lambda_j = \sum_{i=j}^{n-1} a_i + \frac{1}{2} a_n = \sum_{i=i}^{n-1} a_i + \frac{1}{2}(a_n - \varepsilon) + \varepsilon \frac{1}{2}$$

with $\varepsilon = 1$ or 0 depending on the parity of a_n, so that to every dominant integral weight λ we can associate a diagram obtained by concatenating a special "half width" column of height n and the Young diagram of shape μ with i-th part $\lambda_i - \varepsilon \frac{1}{2} \in \mathbb{N}$. In particular, $(1^n) = 2\omega_n^{B_n}$.

As a consequence, any highest weight representation $V(\lambda)$ arises as a summand of tensor products of the *standard representation* $V(\omega_1)$ and the *spin representation* $V(\omega_n^{B_n})$.

In particular, observe that,

$$\mathcal{B}(\omega_n^{B_n})^{\otimes 2} \simeq \mathcal{B}(0) \oplus \mathcal{B}(2\omega_n^{B_n}) \oplus \bigoplus_{i=1}^{n-1} \mathcal{B}(\omega_i^{B_n}), \tag{25}$$

$$\mathcal{B}((1^k)) \otimes \mathcal{B}(\omega_n^{B}) \simeq \mathcal{B}(\omega_n^{B_n}) \oplus \bigoplus_{i=1}^{k} \mathcal{B}((1^i) + \omega_n^{B_n}),\ 1 \leq k \leq n. \tag{26}$$

For $\mathfrak{g} = \mathfrak{sp}_{2n}$, and $\lambda \in P^+$, we write $\lambda = \sum_{i=1}^n a_i \omega_i^{C_n}$, where $\omega_i^{C_n} = \epsilon_1 + \cdots + \epsilon_i$ for all $1 \leq i \leq n$. In this case, dominant weights λ may be identified with partitions with at most n parts, and any highest weight representation $V(\lambda)$ arises as a summand of tensor products of the standard representation $V(\omega_1)$.

In order to combinatorially model these representations for $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ we need to define certain fillings of λ, which we proceed to explain in the next section.

4.1. **Kashiwara-Nakashima tableaux.** Consider the alphabets for Lie types B_n and C_n below,

$$\mathsf{B}_n := \{1 \prec \cdots \prec n \prec 0 \prec \bar{n} \cdots \prec \bar{1}\}$$
$$\mathsf{C}_n := \{1 \prec \cdots \prec n \prec \bar{n} \cdots \prec \bar{1}\}.$$

Given any filling T of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ in an alphabet A, the ***reading word*** of T, denoted $w(T)$, is the word in A obtained by reading the entries of T down each column, from left to right.

Definition 4.1. A ***type B_n column Kashiwara-Nakashima (KN) tableau of height*** k is a filling C of the shape (1^k) for some $1 \leq k \leq n$ with entries in B_n such that

- (i) all entries in C are strictly increasing from top to bottom with the sole exception that 0 may be repeated, and
- (ii) if both z and $\bar{z}$ appear in C with z in the p^{th} box from the top and $\bar{z}$ in the q^{th} box from the bottom, then $N(z) := p + q \leq z$.

Denote by $\mathrm{KN}_n^B(1^k)$ the set of all type B_n Kashiwara-Nakashima tableaux of shape (1^k).

We define the **weight** of a type B_n column Kashiwara-Nakashima tableau C to be the integer vector

$$\mathrm{wt}(C) = (b_1, \ldots, b_n),$$

where b_i equals the number of $i's$ in S minus the number of $\bar{i}'s$.

Whenever both z and $\bar{z}$ appear in C we say that C contains the pair $(z, \bar{z})$ and consider each individual 0 a pair with itself, sometimes written $(0, \bar{0})$ for convenience.

Given $C \in \mathrm{KN}_n^B(1^k)$, consider the subset

$$I(C) := \{z_1, \ldots, z_s\} \subseteq \{1, \ldots, n\} \tag{27}$$

consisting of all letters $z_i \prec 0$ such that $(z_i, \bar{z}_i)$ is a pair in C with $z_i \succ z_{i+1}$ for all $1 \leq i \leq s$. Similarly, set

$$J(C) := (t_1, \cdots, t_s) \tag{28}$$

where we let $t_1 = \max\{t \in \mathsf{B}_n : t \prec z_1, t \notin C\}$, and recursively for $i = 2, \cdots, n$ define

$$t_i = \max\{t \in \mathsf{B}_n : t \notin C, \bar{t} \notin C, t \prec t_{i-1}, t \prec z_i\}.$$

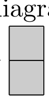
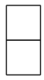
In particular, the definition of a column guarantees the existence of the set $J(C)$ [Lec03].

Definition 4.2. Given $C \in \mathrm{KN}_n^B(1^k)$ with $J(C)$ as in (28), let rC and lC be the columns obtained from C by replacing $\bar{z}_i$ with $\bar{t}_i$ and z_i with t_i, respectively. The ***splitting*** of C is the tableau of shape (2^k),

$$\mathrm{split}(C) := lCrC,$$

with entries lC in the left column and rC in the right column.

Example 4.3. Let $n = 9$ and C be a column with $w(C) = 246900\bar{9}\bar{4}\bar{2}$. Then $I(C) = (0, 0, 9, 4, 2)$, $J(C) = (8, 7, 5, 3, 1)$, $rC = 135678\bar{9}\bar{4}\bar{2}$ and $lC = 2469\bar{8}\bar{7}\bar{5}\bar{3}\bar{1}$.

Remark 4.4. Conventionally, spin tableaux are denoted by half-width columns to represent their half integer weights. In this article we choose to denote spin tableaux as shaded columns of regular width instead for legibility of the labels. That is, the fundamental weight $\omega_n^{B_n}$ is represented by the shaded Young diagram of shape (1^n), called *n-height spin column Young diagram*, while $2\omega_n^{B_n}$ is identified with the usual non-shaded Young diagram of shape (1^n). For instance, for $n = 2$, it means that $\omega_2^{B_n}$ is represented by the shaded Young diagram  of shape (1^2), while $2\omega_2^{B_n}$ is represented by the regular non shaded Young diagram  of shape (1^2).

Definition 4.5. A ***spin Kashiwara-Nakashima (KN) tableau of height*** n is a filling S of shaded shape (1^n) with entries in C_n such that

- (i) all entries in S are strictly increasing from top to bottom, and
- (ii) S contains no pairs $(z, \bar{z})$ for any $z \in \mathsf{C}_n$.

We define the **weight** of a spin Kashiwara-Nakashima tableau S to be the vector

$$\mathrm{wt}(S) = \frac{1}{2}(b_1, \ldots, b_n),$$

where b_i equals the number of $i's$ in S minus the number of $\bar{i}'s$. Note b_i is 1 or -1. Denote by $s\mathrm{KN}_n^B = \mathrm{KN}_n^B(\omega_n^{B_n})$ the set of all such spin tableaux. Since $\omega_n^{B_n}$ is minuscule, the elements of $s\mathrm{KN}_n^B$ are completely characterized by the 2^n weights $(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$, and $s\mathrm{KN}_n^B = O(\omega_n^{B_n})$.

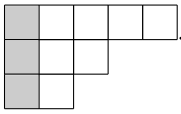
Example 4.6. Let $n = 2$, then the set of all spin Kashiwara-Nakashima tableaux $\mathrm{KN}_2^B(\omega_2^B)$ with shaded shape $(1, 1)$ and corresponding weights $(\pm\frac{1}{2}, \pm\frac{1}{2})$ are:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad (\frac{1}{2}, \frac{1}{2}), \quad \begin{array}{|c|}\hline 1 \\ \hline \bar{2} \\ \hline\end{array} \quad (\frac{1}{2}, -\frac{1}{2}), \quad \begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline\end{array} \quad (-\frac{1}{2}, \frac{1}{2}), \quad \begin{array}{|c|}\hline \bar{2} \\ \hline \bar{1} \\ \hline\end{array}, \quad (-\frac{1}{2}, -\frac{1}{2}).$$

From the discussion above, for any $\lambda \in P^+$ we can write either $\lambda = \mu$ a partition, or $\lambda = (\mu_0|\mu)$ with $\mu_0 = (1^n)$ a height n column.

We will refer to the shape $\lambda = (\mu_0|\mu)$ as a *spin partition* represented by the *spin Young diagram* of shape λ consisting of the n-height spin column Young diagram, shaded, followed with the Young diagram of shape μ.

Example 4.7. Let $n = 3$. Then the spin partition $\lambda = ((1^3))|(4, 2, 1))$ is represented by the following spin Young diagram of shape λ:



Let $\tilde{\mu}$ be the conjugate partition of μ. Then the i-th column of the Young diagram of μ has height $\tilde{\mu}_i$ the i-th entry of $\tilde{\mu}$. Putting Definitions 4.1 and 4.5 together we arrive at the following.

Definition 4.8. A type B_n ***Kashiwara–Nakashima tableau of shape partition*** λ is a filling T with entries in B_n of shape $\lambda = (\mu_0|\mu)$ with μ_0 either empty or (1^n) and μ a partition, such that if C_i denotes the i^{th} column of T, then:

- C_0 is a spin KN-tableau of height n,
- C_i is a column KN-tableau of height $\tilde{\mu}_i$,
- every row is weakly increasing but with no repeated zeros, and
- $\mathrm{split}(T) = \mathrm{split}(C_0)\ \mathrm{split}(C_1) \cdots \mathrm{split}(C_k)$ is a semistandard tableau.

The split of the spin column C_0 is just itself, un-shaded. We denote by $\mathrm{KN}_n^B(\lambda)$ the set of all type B_n Kashiwara-Nakashima tableaux of shape λ.

Given a partition μ, a ***type*** C_n ***Kashiwara-Nakashima tableau*** of shape μ is a type B_n KN-tableau with entries exclusively in C_n and no spin column. Henceforth, for $T \in \mathrm{KN}_n^B(\lambda)$, $\mathrm{split}(T) \in \mathrm{KN}_n^C(2\lambda)$.

Definition 4.9. We define the ***weight*** of a type B_n KN-tableau T to be the vector

$$\mathrm{wt}(T) = \sum_{i=0}^{k} \mathrm{wt}(C_i).$$

In type C_n the weight is defined analogously (with no spin part). Consequently $\mathrm{wt}(\mathrm{split}(T)) = 2\mathrm{wt}(T)$.

Remark 4.10. [Lec03] The previous definitions extend to skew shapes. A skew orthogonal tableau $\mathfrak{T}$ is a skew Young diagram (potentially with shaded leftmost column) filled by letters of B_n whose columns are admissible of type B and such that the rows of its split form $\mathrm{split}(\mathfrak{T})$ (obtained by splitting its columns) are weakly increasing from left to right.

Example 4.11. Let $n = 3$ and $\lambda = \omega_1^B + \omega_2^B + 3\omega_3^B$, which corresponds to the spin partition $(1^3|3, 2, 1)$, and let $\mathfrak{T} \in \mathrm{KN}_n^B(\lambda)$ with $\mathrm{wt}(\mathfrak{T}) = 1/2(1, -1, 1) + (0, 0, 0)$,

$$\mathfrak{T} = [\mathfrak{C}, T] = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 0 & \bar{3} \\ \hline 3 & 3 & 0 \\ \cline{1-3} \bar{2} & \bar{2} \\ \cline{1-2}\end{array} \xrightarrow{\text{split}} \text{split}(\mathfrak{T}) = [\text{split}(\mathfrak{C}), \text{split}(T)] = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & 3 & 3 & \bar{2} \\ \cline{1-5} \bar{2} & \bar{2} & \bar{1} \\ \cline{1-3}\end{array} \in \text{KN}_n^C(2\lambda^C),$$

$$2\lambda = 2\omega_1^C + 2\omega_2^C + 3\omega_3^C, \ \text{wt}(\text{split}(\mathfrak{T})) = (1, -1, 1) = 2\text{wt}(\mathfrak{T}).$$

4.2. **Kashiwara-Nakashima crystal operators.** The crystal graph $\mathcal{B}(\lambda)$ of a highest weight representation $V(\lambda)$ has a realization on KN-tableaux.

Example 4.12. The type B_n, $n \geq 2$, and type C_n crystals, $\mathcal{B}(\omega_1^B)$ and $\mathcal{B}(\omega_1^C)$, corresponding to the standard representations of $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$, respectively, are given below:

$$\mathcal{B}(\omega_1^B): \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-2} \boxed{n-1} \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \boxed{\overline{n-1}} \xrightarrow{n-2} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

$$\mathcal{B}(\omega_1^C): \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-2} \boxed{n-1} \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \boxed{\overline{n-1}} \xrightarrow{n-2} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

where $\text{wt}\left(\boxed{i}\right) = \epsilon_i$, $\text{wt}\left(\boxed{\bar{i}}\right) = -\epsilon_i$ for all $1 \leq i \leq n$ and $\text{wt}\left(\boxed{0}\right) = 0$ (for type B_n only).

Example 4.13. For $n = 1$, the B_1-spin crystal $\mathcal{B}(\omega_1^B)$ is given by:

$$\mathcal{B}(\omega_1^B): \boxed{1} \xrightarrow{f_1^B} \boxed{\bar{1}},$$

and for $n = 2$, the B_2-spin crystal $\mathcal{B}(\omega_2^B)$ is given by:

$$\mathcal{B}(\omega_2^B): \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \xrightarrow{f_2^B} \begin{array}{|c|}\hline 1 \\ \hline \bar{2} \\ \hline\end{array} \xrightarrow{f_1^B} \begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline\end{array} \xrightarrow{f_2^B} \begin{array}{|c|}\hline \bar{2} \\ \hline \bar{1} \\ \hline\end{array}.$$

The B_n spin crystal is

$$\mathcal{B}(\omega_n^{B_n}) = \left\{ \begin{array}{|c|}\hline t_1 \\ \hline \vdots \\ \hline t_n \\ \hline\end{array} \mid t_i \in \mathcal{C}_n, t_1 \prec \cdots \prec t_n, i, \bar{i} \text{ do not appear together }, 1 \leq i \leq n \right\} \tag{29}$$

where the action of the crystal operators f_i, $1 \leq i \leq n$, is given by

$$\begin{array}{|c|}\hline \\ \hline i \\ \hline \vdots \\ \hline \overline{i+1} \\ \hline\end{array} \xrightarrow{i} \begin{array}{|c|}\hline \\ \hline i+1 \\ \hline \vdots \\ \hline \bar{i} \\ \hline\end{array} \quad i \neq n, \qquad \begin{array}{|c|}\hline \\ \hline n \\ \hline\end{array} \xrightarrow{n} \begin{array}{|c|}\hline \\ \hline \bar{n} \\ \hline\end{array}, \tag{30}$$

otherwise is 0.

We define the crystal structure on the set $\text{KN}_n^B(\lambda)$ by first defining it on the crystal of reading words of $\text{KN}_n^B(\lambda)$, reading the columns from right to left and top to bottom.

Suppose T is a type B_n KN-tableau of shape $\lambda = (\mu_0|\mu)$ with reading word $w(T) = x_1 \cdots x_m | z_1 \cdots z_n$ with $z_1 \ldots z_n$ the reading word of the (potentially empty) spin column of shape μ_0 and $x_1 \ldots x_m$ the column reading word of the non-spin part of T of shape μ.

Given such a word, define the ***signature*** of $w(T) = x_1 \cdots x_m | z_1 \cdots z_n$ to be the sequential assignment σ_i for each $i \in I$ of some of its entries with expressions in $\{+, -\}$, determined by the following rules:

$$\text{(B) For non-spin entries } x_j\text{:} \begin{cases} \sigma_i(i) = + \text{ and } \sigma_i(\bar{i}) = - & \text{for } i \neq n \\ \sigma_i(i+1) = - \text{ and } \sigma_i(\overline{i+1}) = + & \text{for } i \neq n \\ \sigma_n(n) = ++, \sigma_n(0) = -+ \quad \text{and } \sigma_n(\bar{n}) = -- \end{cases}$$

(C) For spin entries z_j: $\begin{cases} \sigma_i(i) = + \text{ and } \sigma_i(\bar{i}) = - & \text{for all } i \\ \sigma_i(i+1) = - \text{ and } \sigma_i(\overline{i+1}) = + & \text{for } i \neq n \end{cases}$

where σ_i assigns nothing to an entry $y \neq i, i+1, \bar{i}, \overline{i+1}$ (or $y \neq 0, n, \bar{n}$ in the case of $i = n$).

We note that the signature rule for spin entries above is nothing more than the type C_n signature rule. Thus, for a type C_n KN-tableau T we can define the signature of $w(T)$ similarly but use (C) instead of (B) for every entry.

Definition 4.14. For $T \in \mathrm{KN}_n^B(\lambda)$ and $i \in I$, define the ***i-pairing*** of entries $w(T) = x_1 \dots x_m | z_1 \dots z_n$ as follows. Provided the sequence of $+'$s and $-'$s given by σ_i, iteratively i-pair any unpaired $+$ with a $-$ to its right whenever all signs $+, -$ between them are already i-paired.

If, after i-pairing, there exists an entry y in $w(T)$ with signature $+$ such that this sign is not i-paired with any $-$ to its right, we call y an *unpaired* entry of $w(T)$.

Definition 4.15. For each $i \in I$ and $T \in \mathrm{KN}_n^B(\lambda)$, define the ***lowering operator*** f_i^B on each $w(T) = x_1 \dots x_m | z_1 \dots z_n$ as follows:

- if $w(T)$ has no unpaired entries (with respect to σ_i), then $f_i^B(T) = 0$.
- otherwise, f_i^B will act on the *leftmost* unpaired entry y of $w(T)$ via the following assignment:
  - if y lies in a nonspin column then f_i^B will send $y = i \mapsto i+1$ or $y = \overline{i+1} \mapsto \bar{i}$ if $i \neq n$ or send $y = n \mapsto 0$ or $y = 0 \mapsto \bar{n}$ if $i = n$, and leave all other entries unchanged.
  - if $y = i \neq n$ lies in a spin column with an entry $z = \overline{i+1}$ below it, then f_i^B will simultaneously send $y = i \mapsto i+1$ and $z = \overline{i+1} \mapsto \bar{i}$. If $y = i \neq n$ but $\overline{i+1}$ is not contained in the spin column, then $f_i^B(T) = 0$. If $i = n$ then f_n^B maps $y = n \mapsto \bar{n}$, leaving all other entries unchanged.

Similarly, for $T \in \mathrm{KN}_n^C(\mu)$ the lowering operator f_i^C acts on the leftmost unpaired entry of $w(T)$ as in the nonspin case above with the only change being that for $i = n$, f_n^C maps $n \mapsto \bar{n}$. In the non-spin case, $f_n^B f_n^B(n) = f_n^C(n)$.

Example 4.16. Let $n = 8$ and let us consider a word without spin part,

$$800 \xrightarrow{\sigma_8} ++-+-+ \xrightarrow{\textit{pairing}} +(+-)(+-)+ \xrightarrow{f_8^B} -+-+-+ \xrightarrow{\sigma_8^{-1}} \mathbf{0}00$$

$$000 \xrightarrow{f_8^B} 00\bar{8} \rightarrow -(+-)(+-)- \xrightarrow{f_8^B} 0$$

For $n = 7$, $f_6^B(247007\bar{7}\bar{4}) = f_6^C(247007\bar{7}\bar{4}) = 247006\bar{6}\bar{4}$, where $f_6^B(7\bar{7}) = f_6^C(7\bar{7}) = 7\bar{6}$,

$$7\bar{7} \longrightarrow -+ \xrightarrow{f_6^B}\longrightarrow -- \longrightarrow 7\bar{6}.$$

Thus, define the lowering operators f_i on $T \in \mathrm{KN}_n^B(\lambda)$ as the induced operators from those on $w(T)$, where if the entry y in $w(T)$ is modified under the action of f_i, then the corresponding entry in T is modified in the same way. It is a theorem of Kashiwara and Nakashima [KN94] that this endows $\mathrm{KN}_n^B(\lambda)$ (resp. $\mathrm{KN}_n^C(\lambda)$) with the structure of an $\mathfrak{so}_{2n+1}$-crystal (resp. $\mathfrak{sp}_{2n}$-crystal).

Example 4.17. Let $n = 3$ and $\lambda = \omega_1^B + \omega_2^B + 3\omega_3^B = \omega_3^B + (3,2,1)$ be a spin partition and let $\mathfrak{T} \in \mathcal{B}(\lambda)$ with $\mathrm{wt}(\mathfrak{T}) = 1/2(1,-1,1) + (0,0,0)$,

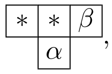
$$\mathfrak{T} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 0 & \bar{3} \\ \hline 3 & 3 & 0 \\ \cline{1-3} \bar{2} & \bar{2} \\ \cline{1-2} \end{array} \xrightarrow{f_3} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 0 & \bar{3} \\ \hline 3 & 3 & \bar{3} \\ \cline{1-3} \bar{2} & \bar{2} \\ \cline{1-2} \end{array} \xrightarrow{f_3} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 0 & \bar{3} \\ \hline 3 & 0 & \bar{3} \\ \cline{1-3} \bar{2} & \bar{2} \\ \cline{1-2} \end{array} \xrightarrow{f_3} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 0 & \bar{3} \\ \hline 3 & \bar{3} & \bar{3} \\ \cline{1-3} \bar{2} & \bar{2} \\ \cline{1-2} \end{array} \xrightarrow{f_3} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 0 & \bar{3} \\ \hline \bar{3} & \bar{3} & \bar{3} \\ \cline{1-3} \bar{2} & \bar{2} \\ \cline{1-2} \end{array} \in \mathcal{B}(\lambda)$$

Remark 4.18. The rectification map, defined by the B_n-spin insertion or $B_n(C_n)$-insertion scheme, $w \otimes w' \mapsto [\emptyset \leftarrow w \leftarrow w']$ is a B_n (C_n)-crystal isomorphism [Bak00a, Lec03]. In type C_n it coincides with the rectification defined by SJDT on the diagonal skew tableau with reading word $w \otimes w'$.

4.3. **Symplectic jeu de taquin.**

4.3.1. *Lecouvey–Sheats symplectic jeu de taquin.* In this section we briefly recall the symplectic jeu de taquin procedure [She99, Lec02, San21a].

Let T be a punctured symplectic KN skew tableau with two columns C_1 and C_2 with the puncture in C_1 and split form $spl(T) = lC_1rC_1lC_2rC_2$. Let α be the entry under the puncture of rC_1, and β the entry to the right of the puncture of rC_1, that is, locally, the tableau $spl(T) = lC_1rC_1lC_2rC_2$ looks like:



where α or β may not exist. The elementary steps of the symplectic jeu de taquin, or SJDT for short, are the following:

A. If $\alpha \leq \beta$ or β does not exist, then the puncture of T will change its position with the cell beneath it. This is called a vertical slide.

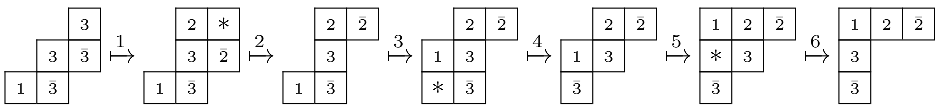
B. If the slide is not vertical, then we say it is horizontal. So we have $\alpha > \beta$ or α does not exist. Let C_1' and C_2' be the columns obtained after the slide. We have two subcases, depending on the sign of β:
   (1) If β is barred, we will have an horizontal slide of the puncture, getting $C_2' = C_2 \setminus \{\beta\} \sqcup \{*\}$ and $C_1' = \Phi^{-1}(\Phi(C_1) \setminus * \sqcup \{\beta\})$.
   (2) If β is unbarred, the we have $C_1' := C_1 \setminus * \sqcup \{\beta\}$ and $C_2' := \Phi^{-1}(\Phi(C_2) \setminus \{\beta\} \sqcup *)$. However, in this case it may happen that C_1' is no longer admissible. In this situation, if i is the lowest entry such that $i, \bar{i}$ appear in C_1' and $N(i) > i$, we erase both i and $\bar{i}$ from the column and remove a cell from the bottom and from the top column, and place all the remaining cells in order.

If one applies elementary SJDT slides successively, the puncture will eventually be a cell such that α and β do not exist. In this case we redefine the shape to not include this cell and the *jeu de taquin* ends.

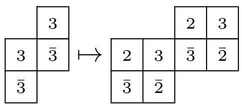
Given a symplectic KN skew tableau T, the rectification of T consists of applying the SJDT, until we get a tableau of shape λ, for some partition λ. The rectification is independent of the order in which the inner corners are filled [Lec02, Corollary 6.3.9].

Remark 4.19. If the columns C_1 and C_2 do not have negative entries then the SJDT coincides with the *jeu de taquin* on semistandard tableaux.

Example 4.20. The following computation shows an example of rectification using SJDT.

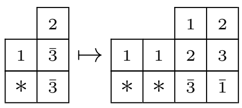


We spell out a few steps in detail. To compute step 1, first we perform the splitting



and get $\alpha = 3$, $\beta = 2$. This is in case B(2). Then $C_1' = 23\bar{3}$, $C_2' = \Phi^{-1}(\Phi(3\bar{3}) \setminus \{2\} \cup *) = \Phi^{-1}(2\bar{2} \setminus \{2\} \cup *) = \Phi^{-1}(*\bar{2}) = *\bar{2}$.

To compute step 4, first we perform the splitting



and get $\beta = \bar{3}$, α is nonexistent. This is in case B(1). Then $C_1' = \Phi^{-1}(\Phi(1*) \setminus * \sqcup \{\bar{3}\}) = \Phi^{-1}(1\bar{3}) = 1\bar{3}$ and $C_2' = 23$.

4.4. **Virtualization and jeu de taquin in type B.** We build on the virtualization procedure described in [PPSS23] where virtual crystals for spin and vector representation (or standard) of type B_n into type C_n are provided. The procedure for spin representation of type B_n into type C_n, that is, the *virtualization* of the spin column tableau is just its un-shade or split, $\Upsilon : s\mathrm{KN}_n^B \hookrightarrow \mathrm{KN}_n^C(\omega_n^C)$, $\mathfrak{C} \mapsto \Upsilon(\mathfrak{C}) = \mathrm{split}(\mathfrak{C})$ where

$2\omega_n^{B_n} = \omega_n^C$ and the virtual crystal operators are $f_i^v = f_i^{C_n} \circ f_i^{C_n}$, for $1 \le i < n$ and $f_n^{C_n}$ for $i = n$ (Figure 1). One also has $\mathbb{D}_2 : s\mathrm{KN}_n^B \hookrightarrow \mathrm{KN}_n^B(2\omega_n^{B_n})$, $\mathfrak{C} \mapsto \mathbb{D}_2(\mathfrak{C}) = \mathrm{split}(\mathfrak{C})$ (cf. [Lec03, Proposition 3.1.9]). Computationally $\Upsilon(\mathfrak{C}) = \mathbb{D}_2(\mathfrak{C})$ but yields a different crystal embedding.

Example 4.21. Let $n = 1$. Consider the B_1-spin crystal $\mathcal{B}(\omega_1^{B_1}) : \boxed{1} \xrightarrow{f_1^B} \boxed{\bar{1}}$ and observe its 2-dilation

$$\mathbb{D}_2(\mathcal{B}(\omega_1^{B_1})) : \boxed{1} \xrightarrow{f_1^B f_1^B} \boxed{\bar{1}} \text{ embedded into the } B_1 \text{ non-spin crystal } \mathcal{B}(2\omega_1^{B_1}) : \boxed{1} \xrightarrow{1} \boxed{0} \xrightarrow{1} \boxed{\bar{1}}.$$

On the other hand, $\Upsilon(\mathcal{B}(\omega_1^{B_1})) : \boxed{1} \xrightarrow{f_1^C} \boxed{\bar{1}}$ is the C_1 crystal $\mathcal{B}(\omega_1^{C_1})$.

For $n = 2$, consider the B_2-spin crystal

$$\mathcal{B}(\omega_2^B) : \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{f_2^B} \begin{array}{|c|}\hline 1\\\hline \bar{2}\\\hline\end{array} \xrightarrow{f_1^B} \begin{array}{|c|}\hline 2\\\hline \bar{1}\\\hline\end{array} \xrightarrow{f_2^B} \begin{array}{|c|}\hline \bar{2}\\\hline \bar{1}\\\hline\end{array}.$$

and its 2-dilation embedded into the B_2 non spin crystal $\mathcal{B}(2\omega_2^{B_2})$

$$\mathbb{D}_2(\mathcal{B}(\omega_2^{B_2})) = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{f_2^B f_2^B} \begin{array}{|c|}\hline 1\\\hline \bar{2}\\\hline\end{array} \xrightarrow{f_1^B f_1^B} \begin{array}{|c|}\hline 2\\\hline \bar{1}\\\hline\end{array} \xrightarrow{f_2^B f_2^B} \begin{array}{|c|}\hline \bar{2}\\\hline 1\\\hline\end{array} \hookrightarrow \mathcal{B}(2\omega_2^{B_2})$$

On the other hand, $\Upsilon((\mathcal{B}(\omega_2^{B_2}))$ is embedded into the C_2 crystal $\mathcal{B}(\omega_2^C)$,

$$\Upsilon((\mathcal{B}(\omega_2^{B_2})) = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{f_2^C} \begin{array}{|c|}\hline 1\\\hline \bar{2}\\\hline\end{array} \xrightarrow{f_1^C f_1^C} \begin{array}{|c|}\hline 2\\\hline \bar{1}\\\hline\end{array} \xrightarrow{f_2^C} \begin{array}{|c|}\hline \bar{2}\\\hline 1\\\hline\end{array} \hookrightarrow \mathcal{B}(\omega_2^C)$$

The procedure on the non-spin part of an orthogonal tableau, is induced by the map defined by

$$i \mapsto ii, \bar{i} \mapsto \overline{ii}, 0 \mapsto \bar{n}n \text{ for all } 1 \le i \le n \tag{31}$$

on words, followed by *symplectic* insertion [Bak00b, Lec02], or SJDT on the diagonal skew tableau with that reading word [Lec02, Corollary 6.3.9].

Remark 4.22. Observe that the assignment (31) on the alphabet $\mathcal{B}_n$ is exactly the splitting

$$\mathrm{spl} : \mathcal{B}(\omega_1^{B_n}) \hookrightarrow \mathcal{B}(2\omega_1^{C_n})$$

of the orthogonal tableaux (consisting of a sole letter) in the B_n standard crystal $\mathcal{B}(\omega_1^{B_n})$. The virtual crystals for the vector and spin representations of type B_n into C_n defined in [PS18] coincide with the splitting. Taking into account [Lec03, Proposition 3.1.9], one also has in those cases that the virtualization map $\Upsilon = \mathrm{spl} = \mathbb{D}_2$.

More precisely, let

$$\zeta : \mathcal{B}_n^* \sqcup \{0\} \hookrightarrow \mathcal{C}_n^* \sqcup \{0\}$$

be the injection of monoids induced by (31) and $\zeta(0) = 0$.

Clearly, for $u \in \mathcal{B}_n^*$, one has

$$\mathrm{wt}^C(\zeta(u)) = 2\mathrm{wt}^B(u). \tag{32}$$

For $\mathfrak{T} = \mathfrak{C}|T \in \mathrm{KN}_n^B(\lambda)$ with $\mathfrak{C}$ the spin part and $\lambda = (\mu_0|\mu)$, let $w(T) = x_1 \cdots x_m = x_1 \otimes \cdots \otimes x_m$ and recall that $T = \emptyset \leftarrow x_1 \leftarrow \cdots \leftarrow x_m$ using the B_n-insertion scheme [Bak00a, Lec03]. Now, by Remark 4.22, ζ is the virtualization map from the crystal of words $\mathcal{B}_n^*$ into the crystal of words $\mathcal{C}_n^*$

$$\zeta(x_1 \otimes \cdots \otimes x_m) = v_1 \otimes \cdots \otimes v_{2m} = \Upsilon(x_1) \otimes \cdots \otimes \Upsilon(x_m) = \Upsilon^{\otimes m}(x_1 \otimes \cdots \otimes x_m) \in \mathcal{B}(2\omega_1^C)^{\otimes m}.$$

Then, from Remarks 4.18 and 2.7, the *virtualization of* T is given by:

$$\Upsilon(T) = \emptyset \leftarrow v_1 \leftarrow \cdots \leftarrow v_{2m}, \tag{33}$$

where $\emptyset \leftarrow v$ denotes the C_n insertion [Bak00b, Lec02].

Remark 4.23. A word $w \in \mathcal{C}_n^*$ is a highest weight word if and only if the weight of all its prefixes (including itself) is a partition [San21b]. A word $u \in \mathcal{B}_n^*$ is a highest weight word if and only if $\zeta(u) \in \mathcal{C}_n^*$ is a highest weight word. Then, from (32), a highest weight word in $\mathcal{B}_n^*$ is similarly characterized.

Theorem 4.24. *Let $\mathfrak{C}|T \in \mathrm{KN}_n^B(\lambda)$ be a Kashiwara–Nakashima tableau of type B_n with $\lambda = (\mu_0, \mu)$. Then*

(1) $\Upsilon(T) = \mathrm{split}(T) \in \mathrm{KN}_n^C(2\mu)$ *the splitting of T, and* $\Upsilon(T) = \mathrm{split}(T) = \mathbb{D}_2(T)$.
(2) $\Upsilon(\mathfrak{C}|T) = \mathrm{split}(\mathfrak{C}|T) \in \mathrm{KN}_n^C(\omega_n^C|2\mu)$ *the splitting of* $\mathfrak{C}|T$*, and* $\Upsilon(\mathfrak{C}|T) = \mathrm{split}(\mathfrak{C}|T) = \mathbb{D}_2(\mathfrak{C}|T)$.
(3) $\mathfrak{C}|T$ *is a B_n key if and only if the columns of $\mathfrak{C}|T$ are nested and the letters i and $-i$, for any $i \in \mathsf{B}_n$, do not appear simultaneously as entries in a given column.*

Proof. (1) Let T be a KN tableau of type B_n (without spin part) and let $w(T)$ be its word. Construct a skew tableau of staircase shape, placing the entries of $w(T)$ along the diagonal, in order, starting from the top right-most corner, and call it T'. Now, the rectification of T' coincides with T by Proposition 3.5.3 in [Lec03]. Recall that to rectify an orthogonal skew KN tableau, one needs perform type C rectification to its splitting and then "unsplit" the obtained result. Note that the splitting of a tableau consisting of one single letter coincides trivially with the virtualization map of [PPSS23]. Therefore, we conclude that the type C_n rectification of $\mathrm{split}(T')$ coincides with $\mathrm{split}(T)$. Since the type C rectification of $\mathrm{split}(T')$ in turn coincides with $\Upsilon(T)$, we may conclude the proof.

(2) From [Lec03, Proposition 3.1.9] and its iteration in Proposition 3.5, we may also observe that $\Upsilon(T) = \mathrm{split}(T) = \mathbb{D}_2(T)$. Hence, from (1),

$$\begin{aligned}\mathrm{split}(\mathfrak{C}|T) = \mathrm{split}T \otimes \mathrm{split}\mathfrak{C} =& \mathbb{D}_2(T) \otimes \mathbb{D}_2(\mathfrak{C}) \\ =& \Upsilon(T) \otimes \Upsilon(\mathfrak{C}) \\ =& \Upsilon(T \otimes \mathfrak{C}) \\ =& \Upsilon(\mathfrak{C}|T) \in \mathrm{KN}_n^C(2\lambda).\end{aligned}$$

(3) By Corollary 3.21, $\mathfrak{C}|T$ is a B_n key if and only if $\mathrm{split}(\mathfrak{C}|T)$ is a C_n key. A symplectic key in type C_n is a C_n KN tableau where columns are nested and the letters i and -i, for any i, do not appear simultaneously as entries in a column [San21b]. Hence $\mathfrak{C}|T$ is a B_n key if and only if the columns of $\mathrm{split}(\mathfrak{C}|T)$ are nested. That is, the columns of $\mathfrak{C}|T$ are nested and the letters i and $-i$, for any $i \in \mathsf{B}_n$, do not appear simultaneously as entries in a column.

□

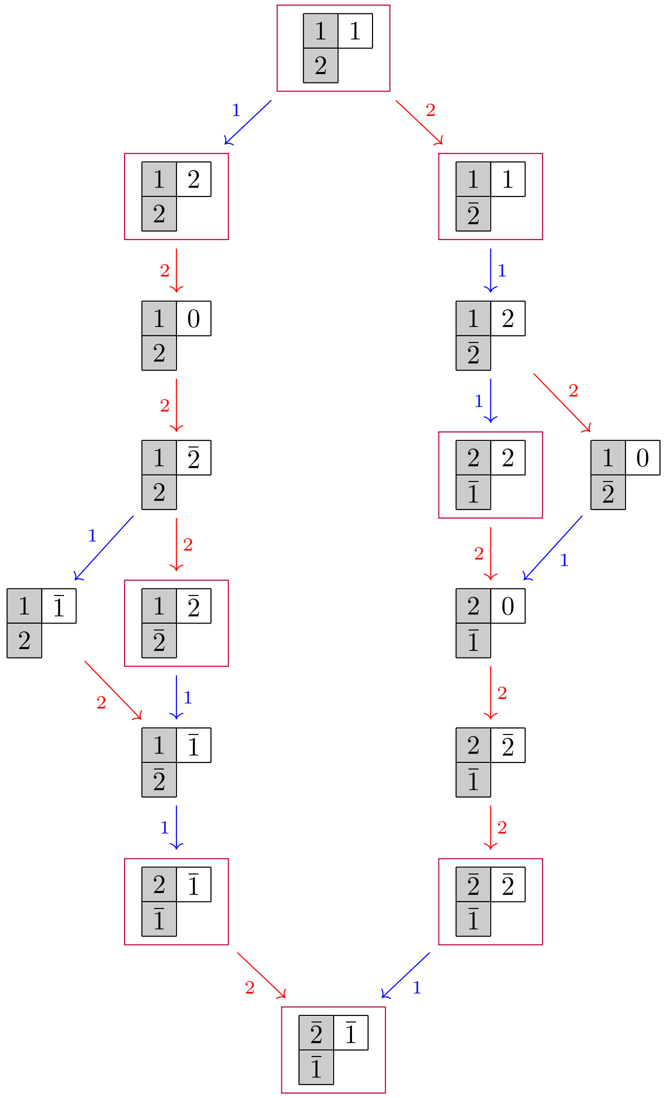
4.5. **Evacuation of KN tableaux.** Let $T \in \mathrm{KN}_{B_n}(\lambda)$ be a KN tableau of type B_n. We first compute the type C_n evacuation to $\mathrm{split}(T)$, $\mathrm{evac}^C(\mathrm{split}(T))$ as in [San21a]. That is, we perform π rotation on $\mathrm{split}(T)$, and apply the longest element of the Weyl group (the hyperoctahedral group, consisting of signed permutations) to each of the entries in $\mathrm{split}(T)$, which means that we just bar, respectively unbar the entries; and then we apply the type C_n rectification, using SJDT from Subsection 4.3.1 or the type C_n insertion [Bak00b, Lec02] to $\mathrm{split}(T)$. Lastly we unsplit $\mathrm{evac}^C(\mathrm{split}(T))$. Let us call this final tableau $\mathrm{evac}^B(T)$. It follows from by Theorem 4.24 and Proposition 2.9that $\mathrm{evac}^B(T)$ is the result of applying the *Lusztig involution* ξ to T.

Example 4.25. For $n = 2$ and $\mathfrak{T} = \begin{array}{|c|c|}\hline 2 & 0 \\ \hline \bar{1} & \\ \cline{1-1}\end{array} \in \mathcal{B}(\omega_2^B + \omega_1^B)$, we have $\mathrm{evac}^B(\mathfrak{T}) = \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 & \\ \cline{1-1}\end{array}$.

Appendix A.

A.1. **Example for virtualization from B_n into C_n and dilation.** In this section we provide an example for Theorem 3.19 and Corollary 3.21 using virtualization from type B to type C.

Example A.1. We give a full example of a B_2- crystal $\mathcal{B}(\lambda) = \mathrm{KN}_2^B(\lambda)$ for $\lambda = \omega_2^{B_2} + \omega_1^{B_2}$. The boxed entries are the keys of $\mathcal{B}(\lambda)$, as defined in Section 2.3 or in Theorem 4.24.

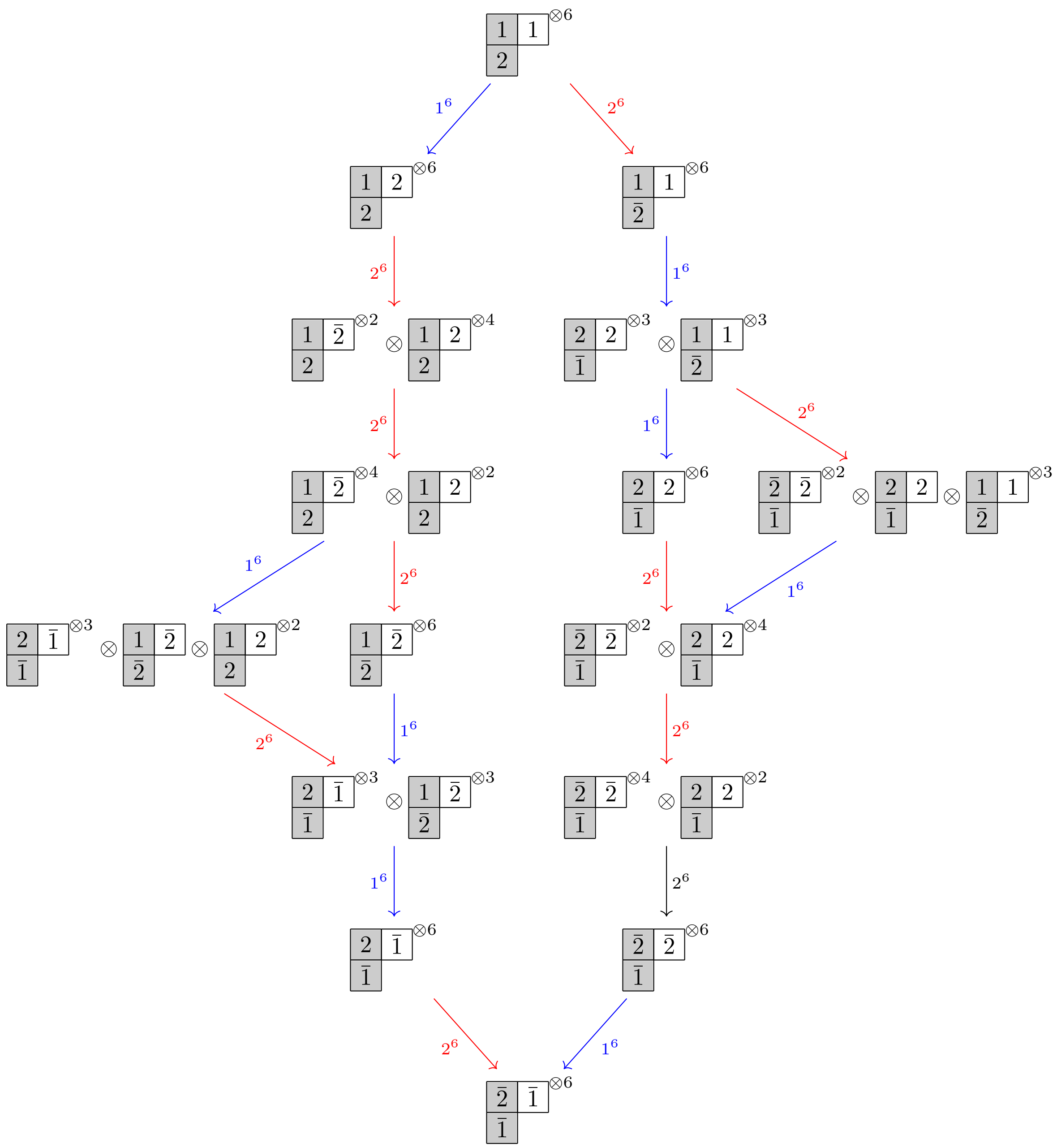


We give an example of a Demazure atom. For $w = s_2 s_1 \in W$, we have $w\lambda = (1, -2)$ and $b_{w\lambda} = \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline \bar{2} & \\ \cline{1-1}\end{array}$. Then

$$\mathring{\mathcal{B}}_w(\omega) = \{T \in B(\lambda) : K^+(T) = b_{w\lambda}\} = \left\{ \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 2 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline \bar{2} & \\ \cline{1-1}\end{array} \right\},$$

where the right keys can be computed from $\mathbb{D}_6\mathcal{B}(\omega_2^{B_2} + \omega_1^{B_2})$ or from virtualization of type B_n into C_n. The latter is shown in Figure 2. It can be computed using either the SJDT or the Willis like direct way presented in [San21a, AS24].

Example A.2.



This picture shows the 6-dilated crystal $\mathbb{D}_6\mathrm{KN}_2^B(\omega_2^{B_2}+\omega_1^{B_2})$ embedded into $\mathrm{KN}_2^B(K(\omega_2^{B_2}+\omega_1^{B_2})^{\otimes 6}) \simeq \mathrm{KN}_2^B(6\omega_2^{B_2}+6\omega_1^{B_2})$ where $K(\omega_2^{B_2}+\omega_1^{B_2})$ denotes the B_2 key of weight $\omega_2^{B_2}+\omega_1^{B_2}$. where $K(\omega_2^{B_2}+\omega_1^{B_2}) =$ $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}$ denotes the type B_2 key of weight $\omega_2^{B_2}+\omega_1^{B_2}$. We have:

$$\Upsilon\Theta_6(K(\omega_2^{B_2}+\omega_1^{B_2})) = \Upsilon(K(\omega_2^{B_2}+\omega_1^{B_2})^{\otimes 6}) = \Theta_6 K(\omega_2^{C_2}+\omega_1^{C_2}) = \Theta_6\Upsilon(K(\omega_2^{B_2}+\omega_1^{B_2}))$$

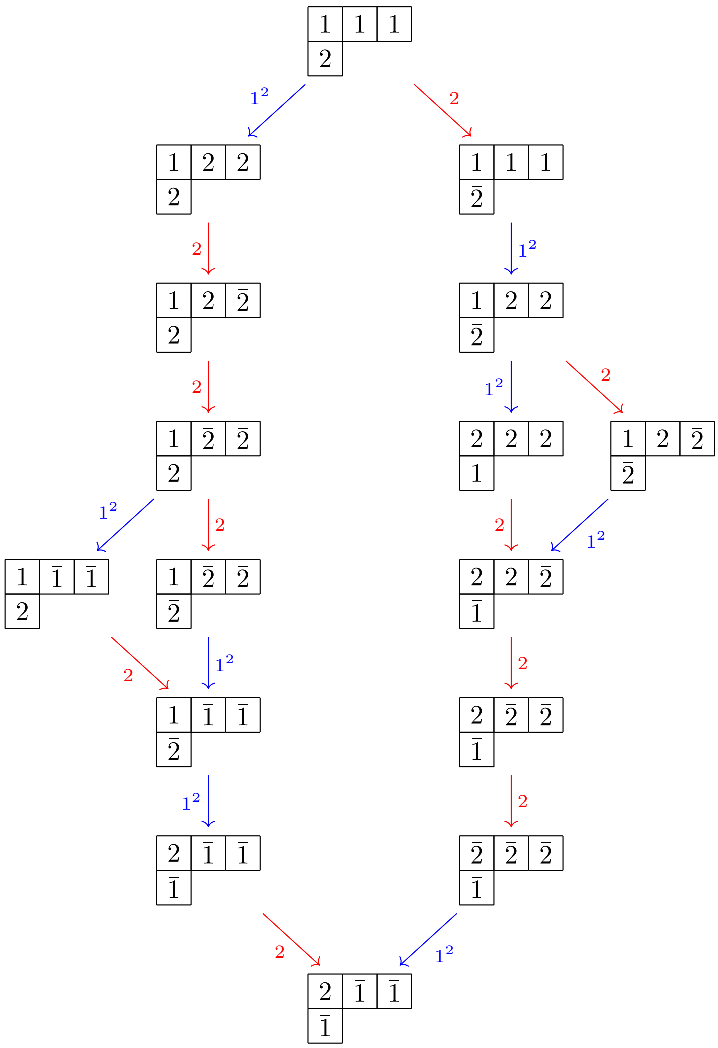


FIGURE 2. The virtual crystal $\Upsilon \mathrm{KN}_2^B(\omega_2^{B_2}+\omega_1^{B_2})$ embedded into $\mathrm{KN}_2^C(\omega_2^C+\omega_1^C)$

and

$$\Upsilon\Theta_6 \begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline 2 \\ \cline{1-1}\end{array} = \Upsilon\Big(\begin{array}{|c|c|}\hline 2 & \bar{1} \\ \hline \bar{1} \\ \cline{1-1}\end{array}^{\otimes 3} \otimes \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline \bar{2} \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}^{\otimes 2} \Big) = \begin{array}{|c|c|c|}\hline 2 & \bar{1} & \bar{1} \\ \hline \bar{1} \\ \cline{1-1}\end{array}^{\otimes 3} \otimes \begin{array}{|c|c|c|}\hline 1 & \bar{2} & \bar{2} \\ \hline \bar{2} \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}^{\otimes 3}$$

$$= \Theta_6 \begin{array}{|c|c|c|}\hline 1 & \bar{1} & \bar{1} \\ \hline 2 \\ \cline{1-1}\end{array} = \Theta_6 \Upsilon \begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline 2 \\ \cline{1-1}\end{array}.$$

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