A COMPARISON BETWEEN WEAKLY PROTOMODULAR AND PROTOMODULAR OBJECTS IN UNITAL CATEGORIES

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Abstract. We compare the concepts of *protomodular* and *weakly protomodular* objects within the context of unital categories. Our analysis demonstrates that these two notions are generally distinct. To establish this, we introduce *left pseudocancellative unital magmas* and characterise weakly protomodular objects within the variety of algebras they constitute. Subsequently, we present an example of a weakly protomodular object that is not protomodular in this category.

1. Introduction

Many of the intrinsic properties of non-abelian algebraic structures, such as groups, rings, Lie algebras and several others, have been successfully described in categorical terms thanks to the notion of *semi-abelian category* [11]. Semi-abelian categories play a central role in many recent developments of Categorical Algebra, such as non-abelian homology and cohomology (see, for instance, [5, 7, 10, 13, 6]).

A fundamental ingredient in the definition of a semi-abelian category is *protomodularity* [2]. A pointed category is protomodular if and only if the *Split Short Five Lemma* holds in it. There are other equivalent ways of defining protomodularity. One of them makes use of the notion of *jointly extremal-epimorphic* pair of morphisms: two morphisms $f\colon A \to C$ and $g\colon B \to C$ with the same codomain are jointly extremal-epimorphic if, whenever they factor through a common monomorphism $m\colon M \rightarrowtail C$, then m is an isomorphism. In concrete algebraic contexts, this means that the object C is generated by the images of f and g. A *point*, namely a pair (f, s) in which f is a split epimorphism and s is a chosen section of f, is said to be *strong* if the section s and the kernel k of f are jointly extremal-epimorphic (here we are supposing that the category is finitely complete and pointed, although the definition of a strong point can be formulated in non-pointed categories as well). The point (f, s) is called *stably strong* if every point obtained as a pullback, along any morphism, of (f, s) is strong. A pointed finitely complete category is protomodular if and only if every point in it is strong or, equivalently, every point is stably strong (see e.g. [1] for a proof of this fact).

In order to find a purely categorical characterisation of groups amongst monoids, in [12] the authors considered a local version of protomodularity, introducing the notion of *protomodular object* inside a (not necessarily protomodular) category.

2020 *Mathematics Subject Classification.* 18E13, 18E99, 18C05, 08C05.

Key words and phrases. Protomodular object; weakly protomodular object; unital category; left pseudocancellative magma.

The first author is supported by Ministerio de Ciencia e Innovación (Spain), with grant number PID2021-127075NA-I00. The second author is member of the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni (GNSAGA) dell'Istituto Nazionale di Alta Matematica "Francesco Severi". The third author acknowledges partial financial support by *Centro de Matemática da Universidade de Coimbra* (CMUC), funded by the Portuguese Government through FCT/MCTES, DOI 10.54499/UIDB/00324/2020. The fourth author is a Senior Research Associate of the Fonds de la Recherche Scientifique–FNRS.

An object X in a pointed finitely complete category is such if every point over X, i.e., every point (f, s) in which X is the codomain of the split epimorphism f, is stably strong. Obviously, a category is protomodular if and only if every object in it is protomodular. The class of protomodular objects in a category satisfies strong algebraic properties, as shown in [12]. In the category of monoids, the protomodular objects are precisely the groups; similarly, in the category of semirings, the protomodular objects are the rings. Note that the protomodularity of an object depends on the category where it is considered. For instance, any group is a protomodular object in the category Grp of groups and in the category Mon of monoids, but it is not protomodular in the category of unital magmas unless it is trivial—see Theorem 2.1 for an explicit proof of this fact.

Later, in [8], the author considered a weaker notion, the concept of *weakly protomodular object*: an object X is such when every point over X is strong. He showed that in Mon the weakly protomodular objects are precisely the groups, so protomodular and weakly protomodular objects coincide in Mon. Similarly, one can show that the same happens for semirings: protomodular and weakly protomodular objects are rings.

It is easy to see that every protomodular object is a weakly protomodular object. The converse is false, since there are weakly protomodular objects that are not protomodular. For example, in every pointed finitely complete category the zero object is weakly protomodular, while the condition that the zero object is protomodular is equivalent to the category being unital in the sense of [3]. Recall that a pointed finitely complete category is a *unital category* if, for every pair of objects (X, Y) the canonical morphisms $\langle 1_X, 0\rangle\colon X \to X \times Y$ and $\langle 0, 1_Y\rangle\colon Y \to X \times Y$ are jointly extremal-epimorphic. As observed in [12, Proposition 7.12], this condition is equivalent to the protomodularity of the zero object. So, for example, in the category of pointed sets the singleton is a weakly protomodular object but not a protomodular object, since this category is not unital.

In all previously studied examples of unital categories, protomodular and weakly protomodular objects have been found to coincide. This includes cases such as monoids and semirings, as mentioned above, but also cocommutative bialgebras over an algebraically closed field, where (weakly) protomodular objects are precisely the cocommutative Hopf algebras, as demonstrated in [9]. This consistency raises the question of whether protomodular and weakly protomodular objects always coincide in unital categories.

In this paper we give a negative answer to this question. To this aim, we introduce the algebraic structure of *left pseudocancellative unital magmas* (briefly LPM). Left pseudocancellative unital magmas and morphisms between them form a unital category, denoted by LPM. We characterise the weakly protomodular objects in LPM (Theorem 2.5) and we show that all the subobjects of a protomodular object must be weakly protomodular (Theorem 2.8). Then we exhibit a weakly protomodular object in LPM with a subobject which is not weakly protomodular, showing that it is not a protomodular object. In conclusion, the notions of protomodular and weakly protomodular objects do not coincide in all unital categories.

2. The case of left pseudocancellative unital magmas

We start by proving a claim made in the Introduction:

Theorem 2.1. *The variety of unital magmas does not admit non-trivial (weakly) protomodular objects.*

Proof. Let M be a unital magma with unit denoted by e. Suppose $m \in M$. Let F be the free unital magma on one element x, and $\overline{m}\colon F \to M$ the morphism sending

x to m. Consider the split epimorphism $(\overline{m}\ 1_M)\colon F+M \to M$ and the induced point determined by the coproduct inclusion $\iota_M\colon M \to F+M$. If this point is strong, then x in $F+M$ can be written as a product of elements in the kernel K of $(\overline{m}\ 1_M)$ with elements in M. Yet, the only ways to write x as a product in $F+M$ are $1x$, $x1$, $1(x1)$, $1(1x)$, $(x1)1$, etc., where 1 denotes the unit of $F+M$. Necessarily then, $x \in K$, which implies that $m = (\overline{m}\ 1_M)(x) = e$. Hence M is trivial. □

We recall that a *left quasigroup* is a set Q equipped with a binary operation $*\colon Q \times Q \to Q$, such that for all $y \in Q$, the *left multiplication map*

$$M_y\colon Q \to Q\colon x \mapsto y * x$$

is bijective. Left quasigroups form a variety of universal algebras. In fact, a left quasigroup can be equivalently defined as a set Q with two binary operations: a multiplication denoted by $*$ and a left division denoted by $\backslash$, satisfying the following identities:

$$y = x * (x\backslash y) \tag{1}$$

$$y = x\backslash(x * y) \tag{2}$$

A left quasigroup with an *identity element*, i.e., an element $e \in Q$ such that

$$x = e * x = x * e, \tag{3}$$

for all $x \in Q$, it is called a *left loop*.

If Q is a left loop, then for each $x \in Q$, we have that

$$x\backslash x = x\backslash(x * e) \overset{(2)}{=} e. \tag{4}$$

It is known that the category of left loops is semi-abelian [4], thus it is a protomodular category. Consequently, all of its objects are (weakly) protomodular.

In this work we consider a weakening of the concept of left loop. We will say that a set X with two binary operations $*$ and $\backslash$ and a nullary operation e is a *left pseudocancellative unital magma* (*LPM* for short) if it satisfies identities (1) and (3). We have not found any reference of this structure in the literature; the name is inspired by *left cancellative magmas*, which are sets with two operations $*$ and $\backslash$ satisfying identity (2). Note that, in general, the identity $x\backslash x = e$ need not hold in an LPM: see Proposition 2.9.

Proposition 2.2. *Let X be an LPM. Then:*

(i) *all maps $M_y\colon X \to X\colon x \mapsto y * x$ are surjective;*
(ii) *all maps $D_y\colon X \to X\colon x \mapsto y\backslash x$ are injective, and $D_e = 1_X$;*
(iii) *if $x\backslash y = e$, then $x = y$.*

Proof. (i) Given any $z \in X$, we have $M_y(y\backslash z) = y * (y\backslash z) \overset{(1)}{=} z$.

(ii) Let us assume that $y\backslash x = y\backslash x'$; then

$$x \overset{(1)}{=} y * (y\backslash x) = y * (y\backslash x') \overset{(1)}{=} x'.$$

Moreover, $D_e(x) = e\backslash x \overset{(3)}{=} e * (e\backslash x) \overset{(1)}{=} x$.

(iii) If $x\backslash y = e$, then

$$x \overset{(3)}{=} x * e = x * (x\backslash y) \overset{(1)}{=} y.$$ □

Corollary 2.3. *Any finite LPM is a left loop.*

Proof. A surjection from a finite set to itself is automatically injective. □

Proposition 2.4. *Any set X with a chosen element e and a binary operation $\backslash$ such that conditions (ii) and (iii) of Proposition 2.2 hold, admits an LPM structure by defining the $*$ operation via*

$$y * x = \begin{cases} D_y^{-1}(x) & \text{if } x \in \operatorname{Im} D_y \\ y & \text{if } x \notin \operatorname{Im} D_y. \end{cases}$$

Proof. Note that the operation $*$ is well defined by the injectivity of each map D_y.

To prove (1): given any x, $y \in X$, $y\backslash x \in \operatorname{Im} D_y$ with $D_y^{-1}(y\backslash x) = x$. Thus, $y * (y\backslash x) = x$.

To prove (3): $e * x = e * D_e(x) = e * (e\backslash x) \overset{(1)}{=} x$. On the other hand, if $e \in \operatorname{Im} D_x$, then $e = x\backslash a$, for some $a \in X$. By the assumption that condition (iii) of Proposition 2.2 holds, $x = a$. Then, $x * e = x * (x\backslash x) \overset{(1)}{=} x$. If $e \notin \operatorname{Im} D_x$, then $x * e = x$ by definition. □

Let us denote the category of all LPM, and morphisms between them, by LPM. Since LPM form a variety of universal algebras, LPM is a complete and cocomplete category. Moreover, since LPM have a unique constant in their signature, LPM is a pointed category. In fact, it is a unital category since it is a variety containing, among its operations, those of unital magmas. It was shown in [1] that this is enough to prove that LPM is a unital category. LPM is not a protomodular category, as we will see below by proving that not every object in it is protomodular. Let us characterise weakly protomodular objects in LPM.

Theorem 2.5. *Let X be an LPM. The following statements are equivalent:*

(i) *X is a weakly protomodular object in LPM;*
(ii) *for any $x \in X$, there exist $x_1, \ldots, x_n \in X$ such that*

$$x_1\backslash(x_2\backslash(\cdots\backslash(x_n\backslash x)\cdots)) = e. \tag{5}$$

Proof. (ii) ⇒ (i) We consider any split extension

$$0 \longrightarrow K \overset{k}{\rightarrowtriangle} Y \underset{s}{\overset{f}{\rightleftarrows}} X \longrightarrow 0$$

and we take any $y \in Y$. Let $x = f(y)$ and let x_i be the elements whose existence is guaranteed by (ii). Then by identity (1),

$$y = s(x_n) * \big(\cdots * \big(s(x_1) * \big(s(x_1)\backslash(s(x_2)\backslash(\cdots\backslash(s(x_n)\backslash y)\cdots))\big)\big)\cdots\big),$$

where clearly $s(x_1)\backslash(s(x_2)\backslash(\cdots\backslash(s(x_n)\backslash y)\cdots))$ belongs to the kernel K of f, thanks to (5). This proves that any element $y \in Y$ can be written as a product of elements in K and elements of the image of s. Hence the point (f, s) is strong and, consequently, X is a weakly protomodular object.

(i) ⇒ (ii) We choose any $x \in X$ and we consider the split extension

$$0 \longrightarrow K \overset{k}{\rightarrowtriangle} \mathrm{F}(z) + X \underset{\iota_X}{\overset{(\overline{x}\ 1_X)}{\rightleftarrows}} X \longrightarrow 0,$$

where $\mathrm{F}(z)$ denotes the free LPM on one generator z, $\overline{x} \colon \mathrm{F}(z) \to X$ denotes the universal map taking z to x, $\iota_X \colon X \to \mathrm{F}(z) + X$ is the coproduct inclusion, and $(\overline{x}\ 1_X)$ is induced by the universal property of the coproduct.

Since X is weakly protomodular, $z \in \mathrm{F}(z) + X$ can be written as a word w consisting of elements of X and elements of the kernel K. In this combination of elements, the rightmost element has to be a z, since it is impossible to cancel it in the coproduct. Since $z \notin X$, some part on the right of the word w must belong to K. Therefore, we have a word w' ending in z that belongs to K, and coincides with the end of the word w. (w' is a "rightmost part" of the word w.) Now if we substitute all but the rightmost z that appear in the word w' with x and make all possible cancellations, we obtain a word in K of the form

$$x_1 *_1 \big(x_2 *_2 \cdots *_{n-1} (x_n *_n z) \cdots\big) \tag{6}$$

where $x_i \in X$ and $*_i \in \{*, \backslash\}$. The word (6) is a rightmost part of a word in the coproduct $\mathrm{F}(z) + X$ which is equal to z. Since the only possible cancellations in the free product $\mathrm{F}(z) + X$ come from axiom (1), we conclude that all the operations $*_i$ must be $\backslash$. Applying $(\overline{x}\ \ 1_X)$ to such an element gives (5). □

Corollary 2.6. *Any left loop is a weakly protomodular object in* LPM.

Proof. By identity (4) and Theorem 2.5. □

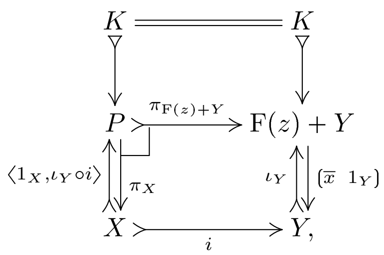
Example 2.7. Let $\mathbb{N}$ be the set of natural numbers with the operations

$$x\backslash y = \begin{cases} y & \text{if } x = 0 \\ y+1 & \text{if } x > 0 \end{cases} \qquad\qquad x * y = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ y-1 & \text{if } x > 0 \text{ and } y > 0. \end{cases}$$

It is easy to see that $(\mathbb{N}, *, \backslash, 0)$ forms an LPM which is not weakly protomodular, since any $x > 0$ does not satisfy condition (ii) in Theorem 2.5. Therefore, the category LPM is not protomodular.

Theorem 2.8. *In the category* LPM*, the subalgebras of a protomodular object are weakly protomodular.*

Proof. Let Y be a protomodular object in LPM and X a subalgebra of Y. Given an element $x \in X$, we show that condition (ii) in Theorem 2.5 holds. Let us consider the pullback diagram



where i is the inclusion, $\mathrm{F}(z)$ is the free LPM on one generator z, $\overline{x}$ denotes the universal map sending z to x, ι_Y is the coproduct inclusion, $(\overline{x}\ \ 1_Y)$ is induced by the universal property of the coproduct, and K is the kernel of the vertical split epimorphisms. Note that the pullback P consists of those words of $F(z) + Y$ such that, when z is replaced by x and the operations are computed, the resulting word belongs to X. Since we are assuming that Y is a protomodular object, the left vertical point is strong. Therefore $z \in P$ can be written as a word w which is a combination of elements from X and from K. Using the same argument as in the proof of Theorem 2.5, we end up with a word of the form

$$x_1\backslash\big(x_2\backslash \cdots \backslash(x_n\backslash z)\cdots\big)$$

which belongs to the kernel K of $(\overline{x}\ \ 1_Y)$. Hence, replacing z with x, we see that x satisfies condition (ii) in Theorem 2.5. □

Proposition 2.9. *Let X be an LPM such that $x\backslash x = e$ for all $x \in X$. Then X is a protomodular object in* LPM.

Proof. Note that X is obviously a weakly protomodular object, by Theorem 2.5. Let us prove that X is, moreover, a protomodular object by showing that any point (f, s) over X is stably strong. Consider the following pullback of (f, s) along an arbitrary morphism g:

$$\begin{array}{ccc} K & = & K \\ \downarrow & & \downarrow \\ P & \xrightarrow{\pi_Z} & Z \\ {\scriptstyle \langle 1_Y, s\circ g\rangle}\uparrow\downarrow{\scriptstyle \pi_Y} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & X. \end{array}$$

Here K is the kernel of the vertical split epimorphisms. A pair $(y, z) \in Y \times Z$ belongs to P if an only if $g(y) = f(z)$. Then, thanks to our assumption, which implies that

$$f(sg(y)\backslash z) = g(y)\backslash f(z) = e = g(e),$$

we can decompose (y, z) in P as

$$(y, z) = (y, sg(y)) * (e, sg(y)\backslash z),$$

where the first pair belongs to the image of $\langle 1_Y, s \circ g\rangle$ and the second one belongs to the kernel K of π_Y. This proves that the point $(\pi_Y, \langle 1_Y, s \circ g\rangle)$ is strong; thus (f, s) is a stably strong point. □

Corollary 2.10. *Any left loop is a protomodular object in* LPM.

Proof. By (4) and Proposition 2.9. □

Now we can show that there exists a weakly protomodular object in LPM which is not protomodular. To do that, we consider the following example.

Example 2.11. Let $\mathbb{Z}$ be the set of integers with the operations

$$x\backslash y = \begin{cases} y & \text{if } x = 0 \\ y+1 & \text{if } x > 0 \text{ and } y \geqslant 0 \\ y & \text{if } x > 0 > y \\ -2y-1 & \text{if } x < 0 \leqslant y \\ 2y & \text{if } x,\, y < 0 \text{ and } x \neq y \\ 0 & \text{if } x = y < 0 \end{cases} \qquad x * y = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ y-1 & \text{if } x,\, y > 0 \\ y & \text{if } x > 0 > y \\ \frac{-y-1}{2} & \text{if } x < 0,\, y \text{ odd} \\ \frac{y}{2} & \text{if } x < 0 \neq y \text{ even.} \end{cases}$$

It can be routinely checked that $(\mathbb{Z}, *, \backslash, 0)$ forms an LPM. Moreover, it is weakly protomodular, since it satisfies condition (ii) in Theorem 2.5:

$$0 = \begin{cases} x\backslash x & \text{if } x \leqslant 0 \\ (-2x-1)\backslash(-1\backslash x) & \text{if } x > 0. \end{cases}$$

However, the non–weakly protomodular LPM of Example 2.7 is a subalgebra of this LPM. Therefore, by Theorem 2.8 we have found an LPM which is weakly protomodular but not protomodular.

The next example shows that there exists an LPM X satisfying the identity $x\backslash x = e$—thus, by Proposition 2.9, a protomodular object in LPM—that is not a left loop.

Example 2.12. Let $\mathbb{N}$ be the set of natural numbers with the operations

$$x\backslash y = \begin{cases} y & \text{if } x = 0 \\ 0 & \text{if } x = y \\ y+1 & \text{if } x > 0 \text{ and } x \neq y \end{cases} \qquad x * y = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ y-1 & \text{if } x, y > 0. \end{cases}$$

It is easy to check that $(\mathbb{N}, *, \backslash, 0)$ is an LPM. Thanks to Proposition 2.9, it is a protomodular object in LPM: for any element $x \in \mathbb{N}$, we have that $x\backslash x = 0$. Nevertheless, it is not a left loop, since

$$1\backslash(1 * 2) = 1\backslash 1 = 0 \neq 2.$$

This, together with Example 2.11, proves that the classes of left loops, weakly protomodular objects in LPM and protomodular objects in LPM are strictly included in each other:

$$\{\text{left loops}\} \subsetneq \{\text{protomodular objects}\} \subsetneq \{\text{weakly protomodular objects}\}.$$

Acknowledgements

The authors wish to express their gratitude to Manuel Ladra and the University of Santiago de Compostela for their kind hospitality.

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