# On Holland's inequalities for the coefficients of the power series of the harmonic mean

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Abstract: Finbarr Holland showed in the case that $p_1 = \cdots = p_k = 1/k$ that expanding the harmonic mean $(p_1(1-x_1t)^{-1}+p_2(1-x_2t)^{-1}+\cdots+p_k(1-x_kt)^{-1})^{-1}$ into a power series in $t$, of the form $\sum_{l\geq 0} q_l(x_1,...,x_k)t^l$, the coefficient polynomials $q_l = q_l(\underline{p}, \underline{x})$ are nonpositive on the nonnegative orthant $\mathbb{R}^k_{\geq 0}$. We show that even under the more general hypothesis that $\underline{p} = (p_1, ..., p_k)$ is an arbitrary probability vector a stronger conclusion can be drawn: writing, say, $x_i = h_i + \cdots + h_k$, and expanding $q_l$ in variables $h_i$ results in a polynomial $h_1, ..., h_k$ with only negative coefficients. The proof makes use of results in three earlier preprints.



## 0. Introduction: Statement of Main Result and Thread of Reasoning

Let us begin by citing the following theorem

**Theorem 0.** ( [Hol, Proposition 2.2]) *If $x_1, ..., x_k$ are positive real numbers then the power series expansion about $t = 0$ of*

$$A(t) = 1 - \frac{k}{\frac{1}{1-x_1t} + \frac{1}{1-x_2t} + \cdots + \frac{1}{1-x_kt}}$$

*has nonnegative coefficients.*

Since the power series $(1-xt)^{-1} = 1 + x^1t^1 + x^2t^2 + \cdots$ has coefficient of $t^0$ equal to 1 it is easily seen that every power series $(p_1(1-x_1t)^{-1}+p_2(1-x_2t)^{-1}+\cdots+p_k(1-x_kt)^{-1})^{-1}$ with $(p_1, ..., p_k)$ nonnegative real numbers of sum 1 must be of form $1 + \sum_{n\geq 1} q_n(x_1, ..., x_k)t^n$. Holland' s theorem is therefore contained in the following main result that will be proved in the present paper.

**Theorem 1.**(Main Result) *Let $(p_1, ..., p_k)$ be an arbitrary probability vector and consider the expansion*

$$(p_1(1-x_1t)^{-1}+p_2(1-x_2t)^{-1}+\cdots+p_k(1-x_kt)^{-1})^{-1} = 1 + \textstyle\sum_{l\geq 1} q_l(x_1, ..., x_k)t^l.$$

*a. Given any permutation $\pi \in S_k$, writing $x_{\pi i} = h_i + h_{i+1} + \cdots + h_k$ for $i = 1, ..., k$, and expanding $q_l$ in terms of the variables $h_i$ the corresponding polynomial has only negative coefficients.*
*b. In particular the $q_l$ are polynomials nonpositive on $\mathbb{R}^k_{\geq 0}$.*

If a polynomial $p \in \mathbb{R}[x_1, ..., x_k]$ has the property that after doing the replacement $x_{\pi i} = h_i + h_{i+1} + \cdots + h_k$ for $i = 1, ..., k$, and expanding there results a polynomial in the $h_i$ with only positive coefficients we shall say it has the property $\mathsf{pos}$ for $\pi$; if $\mathsf{pos}$ for $\pi$ holds for $-p$ we shall say that $p$ is $-\mathsf{pos}$ for $\pi$. If $p$ is $\mathsf{pos}$ for all $\pi \in S_k$ then it is sure that it is nonnegative on $\mathbb{R}^k_{\geq 0}$.

Example. The polynomial $x_2^6 + x_1^4x_3^2 + x_1^2x_3^4 - 3x_1^2x_2^2x_3^2$ (a variant of the so called Motzkin polynomial) can be seen via the arithmetic geometric mean inequality nonnegative for all real $x_1, x_2, x_3$ but if we do the replacements $x_1 \to h_1+h_2+h_3$, $x_2 \to h_2+h_3$, and $x_3 \to h_3$ we get a polynomial in $h_1, h_2, h_3$ which has some negative coefficients, namely $h_2^6 + 6h_2^5h_3 + h_1^4h_3^2 + 4h_1^3h_2h_3^2 + 3h_1^2h_2^2h_3^2 - 2h_1h_2^3h_3^2 + 13h_2^4h_3^2 + 4h_1^3h_3^3 + 6h_1^2h_2h_3^3 - 6h_1h_2^2h_3^3 + 12h_2^3h_3^3 + 4h_1^2h_3^4 - 4h_1h_2h_3^4 + 4h_2^2h_3^4$. This shows that the polynomial is not $\mathsf{pos}$ for the identity permutation. So being $\mathsf{pos}$ for all permutations ('absolutely pos' ) is strictly stronger than simply nonnegativity.

Because of the symmetries of the harmonic mean, it is not hard to see that it is enough to prove part a of Theorem 1 for the case that $\pi$ is the identity permutation. This will imply that the $q_l$ are absolutely $-\mathsf{pos}$.

Example. The reader may verify on his computer that the coefficients of $t^0, t^1, t^2, t^3$ of

$$(\tfrac{1}{5}(1-x_1t)^{-1} + \tfrac{3}{4}(1-x_2t)^{-1} + \tfrac{1}{20}(1-x_3t)^{-1})^{-1}$$

are as given in the table below: first in the original variables, then in the variables $h$ defined by

$x_1 = h_1 + h_2 + h_3,\ x_2 = h_2 + h_3,\ x_3 = h_3.$

$$
\begin{aligned}
t^0:&\quad 1\\
t^1:&\quad 1/20(-4x_1-15x_2-x_3)\\
&\quad = -\frac{h_1}{5}-\frac{19h_2}{20}-h_3\\
t^2:&\quad 1/400(-64x_1^2+120x_1x_2-75x_2^2+8x_1x_3+30x_2x_3-19x_3^2)\\
&\quad = -\frac{4h_1^2}{25}-\frac{h_1h_2}{50}-\frac{19h_2^2}{400}\\
t^3:&\quad (1/8000)(-1024x_1^3+1680x_1^2x_2-300x_1x_2^2-375x_2^3+112x_1^2x_3-360x_1x_2x_3-75x_2^2x_3+\\
&\quad +148x_1x_3^2+555x_2x_3^2-361x_3^3)\\
&\quad = -\frac{16h_1^3}{125}-\frac{87h_1^2h_2}{500}-\frac{3h_1h_2^2}{2000}-\frac{19h_2^3}{8000}-\frac{4h_1^2h_3}{25}-\frac{h_1h_2h_3}{50}-\frac{19h_2^2h_3}{400}
\end{aligned}
$$

The proof of Theorem 1 rests on the results of the preprints [K1],[K2],[K3]. Keeping track of the reasoning will be easier by presenting here briefly the history of these developments more details of which are given in sections 1,2,3.

In [K1] we used a simple method to prove the positive semidefiniteness of individual multivariate polynomials $p(\underline{x}) \in \mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_k]$ in subsets of $\mathbb{R}^k$ of the form $x_1 \geq x_2 \geq \cdots \geq x_k \geq r$, by introducing new variables $h_i = x_i - x_{i+1}$, $i = 1, ..., k$, $x_{k+1} = r$, expressing the $x_i$ via the $h_i$, and to show that the polynomial $p$ in the $h_i$ has only nonnegative coefficients. We showed how such representations can be obtained systematically using partial derivatives and then be used for families of polynomials. In particular we gave an alternative proof to a theorem by Thomas Laffey [L] saying that for real nonnegative $\alpha_1, \ldots, \alpha_k$ of sum $\leq 1$, the coefficients of the $t^l$, $l \geq 1$, of the geometric mean $(1 - x_1t)^{\alpha_1}(1 - x_2t)^{\alpha_2} \cdots (1 - x_kt)^{\alpha_k}$, when expanded into a power series, are nonpositive for nonnegative $x_i$. This had an impact on the so-called nonnegative inverse eigenvalue problem in Linear Algebra; see also [LLS]. At that time we learned about Holland's theorem for the harmonic mean. But while the coefficients of the powers of $t$ in Laffey's and the author's proof for the geometric mean are easily seen to be absolutely -pos, one cannot discern a similar property in the beautiful proof of Holland of his theorem which is based on a 1928 theorem of Theodor Kaluza [Kal]. So we began in the penultimate section of [K1] with some thoughts trying to strengthen it in the form explained above. In that investigation the development

$$q_l(\underline{p}, \underline{x}) = -s_l(\underline{p}, \underline{x}) + \sum_{l_1+l_2=l} s_{l_1}(\underline{p}, \underline{x})s_{l_2}(\underline{p}, \underline{x}) - \sum_{l_1+l_2+l_3=l} s_{l_1}(\underline{p}, \underline{x})s_{l_2}(\underline{p}, \underline{x})s_{l_3}(\underline{p}, \underline{x}) + \cdots + (-1)^l s_1(\underline{p}, \underline{x})^l,$$

via the pseudo-symmetric power sums $s_j = s_j(\underline{p}, \underline{x}) = \sum_{i=1}^k p_i x_i^j$ occurs in a natural manner. The $l_i$ are integers $\geq 1$. One of the subproblems we had was to find a 'nice formula' for the result $R$ of the computation $s_{l_1}s_{l_2}\cdots s_{l_k} \xrightarrow{\text{to } i_1,\partial_{i_1},\text{to } i_2,\, \partial_{i_2},\cdots,\text{to } i_l,\, \partial_{i_l}} R$, where the 'to $i, \partial_i$' are certain linear operators the details of which are explained in Section 2. The solution to this was a formula found in [K2] in which the expressions $l_1l_2\cdots l_k \sum_{\sigma\in S_k}\prod_{i=1}^k (l_{\sigma 1}+\cdots+l_{\sigma i}-\nu_i)^{\underline{l_{\sigma i}-1}}$ with $S_k$ the symmetric group on $\{1, 2, ..., k\}$ played an important role. $R$ is a polynomial and the expression is the coefficient of a certain monomial determined by the $\nu_i$. At the other hand computational experiences led, independently, to the (by us so called) reduced polynomials $q_l^{\text{red}}$, through the same sequence of operations but applied to the $q_l$ (so $q_l \xrightarrow{\text{to } i_1,\partial_{i_1},\text{to } i_2,\, \partial_{i_2},\cdots,\text{to } i_s,\, \partial_{i_s}} -a_0q_{l-1}^{\text{red}}$ ). The sequence of the $q_{l-1}^{\text{red}}$ are inhomogeneous affine polynomials in $l-1$ variables $a_1, a_2, ..., a_{l-1}$ of degree $l-1$ of which we established that if we could prove they are $\geq 0$ on the region $a_1 \leq a_2 \leq \cdots \leq a_{l-1} \leq 1$, then we could infer Holland' s inequality in the refined form of Theorem 1a above. Already in [K1] we found hints and an easy method that apparently worked for each individual polynomial $q_{l-1}^{\text{red}}$ to establish this. But to establish this for the totality of all polynomials $q_{l-1}^{\text{red}}$ we found it necessary first to find an efficient way to describe the $q_{l-1}^{\text{red}}$ in the hope to be able to do this inductively. After much pondering we established in [K3] a conjecture involving a recursion for the $q_{l-1}^{\text{red}}$ and showed that *if* the conjectured recursion was correct then indeed the $q_{l-1}^{\text{red}}$ on the region $a_1 \leq a_2 \leq \cdots \leq a_{l-1} \leq 1$

were nonnegative. This left us to prove that indeed the reduction process of the $q_l$ always leads to the $q_{l-1}^{\text{red}}$ as conjectured by the recursion. It is this step that is achieved in the present paper permitting finally to conclude the investigations on the refinement of Holland' s inequality.

As to the organization of the paper, we present the preliminaries more or less in the order they were found so that Sections 1, 2, 3 correspond roughly to the preprints [K1, K2, K3], respectively and Section 4 presents the last piece of the puzzle. We shall explicitly cite the theorems needed almost in the forms they are found and proved in the preprints and illustrate them with examples. References like 'Lemma n.i' refer to Section $n$, provided made outside of that section.

## 1. The origin of the expansion of the $q_l$ in terms of the $s_l$.

Along this paper we use $x_1, ..., x_k$ as well for indeterminates as for real numbers in such a way that context will make clear the intended meaning. We sometimes write $\underline{x} = (x_1, ..., x_k)$, and for $1 \leq i \leq j \leq k$ we may write $x_{i:j} := (x_i, ..., x_j)$. Also the notation $Sx_{i:j} = x_i + \cdots + x_j$ will help to lighten notation. Similar observations go for letters other than $x$. We often will rewrite any polynomial $p \in \mathbb{R}[x_1, ..., x_k]$ as a polynomial in the quantities $h_i = x_i - x_{i+1}$, $i = 1, ..., k-1$, $h_k = x_k$. To do so note $x_i = h_i + h_{i+1} + \cdots + h_k$ and expand $p$. So, defining $\sigma(\underline{h}) := \sigma(h_1, ..., h_k) := (Sh_{1:k}, Sh_{2:k}, ..., Sh_{k-1:k}, h_k)$, the expansion $(p \circ \sigma)(\underline{h}) = p(\sigma(\underline{h}))$ is the sought presentation.

For nonnegative reals $p_1, ..., p_k$ of sum equal to 1, denote by $s_l = s_l(\underline{p}, \underline{x}) = \sum_{i=1}^k p_i x_i^l$ the $\underline{p}$-weighted $l$-th powersum of $x_1, x_2, ..., x_k$. For the harmonic mean as introduced in Section 0 we may write

$$(p_1(1-x_1t)^{-1} + p_2(1-x_2t)^{-1} + \cdots + p_k(1-x_kt)^{-1})^{-1} = \left(\sum_{i=1}^k p_i \sum_{l\geq 0}(x_it)^l\right)^{-1}$$

$$= \left(1 + \sum_{i=1}^k p_i \sum_{l\geq 1} x_i^l t^l\right)^{-1} = \left(1 + \sum_{l\geq 1}(\sum_{i=1}^k p_i x_i^l)t^l\right)^{-1} = \left(1 + \sum_{l\geq 1} s_l(p, \underline{x})t^l\right)^{-1}$$

$$= 1 - \sum_{l\geq 1} s_l(\underline{p}, \underline{x})t^l + (\sum_{l\geq 1} s_l(\underline{p}, \underline{x})t^l)^2 - (\sum_{l\geq 1} s_l(\underline{p}, \underline{x})t^l)^3 + \cdots .$$

From this follows that the coefficient of $t^l$, $l \geq 1$ is

$$q_l(\underline{p}, \underline{x}) = -s_l(\underline{p}, \underline{x}) + \sum_{l_1+l_2=l} s_{l_1}(\underline{p}, \underline{x})s_{l_2}(\underline{p}, \underline{x}) - \sum_{l_1+l_2+l_3=l} s_{l_1}(\underline{p}, \underline{x})s_{l_2}(\underline{p}, \underline{x})s_{l_3}(\underline{p}, \underline{x}) + \cdots + (-1)^l s_1(\underline{p}, \underline{x})^l,$$

with $l_i \in \mathbb{Z}_{\geq 1}$. It is clear that $q_l$ is a homogeneous polynomial of degree $l$ in $x_1, ..., x_k$. Our aim will be accomplished as soon as we can show that $q_l(\underline{p}, \sigma(\underline{h}))$ is a polynomial in the $h_i$ with only negative coefficients.

Example. For $l = 3$ we have

$$q_3(\underline{p}, \underline{x}) = -s_3(\underline{p}, \underline{x}) + 2s_1(\underline{p}, \underline{x})s_2(\underline{p}, \underline{x}) - s_1(\underline{p}, \underline{x})^3.$$

Assuming $\underline{x} = (x_1, x_2, ..., x_6)$ and $\underline{p} = \frac{1}{6}\mathbb{1} = \frac{1}{6}(1, 1, 1, 1, 1, 1)$ results in the more explicit form

$$q_3(\frac{1}{6}\mathbb{1}, x_{1:6}) = (-25\sum_i x_i^3 + 9\sum_{i\neq j} x_i^2 x_j - 6\sum_{i<j<k} x_i x_j x_k)/216.$$

and finally

$q_3(\frac{1}{6}\mathbb{1}, \sigma(\underline{h})) = 1/216(-25h_1^3 - 66h_1^2h_2 - 48h_1h_2^2 - 32h_2^3 - 57h_1^2h_3 - 84h_1h_2h_3 - 84h_2^2h_3 - 27h_1h_3^2 - 54h_2h_3^2 - 27h_3^3 - 48h_1^2h_4 - 72h_1h_2h_4 - 72h_2^2h_4 - 48h_1h_3h_4 - 96h_2h_3h_4 - 72h_3^2h_4 - 12h_1h_4^2 - 24h_2h_4^2 - 36h_3h_4^2 - 16h_4^3 - 39h_1^2h_5 - 60h_1h_2h_5 - 60h_2^2h_5 - 42h_1h_3h_5 - 84h_2h_3h_5 - 63h_3^2h_5 - 24h_1h_4h_5 - 48h_2h_4h_5 - 72h_3h_4h_5 - 48h_4^2h_5 - 3h_1h_5^2 - 6h_2h_5^2 - 9h_3h_5^2 - 12h_4h_5^2 - 5h_5^3 - 30h_1^2h_6 - 48h_1h_2h_6 - 48h_2^2h_6 - 36h_1h_3h_6 - 72h_2h_3h_6 - 54h_3^2h_6 - 24h_1h_4h_6 - 48h_2h_4h_6 - 72h_3h_4h_6 - 48h_4^2h_6 - 12h_1h_5h_6 - 24h_2h_5h_6 - 36h_3h_5h_6 - 48h_4h_5h_6 - 30h_5^2h_6).$

The following proposition is essentially [K1, Theorem 2.1,Corollary 2.2] (by means of which we there reproved Laffey's result) but was recast in a different language in [K2]. There we introduced the arrows $\xrightarrow{\text{to } i}$ and $\xrightarrow{\partial_i}$. These serve to indicate certain operations on polynomials. Write '$\xrightarrow{\text{to } i}$' for saying that the currently existing variables of index $\le i$ should be mapped to $x_i$. For example $-3x_1 + x_1x_2^2 + x_3 \xrightarrow{\text{to } 2} -3x_2 + x_2^3 + x_3$. Similarly write $\xrightarrow{\partial_i}$ to indicate a partial derivative w.r.t. variable $x_i$. We can concatenate such arrows in the obvious way and have proved the following.

**Proposition 1.** ([K2, Corollary 1.1]) a. *Let $p$ be homogeneous of degree $l$ in $\mathbb{R}[x_{1:k}]$ and assume $l_1 + \cdots + l_k = l$. Then the coefficient of $h_1^{l_1} \cdots h_k^{l_k}$ in the development of $(p \circ \sigma)(\underline{h})$ is obtained by applying $l_1$ operators $\xrightarrow{\text{to } 1, \partial_1}$, followed by $l_2$ operators $\xrightarrow{\text{to } 2, \partial_2}$, ... followed by $l_k$ operators $\xrightarrow{\text{to } k, \partial_k}$, and dividing the result by $l_1! \cdots l_k!$.*
b. *In particular, if all such operations yield a nonnegative real number then the polynomial $p$ is nonnegative on the region $x_1 \ge x_2 \ge \cdots \ge x_k \ge 0$; and if $p$ is symmetric then $p$ is nonnegative on the nonnegative orthant $\mathbb{R}^n_{\ge 0}$.*

Example. Consider the (symmetric) polynomial $p(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$. Then

$$p(x_1, x_2, x_3) \xrightarrow{\text{to } 1, \partial_1} 3x_1^2 - 3x_2x_3 \xrightarrow{\text{to } 1, \partial_1} 6x_1 \xrightarrow{\text{to } 3, \partial_3} 6.$$

The coefficient of $h_1^2h_3$ in $(p \circ \sigma)(\underline{h})$ therefore is $6/(2!0!1!) = 3$. Indeed one easily computes that

$$(p \circ \sigma)(\underline{h}) = h_1^3 + 3h_1^2h_2 + 3h_1h_2^2 + 2h_2^3 + 3h_1^2h_3 + 3h_1h_2h_3 + 3h_2^2h_3.$$

And this turns nonnegativity of $p|\mathbb{R}^3_{\ge 0}$ evident in a most satisfactory way.

## 2. Reducing products of pseudo-symmetric power sums

Given that the polynomials $q_l(\underline{p}, \underline{x})$ born from the harmonic mean are linear combinations of products of polynomials $s_l(\underline{p}, \underline{x})$, we tried in [K2] also to find the result of applying a sequence of operators of the form mentioned to such products. This somewhat subtle investigation resulted in the following theorem. In it we use with [GKP] the notation $m^{\underline{k}} = m(m-1)\cdots(m-k+1)$ for falling factorials, where $m \in \mathbb{Z}, k \in \mathbb{Z}_{\ge 0}$.

**Theorem 1.** ([K2, Theorem 3.2 and Corollary 3.4]) *Let $l = l_1 + l_2 + \cdots + l_k$ (so it is the degree of the product of the powersums below) and assume $1 \le i_1 < i_2 < \cdots < i_l \le \#$of variables. Then the result $R$ of the computation*

$$s_{l_1}s_{l_2}\cdots s_{l_k} \xrightarrow{\text{to } i_1, \partial_{i_1}, \text{to } i_2, \partial_{i_2}, \cdots, \text{to } i_l, \partial_{i_l}} R$$

*is a homogeneous polynomial of degree $k$ in the variables $Sp_{1:i_1}, ..., Sp_{1:i_l}$. Each monomial is of form $Sp_{1:i_1}Sp_{1:i_{\nu_2}} \cdots Sp_{1:i_{\nu_k}}$ with $1 = \nu_1 < \nu_2 < \cdots < \nu_k \le l$. The coefficient of this particular monomial equals the positive integer*

$$l_1l_2\cdots l_k \sum_{\sigma \in S_k} \prod_{i=1}^{k} (l_{\sigma 1} + \cdots + l_{\sigma i} - \nu_i)^{\underline{l_{\sigma i} - 1}}.$$

It will be useful to note that the coefficient does not depend on $i_1, i_2, ..., i_l$. So all the information that can be gotten from the theorem can be gotten by selecting $i_1 = 1, i_2 = 2, ..., i_l = l$.

Example. $s_1s_2$ is a degree 3 polynomial. Let us compute in above sense $s_1s_2 \xrightarrow{\text{to } i_1, \partial_{i_1}, \text{to } i_2, \partial_{i_2}, \text{to } i_3, \partial_{i_3}} R$

$$\begin{aligned}
s_1(\underline{p}, \underline{x})s_2(\underline{p}, \underline{x}) &\xrightarrow{\text{to } i_1} (Sp_{1:i_1}x_{i_1} + s_1(p_{i_1+1:k}, x_{i_1+1:k}))(Sp_{1:i_1}x_{i_1}^2 + s_2(p_{i_1+1:k}, x_{i_1+1:k}))\\
&\xrightarrow{\partial_{i_1}} Sp_{1:i_1}(Sp_{1:i_1}x_{i_1}^2 + s_2(p_{i_1+1:k}, x_{i_1+1:k})) + (Sp_{1:i_1}x_{i_1} + s_1(p_{i_1+1:k}, x_{i_1+1:k}))2Sp_{1:i_1}x_{i_1}\\
&\xrightarrow{\text{to } i_2} Sp_{1:i_1}(Sp_{1:i_2}x_{i_2}^2 + s_2(p_{i_2+1:k}, x_{i_2+1:k})) + (Sp_{1:i_2}x_{i_2} + s_1(p_{i_2+1:k}, x_{i_2+1:k}))2Sp_{1:i_1}x_{i_2}\\
&\xrightarrow{\partial_{i_2}} 2Sp_{1:i_1}Sp_{1:i_2}x_{i_2} + Sp_{1:i_2} \cdot 2Sp_{1:i_1}x_{i_2} + (Sp_{1:i_2}x_{i_2} + s_1(p_{i_2+1:k}, x_{i_2+1:k}))2Sp_{1:i_1}\\
&\xrightarrow{\text{to } i_3} 2Sp_{1:i_1}Sp_{1:i_2}x_{i_3} + Sp_{1:i_2} \cdot 2Sp_{1:i_1}x_{i_3} + (Sp_{1:i_3}x_{i_3} + s_1(p_{i_3+1:k}, x_{i_3+1:k}))2Sp_{1:i_1}\\
&\xrightarrow{\partial_{i_3}} 4Sp_{1:i_1}Sp_{1:i_2} + 2Sp_{1:i_1}Sp_{1:i_3}.
\end{aligned}$$

So we showed $s_1 s_2 \xrightarrow{\text{to } i_1\partial_{i_1}, \text{to } i_2\partial_{i_2}, \text{to } i_3\partial_{i_3}} 4Sp_{1:i_1}Sp_{1:i_2}+2Sp_{1:i_1}Sp_{1:i_3}$. In the notation of the theorem $l_1 = 1, l_2 = 2, k = 2, \nu_1 = 1$ and the coefficient of $Sp_{1:i_{\nu_1}}Sp_{1:i_{\nu_2}}$ is $l_1 l_2((l_1-\nu_1)^{\underline{l_1-1}}(l_1+l_2-\nu_2)^{\underline{l_2-1}}+(l_2-\nu_1)^{\underline{l_2-1}}(l_2+l_1-\nu_2)^{\underline{l_1-1}}) = 1\cdot 2((1-1)^{\underline{0}}(3-\nu_2)^{\underline{1}}+(2-1)^{\underline{1}}(3-\nu_2)^{\underline{0}}) = 2((3-\nu_2)^{\underline{1}}+1)$. From this formula one obtains according to the cases $\nu_2 = 2$ and $\nu_2 = 3$ indeed the values 4 and 2 for the coefficients as our detailed calculation above shows.

A direct consequence of the Theorem 1 and the definition of the polynomials $q_l(\underline{p}, \underline{x})$ is the following:

**Corollary 2.** *If we subject $q_l$ to the operation $\xrightarrow{\text{to } i_1, \partial_{i_1}, \text{to } i_2, \partial_{i_2}, \cdots, \text{to } i_l, \partial_{i_l}}$ we get an inhomogeneous polynomial of degree $l$ in the variables $Sp_{1:i_1}, ..., Sp_{1:i_l}$. The homogeneous component of degree $k$ has the sign $(-1)^k$ ; and each degree $k = 1, 2, ..., l$ is present.*

Example (continued). Applying the same arrow as before we get $s_3 \longrightarrow 6Sp_{1:i_1}$ and $s_1^3 \longrightarrow 6Sp_{1:i_1}Sp_{1:i_2}Sp_{1:i_3}$. Consequently $q_3 \longrightarrow -6Sp_{1:i_1}+8Sp_{1:i_1}Sp_{1:i_2}+4Sp_{1:i_1}Sp_{1:i_3}-6Sp_{1:i_1}Sp_{1:i_2}Sp_{1:i_3}$.

## 3. The sequence of reductions of polynomials $q_l$ and the property pos.

To get rid of heavy notation and trivial transformations we define the *reduction* of $q_l(\underline{p}, \underline{x})$ as the result of the following operations:

- · Determine $R$ from $q_l(\underline{p}, \underline{x}) \xrightarrow{\text{to } i_1, \partial_{i_1}, \text{to } i_2, \partial_{i_2}, \cdots, \text{to } i_l, \partial_{i_l}} R$.
- · Cancel herein $Sp_{1:i_1}$ and change the sign.
- · Replace $Sp_{1:i_\nu}$ by the letter $a_{\nu-1}$.

This is an inhomogeneous polynomial of degree $l-1$ in $l-1$ variables $a_1, a_2, ..., a_{l-1}$. Call it $q_{l-1}^{\text{red}}$. With this definition we have

$$q_l(\underline{p}, \underline{x}) \xrightarrow{\text{to } i_1, \partial_{i_1}, \text{to } i_2, \partial_{i_2}, \cdots, \text{to } i_l, \partial_{i_l}} -a_0 q_{l-1}^{\text{red}}.$$

With the two intentions to confirm at the one hand our conjecture that in general $q_l(\underline{p}, \sigma(h))$ has only negative coefficient coefficients and at the other hand to find a pattern according to which the $q_l^{\text{red}}$ develops with $l$, we computed via Mathematica© a number of further reductions. The first few are the following ones.

$$\begin{aligned}
q_0^{\text{red}} &= 1\\
q_1^{\text{red}} &= 2 - 2a_1\\
q_2^{\text{red}} &= 6 - 8a_1 - 4a_2 + 6a_1a_2\\
q_3^{\text{red}} &= 24 - 40a_1 - 20a_2 + 36a_1a_2 - 12a_3 + 24a_1a_3 + 12a_2a_3 - 24a_1a_2a_3\\
q_4^{\text{red}} &= 120 - 240a_1 - 120a_2 + 252a_1a_2 - 72a_3 + 168a_1a_3 + 84a_2a_3 - 192a_1a_2a_3 - 48a_4 + 120a_1a_4\\
&\quad +60a_2a_4 - 144a_1a_2a_4 + 36a_3a_4 - 96a_1a_3a_4 - 48a_2a_3a_4 + 120a_1a_2a_3a_4.
\end{aligned}$$

Since $p = (p_1, ..., p_k)$ is a probability vector we know furthermore $0 \le Sp_{1:i_1} \le Sp_{1:i_2} \le \cdots \le Sp_{1:i_k} \le 1$; in other words we wish to prove that the polynomial $q_l^{\text{red}}$ is for $a_1 \le a_2 \le \cdots \le a_l \le 1$ nonnegative; it turns out this is even true if some $a_i$ are negative. Indeed to show this for some individual of these polynomials we employed once more the technique above. For example to show $q_3^{\text{red}}|\Delta_3 \ge 0$, where $\Delta_l = \{(a_1, a_2, \ldots, a_l) : a_1 \le a_2 \le \ldots a_l \le 1\}$, we write $a_3 = 1 - h_1, a_2 = 1 - h_1 - h_2, a_1 = 1 - h_1 - h_2 - h_3$, assuming the $h_i \ge 0$ and substitute these expressions in the $h$s for the $a_i$ in $q_3^{\text{red}}$ and expand. The result is $24h_1^3 + 12h_1h_2 + 48h_1^2h_2 + 12h_2^2 + 24h_1h_2^2 + 4h_3 + 12h_1h_3 + 24h_1^2h_3 + 12h_2h_3 + 24h_1h_2h_3$ proving the pretended inequality. This method worked for every individual $q_l^{\text{red}}$ that we tried. However, how to show for all $l$ that $q_l^{\text{red}}|\Delta_l \ge 0$? We thought the best path forward would be some type of inductive proof; and for this, after a prolonged search, we conjectured in [K3] the recursion RC below and showed there that the conjectured $q_l^{\text{red}}$ are indeed nonnegative on $\Delta_l$. Given any polynomial $q$ in various variables written in standard form as a linear combination of monomials, define $T(q) = \{\text{terms of polynomial } q\}$. For example $T(q_2^{\text{red}}) = \{6, -8a_1, -4a_2, 6a_1a_2\}$. For $t \in T(q)$, let $\deg t :=$ degree of $t$; for example $\deg(6a_1a_2) = 2$.

**RC.** The $q_l^{\text{red}}$ can be computed by the following recursion.

$$\begin{aligned} q_0^{\text{red}} &= 1. \\ q_{n+1}^{\text{red}} &= \sum_{t\in T(q_n^{\text{red}})} (2+n+\deg t)\cdot t - \sum_{t\in T(q_n^{\text{red}})} (2+\deg t)\cdot t\cdot a_{n+1} = \sum_{t\in T(q_n^{\text{red}})} ((1-a_{n+1})(2+\deg t)+n)t \end{aligned}$$

The reader is invited to check the recursion RC against the polynomials above. We shall prove in Section 4 that the reduction process delivers in fact the inductively defined sequence of polynomials.

If $q = q(a_1, a_2, ..., a_n)$ is any real polynomial in $n$ variables $a_1, ..., a_n$ which we wish to prove is nonnegative on $\Delta_n$, then as exemplified above, there is a good chance to show this by expressing it in certain other variables; namely we introduce $h_1 = 1 - a_n, h_2 = a_n - a_{n-1}, \cdots , h_n = a_2 - a_1$ from which it follows that $a_j = 1 - h_1 - h_2 - \cdots - h_{n-j+1}$. Then any monomial (i.e. any product of variables) of $q$ is a certain product of some of the factors in $\prod_{i=0}^{n}(1 - h_1 - h_2 - \cdots - h_i)$. By substituting such products for the monomials and expanding we get $q$ as a polynomial in the $h_i$. We shall call this polynomial the *h-form* of $q$ while the original form in which $q$ is written is its *a-form.* It is evident that the coefficient of a monomial $h^{\mathbf{j}} = h^{(j_1,...,j_n)} := h_1^{j_1} h_2^{j_2} \cdots h_n^{j_n}$ in the $h$-form of $q$ is a linear combination of the coefficients of $q$ in the $a$-form. Evidently $(a_1, a_2, ..., a_n) \in \Delta_n$ if and only if $h_1, h_2, ..., h_n$ are all nonnegative. It follows that the nonnegativity of the referred linear combinations of the coefficients of $q$ that occur writing $q$ in $h$-form is a sufficient condition for having $q|\Delta_n \geq 0$.

Note that the polynomials $q_n^{\text{red}}$ of the conjecture are all multilinear (or affine). Progress only came when instead of looking for positivity of our particular polynomials $q_n^{\text{red}}$ we introduced generic coefficients $c_I$ and looked at how these relate in the $h$-forms of $q_n^{\text{red}}$ and $q_{n+1}^{\text{red}}$.

The following lemma and theorem expresses the relation between such coefficients slightly more generally than we shall need. Let $n \in \mathbb{Z}_{\geq 0}$ and $a, b \in \mathbb{R}$, and consider the affine polynomials in $a$-forms

$$q = \sum_{I\subseteq[n]} c_I \prod_{i\in I} a_i \quad \text{and} \quad \tilde{q} = \sum_{I\subseteq[n+1]} \tilde{c}_I \prod_{i\in I} a_i$$

with $\tilde{q}$ defined via $q$ by

$$\tilde{q} = \sum_{t\in T(q)} (a+\deg t)\cdot t - \sum_{t\in T(q)} (b+\deg t)\cdot t\cdot a_{n+1} = \sum_{t\in T(q)} ((1-a_{n+1})(b+\deg t)+a-b)t.$$

**Lemma 1.** [K3, Lemma 4.1] *The coefficients $\tilde{c}_I$ can be computed from the $c_I$ according to the rule*

$$\tilde{c}_I = \begin{cases} (a+|I|)c_I & \text{if } n+1 \notin I \\ -(b-1+|I|)c_{I\setminus\{n+1\}} & \text{if } n+1 \in I. \end{cases}$$

This was one of the pieces - we will use it later again - which allowed us to prove

**Theorem 2.** [K3, Theorem 1.1] *Provided $a - b - n \geq 0$ and $b \geq 0$, then the coefficients of the h-form of $\tilde{q}$ are nonnegative linear combinations of at most two of the coefficients of the h-form of q. Consequently, if the h-form of q has only nonnegative coefficients, then the h-form of $\tilde{q}$ has only nonnegative coefficients, and hence $\tilde{q}|_{\Delta_{n+1}} \geq 0$.*

Example. Assume $a = 4, b = 2$. If $n = 2$ then the $a$-form and the $h$-form of $q$ are

$$\begin{aligned} q &= c_\emptyset + c_1a_1 + c_2a_2 + c_{12}a_1a_2 \\ &= (c_\emptyset + c_1 + c_2 + c_{12}) + (-c_1 - c_2 - 2c_{12})h_1 + c_{12}h_1^2 + (-c_1 - c_{12})h_2 + c_{12}h_1h_2. \end{aligned}$$

(Here and in other examples we often write strings of numbers instead of sets; e.g. $c_{12}$ instead of $c_{\{1,2\}}$.) The $a$-form and the $h$-form of $\tilde{q}$ are

$$\begin{aligned} \tilde{q} &= 4c_\emptyset + 5c_1a_1 + 5c_2a_2 + 6c_{12}a_1a_2 - 2c_\emptyset a_3 - 3c_1a_1a_3 - 3c_2a_2a_3 - 4c_{12}a_1a_2a_3 \\ &= (2c_\emptyset + 2c_1 + 2c_{12} + 2c_2) + (2c_\emptyset + c_1 + c_2)h_1 + (-3c_1 - 6c_{12} - 3c_2)h_1^2 + 4c_{12}h_1^3 \\ &\quad +(-2c_1 - 4c_{12} - 2c_2)h_2 + (-3c_1 - 4c_{12} - 3c_2)h_1h_2 + (8c_{12})h_1^2h_2 + (2c_{12})h_2^2 + (4c_{12})h_1h_2^2 \\ &\quad +(-2c_1 - 2c_{12})h_3 + (-3c_1 - 2c_{12})h_1h_3 + 4c_{12}h_1^2h_3 + 2c_{12}h_2h_3 + 4c_{12}h_1h_2h_3. \end{aligned}$$

As an example the coefficient of $h^{(1,1,0)} = h_1h_2$ of $\tilde{q}$ in $h$-form is indeed a nonnegative linear

combinations of two of the coefficients of the $h$-form of $q$:
(coefficient of $h_1h_2$ in $\tilde{q}$) $= -3c_1 - 3c_2 - 4c_{12} = 3\cdot$(coefficient of $h_1$ in $q$) $+ 2\cdot$(coefficient of $h_1^2$ in $q$)

To see that the theorem above indeed implies for the above conjectured sequence $q_n^{\text{red}}$ that $q_n^{\text{red}}|\Delta_n \geq 0$, choose $b = 2$ and make $a$ dependent on $n$, putting $a = 2+n$. Then beginning with $q = q_0^{\text{red}} = 1 = c_\emptyset$, the theorem applied with $n = 0$ yields as $\tilde{q}$ the polynomial $q_1^{\text{red}}$; now applying it with $q = q_1^{\text{red}}$ and $n = 1$, we get $\tilde{q} = q_2^{\text{red}}$, etc. The claim follows as $q_0^{\text{red}} \geq 0$.

## 4. The polynomial sequence $q_l^{\text{red}}$, $l = 0, 1, 2, \ldots$ satisfies the recursion RC

We finally make the last step in the proof of the main result. We establish that the polynomial sequence $q_{l-1}^{\text{red}}$ obtained by the reduction process applied to each element of the sequence $q_l$, $l = 1, 2, \ldots$ indeed satisfies the recursion RC defined in the section before.

Again we need some preparation. For two sets $I, J$ of integers let us agree to write $I < J$ if $\forall i \forall j\ (i \in I \,\&\, j \in J) \Rightarrow i < j$. It follows in particular that $\emptyset < J$ as well as $J < \emptyset$. Also for $I \subseteq \mathbb{Z}$ and $a \in \mathbb{Z}$ we define $a + I = \{a + i : i \in I\}$. In particular then $a + \emptyset = \emptyset$. Let now $[a, b] = \{a, a+1, \ldots, b\}$ be an interval of integers and let $I \subseteq [a, b]$. A *chain of $I$ in $[a, b]$* is a family of subsets $E_i, I_i$ of $[a, b]$ such that $E_0 < I_1 < E_1 < \ldots < I_k < E_k$, the $I$s are nonempty sets whose union is $I$, and the union of the $E$s is the complement $I^c$ of $I$ in $[a, b]$. Clearly the $I$s and $E$s are intervals and some $E$s may be empty; if nonempty $E_1, \ldots, E_{k-1}$ are required then the chain is unique.

As mentioned at the end of Section 3, the specialization $a = 2 + n$, $b = 2$ in Lemma 3.1 gives us the particular polynomial $q_{n+1}^{\text{red}} = \sum_{I \subseteq [n+1]} \tilde{c}_I \prod_{i \in I} a_i$ from $q_n^{\text{red}}$. We obtain

$$\tilde{c}_I = \begin{cases} (2 + n + |I|)c_I & \text{if } n+1 \notin I \\ -(1 + |I|)c_{I \setminus \{n+1\}} & \text{if } n+1 \in I. \end{cases}$$

This gives a recursive possibility to compute coefficient $c_I$ of $q_n^{\text{red}}$.

**Lemma 1.** *Assume $I \subseteq [n]$ has in $[n]$ the chain $E_0 < I_1 < E_1 < \cdots E_{k-1} < I_k < E_k$. Define the sets*

$$A = \bigcup_{l=1}^{k+1} (1 + \sum_{\nu=1}^{l-1} |I_\nu| + E_{l-1}) \quad \textit{and} \quad B = \{-2, -3, \ldots, -(1 + |I|)\}.$$

*Then the set $A$ does not depend on the chain selected for $I$ and the coefficient $c_I$ of $q_n^{\text{red}}$ equals the product of the elements in $A \cup B$; so*

$$c_I = (-1)^{|I|}(1 + |I|)! \cdot \prod_{a \in A} a.$$

Proof. We first show that the pseudocode below at the left produces the sets $A$ and $B$.

```
A = ∅, B = ∅
for j = n step −1 to 1 do
 if j ∉ I put 1 + j + |I| into A.
 if j ∈ I put −(1 + |I|) into B
    and redefine I = I \ {j}
end
```

We have the partition $[n] = E_0 \uplus I_1 \uplus \cdots \uplus E_{k-1} \uplus I_k \uplus E_k$. $I$ will be in the analysis below a dynamic variable. So we write $I^o$ to refer to the original set $I$. As we enter the code, $j$ will first run decreasingly and beginning in $n$, through the elements of $E_k$ and since $j \notin I$ concomitantly produce the elements $1 + |I| + e$ for $e \in E_k$ and put them into $A$. We may write this as $1 + |I| + E_k$. Next $j$ enters $I_k$ while the set $I$ is still unaltered.

$j$ will now assume the values of the elements in $I_k$ and the $|I_k|$ numbers $-(1 + |I^o|), -(1 + |I^o| - 1), \ldots, -(1 + |I^o| - (|I_k| - 1)) = -(2 + |I^o| - |I_k|)$ are put into $B$. Next $j$ leaves $I_k$ and enters $E_{k-1}$. At that point $I$ has cardinality $|I^o| - |I_k|$. The set $1 + |I^o| - |I_k| + E_{k-1}$ is produced and put into $A$ while $I$ is not altered. $j$ next enters $I_{k-1}$ with the current cardinality of $I$ being $|I^o| - |I_k|$ and hence the $|I_{k-1}|$ numbers $-(1+|I^o|-|I_k|), -(1+|I^o|-|I_k|-1), \ldots, -(1+|I^o|-|I_k|-(|I_{k-1}|-1)) = -(2 + |I^o| - |I_k| - |I_{k-1}|)$ are put into $B$. The scheme that leads to the lemma should now be clear in its essentials. We still have to analyse the final steps of the algorithm. Towards the end $j$ ranges still through the sets $I_1 \subseteq I$ and then $E_0$. As $j$ enters $I_1$ the set $I$ has cardinality

$|I^o| - |I_k| - ... - |I_2|$ and so $-(1 + |I^o| - |I_k| - ... - |I_2|)$ as well as the negatives of the numbers produced from the next $|I_1| - 1$ numbers smaller than $|I^o| - |I_k| - ... - |I_2|$ are put into $B$. So the smallest number in $(\dots)$ is $1 + |I^o| - |I_k| - ... - |I_2| - (|I_1| - 1) = 2$, because the cardinality of the original $I$, i.e. $I^o$ is precisely $|I_1| + \cdots + |I_k|$. This gives the claim concerning $B$. Finally the last time $j$ enters the complement of $I$ is when it enters $E_0$. We see that then the number set $(1 + |I^o| - |I_k| - \cdots - |I_1| + E_0) = (1 + E_0)$ is produced and put into $A$. This last set is the one referred in the claim of the lemma as pertaining to $l = 1$ in the union given for $A$. Summarizing we see that $B$ is as claimed and $A = \bigcup_{l=1}^{k+1} (1 + |I^o| - |I_k| - \cdots - |I_l| + E_{l-1})$. This can evidently be written as indicated above. The independence of $A$ from the chain chosen follows by noting that empty sets $E$ contribute with empty sets in the union written for $A$. Finally, after writing the formula before the lemma for $q_n^{\text{red}}$ in place of $q_{n+1}^{\text{red}}$, the remainder of the proof follows by recalling that $c_\emptyset$ in $q_0^{\text{red}}$ is 1, seeing that $A$ and $B$ are obviously disjoint, and noting that the numbers occurring in $A \cup B$ are precisely the numbers used to recursively produce the $c_I$ by successive multiplications. □

Example. a. We take $n = 3$ and $I = \{1, 2\}$. Then $[n] = [3] = E_0 \cup I_1 \cup E_1$ with $E_0 = \emptyset, I_1 = \{1, 2\}, E_1 = \{3\}$. Therefore $k = 1$ and $A = \bigcup_{l=1}^{2} (1 + \sum_{\nu=1}^{l-1} |I_\nu| + E_{l-1}) = (1 + 0 + \emptyset) \cup (1 + |I_1| + E_1) = \emptyset \cup (1 + 2 + \{3\}) = \{6\}$, while $B = \{-2, -3\}$ consequently $A \cup B = \{6, -2, -3\}$ and the product of these elements is 36 as it should be; see the coefficient of $a_1 a_2$ of $q_3^{\text{red}}$.
b. Now assume we split $\{1, 2\}$ and consider the chain $E_0 < \{1\} < E_1 < \{2\} < E_2$ with $E_0 = E_1 = \emptyset$. In this case $k = 2$ and the associated $A = \bigcup_{l=1}^{3} (1 + \sum_{\nu=1}^{l-1} |I_\nu| + E_{l-1}) = (1 + 0 + \emptyset) \cup (1 + |I_1| + E_1) \cup (1 + |I_1| + |I_2| + E_2) = \emptyset \cup \emptyset \cup (3 + \{3\}) = \{6\}$, as before.

We need still a supplement to the lemma before.

**Supplement** (to Lemma 1). *Assume sets $I \subseteq [n]$ and $A$ as in Lemma 1, but now $A$ given by its individual elements, $I = \{i_1, i_2, ..., i_{|I|}\}$. Define additionally $i_0 = 0, i_{1+|I|} = n + 1$. Then we shall have The set $1, n + 1 \in -1 + A \subset [1, n + |I|]$ and its complement in $[1, n + |I|]$, that is $(-1 + A)^c = [1, n + |I|] \setminus (-1 + A)$ can be written, respectively,*

$$-1 + A = \bigcup_{l=1}^{1+|I|} (l - 1 + ]i_{l-1}, i_l[); \quad \text{and} \quad (-1 + A)^c = \bigcup_{\nu=1}^{|I|} \{i_\nu + \nu - 1, i_\nu + \nu\}.$$

Proof. The containment claims are clear because $1 \in 1 - 1 + ]i_0, i_1[$ and $n + |I| \in (1 + |I| - 1 + ]i_{|I|}, i_{1+|I|}[$.

The set $I$ gives rise to the 'atomic' chain $E_0\{i_1\}E_1\{i_2\}E_2\{i_3\}E_3...\{i_{|I|}\}E_{|I|}$ from where we see in earlier notation $I_\nu = \{i_\nu\}$ and $E_l = ]i_l, i_{l+1}[,\ l = 1, ..., 1 + |I|$; and $\sum_{\nu=1}^{l-1} |\{i_\nu\}| = l - 1$. This yields the formula for $-1 + A$ directly from the lemma. What concerns the complement of $-1 + A$, we write $-1 + A = \bigcup_{l=1}^{1+|I|} (l - 1 + E_{l-1})$. Take two integers $l < k$ such that $E_l, E_k$ are nonempty but $E_{l+1}, ..., E_{k-1}$ are empty. Then $l + E_l$ and $k + E_k$ are nonempty subsets of $-1 + A$ and we have a situation $\{i_l\} < E_l < \{i_{l+1}, ..., i_k\} < E_k$, where $\{i_{l+1}, ..., i_k\}$ is actually an interval consisting of $k - l$ consecutive numbers. It can be written $\{i_{l+1}, i_{l+1} + 1, i_{l+1} + 2, ..., i_{l+1} + k - l - 1\}$. So for $l + 1 \leq \nu \leq k$ we have $i_\nu = i_{l+1} + \nu - l - 1$. The interval $[\max(l + E_l) + 1, \min(k + E_k) - 1] = [l + (i_{l+1} - 1) + 1, k + (i_k + 1) - 1] = [i_{l+1} + l, k + i_k]$ comprises the gap between $l + E_l$ and $k + E_k$ and is subset of $(-1 + A)^c$. Using the relation between the $i_\nu$ and $i_{l+1}$ we see this gap can be written $\{i_{l+1} + l, i_{l+1} + l + 1, i_{l+2} + l + 1, i_{l+2} + l + 2, , ..., i_k + k - 1, i_k + k\}$. Using this type of analysis for any two successive nonempty sets $E_l, E_k$, the claim follows. □

Example. View $I = \{6, 10, 11\}$ as a subset of $[1, 13]$. So $|I| = 3$ and the following is an atomic chain of $I$.

$$\underbrace{\{1,2,3,4,5\}}_{E_0} < \underbrace{\{6\}}_{I_1} < \underbrace{\{7,8,9\}}_{E_1} < \underbrace{\{10\}}_{I_2} < \underbrace{\{\}}_{E_2} < \underbrace{\{11\}}_{I_3} < \underbrace{\{12,13\}}_{E_3}.$$

We have $-1+A = \bigcup_{l=1}^{4}(l-1+E_{l-1}) = (0+E_0)\cup(1+E_1)\cup(2+E_2)\cup(3+E_3) = \{1,2,3,4,5\}\cup\{8,9,10\}\cup\emptyset\cup\{15,16\} = \{1,2,3,4,5,8,9,10,15,16\} \subseteq [1,13+|I|] = [1,16]$. Hence $(-1+A)^c = \{6,7,11,12,13,14\} = \{6,6+1,10+1,10+2,11+2,11+3\}$. □

From the reduction arrow $q_l \xrightarrow{\qquad\qquad} -a_0 q_{l-1}^{\text{red}}$ mentioned at the beginning of Section 3 we get that our Theorem 0.1 will be established provided we can prove that for any positive integers $l$ and $k \le l$, there holds

$$\sum_{l_1+l_2+\cdots+l_k=l} s_{l_1}s_{l_1}\cdots s_{l_k} \xrightarrow{\text{to } i_1,\partial_{i_1},\text{to } i_2,\,\partial_{i_2},\cdots,\text{to } i_l,\,\partial_{i_l}} \begin{cases}\text{unsigned homogeneous}\\ \text{component of degree } k \text{ of } -a_0q_{l-1}^{\text{red}}.\end{cases}$$

This is a direct consequence of part a of the Theorem in Section 2 and the linearity of the reduction operations occurring in ' $\xrightarrow{\cdots}$ ' . By $q_{l-1}^{\text{red}}$ is meant the respective polynomial obtained by the recursion of Section 3.

We next reduce our conjecture to the proof of a combinatorial identity.

By Theorem 2.1 we know, given $1=\nu_1<\nu_2<\ldots<\nu_k\le d$, that

(Coefficient of $Sp_{1:i_{\nu_1}}Sp_{1:i_{\nu_2}}\cdots Sp_{1:i_{\nu_k}}$ of reduction R of $\sum_{l_1+\cdots+l_k=l} s_{l_1}s_{l_2}\cdots s_{l_k}$)

$$= \sum_{l_1+\cdots+l_k=l} l_1\cdots l_k \sum_{\sigma\in S_k}\prod_{i=1}^{k}(l_{\sigma 1}+\cdots+l_{\sigma i}-\nu_i)^{\underline{l_{\sigma i}-1}}.$$

At the other hand by the conjecture for the $q_l^{\text{red}}$ in Section 3 this coefficient is the coefficient of $a_0a_{\nu_2-1}a_{\nu_3-1}\cdots a_{\nu_k-1}$ in $a_0q_{l-1}^{\text{red}}$ and therefore the coefficient of $a_{\nu_2-1}a_{\nu_3-1}\cdots a_{\nu_k-1}$ of $q_{l-1}^{\text{red}} = q_{l-1}^{\text{red}}(a_1,\ldots,a_{l-1})$. (Recall, see Section 3, that we introduced the letters $a_0=Sp_{1:i_1}, a_1=Sp_{1:i_2},\ldots,a_{l-1}=Sp_{1:i_l}$.) To get the mentioned coefficient explicitly we consider the atomic chain given by $E_0 < \{\nu_2-1\} < E_1 < \{\nu_3-1\} < E_2 < \cdots < \{\nu_k-1\} < E_{k-1} = [\nu_k, l-1]$ so that $E_0 = ]\nu_1-1,\nu_2-1[ = [1,\nu_2-1[$, $E_l = ]\nu_{l+1}-1,\nu_{l+2}-1[$, for $l=1,2,\ldots,k-2$, $E_{k-1} = ]\nu_k-1,l-1]$. Clearly $|I|=k-1$. Define $\nu_{k+1} := l$. By Lemma 1 the coefficient of $a_{\nu_2-1}a_{\nu_3-1}\cdots a_{\nu_k-1}$ in $q_{l-1}^{\text{red}}$ as conjectured in the recursion equals

$$(*_0)\ (1+|I|)!\times\text{the product of the elements of the set } \bigcup_{l=1}^{1+|I|}(1+(l-1)+]\nu_l-1,\nu_{l+1}-1[)$$

$$= (1+|I|)!\times\text{the product of the elements of the set } 1+\bigcup_{l=1}^{k}((l-1)+]\nu_l-1,\nu_{l+1}-1[).$$

The union $\bigcup_{l=1}^{k}(\ldots)$ here is subset of $[1,l-1+|I|]$, where it has complement $\{\nu_2-1,\nu_2-1+1,\ldots,\nu_k-1+(k-1)-1,\nu_k-1+k-1\}$. The set $1+\bigcup_{l=1}^{k}((l-1)+]\nu_l-1,\nu_{l+1}-1[)$ thus has in $[2,l+|I|]$ the complement $\{\nu_2,\nu_2+1,\ldots,\nu_k-1+(k-1),\nu_k-1+k\}$. It follows that the product $(*_0)$ has the value of the right hand side of the following theorem which we will prove briefly and which in view of the the above discussion establishes in particular our main result, Theorem 0.1.

**Theorem 2.** *Let $l\ge 2$ be an integer. Then for any $k$ integers $1=\nu_1<\nu_2<\ldots<\nu_k\le l$, there holds the identity*

$$\sum_{l_1+\cdots+l_k=l} l_1\cdots l_k \sum_{\sigma\in S_k}\prod_{i=1}^{k}(l_{\sigma 1}+\cdots+l_{\sigma i}-\nu_i)^{\underline{l_{\sigma i}-1}} = \frac{k!(l+k-1)!}{\nu_2(\nu_2+1)(\nu_3+1)(\nu_3+2)\cdots(\nu_k+k-2)(\nu_k+k-1)}.$$

Proof. Given a function $f$ defined on $\mathbb{Z}_{\ge 1}^k$, for any $\sigma\in S_k$ the multiset of $k$-uples $\{f(l_{\sigma 1},\ldots,l_{\sigma k}) : l_1+\cdots+l_k=l\}$ is the same simply because the multiset of underlying $k$-uples $(l_{\sigma 1},\ldots,l_{\sigma k})$ remains the same. Now we can write the expression on the left hand side of the theorem as

$$\sum_{\sigma\in S_k}\sum_{l_1+\cdots+l_k=l} f(l_{\sigma 1},\ldots,l_{\sigma k}) \quad \text{where} \quad f(l_1,\ldots,l_k) = l_1\cdots l_k\prod_{i=1}^{k}(l_1+\cdots+l_i-\nu_i)^{\underline{l_i-1}}.$$

As the inner sum remains invariant under each $\sigma$ and since $|S_k| = k!$ the theorem will follow from the identity

$$*_2: \sum_{l_1+\cdots+l_k=l} l_1\cdots l_k \prod_{i=1}^{k}(l_1+\cdots+l_i-\nu_i)^{\underline{l_i-1}} = \frac{(l+k-1)!}{\nu_2(\nu_2+1)(\nu_3+1)(\nu_3+2)\cdots(\nu_k+k-2)(\nu_k+k-1)}.$$

To prove this we first show two claims:

Claim 1. For any nonnegative integer $\nu$ there holds the following polynomial identity in $\mathbb{R}[X]$

$$*: \sum_{i=0}^{\nu+1} i\nu^{\underline{i-1}}(X-i)^{\underline{\nu+1-i}} = (X+1)^{\underline{\nu}}.$$

⊳ We prove the claim by induction on $\nu$. In the case $\nu = 0$, the left hand side collapses to the expression $0^{\underline{0}}(X-1)^{\underline{0}}$ which is 1 by conventions. This is also the value of the empty product at the right. Recall that the forward difference operator $\Delta$ defined as $\Delta f(X) = f(X+1) - f(X)$ has the property that $\Delta X^{\underline{m}} = mX^{\underline{m-1}}$ and is linear. Let now $\nu \geq 1$ and assume that above equation holds for $\nu - 1$ in place of $\nu$. Applying $\Delta$ at the left hand side we get

$$\begin{aligned}
\Delta(\mathrm{lhs}(*)) &= \sum_{i=0}^{\nu+1} i\nu^{\underline{i-1}}\,\Delta((X-i)^{\underline{\nu+1-i}}) = \sum_{i=0}^{\nu+1} i\nu^{\underline{i-1}}(\nu+1-i)(X-i)^{\underline{\nu-i}} \\
&= \nu\sum_{i=0}^{\nu+1} i(\nu-1)^{\underline{i-1}}\,(X-i)^{\underline{\nu-i}} = \sum_{i=0}^{\nu} i(\nu-1)^{\underline{i-1}}\,(X-i)^{\underline{\nu-i}} \\
&= \nu(X+1)^{\underline{\nu-1}} = \Delta(X+1)^{\underline{\nu}} = \Delta(\mathrm{rhs}(*)).
\end{aligned}$$

The fact $\Delta(\mathrm{lhs}(*) - \mathrm{rhs}(*)) = 0$ implies that lhs $-$ rhs is a constant. Now, except in the case $\nu+1 = i$, in the polynomial $(X-i)^{\underline{\nu+1-i}}$ occurs the factor $(X-i)-(\nu+1-i)+1 = X-\nu$. Thus $\mathrm{lhs}(*)|_{X=\nu} = (\nu+1)\nu^{\underline{\nu}} = (\nu+1)! = (\nu+1)^{\underline{\nu}} = \mathrm{rhs}(*)|_{X=\nu}$. So the mentioned constant is 0 and we have proved the claimed equality. ◁

Claim 2. For integers $1 \leq \nu \leq l$ there holds

$$*_1: \sum_{l_1+l_2=l} l_2 l_1!(l_1+l_2-\nu)^{\underline{l_2-1}} = \sum_{l_2=1}^{l} l_2(l-\nu)^{\underline{l_2-1}}(l-l_2)! = \tfrac{(l+1)!}{\nu(\nu+1)}$$

⊳. We have

$$\nu\cdot(\nu+1)\cdot\mathrm{lhs}(*_1) = \nu\cdot(\nu+1)\sum_{l_1=1}^{l-1} l_1!(l-l_1)(l-\nu)^{\underline{l-l_1-1}} \overset{1}{=} \nu\cdot(\nu+1)\sum_{l_1=\nu-1}^{l-1} l_1!(l-l_1)(l-\nu)^{\underline{l-l_1-1}}.$$

To show that the right hand side equals $(l+1)!$ for any $\nu$ with $1 \leq \nu \leq l$ is equivalent to showing that after replacing $\nu$ by $l-\nu$ that right hand side is equal to $(l+1)!$ provided $0 \leq \nu \leq l-1$. Now for the replacement we have

$$\begin{aligned}
&(l-\nu)(l-\nu+1)\sum_{l_1=l-\nu-1}^{l-1} l_1!(l-l_1)\,\nu^{\underline{l-l_1-1}} = \\
&\qquad = (l-\nu)(l-\nu+1)(l-\nu-1)!\sum_{l_1=l-\nu-1}^{l-1}(l-l_1)\,l_1^{\underline{l_1-l+\nu+1}}\,\nu^{\underline{l-l_1-1}} \\
&\qquad \overset{2}{=} (l+1-\nu)!\sum_{i=0}^{\nu+1} i(l-i)^{\underline{\nu+1-i}}\,\nu^{\underline{i-1}} \overset{3}{=} (l+1)!
\end{aligned}$$

In '$\overset{2}{=}$' we introduced the summation index $i = l - l_1$, while '$\overset{3}{=}$' follows from the lemma by noting $(l+1)!/(l+1-\nu)! = (l+1)^{\underline{\nu}}$ and replacing $X$ by $l$. ◁

Proof of the identity ($*_2$).
Case $k = 1$ : This case reduces to the simple statement $l(l-1)^{\underline{l-1}} = l!$ which is obviously true.

Step $k-1 \to k$ assuming $k \geq 2$. By the induction hypothesis we have

$$\begin{aligned} \text{lhs}(*_2) &= \sum_{l_k=1}^{l} l_k (l-\nu_k)^{\underline{l_k-1}} \sum_{l_1+\cdots+l_{k-1}=l-l_k} l_1 l_2 \cdots l_{k-1} \prod_{i=1}^{k-1} (\sum_{h=1}^{i} l_h - \nu_i)^{\underline{l_i-1}}) \\ &= \sum_{l_k=1}^{l} l_k (l-\nu_k)^{\underline{l_k-1}} \cdot \frac{(l-l_k+k-2)!}{\nu_2(\nu_2+1)\cdots(\nu_{k-1}+k-3)(\nu_{k-1}+k-2)} \end{aligned}$$

This is indeed equal to rhs$(*_2)$ because

$$\sum_{l_k=1}^{l} l_k (l-\nu_k)^{\underline{l_k-1}} (l-l_k+k-2)! = \frac{(l+k-1)!}{(\nu_k+k-2)(\nu_k+k-1)},$$

as follows from Claim 2 above by replacing $l_2$ by $l_k$, $l$ by $l+k-2$, and $\nu$ by $\nu_k+k-2$. We find precisely the claim we wish to prove; except that the upper limit for $l_k$ now reads $l+k-2$. But note that if $l_k > l$, then $l_k - 1 > l - \nu$ and then $(l-\nu)^{\underline{l_k-1}} = 0$. It follows we are allowed to restrict the upper limit for $l_k$ to $l$ without changing the end result. □

This now together with DMUC preprints [K1-3] finishes our work on strengthening Holland's inequalities for the harmonic mean.


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