FUNCTORS PRESERVING EFFECTIVE DESCENT MORPHISMS

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ABSTRACT. Effective descent morphisms, originally defined in Grothendieck descent theory, form a class of special morphisms within a category. Essentially, an effective descent morphism enables bundles over its codomain to be fully described as bundles over its domain endowed with additional algebraic structure, called descent data. Like the study of epimorphisms, studying effective descent morphisms is interesting in its own right, providing deeper insights into the category under consideration. Moreover, studying these morphisms is part of the foundations of several applications of descent theory, notably including Janelidze-Galois theory, also known as categorical Galois theory.

Traditionally, the study of effective descent morphisms has focused on investigating and exploiting the reflection properties of certain functors. In contrast, we introduce a novel approach by establishing general results on the preservation of effective descent morphisms. We demonstrate that these preservation results enhance the toolkit for studying such morphisms, by observing that all Grothendieck (op)fibrations satisfying mild conditions fit our framework. To illustrate these findings, we provide several examples of Grothendieck (op)fibrations that preserve effective descent morphisms, including topological functors and other forgetful functors of significant interest in the literature.

1. INTRODUCTION

Given an abstract notion of categories of bundles over each object e of a category $\mathcal{A}$ – usually provided by a fibration $\mathcal{F}$ over $\mathcal{A}$ – the *effective descent morphisms* are those morphisms $p\colon e \to b$ for which the corresponding change-of-base functor witnesses the bundles over b as bundles over e with additional algebraic structure – which we call *descent data*.

For any such notion of bundle, studying and characterizing effective descent morphisms is compelling in its own right, in addition to being a fundamental aspect of Grothendieck descent theory and its applications, including Janelidze-Galois theory [2, 17]. For general and comprehensive introductions to descent theory and effective descent morphisms, we refer the reader to [20, 35, 48, 51].

For the sake of clarity and to present our main motivating examples, we decided to focus on the quintessential context where morphisms with codomain e are the bundles over e; more precisely, the categories of bundles over e are just the comma categories $\mathcal{A} \downarrow e$. Although we claim that our work can be applied to more general notions of bundles – provided by more general settings following [48], or, more generally, by pseudocosimplicial categories in the spirit of [31] – our results are already noteworthy and relevant within our scope.

We study a slight generalization of the classical notion of effective descent morphisms with respect to the *codomain fibration* relaxing the traditional requirement that the category must have all pullbacks. We outline our setting below.

Let $\mathcal{A}$ be a category and $p\colon e \to b$ a morphism in $\mathcal{A}$. We assume that morphisms along p exist. We, then, consider the equivalence relation $\mathtt{Eq}(p)$, given by Diagram (1.a), induced by the *kernel pair* of p.

$$e \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; e \times_b e \; \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \; e \times_b e \times_b e \tag{1.a}$$

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Moreover, we consider the usual *change-of-base functor*

$$p^* \colon \mathcal{A} \downarrow b \to \mathcal{A} \downarrow e \tag{1.b}$$

defined by taking pullbacks along p. Under our assumptions, we can similarly define change-of-base functors for each of the morphisms appearing in Diagram (1.a). In other words, we get the (truncated) pseudocosimplicial category

$$\begin{tikzcd} \mathcal{A} \downarrow e \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{A} \downarrow (e \times_b e) \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{A} \downarrow (e \times_b e \times_b e) \end{tikzcd} \tag{1.c}$$

by taking the respective change-of-base functors. The conical bilimit of (1.c) – called *descent category* or *category of descent data* – induces a factorization

$$\begin{tikzcd} \mathcal{A} \downarrow b \arrow[rr, "p^*"] \arrow[dr, "\mathcal{K}_p"'] & & \mathcal{A} \downarrow e \\ & \mathsf{Desc}(p) \arrow[ur] & \end{tikzcd} \tag{1.d}$$

called the *descent factorization of* p – see, for instance, [34, Sections 3 and 4] or [30, 31] for this approach, and [35, 58] for details on the definitions of *descent category* within our setting. We say that *p is of effective descent* if the comparison functor $\mathcal{K}_p$ is an equivalence.

By the so-called Bénabou-Roubaud theorem [4], even in this generalized framework (see [31, Theorem 8.5] for the generalized version), the Diagram (1.d) coincides with the Eilenberg-Moore factorization

$$\begin{tikzcd} \mathcal{A} \downarrow b \arrow[rr, "p^*"] \arrow[dr] & & \mathcal{A} \downarrow e \\ & T^p\text{-}\mathsf{Alg} \arrow[ur] & \end{tikzcd}$$

of the adjunction $p_! \dashv p^*$, up to a suitable pseudonatural equivalence. For a detailed exposition of the considerations and perspectives outlined above, we refer the interested readers to [31, 34, 35].

Henceforth, we only need to keep in mind the following consequence, which we *adopt* herein as the definition of effective descent morphisms:

p is of effective descent morphism if and only if p^* is monadic.

Characterization results are available for well-behaved classes of categories, such as *Barr-exact* categories and *locally cartesian closed* categories, where the effective descent morphisms are precisely the regular epimorphisms [18, 19, 27]. However, in general, within the setting of our present work, characterizing effective descent morphisms often poses a challenging problem even for specific instances of categories $\mathcal{A}$ – see, for example, earlier works like [19, 18, 21] and more recent studies such as [9, 53]. A celebrated illustration of this complexity is the category Top of topological spaces and continuous maps. The characterization of effective descent morphisms in Top, solved in [54] and later refined in [5, 7, 8], demonstrates the involved nature of the problem.

In the literature, the study of effective descent morphisms $\mathcal{A}$ has predominantly relied on *reflection* results – for an overview of the classical results in this area, see [31, Section 1]. For instance, classical reflection-based approaches have been employed in works such as [6, 10, 54]. Additionally, [31] introduced novel techniques to this framework, which have been applied in recent studies like [9, 52, 53].

An overview of the reflection-based paradigm can be given as follows. In order to study the effective descent morphisms of a category $\mathcal{A}$, we start by looking for a category $\mathcal{B}$ whose effective descent morphisms are better understood and a pullback-preserving functor $U \colon \mathcal{A} \to \mathcal{B}$. In this setting, fall under one of the following two situations:

(1) The functor $U \colon \mathcal{A} \to \mathcal{B}$ is fully faithful. Most classical descent results fall under this setting, which is justified by [19, Corollary 2.7.2] or [31, Corollary 9.9] for the general case. In our setting, Theorem 1 plays the fundamental role of this approach – it characterizes when p is

of effective descent, given that $U(p)$ is; in other words, when the property of being effective descent is reflected.

(2) More generally, the functor $U\colon \mathcal{A} \to \mathcal{B}$ is a forgetful functor, in the sense that the objects of $\mathcal{A}$ can be seen as objects of $\mathcal{B}$ with additional structure. More precisely, we attempt to exhibit $\mathcal{A}$ as a *bilimit* of categories whose descent and effective descent morphisms are reasonably understood– such an approach is justified by the viewpoint of [31] or, more specifically, Corollary 9.5 and *ibid.* Again, these results give conditions under which the property of effective descent morphisms is reflected by U.

Naturally, unless we are endowed with a preservation result, reflection results for a functor $U\colon \mathcal{A} \to \mathcal{B}$ can only provide *sufficient conditions* for a morphism to be effective for descent, and depend on the knowledge of effective descent morphisms in $\mathcal{B}$.

The goal of this paper is to provide *general* preservation results for effective descent morphisms. For the purposes of the characterization problem of effective descent morphisms, we are interested in finding guiding principles that can be applied to a wide variety of categorical settings.

The preservation tools and techniques we present here aim to aid our understanding of effective descent morphisms for a general class of categories, especially in the setting of the study of effective descent morphisms in categories of generalized categorical structures – for instance, in the context of [52, 53, 10, 51, 12, 13, 9].

Outline of the paper: We recall the basic notions of descent theory in Section 2. We also take the opportunity to recall the notion of Grothendieck (op)constructions, and we review the basic definitions needed for our work.

In Section 3, we provide an abstract result on preservation of effective descent morphisms. More specifically, we observe that a comonad with a cartesian counit always preserves effective descent morphisms (Lemma 11). In fact, we say more: for any adjunction $L \dashv U$ with a cartesian counit, if L reflects effective descent morphisms cartesian counit, then U preserves them. By listing some reasonable conditions under which L reflects effective descent morphism, we obtain Theorem 12.

Section 4 contains our main results, Theorems 15 and 19. These provide sufficient conditions for a(n) (op)fibration to preserve effective descent morphisms.

Employing our results on Grothendieck (op)fibrations, in Section 5 we work out several examples of (op)fibrations that preserve effective descent morphisms. These include

– the codomain opfibration induced by any category $\mathcal{A}$ (Subsection 5.1),
– the projection $\mathcal{A} \downarrow \Phi \to \mathcal{B}$ of the scone of a functor $\Phi\colon \mathcal{B} \to \mathcal{A}$ (Subsection 5.1),
– the natural forgetful functor from lax comma categories to base category (Subsection 5.2),
– topological functors (Subsection 5.3),
– co-Kleisli fibrations (Subsection 5.4),
– the (op-)Grothendieck constructions over the op-indexed category of lax comma categories and lax direct images (Subsection 5.5).

In Section 6, we talk about future work, extending the framework to more general notions of bundles (provided by general indexed categories or (truncated) pseudocosimplicial categories in the spirit of [31]). We show how this particular problem can be useful to further understand natural settings that arise from two-dimensional category theory.

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2. Preliminaries

We review the fundamental notions on descent theory and (op)-Grothendieck constructions. In particular, we rework some classical results of descent theory to encompass settings where our category of interest is not guaranteed to have all pullbacks, and we provide (well-known) explicit descriptions of pullbacks in the underlying total categories of both Grothendieck and op-Grothendieck constructions [15, 43]. Additionally, we recall some basic properties of categories with a strict initial object needed for our work on examples.

Descent theory: Let $\mathcal{A}$ be a category, and $p\colon a \to b$ be a morphism in $\mathcal{A}$. The direct image functor

$$\begin{aligned} p_!\colon \mathcal{A}\downarrow a &\to \mathcal{A}\downarrow b \\ f &\mapsto p\circ f \end{aligned}$$

has a right adjoint p^* if, and only if, $\mathcal{A}$ has pullbacks along p. When this is case, we write $\mathsf{Desc}(p)$ for the category of algebras of the monad induced by $p_! \dashv p^*$.

We say that p is an *effective descent* morphism (respectively, *descent* morphism) precisely when p^* exists and is monadic (respectively, premonadic).

Theorem 1 studies conditions for a functor to reflect effective descent morphisms. Its traditional form assumes a stronger hypothesis, requiring existence and preservation of all pullbacks, and is the main technique for studying effective descent morphisms – see, for instance, [54, 18, 31, 52, 9].

Theorem 1. *Let $p\colon a \to b$ be a morphism in $\mathcal{A}$ such that $\mathcal{A}$ has pullbacks along p, and let $L\colon \mathcal{A} \to \mathcal{C}$ be a fully faithful functor that preserves pullbacks along p. If $L(p)$ is an effective descent morphism, then the following are equivalent:*

(a) p is an effective descent morphism,
(b) for all pullback diagrams of the form (2.a),

(2.a)
$$\begin{tikzcd} L(c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & x \arrow[d, "f"] \\ L(a) \arrow[r, "L(p)"'] & L(b) \end{tikzcd}$$

there exists d in $\mathcal{A}$ such that $x \cong L(d)$.

Proof. This is an immediate consequence of [31, Theorem 9.8]; our hypotheses state that the change-of-base functors

$$p^*\colon \mathcal{A}\downarrow b \to \mathcal{A}\downarrow a \qquad \big(L(p)\big)^*\colon \mathcal{C}\downarrow L(b) \to \mathcal{C}\downarrow L(a)$$

exist, and that we have an invertible 2-cell

(2.b)
$$\begin{tikzcd} \mathcal{A}\downarrow b \arrow[r, "p^*"] \arrow[d, "L\downarrow b"'] \arrow[dr, phantom, "\cong"] & \mathcal{A}\downarrow a \arrow[d, "L\downarrow a"] \\ \mathcal{C}\downarrow L(b) \arrow[r, "(L(p))^*"'] & \mathcal{C}\downarrow L(a) \end{tikzcd}$$

since L preserves pullbacks along p. Moreover, we note that the induced functors $L\downarrow b$, $L\downarrow a$ are both fully faithful.

Thus, if $L(p)$ is an effective descent morphism, then Theorem 9.8 *ibid* confirms that p is an effective descent morphism if and only if (2.b) is a pseudopullback diagram. This is the case if and only if (b) holds. □

It should be noted that, under the hypothesis established above, Theorem 1 characterizes the class of morphisms for which the property

$$L(p) \text{ effective descent morphism } \implies p \text{ effective descent morphism.}$$

Motivated by this result, we introduce the following.

Definition 2. Let $L : \mathcal{A} \to \mathcal{C}$ be a functor. We assume that $\mathscr{C}$ is a class of morphisms in $\mathcal{A}$. We say that:

– L *reflects effective descent* $\mathscr{C}$*-morphisms* if
$$L(p) \text{ effective descent morphism } \implies p \text{ effective descent morphism,}$$
provided that p in $\mathscr{C}$;

– L *preserves effective descent* $\mathscr{C}$*-morphisms* if
$$p \text{ effective descent morphism } \implies L(p) \text{ effective descent morphism,}$$
whenever p in $\mathscr{C}$.

In both definitions above, whenever we do not mention the classes, we mean the class of all morphisms in their respective categories.

Following the terminology established above, by Theorem 1 we can get the following:

Corollary 3. *Let $L : \mathcal{A} \to \mathcal{C}$ be a fully faithful functor. We consider the following conditions on a class of morphisms of $\mathcal{A}$:*

(a) for any morphism p in $\mathscr{C}$, the category $\mathcal{A}$ has all pullbacks along p, and L preserves them;
(b) for any morphism p in $\mathscr{C}$, $L(p)$ satisfies (b) of Theorem 1;
(c) L reflects effective descent $\mathscr{C}$-morphisms.

Let $\mathscr{D}$ be the class of morphisms in $\mathcal{A}$ satisfying (a) and (b). We conclude that $\mathscr{D}$ satisfies (c) and, furthermore, $\mathscr{D}$ is maximal w.r.t. satisfying the properties (a) and (c).

Theorem 4. *Let $L \dashv U : \mathcal{C} \to \mathcal{A}$ be an adjunction with L fully faithful and counit ε, and let $p : a \to b$ be a morphism in $\mathcal{A}$.*

If $L(p)$ is an effective descent morphism, then p is an effective descent morphism if and only if for all pullback diagrams (2.a)*, we have ε_x monomorphic.*

Proof. For clarity, we recall that, under our hypothesis, if $\mathcal{C}$ has pullbacks along $L(p)$, then $\mathcal{A}$ has pullbacks along p. Indeed, if we have a morphism $f : c \to b$ in $\mathcal{A}$, then we consider the following pullback diagram:

(2.c)
$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L(c) \arrow[d, "L(f)"] \\ L(a) \arrow[r, "L(p)"'] & L(b) \end{tikzcd}$$

By composing U with Diagram (2.c), and noting that $UL \cong \mathsf{id}$, we conclude that $\mathcal{A}$ has the pullback of f along p.

Since effective descent morphisms are stable under pullback, the induced morphism $L(c) \to a$ in Diagram (2.a) is effective for descent (and is, in particular, an extremal epimorphism). This morphism factors as $\varepsilon_x \circ L(q)$ for a suitable q, hence, if ε_x is a monomorphism, it must be an isomorphism. This implies $LU(x) \cong x$, and Theorem 1 may be applied. ☐

Grothendieck construction: Since it is a two-dimensional notion, the *Grothendieck construction* over an indexed category (which strictly implies that our domain is a one-dimensional category) has four duals (see, for instance, [26, 35, 32]), which we describe below.

For clarity, we fix two pseudofunctors

$$\mathcal{F} : \mathcal{A}^{\mathsf{op}} \to \mathsf{CAT}, \qquad \mathcal{H} : \mathcal{A} \to \mathsf{CAT}.$$

We recall that the *Grothendieck construction* of $\mathcal{F}$ is a fibration $\int_{\mathcal{A}} \mathcal{F} \to \mathcal{A}$, whose *total category* $\int_{\mathcal{A}} \mathcal{F}$ has set of objects $\sum_{a \in \mathsf{ob}\,\mathcal{A}} \mathsf{ob}\,\mathcal{F}_a$, and hom-sets

$$\int_{\mathcal{A}} \mathcal{F}\big((a,x),(b,y)\big) = \sum_{f \in \mathcal{A}(a,b)} \mathcal{F}_a\big(x, \mathcal{F}_f(y)\big). \tag{2.d}$$

Dually, herein, the *op-Grothendieck construction*[1] of $\mathcal{H}$ is an opfibration $\int^{\mathcal{A}} \mathcal{H} \to \mathcal{A}$, whose *total category* $\int^{\mathcal{A}} \mathcal{H}$ has set of objects $\sum_{a \in \mathsf{ob}\,\mathcal{A}} \mathsf{ob}\,\mathcal{H}_a$, and hom-sets

$$\int^{\mathcal{A}} \mathcal{H}\big((a,x),(b,y)\big) = \sum_{f \in \mathcal{A}(a,b)} \mathcal{H}_a\big(\mathcal{H}_f(x), y\big). \tag{2.e}$$

The total categories of the codual of the op-Grothendieck and Grothendieck constructions presented above are, respectively, given by:

$$\left(\int^{\mathcal{A}} \mathcal{H}\right)^{\mathsf{op}} \text{ and } \left(\int_{\mathcal{A}} \mathcal{F}\right)^{\mathsf{op}}.$$

Remark 5. *If $\mathcal{F}_a = \mathcal{H}_a$ for all objects a in $\mathcal{A}$ and $\mathcal{H}_f \dashv \mathcal{F}_f$, then it is immediate that the Grothendieck construction of $\mathcal{F}$ is isomorphic to the op-Grothendieck construction of $\mathcal{H}$ – so that $\int_{\mathcal{A}} \mathcal{F} \to \mathcal{A}$ is a bifibration.*

Moreover, the op-Grothendieck construction of $\mathcal{H}$ can be obtained from the Grothendieck construction, via

$$\int^{\mathcal{A}} \mathcal{H} = \Big(\int_{\mathcal{A}^{\mathsf{op}}} (-)^{\mathsf{op}} \circ \mathcal{H}^{\mathsf{co}} \Big)^{\mathsf{op}},$$

where $(-)^{\mathsf{op}} \colon \mathsf{CAT}^{\mathsf{co}} \to \mathsf{CAT}$ is the dualization 2-functor, and $\mathcal{H}^{\mathsf{co}} \colon \mathcal{A} \to \mathsf{CAT}^{\mathsf{co}}$ is the codual of $\mathcal{H}$.

The following results characterizing pullbacks in $\int_{\mathcal{A}} \mathcal{F}$ and $\int^{\mathcal{A}} \mathcal{H}$ are special cases of familiar characterizations for limits in (op-)Grothedieck constructions, appearing in the literature at least since [14]. See, for instance, [43, Lemma 1, Lemma 2] for the statements and proofs.

Lemma 6. *Let $\mathcal{F} \colon \mathcal{A}^{\mathsf{op}} \to \mathsf{CAT}$ be a pseudofunctor. If we have a commutative square*

$$\begin{tikzcd} (a,w) \arrow[r, "{(p,\pi)}"] \arrow[d, "{(q,\chi)}"'] & (b,x) \arrow[d, "{(f,\phi)}"] \\ (c,y) \arrow[r, "{(g,\psi)}"'] & (d,z) \end{tikzcd} \tag{2.f}$$

in $\int_{\mathcal{A}} \mathcal{F}$ such that

$$\begin{tikzcd} a \arrow[r, "p"] \arrow[d, "q"'] & b \arrow[d, "f"] \\ c \arrow[r, "g"'] & d \end{tikzcd}$$

is a pullback diagram, then the following are equivalent:

(a) *Diagram* (2.f) *is a pullback diagram.*
(b) *For every morphism k in $\mathcal{A} \downarrow a$, the following diagram*

$$\begin{tikzcd} \mathcal{F}_k(w) \arrow[rr, "\mathcal{F}_k(\pi)"] \arrow[d, "\mathcal{F}_k(\chi)"'] & & \mathcal{F}_k\mathcal{F}_p(x) \arrow[d, "\mathcal{F}_k\mathcal{F}_p(\phi)"] \\ \mathcal{F}_k\mathcal{F}_q(y) \arrow[r, "\mathcal{F}_k\mathcal{F}_q(\psi)"'] & \mathcal{F}_k\mathcal{F}_q\mathcal{F}_g(z) \cong & \mathcal{F}_k\mathcal{F}_p\mathcal{F}_f(z) \end{tikzcd}$$

is a pullback square.

[1] In the literature, op-Grothendieck construction is usually the codual of what we define herein – see, for instance, [40, 43, 15, 22].

Lemma 7. *Let $\mathcal{H}\colon \mathcal{A} \to \mathsf{CAT}$ be a pseudofunctor. If we have a commutative square*

$$\begin{tikzcd} (a,w) \arrow[r, "(p,\pi)"] \arrow[d, "(q,\chi)"'] & (b,x) \arrow[d, "(f,\phi)"] \\ (c,y) \arrow[r, "(g,\psi)"'] & (d,z) \end{tikzcd} \tag{2.g}$$

in $\int^{\mathcal{A}} \mathcal{H}$ such that

$$\begin{tikzcd} a \arrow[r, "p"] \arrow[d, "q"'] & b \arrow[d, "f"] \\ c \arrow[r, "g"'] & d \end{tikzcd}$$

is a pullback diagram, then the following are equivalent:

(a) Diagram (2.g) *is a pullback.*
(b) For every pair of morphisms $\sigma\colon \mathcal{H}_p(v) \to x$, $\tau\colon \mathcal{H}_q(v) \to y$ such that the diagram

$$\begin{tikzcd} \mathcal{H}_g\mathcal{H}_q(v) \cong \mathcal{H}_f\mathcal{H}_p(v) \arrow[r, "\mathcal{H}_f(\sigma)"] \arrow[d, "\mathcal{H}_g(\tau)"'] & \mathcal{H}_f(x) \arrow[d, "\phi"] \\ \mathcal{H}_g(y) \arrow[r, "\psi"'] & z \end{tikzcd}$$

in $\mathcal{H}_d$ commutes, there exists a unique morphism $\theta\colon v \to w$ such that $\sigma = \pi \circ \mathcal{H}_p(\theta)$ and $\tau = \chi \circ \mathcal{H}_q(\theta)$.

Strict initial objects: Let $\mathcal{A}$ be a category with an initial object $\mathsf{0}$. We say that $\mathsf{0}$ is *strict* if $x \cong \mathsf{0}$ whenever $\mathcal{A}(x,\mathsf{0})$ is non-empty. We note the following:

Lemma 8. *If $\mathcal{A}$ has binary products of every object with $\mathsf{0}$, then the following are equivalent:*

(a) $\mathsf{0}$ is a strict initial object.
(b) $a \times \mathsf{0} \cong \mathsf{0}$ for all a.

Proof. The projection $a \times \mathsf{0} \to \mathsf{0}$ witnesses that (a) $\implies$ (b).

Conversely, we assume that (b) holds, so that $\mathcal{A}(x,a) \times \mathcal{A}(x,\mathsf{0}) \cong \mathcal{A}(x,\mathsf{0})$. If x is an object in $\mathcal{A}$ such that $\mathcal{A}(x,\mathsf{0})$ is non-empty, then

$$\mathcal{A}(x,\mathsf{0}) \times \mathcal{A}(x,\mathsf{0}) \cong \mathcal{A}(x,\mathsf{0})$$

implies that $\mathcal{A}(x,\mathsf{0}) \cong \mathsf{1}$. Hence, we must have

$$\mathcal{A}(x,a) \cong \mathcal{A}(x,a) \times \mathcal{A}(x,\mathsf{0}) \cong \mathcal{A}(x,\mathsf{0}) \cong \mathsf{1}$$

for all a – which entails that $x \cong \mathsf{0}$, confirming (a). ☐

Lemma 9. *If $\mathcal{A}$ has a strict initial object $\mathsf{0}$, then the following commutative square*

$$\begin{tikzcd} \mathsf{0} \arrow[r] \arrow[d] & \mathsf{0} \arrow[d] \\ a \arrow[r, "p"'] & b \end{tikzcd} \tag{2.h}$$

is a pullback diagram for every morphism $p\colon a \to b$ in $\mathcal{A}$.

Proof. If we have a commutative square

$$\begin{tikzcd} w \arrow[r] \arrow[d] & \mathsf{0} \arrow[d] \\ a \arrow[r, "p"'] & b \end{tikzcd}$$

we necessarily have $w \cong \mathsf{0}$, so there is nothing to be done. ☐

3. Guiding principle

In Section 4, we present the foundational observations that led us to develop more structured results. Our guiding principle is that any functor preserving effective descent morphisms induces another functor with the same property, provided it factors through a functor that reflects effective descent morphisms. Given the availability of several techniques in the literature for obtaining functors that reflect morphisms (see, for instance, [54, 18, 31]), we consider this principle to be fruitful. We begin by stating a basic naive general result, followed by more structured versions derived through our comonadic approach.

Lemma 10. *Let* (3.a) *be a diagram of functors, and let $\mathscr{C}$ be a class of morphisms in $\mathcal{A}$.*

$$\mathcal{A} \xrightarrow{U} \mathcal{B} \xrightarrow{L} \mathcal{C} \tag{3.a}$$

We consider the class $U(\mathscr{C}) = \mathscr{D}$ of morphisms in the image of $\mathscr{C}$ by U. In this setting, if

- *LU preserves effective descent $\mathscr{C}$-morphisms,*
- *and L reflects effective descent $\mathscr{D}$-morphisms,*

then U preserves effective descent $\mathscr{C}$-morphisms.

Preservation via comonads. In this subsection, we consider an adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "U"] \end{tikzcd}$$

whose counit is denoted by $\varepsilon\colon LU \to \mathsf{id}$.

Lemma 11. *If the counit $\varepsilon\colon LU \to \mathsf{id}$ is a cartesian natural transformation, then LU preserves effective descent morphisms.*

Proof. By definition, we have a pullback square

$$\begin{tikzcd} LU(x) \arrow[r, "LU(f)"] \arrow[d, "\varepsilon_x"'] \arrow[dr, phantom, "\lrcorner"] & LU(y) \arrow[d, "\varepsilon_y"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

for every morphism $f\colon x \to y$, and since effective descent morphisms are stable under pullback, $LU(f)$ is an effective descent morphism whenever f is. □

From the preliminaries and Lemma 11, we conclude that:

Theorem 12. *We assume that ε is a cartesian natural transformation that is also componentwise a monomorphism.*

If L is fully faithful, then U preserves effective descent $\mathscr{C}$-morphisms, where $\mathscr{C}$ consists of the morphisms p such that L preserves pullbacks along $U(p)$.

Proof. Let p be an effective descent morphism in $\mathcal{C}$ that is in the class $\mathscr{C}$.

By Lemma 11, $LU(p)$ is an effective descent morphism. Now our result follows from Theorem 4: since ε is a componentwise monomorphism, L is fully faithful, and L preserves pullbacks along $U(p)$, we conclude that $U(p)$ is an effective descent morphism. □

4. Preservation results

Many examples of (Grothendieck) fibrations and opfibrations are under the conditions of Theorem 12, and therefore preserve effective descent morphisms – these observations are the main result of this note, which are studied in this section.

This is split into two subsections, which respectively contain our results for fibrations and opfibrations. These results are not dual to each other in any sense – the details are more delicate in the case of opfibrations.

Fibrations. We fix a pseudofunctor

$$\mathcal{F}\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{CAT}$$

and we denote its Grothendieck construction by

$$U\colon \textstyle\int_{\mathcal{A}} \mathcal{F} \to \mathcal{A}.$$

Lemma 13. *If $\mathcal{F}_a$ has an initial object 0_a for every object $a \in \mathcal{A}$, then U has a fully faithful left adjoint L, given on objects by $L(a) = (a, 0_a)$.*

Proof. We have the following natural isomorphism

$$\int_{\mathcal{A}} \mathcal{F}\big(L(a),(b,x)\big) \cong \sum_{f\in\mathcal{A}(a,b)} \mathcal{F}_a\big(0_a, \mathcal{F}_f(x)\big) \cong \mathcal{A}(a,b) = \mathcal{A}\big(a, U(b,x)\big), \tag{4.a}$$

exhibiting the adjunction $L \dashv U$. By taking $x = 0_b$ in (4.a) we conclude that L is fully faithful. ☐

Lemma 14. *If $\mathcal{F}_a$ has a strict initial object for all a in $\mathcal{A}$, and the change-of-base functors $\mathcal{F}_f$ preserve them for every morphism f in $\mathcal{A}$, then*

(i) the counit ε of $L \dashv U$ is a cartesian natural transformation,
(ii) L preserves pullbacks,
(iii) ε is a componentwise monomorphism.

Proof. Via Lemma 6, we may conclude that (i) holds if the following diagram is a pullback

$$\begin{tikzcd} \mathcal{F}_p(0_a) \arrow[r] \arrow[d] & \mathcal{F}_p\mathcal{F}_f(0_b) \arrow[d] \\ \mathcal{F}_p(x) \arrow[r, "\mathcal{F}_p(\phi)"'] & \mathcal{F}_p\mathcal{F}_f(y) \end{tikzcd}$$

for every morphism $(f,\phi)\colon (a,x) \to (b,y)$ in $\int_{\mathcal{A}}\mathcal{F}$, and every morphism $p\colon e \to a$ in $\mathcal{A}$. Since the change-of-base functors preserve initial objects, this is an immediate consequence of Lemma 9.

Let

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[d, "h"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

be a pullback diagram. Since the change-of-base functors preserve initial objects, the following diagram

$$\begin{tikzcd} \mathcal{F}_p(0_a) \arrow[rr, "\cong"] \arrow[d, "\cong"'] & & \mathcal{F}_p\mathcal{F}_f(0_b) \arrow[d, "\cong"] \\ \mathcal{F}_p\mathcal{F}_g(0_c) \arrow[r, "\cong"] & \mathcal{F}_p\mathcal{F}_g\mathcal{F}_k(0_d) \cong & \mathcal{F}_p\mathcal{F}_f\mathcal{F}_h(0_d) \end{tikzcd} \tag{4.b}$$

is a pullback diagram for any $p\colon e \to a$. Thus, we may apply Lemma 6 to conclude that (ii) holds.

To prove (iii), we note that ε is given at (b,y) by the pair

$$(\mathsf{id}_b, u)\colon (b, 0_b) \to (b,y),$$

where $u\colon 0_b \to \mathcal{F}_{\mathsf{id}_b}(y)$ is the unique morphism. If we have

$$(\mathsf{id}_b, u) \circ (h,\zeta) = (\mathsf{id}_b, u) \circ (k,\xi)$$

for morphisms $(h,\zeta),(k,\xi)\colon (a,x) \to (b,0_b)$, it follows that $h = k$, and since change-of-base functors preserve (strict) initial objects, we must have $x \cong 0_a$, and hence $\zeta = \xi$. Thus, we conclude that $\varepsilon_{b,y}$ is a monomorphism. ☐

As a corollary, we obtain that:

Theorem 15. *If $\mathcal{F}_a$ has a strict initial object 0_a for each a in $\mathcal{A}$, and $\mathcal{F}_f$ preserves them for all morphisms f in $\mathcal{A}$, then $U\colon \int_{\mathcal{A}}\mathcal{F} \to \mathcal{B}$ preserves effective descent morphisms.*

Proof. By Lemmas 13 and 14, U has a left adjoint L which enjoys all the properties needed to apply Theorem 12. □

Corollary 16. *Let $\Phi\colon \mathcal{B} \to \mathcal{A}$ be any functor. If $\mathcal{F}_a$ has a strict initial object $\mathsf{0}_a$ for all a in $\mathcal{A}$, and $\mathcal{F}_f$ preserves initial objects for all morphisms f in $\mathcal{A}$, then the Grothendieck construction*

$$\textstyle\int_{\mathcal{B}} \mathcal{F} \circ \Phi^{\mathsf{op}} \to \mathcal{B}$$

preserves effective descent morphisms as well.

Opfibrations. We fix a pseudofunctor

$$\mathcal{H}\colon \mathcal{A} \to \mathsf{CAT},$$

and we denote its Grothendieck construction by

$$V\colon \int^{\mathcal{A}} \mathcal{H} \to \mathcal{A}.$$

Lemma 17. *If $\mathcal{H}_a$ has an initial object $\mathsf{0}_a$ for every object $a \in \mathcal{A}$, and the change-of-base functors $\mathcal{H}_f$ preserve them for every morphism f in $\mathcal{A}$, then V has a fully faithful left adjoint K, given on objects by $K(a) = (a, \mathsf{0}_a)$.*

Proof. This is similar to the proof of 13; since $\mathcal{H}_f(\mathsf{0}_a) \cong \mathsf{0}_b$, we have

$$\int^{\mathcal{A}} \mathcal{H}\big(K(a),(b,x)\big) \cong \sum_{f\in\mathcal{A}(a,b)} \mathcal{H}_b\big(\mathcal{H}_f(\mathsf{0}_a),x\big) \cong \sum_{f\in\mathcal{A}(a,b)} \mathcal{H}_b\big(\mathsf{0}_b,x\big) \cong \mathcal{A}(a,b) = \mathcal{A}\big(a,V(b,x)\big), \tag{4.c}$$

which witnesses $K \dashv V$. Taking $x = \mathsf{0}_b$ in (4.c) confirms that K is fully faithful. □

Lemma 18. *If $\mathcal{H}_a$ has a strict initial object $\mathsf{0}_a$ for all a in $\mathcal{A}$ that is preserved by the change-of-base functors $\mathcal{H}_f$ for all morphisms f in $\mathcal{A}$, then*

- *(a) the naturality square of ν at a morphism (f,ϕ) is a pullback diagram if and only if $\mathcal{H}_f$ reflects initial objects,*
- *(b) the left adjoint K preseves the pullback of a pair of arrows f, g if and only if the change-of-base functor induced by the diagonal of the pullback square reflects initial objects,*
- *(c) ν is a componentwise monomorphism.*

Proof. By Lemma 7, the naturality square of ν at a morphism $(f,\phi)\colon (a,x) \to (b,y)$ is a pullback diagram if and only if for every commutative diagram

$$\begin{tikzcd} \mathcal{H}_f(v) \arrow[r] \arrow[d, "\mathcal{H}_f(\gamma)"'] & \mathsf{0}_b \arrow[d] \\ \mathcal{H}_f(x) \arrow[r, "\phi"'] & y \end{tikzcd}$$

we have $v \cong \mathsf{0}_a$. Since $\mathsf{0}_b$ is strict, every such diagram commutes if and only if $\mathcal{H}_f(v) \cong \mathsf{0}_b$, so (a) holds.

If the following diagram

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "k"'] & b \arrow[d, "f"] \\ c \arrow[r, "g"'] & d \end{tikzcd}$$

is a pullback in $\mathcal{A}$, then by Lemma 7, K preserves this pullback if and only if for every commutative square

$$\begin{tikzcd} \mathcal{H}_g\mathcal{H}_k(v) \cong \mathcal{H}_f\mathcal{H}_h(v) \arrow[r] \arrow[d] & \mathcal{H}_f(\mathsf{0}_b) \arrow[d] \\ \mathcal{H}_g(\mathsf{0}_c) \arrow[r] & \mathsf{0}_d \end{tikzcd}$$

we have $v \cong 0_a$. Since 0_d is strict, and change-of-base functors preserve initial objects, every such diagram commutes if and only if $\mathcal{H}_f\mathcal{H}_h(v) \cong \mathcal{H}_g\mathcal{H}_k(v) \cong 0_d$, which confirms (b).

ν is given at (b, y) by the pair

$$(\mathsf{id}_b, u)\colon (b, 0_b) \to (b, y),$$

where $u\colon \mathcal{H}_{\mathsf{id}_b}(0_b) \to y$ is the unique morphism, since $\mathcal{H}_{\mathsf{id}_b}(0_b) \cong 0_b$. If

$$(\mathsf{id}_b, u) \circ (h, \zeta) = (\mathsf{id}_b, u) \circ (k, \xi)$$

for morphisms $(h, \zeta), (k, \xi)\colon (a, x) \to (b, 0_b)$, it follows that $h = k$, and since initial objects are strict, we must have $\zeta = \xi\colon \mathcal{H}_f(x) \cong 0_b$. Thus, we conclude that (c) holds. ☐

As an immediate corollary, we obtain:

Theorem 19. *If $\mathcal{H}_a$ has a strict initial object 0_a for all a, which is created by every change-of-base functor, then $V\colon \int^{\mathcal{A}} \mathcal{H} \to \mathcal{A}$ preserves effective descent morphisms.*

Corollary 20. *Let $\Phi\colon \mathcal{B} \to \mathcal{A}$ be any functor. If $\mathcal{H}_a$ has a strict initial object 0_a for all a in $\mathcal{A}$, and $\mathcal{H}_f$ creates them for all morphisms f in $\mathcal{A}$, then the op-Grothendieck construction*

$$\int^{\mathcal{B}} \mathcal{H} \circ \Phi \to \mathcal{B}$$

preserves effective descent morphisms as well.

5. Examples

5.1. **Codomain bifibration:** Let $\mathcal{A}$ be a category with a strict initial object, and we consider the "direct image" pseudofunctor

$$\begin{aligned} \mathcal{H}\colon \mathcal{A} &\to \mathsf{CAT} \\ a &\mapsto \mathcal{A} \downarrow a \\ p &\mapsto p_! \end{aligned}$$

where $p_!\colon \mathcal{A} \downarrow a \to \mathcal{A} \downarrow b$ is given by $f \mapsto p \circ f$. We denote its op-Grothendieck construction by $\mathsf{cod}\colon \mathcal{A}^2 \to \mathcal{A}$ – the *codomain opfibration*.

Lemma 21. *The codomain opfibration preserves effective descent morphisms.*

Proof. The unique morphism $0 \to a$ is a strict initial object in $\mathcal{A} \downarrow a$ for every a, and the change-of-base functors $p_!\colon \mathcal{A} \downarrow a \to \mathcal{A} \downarrow b$ create them. The result is therefore confirmed by Theorem 19. ☐

Let $\Phi\colon \mathcal{B} \to \mathcal{A}$, and consider the Artin gluing (also known as *scone*) $\mathcal{A} \downarrow \Phi$, together with the projection $U\colon \mathcal{A} \downarrow \Phi \to \mathcal{B}$. This fibration, called herein the *scone forgetful functor*, has been extensively studied in the literature for its rich properties, particularly as a categorical viewpoint on logical relations techniques [47, 16, 39, 40, 41]. As an application of our framework, we find that this fibration is particularly relevant to our context in descent theory. Specifically, we obtain the following result:

Lemma 22. *The scone forgetful functor U preserves effective descent morphisms.*

Proof. We apply Corollary 16 to the composite

$$\mathcal{B} \xrightarrow{\Phi} \mathcal{A} \xrightarrow{\mathcal{H}} \mathsf{CAT}$$

and we note that $\int^{\mathcal{B}} \mathcal{H} \circ \Phi \simeq \mathcal{A} \downarrow \Phi$. ☐

We also consider the Grothendieck construction $\int_{\mathcal{A}^{\mathsf{op}}} \mathcal{H} \to \mathcal{A}^{\mathsf{op}}$ of $\mathcal{H}$; the objects of $\int_{\mathcal{A}^{\mathsf{op}}} \mathcal{H}$ still are the morphisms of $\mathcal{A}$, while a morphism $f \to g$ is a pair (p, q) of morphisms such that the following square commutes:

$$\begin{tikzcd} a \arrow[r, "p"] \arrow[d, "f"'] & b \arrow[d, "g"] \\ c & d \arrow[l, "q"] \end{tikzcd}$$

By Theorem 15, we obtain:

Lemma 23. *The fibration $\int_{\mathcal{A}^{\mathsf{op}}} \mathcal{H} \to \mathcal{A}^{\mathsf{op}}$ preserves effective descent morphisms.*

5.2. **Co-Kleisli categories for writer comonads:** Let $\mathcal{A}$ be a category with binary products and a strict initial object. For each $a \in \mathcal{A}$, the functor $a \times -$ underlies a comonad on $\mathcal{A}$. We denote its co-Kleisli category by $\mathsf{CoKl}(a \times -)$, and we recall it may be given as follows:

- $\mathsf{ob}\,\mathsf{CoKl}(a \times -) = \mathsf{ob}\,\mathcal{A}$,
- $\mathsf{CoKl}(a \times -)(x, y) = \mathcal{A}(a \times x, y)$,
- Identities are the product projections $a \times x \to x$,
- Composition of morphisms $f : a \times x \to y$, $g : a \times y \to z$ is given by

$$a \times x \xrightarrow{\Delta \times \mathsf{id}_x} a \times a \times x \xrightarrow{\mathsf{id}_a \times f} a \times y \xrightarrow{g} z$$

For each morphism $f : a \to b$, we let $\mathcal{F}_f : \mathsf{CoKl}(b \times -) \to \mathsf{CoKl}(a \times -)$ be given by the identity on objects, and

$$\begin{aligned} \mathcal{A}(b \times x, y) &\to \mathcal{A}(a \times x, y) \\ p &\mapsto p \circ (f \times \mathsf{id}_x) \end{aligned}$$

on hom-sets. This defines a pseudofunctor $\mathcal{F} : \mathcal{A}^{\mathsf{op}} \to \mathsf{CAT}$.

The total category of the Grothendieck construction $U : \int_{\mathcal{A}} \mathcal{F} \to \mathcal{A}$ of $\mathcal{F}$ is the full subcategory of $\mathcal{A}^2$ consisting of the product projections. To be precise, the objects of $\int_{\mathcal{A}} \mathcal{F}$ are pairs of objects (a, x), corresponding to product projections $a \times x \to a$, and a morphism $(f, g) : (a, x) \to (b, y)$ consists of morphisms $f : a \times x \to b \times y$, $g : a \to b$ such that the following diagram commutes:

$$\begin{tikzcd} a \times x \arrow[r, "f"] \arrow[d] & b \times y \arrow[d] \\ a \arrow[r, "g"'] & b \end{tikzcd}$$

To ensure that $\mathcal{F}$ is within the scope of Theorem 15, we have

Lemma 24. *The co-Kleisli categories $\mathsf{CoKl}(a \times -)$ have strict initial objects, and the change-of-base functors $\mathcal{F}_f$ preserve them.*

Proof. We recall that the category of coalgebras for $a \times -$ is equivalent to $\mathcal{A} \downarrow a$ – this allows us to view the co-Kleisli category $\mathsf{CoKl}(a \times -)$ as the full subcategory of $\mathcal{A} \downarrow a$ whose objects are the product projections $d_0 : a \times x \to a$.

Since $a \times 0 \cong 0$ for all a, this implies that $\mathsf{CoKl}(a \times -)$ has a strict initial object – 0 itself – for all a. These are preserved by the identity-on-objects change-of-base functors. ☐

Thus, as a corollary, we obtain:

Lemma 25. *The fibration $U : \int_{\mathcal{A}} \mathcal{F} \to \mathcal{A}$ preserves effective descent morphisms.*

We may also consider the op-Grothendieck construction $\int^{\mathcal{A}^{\mathsf{op}}} \mathcal{F} \to \mathcal{A}^{\mathsf{op}}$ of $\mathcal{F}$. A morphism $(a, x) \to (b, y)$ in its total category corresponds to a pair of morphisms $f : a \times x \to b \times y$, $g : b \to a$ such that the following diagram commutes:

$$\begin{tikzcd} a \times x \arrow[r, "f"] \arrow[d] & b \times y \arrow[d] \\ a & b \arrow[l, "g"] \end{tikzcd}$$

Moreover, we note that the identity-on-objects change-of-base functors create initial objects, whence we may apply Lemma 18 to conclude that

Lemma 26. *The opfibration $\int^{\mathcal{A}^{\mathsf{op}}} \mathcal{F} \to \mathcal{A}^{\mathsf{op}}$ preserves effective descent morphisms.*

5.3. **Topological functors:** We recall from [59] that any topological functor $U\colon \mathcal{C} \to \mathcal{A}$ (with possibly large fibers) is the Grothendieck construction of a bifibration $\mathcal{F}\colon \mathcal{A}^{\mathsf{op}} \to \mathsf{CAT}$ whose fibers $\mathcal{F}_a$ are (large-)complete thin categories. In particular, the fibers $\mathcal{F}_a$ have strict initial objects.

Lemma 27. *If the change-of-base functors $\mathcal{F}_f\colon \mathcal{F}_b \to \mathcal{F}_a$ preserve initial objects (bottom elements), then $U\colon \mathcal{C} \to \mathcal{A}$ preserves effective descent morphisms.*

As a consequence of Lemma 27, the quintessential topological functor $\mathsf{Top} \to \mathsf{Set}$ preserves effective descent morphisms.

5.4. **Lax comma categories:** Recall that, given a 2-category $\mathbb{A}$ and an object X of $\mathbb{A}$, the lax comma category $\mathbb{A} \Downarrow X$ is defined as the total category of the Grothendieck construction of the representable 2-functor $\mathbb{A}(-, X)\colon \mathbb{A}^{\mathsf{op}} \to \mathsf{CAT}$[2] – see, for instance, [11, 12, 43].

Unpacking the definition, the objects of $\mathbb{A} \Downarrow X$ are morphisms $f\colon A \to X$ with fixed codomain X, while a morphism $(p, \pi)\colon f \to g$ given by a 2-cell $\pi\colon f \to g \cdot p$, which is depicted in the following diagram:

$$\begin{tikzcd} A \arrow[rr, "p"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\pi}{\Longrightarrow}"] & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

The composition of morphisms $(p, \pi)\colon (A, f) \to (B, g)$ and $(q, \chi)\colon (B, g) \to (C, h)$ is given by the pair $(q \cdot p, (\chi \cdot p) \circ \pi)$, which may be obtained diagramatically by pasting the 2-cells as follows:

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, bend right, "f"'] \arrow[dr, phantom, "\overset{\pi}{\Longrightarrow}"] & B \arrow[r, "q"] \arrow[d, "g"] \arrow[dr, phantom, "\overset{\chi}{\Longrightarrow}"] & C \arrow[dl, bend left, "h"] \\ & X & \end{tikzcd}$$

This work was developed from the examination of the core ideas behind the preservation results for the forgetful functor $\mathbb{A} \Downarrow X \to \mathbb{A}$, including the specific cases studied in [10, 12, 13]. For instance, it was shown that $\mathsf{Ord} \Downarrow X \to \mathsf{Ord}$ preserves effective descent morphisms when X is an complete ordered set [10, Theorem 3.3], and that $\mathsf{Cat} \Downarrow \mathcal{A} \to \mathsf{Cat}$ preserves effective descent morphisms when $\mathcal{A}$ has pullbacks and a strict initial object [12, Theorem 3.7]. We will show that both results can be obtained (and generalized) in the present setting.

From this point forward, we assume that X is an object of $\mathbb{A}$ such that $\mathbb{A}(A, X)$ has a strict initial object for every object A in $\mathbb{A}$, and that the change-of-base functor $- \cdot f\colon \mathbb{A}(B, X) \to \mathbb{A}(A, X)$ preserves them for every morphism $f\colon A \to B$ in $\mathbb{A}$. As an immediate consequence of Theorem 15, we obtain the following result:

Lemma 28. *The fibration $\mathbb{A} \Downarrow X \to \mathbb{A}$ preserves effective descent morphisms.*

We recover [10, Theorem 3.3] by taking $\mathbb{A} = \mathsf{Ord}$, and X an ordered set with a bottom element, as well as [12, Theorem 3.7] by taking $\mathbb{A} = \mathsf{Cat}$, and X a category with a strict initial object.

If we have a 2-functor $\Phi\colon \mathbb{B} \to \mathbb{A}$ between 2-categories, we may consider the composite 2-functor $\mathbb{A}(\Phi(-), X)\colon \mathbb{B}^{\mathsf{op}} \to \mathsf{CAT}$ for X in $\mathbb{A}$, and we denote its Grothendieck construction as $\Phi \Downarrow X \to \mathbb{B}$. In case Φ is fully faithful, we denote the total category as $\mathbb{B} \Downarrow X$ instead.

As a consequence of Corollary 16, we conclude that

Lemma 29. *The fibration $\Phi \Downarrow X \to \mathbb{B}$ preserves effective descent morphisms.*

[2]The Grothendieck construction can be defined for pseudofunctors $\mathbb{A}^{\mathsf{op}} \to \mathsf{CAT}$ for $\mathbb{A}$ a 2-category, outputting another 2-category, but for our purposes, we are implicitly taking the domain of the 2-functor to be the underlying category of $\mathbb{A}^{\mathsf{op}}$.

As an important example, we conclude at once that $\mathsf{Fam}(X) \to \mathsf{Set}$ preserves effective descent morphisms for any category X with a strict initial object, by taking Φ to be the "discrete category" functor $\mathsf{Set} \to \mathsf{Cat}$ – it should be noted that $\mathsf{Fam}(X) \simeq \mathsf{Set} \Downarrow X$. This observation is particularly helpful in the characterization of effective descent morphisms in $\mathsf{Fam}(X)$, a task which started in [50].

5.5. **Lax direct image:** Let $\mathbb{A}$ be a 2-category with a strict initial object. We consider the pseudo-functor

$$\begin{aligned} \mathcal{H}\colon \mathbb{A} &\to \mathsf{CAT} \\ x &\mapsto \mathbb{A} \Downarrow x \\ p &\mapsto p_! \end{aligned}$$

where $p_!\colon \mathbb{A} \Downarrow x \to \mathbb{A} \Downarrow y$ is given by composing (whiskering) with p on morphisms (2-cells).

The total category of its op-Grothendieck construction $\int^{\mathcal{A}} \mathcal{H} \to \mathbb{A}$ has

- objects are the morphisms of $\mathbb{A}$,
- morphisms $f \to g$ consist of triples (p, h, π) where $\pi\colon p \cdot f \to g \cdot h$ is a 2-cell, which may be depicted as

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\pi}{\nearrow}"] & b \arrow[d, "g"] \\ x \arrow[r, "p"'] & y \end{tikzcd}$$

Since $\mathcal{H}_a$ has an initial object for all a in $\mathbb{A}$, and the change-of-base functors create them for all $f\colon a \to b$ in $\mathbb{A}$, from Theorem 19 we conclude that:

Lemma 30. *The opfibration $\int^{\mathbb{A}} \mathcal{H} \to \mathbb{A}$ preserves effective descent morphisms.*

The total category of its Grothendieck construction $U\colon \int_{\mathcal{A}^{\mathsf{op}}} \mathcal{H} \to \mathbb{A}^{\mathsf{op}}$ has

- objects: the morphisms of $\mathbb{A}$,
- morphisms $f \to g$: triples (p, h, π) where $\pi\colon g \to p \cdot f \cdot h$ is a 2-cell, which may be depicted as

$$\begin{tikzcd} b \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\pi}{\Rightarrow}"] & a \arrow[d, "f"] \\ y & x \arrow[l, "p"] \end{tikzcd}$$

Since $\mathcal{H}_a$ has an initial object for all a in $\mathbb{A}$, and the change-of-base functors preserve them for all $f\colon a \to b$ in $\mathbb{A}$, we conclude that:

Lemma 31. *The fibration $\int_{\mathcal{A}^{\mathsf{op}}} \mathcal{H} \to \mathcal{A}^{\mathsf{op}}$ preserves effective descent morphisms.*

6. Epilogue

As explained in the introduction, we have restricted our present work to the context where the notion of bundles over an object is given by suitable comma categories. However, there is much to be explored in generalized settings where bundles are defined by alternative notions.

For instance, we highlight the work of [55, 36], where effective descent morphisms with respect to indexed categories of discrete fibrations and the indexed category of split fibrations were studied.

Building upon the work mentioned above, our project on the study of descent for generalized categorical structures [52, 36, 53, 51] aims to investigate effective descent morphisms with respect to generalized notions of bundles of (generalized) categorical structures, as briefly outlined below.

6.1. **Effective descent morphisms w.r.t. an orthogonal factorization systems.** Every orthogonal factorization system induces a notion of bundles in a category. We are particularly interested in the case of the category of categories CAT.

For each category $\mathcal{A}$, we may consider the category $\mathsf{FF}(A)$ which is the full subcategory of $\mathsf{CAT} \downarrow \mathcal{A}$ consisting of the fully faithful functors with codomain $\mathcal{A}$. This gives us an indexed category $\mathsf{FF} \colon \mathcal{A}^{\mathsf{op}} \to \mathsf{CAT}$ and a corresponding notion of effective descent morphism hasn't been studied in the literature.

More generally, the right class of morphisms of any orthogonal factorization system in CAT provides us with such an indexed category and hence a notion of effective descent morphism w.r.t. the given orthogonal factorization. Notably, this was explored by [55] in the case of [57], where the right class of morphisms consists of discrete fibrations. However, following the lines above, the orthogonal factorization defined in [38], whose right class of morphisms consists of *discrete splitting bifibrations* as introduced therein, provides us with a meaningful notion of effective descent morphisms yet to be explored.

6.2. **Effective descent morphisms w.r.t. a 2-monad.** We have a specific aim pertaining to freely generated categorical structures [29, 37, 42] and two-dimensional monad theory – see [1, 45, 3, 28, 33] for the general setting of 2-dimensional monad theory.

We start by observing that any pseudomonad $\mathcal{T}$ on CAT provides us with a relevant notion of bundles, called herein $\mathcal{T}$-bundles over a category $\mathcal{A}$, given by the $\mathcal{T}(\mathcal{A})$. In this setting, we are posed with the following question: assuming that $p \colon a \to b$ has a left (or right) adjoint p_*, we end up with a relevant notion of descent factorization

$$\begin{tikzcd} \mathcal{T}(a) \arrow[rr, "\mathcal{T}(p_*)"] \arrow[dr, "\mathcal{K}_p"'] & & \mathcal{T}(b) \\ & \mathsf{Desc}_{\mathcal{T}}(p) \arrow[ur] & \end{tikzcd}$$

following the recipe explained in the introduction. We, then, are interested in further understanding when p is of effective descent with respect to $\mathcal{T}$-bundles. We are particularly interested in the case where $\mathcal{T}$ is a Kock-Zoberlein pseudomonad, also known as lax idempotent pseudomonads [56, 60, 25, 44, 23] which encompasses most of the free completion pseudomonads [24, 49, 42, 37].

Still, within the setting of 2-monads, we have a particular interest in the notion of bundles provided by freely generated categorical structures, including free distributive, completely distributive, and extensive categories over categories [46, 42, 37].

6.3. **Future work.** Since the contexts above are not encompassed by the present work, we aim to extend our results on preservation properties to the general setting of [31], which provides a framework that covers all the settings and open problems described above.

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