Endpoint Geodesic Formulas on SE_3, $\mathrm{SO}_3 \times \mathbb{R}^3$ and Subspaces of SE_3

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Abstract. The special Euclidean group SE_3 and the closely related group $\mathrm{SO}_3 \times \mathbb{R}^3$ are important spaces in many fields of application. Explicit embeddings into matrix subgroups are being studied and it is shown that these spaces are extrinsic symmetric with respect to their embeddings. Furthermore, endpoint geodesic formulas are derived explicitly in both cases, as well as for certain subspaces of SE_3. Ultimately, the formulas could be used to solve for geodesics that connect two given points on these spaces.

SO_3	Special orthogonal group in 3 dimensions
$\mathfrak{so}_3$	Space of all skew-symmetric (3×3)-matrices
SE_3	Special Euclidean group in 3 dimensions
$\mathfrak{se}_3$	Lie algebra of SE_3
GL_4	Space of all invertible (4×4)-matrices
Sym_3	Space of all symmetric (3×3)-matrices
I_4, 0_3	(4×4)-identity matrix resp. (3×3)-zero matrix
$M \hookrightarrow W$	Submanifold M embedded into vector space W
Ad_G	Adjoint representation of the Lie group G
T_pM, N_pM	Tangent space of M at p resp. normal space of M at p
$\mathrm{d}_p f : T_pM \to T_{f(p)}N$	Derivative of the map $f : M \to N$ at $p \in M$
e^A	Matrix exponential of A
$\exp_p$	Riemannian exponential map at the point p
$\mathrm{Sym}^2(W^*)$	Space of all symmetric bilinear forms on W

Table 1: Table of Notations.

1 Introduction

The special Euclidean group of motions in 3-dimensional space, SE_3, provides a unified framework for dealing with problems involving rotations and translations in contexts where these transformations are inherently coupled. For that reason, it is commonly denote by the semi-direct product of the rotation group SO_3 and the abelian additive group $\mathbb{R}^3$, i.e. $\mathrm{SE}_3 = \mathrm{SO}_3 \ltimes \mathbb{R}^3$. In situations where rotations and translations are independent or only loosely coupled, working with the direct product $\mathrm{SO}_3 \times \mathbb{R}^3$ instead is more realistic. These groups turn out to be Lie groups and also Riemannian manifolds when equipped with a convenient Riemannian metric. Although their constituent subgroups are the same, these two groups differ in how those subgroups interact with each other within the product. Both, the direct and the semi-direct products, find interesting applications in many engineering applications, in particular in robotics and geometric mechanics.

For instance, to control simultaneously the position and orientation of a robot, SE_3 helps to keep track of the correct motions that force the robot to perform the necessary tasks. The literature involving applications of SE_3 is quite extensive and has been used in robotics for decades, as can be seen, for instance, in the books [7], [11], [13] and [18]. There are also many examples where control of orientation and position must be kept independent. Such is the case for robotic arms that are used to control surgical tools in minimally invasive surgeries, [20]. To position the surgery tool accurately in the body a translational motion is represented by a vector in $\mathbb{R}^3$, while rotations, modeled by SO_3, are used for orientation of the tool in accessing specific parts of the anatomy without causing damage. A similar situation arises in the control of remotely operated underwater vehicles that first have to move through the water to a certain location and then change its orientation to inspect the area of interest. When changes in pose are viewed from a space-fixed reference frame, such as controlling the state of aerial vehicles from the ground, direct product operations are more suitable, as explained in [2].

There are many circumstances when the rigid body motion is constrained, thus reducing the six degrees of freedom that correspond to the dimension of SE_3. In such cases, working on subspaces of SE_3 is more realistic since it avoids unnecessary or even forbidden tasks. One example of this situation is, for instance, the motion of a camera on a pan-tilt mechanism while moving horizontally along a linear track. To pan means rotating horizontally, to tilt means rotating vertically up and down, and linear motion means translating. In this case, the configuration space is a 3-dimensional subspace of SE_3 generated by two rotations and one translation. More examples of manipulators and other mechanisms with restricted motion may be found, for instance, in [3] and [22]. The latter also includes a full classification of all non-trivial symmetric subspaces of SE_3.

In all these applications, trajectory planning by smooth interpolating through multiple configurations of a moving robot is particularly important. If the objective is to move only from one configuration to another, the most efficient way will be to follow a geodesic path in the configuration space. However, if it is required to interpolate through multiple configurations while keeping the path smooth, polynomial spline interpolation is the most appropriate. In this case, a geometric procedure that generates polynomial interpolation curves on manifolds is available. This is the de Casteljau algorithm on manifolds, which generalizes Bézier curves in Euclidean spaces, and is based on successive geodesic interpolation, thus requiring explicit formulas for geodesics joining two points. For details concerning the general description of this algorithm and its implementation on some particular manifolds see, for instance, [4], [15] and [23]. But simply solving a two-point boundary problem might be harder than it seems. There is, however, a special class of Riemannian manifolds, called extrinsic symmetric spaces, where explicit formulas for the end-point geodesic problem can be derived after embedding the manifold in a certain vector space and using normal space involutions. This investigation was inspired by the following observation of the sphere case. Consider the circle S^1 embedded in $\mathbb{R}^2$. Fix $P, Q \in S^1$ as shown in Fig. 1 and take another point $Z \in \mathbb{R}^2$ in the embedding space. Now reflect Z on the normal space $N_P S^1 = \operatorname{span}(P)$ to get the point $Z' := \mathrm{R}_P(Z)$ and then reflect this new point on $N_Q S^1 = \operatorname{span}(Q)$ to get $Z'' := \mathrm{R}_Q(Z') = \mathrm{R}_Q(\mathrm{R}_P(Z))$. Then the angle between Z and Z'' equals twice the angle ϕ between P and Q, meaning the composition of two reflections can also be realized as a rotation.

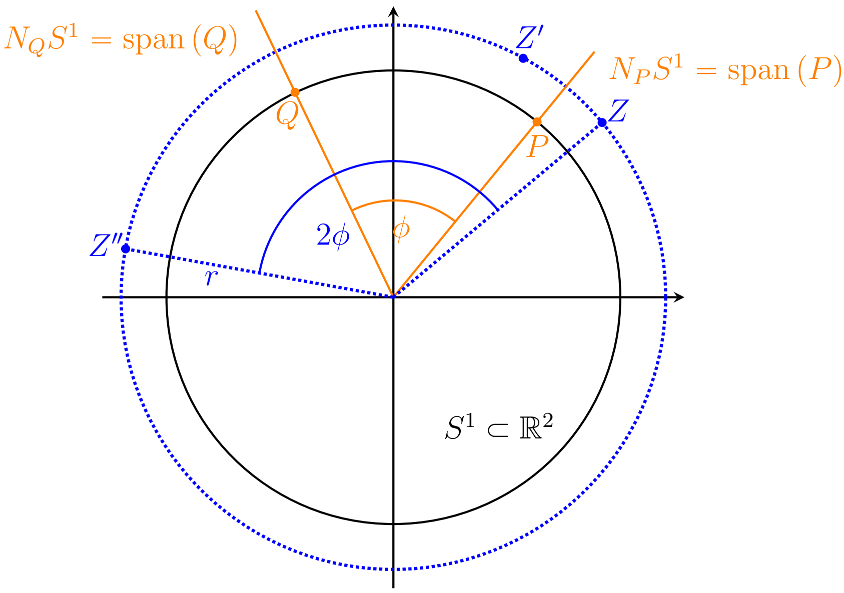

Figure 1: Illustration of (1.1) with $Z' := \mathrm{R}_P(Z)$ and $Z'' := \mathrm{R}_Q(Z') = \mathrm{R}_Q(\mathrm{R}_P(Z))$. The angle between Z and Z'' is exactly twice the angle ϕ between P and Q.

To be more precise, the reflections R_P and R_Q on $N_P S^1$ resp. $N_Q S^1$ are given by

$$\mathrm{R}_P = 2PP^\top - I_2 \quad \text{and} \quad \mathrm{R}_Q = 2QQ^\top - I_2.$$

By means of the formula $Q = \mathrm{e}^{\phi \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]} P$ we get

$$\mathrm{R}_Q \circ \mathrm{R}_P = \left(2QQ^\top - I_2\right) \cdot \left(2PP^\top - I_2\right) = \mathrm{e}^{2\phi \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]}. \tag{1.1}$$

This equation can be solved for ϕ, from which the geodesic connecting P and Q depending only on these points can be derived.

Our objective is to derive endpoint geodesic formulas for SE_3, $\mathrm{SO}_3 \times \mathbb{R}^3$ and also for some subspaces of SE_3. Although the whole construction works, with minor adjustments, for arbitrary dimensions, here we focus on the lower dimensional cases due to their importance in engineering applications. These Lie groups are not semisimple, but we will show that the construction in [19] for semisimple Lie groups still works and the derived formulas follow the same pattern as those obtained in [19]. The general procedure to derive endpoint geodesic formulas starts with the choice of an embedding of the Lie group in a suitable vector space equipped with an appropriate Riemannian metric. Then we can define normal space involutions which are the crucial ingredients for formulating the endpoint geodesic formula. The embedding of SE_3 might look a bit surprising and unmotivated, but was inspired by Kobayashi in [8].

This paper is a significally extended version of [16] and its organization is as follows. We start Section 2 with the necessary background including a matrix representation for the group $\mathrm{SO}_3 \times \mathbb{R}^3$ which, as far as we know, has not appeared before. Sections 3 and 4 are dedicated to derive endpoint geodesic formulas for SE_3 and $\mathrm{SO}_3 \times \mathbb{R}^3$, respectively. In Section 5 we present a classification of all symmetric subspaces of SE_3 that has been derived in [22] and present the endpoint geodesic formulas for two particular examples of this list.

2 Background

In this section, we present the groups SE_3 and $\mathrm{SO}_3 \times \mathbb{R}^3$, introduce the notion of extrinsic symmetric spaces and also recall some results about symmetric subspaces. For more information regarding the theory of Lie groups and Lie algebras we refer to [6].

Definition 2.1. *The semidirect product* $\mathrm{SE}_3 := \mathrm{SO}_3 \ltimes \mathbb{R}^3$ *with the group multiplication*

$$(R_1, v_1) \cdot (R_2, v_2) = (R_1 R_2, v_1 + R_1 v_2) \tag{2.1}$$

for $R_1, R_2 \in \mathrm{SO}_3$ *and* $v_1, v_2 \in \mathbb{R}^3$ *is called the* ***special Euclidean group****.*

One can easily see that the inverse of an element (R, v) is given by $(R, v)^{-1} = (R^\top, -R^\top v)$. As it turns out, group multiplication and inverse are smooth maps, which gives SE_3 the structure of a Lie group.
There is a transitive group action of SE_3 on $\mathbb{R}^3$ given by $((R, v), w) := Rw + v$. Furthermore, for computations it is often more practical to use a matrix representation of SE_3, i.e.

$$\mathrm{SE}_3 = \left\{ \begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_4 \;\middle|\; R \in \mathrm{SO}_3, v \in \mathbb{R}^3 \right\}.$$

This way, the group multiplication (2.1) becomes just matrix multiplication. The Lie algebra of the Lie group SE_3 is

$$\mathfrak{se}_3 := \left\{ \begin{bmatrix} \Omega & \omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, \Omega \in \mathfrak{so}_3, \omega \in \mathbb{R}^3 \right\}$$

equipped with the matrix commutator as Lie bracket. It is a well-known fact that $\mathfrak{so}_3$ and the space of all symmetric (3×3)-matrices Sym_3 give an additive decomposition of $\mathbb{R}^{3\times 3}$, i.e. every matrix can be written uniquely as the sum of a skew-symmetric and a symmetric matrix. This property will be used for calculating tangent and normal spaces. In [6], explicit formulas for the matrix exponential on $\mathfrak{se}_3$ are being found. We will need them for later calculations.

Lemma 2.2. *Let* $A = \left[\begin{smallmatrix} \Omega & \omega \\ 0 & 0 \end{smallmatrix}\right] \in \mathfrak{se}_3$, *with* $\Omega = \left[\begin{smallmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{smallmatrix}\right] \in \mathfrak{so}_3$ *and* $\omega \in \mathbb{R}^3$, *and define* $\theta := \sqrt{a^2 + b^2 + c^2}$. *Then, we have*

$$\mathrm{e}^A = \begin{cases} \begin{bmatrix} I_3 & \omega \\ 0 & 1 \end{bmatrix} & \text{if } \theta = 0 \\ \begin{bmatrix} \mathrm{e}^\Omega & \int_0^1 \mathrm{e}^{t\Omega} \mathrm{d}t\, \omega \\ 0 & 1 \end{bmatrix} & \text{if } \theta \neq 0, \end{cases}$$

with

$$\mathrm{e}^{t\Omega} = I_3 + \frac{\sin t\theta}{\theta}\Omega + \frac{(1 - \cos t\theta)}{\theta^2}\Omega^2$$

and

$$\int_0^1 \mathrm{e}^{t\Omega} \mathrm{d}t = I_3 + \frac{(1 - \cos\theta)}{\theta^2}\Omega + \frac{(\theta - \sin\theta)}{\theta^3}\Omega^2.$$

However, one must be careful to not confuse the matrix exponential used in the above lemma with the Riemannian exponential map. The latter one strongly depends on the choice of metric on SE_3.

Remark 2.3. Throughout this paper, an inner product is just a non-degenerate symmetric bilinear form, whereas a scalar product has to be positive definite, too.

The special Euclidean group becomes a Riemannian manifold when equipped with a Riemannian metric. By the one-to-one correspondence between left-invariant Riemannian metrics of a Lie group and scalar products on its Lie algebra, one obtains a 2-parameter family of left-invariant Riemannian metrics on SE_3 by choosing the scalar product

$$\left\langle \begin{bmatrix} \Omega_1 & \omega_1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \Omega_2 & \omega_2 \\ 0 & 0 \end{bmatrix} \right\rangle_F := \alpha \mathrm{tr}\left(\Omega_1^\top \Omega_2\right) + \beta \omega_1^\top \omega_2 \tag{2.2}$$

for $\Omega_1, \Omega_2 \in \mathfrak{so}_3$, $\omega_1, \omega_2 \in \mathbb{R}^3$ and arbitrary $\alpha, \beta > 0$. We will choose $\alpha = \beta = 4$ to match the Killing form on SO_3 when we set all translations to zero. This way, the scalar product becomes just a multiple of the usual Frobenius scalar product.

Remark 2.4. There exists a bi-invariant metric on SE_3, too. It is a pseudo-Riemannian one, as shown in [10]. Also, see [12] for more details.

For this bi-invariant metric, the Riemannian exponential map agrees with the usual matrix exponential. However, for the Riemannian metric that we are using, they do not. It is known (see for instance [21]) that a geodesic $\gamma : \mathbb{R} \to \mathrm{SE}_3$ with $\gamma(0) = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right]$ and $\dot{\gamma}(0) = \left[\begin{smallmatrix} R\Omega & R\omega \\ 0 & 0 \end{smallmatrix}\right]$ is given by

$$\gamma(t) = \begin{bmatrix} Re^{t\Omega} & tR\omega + v \\ 0 & 1 \end{bmatrix}$$

for all $t \in \mathbb{R}$. Thus, the Riemannian exponential map at $X \in \mathrm{SE}_3$ is given by

$$\exp_X : T_X\mathrm{SE}_3 \to \mathrm{SE}_3 \,, \quad \begin{bmatrix} R\Omega & R\omega \\ 0 & 0 \end{bmatrix} \mapsto \gamma(1) = \begin{bmatrix} Re^{\Omega} & R\omega + v \\ 0 & 1 \end{bmatrix}. \tag{2.3}$$

We will also have a detailed look at the direct product group $\mathrm{SO}_3 \times \mathbb{R}^3$. Consider the following component-wise group multiplication

$$(R_1, v_1) \cdot (R_2, v_2) = (R_1R_2, v_1 + v_2) \quad \text{for } R_1, R_2 \in \mathrm{SO}_3 \text{ and } v_1, v_2 \in \mathbb{R}^3 \tag{2.4}$$

with inverse of (R, v) given by $(R, v)^{-1} = \big(R^\top, -v\big)$. It is closely related to SE_3, but now rotations and translations do not get mixed up anymore, so this case is less complicated as we will see. There are similar results compared to SE_3. In particular, the Lie algebra of $\mathrm{SO}_3 \times \mathbb{R}^3$ is $\mathfrak{so}_3 \times \mathbb{R}^3$ with the Lie bracket being the matrix commutator on $\mathfrak{so}_3$ and zero on $\mathbb{R}^3$. We can also make this Lie Group a Riemannian manifold by choosing the Riemannian metric that is induced by the Killing form of SO_3 and the usual Euclidean scalar product on $\mathbb{R}^3$. In spite of the similarity between this metric and the one we use on SE_3, only the metric on the direct product $\mathrm{SO}_3 \times \mathbb{R}^3$ is bi-invariant due to the simpler group multiplication in this case. Furthermore, there is a matrix embedding for $\mathrm{SO}_3 \times \mathbb{R}^3$, as well. We identify any pair $(R, v) \in M$ with the (7×7)-matrix

$$\left[\begin{array}{c|c} R & 0 \\ \hline 0 & I_4 + V \end{array}\right] \quad \text{with} \quad V = \begin{bmatrix} 0_3 & v \\ 0 & 0 \end{bmatrix}. \tag{2.5}$$

The zeros in the matrices are chosen to be zero matrices of suitable sizes. The identification is chosen in such a way, that the group multiplication (2.4) on M becomes ordinary matrix multiplication of (7×7)-matrices, i.e.

$$\left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & I_4 + V_1 \end{array}\right] \cdot \left[\begin{array}{c|c} R_2 & 0 \\ \hline 0 & I_4 + V_2 \end{array}\right] = \left[\begin{array}{c|c} R_1R_2 & 0 \\ \hline 0 & I_4 + (V_1 + V_2) \end{array}\right]$$

since $V_1V_2 = 0$. Also, $\left[\begin{array}{c|c} I_3 & 0 \\ \hline 0 & I_4 \end{array}\right]$ is the identity element and $\left[\begin{array}{c|c} R & 0 \\ \hline 0 & I_4+V \end{array}\right]^{-1} = \left[\begin{array}{c|c} R^\top & 0 \\ \hline 0 & I_4-V \end{array}\right]$.

To end this section, let us define a special class of symmetric spaces called extrinsic symmetric spaces. We assume that the reader is familiar with basic notions of Riemannian geometry and refer to [9] and [14] for an introduction.

Definition 2.5 (Normal Space Involutions)**.** *Let $M \hookrightarrow W$ be a Riemannian embedded submanifold of a vector space W with Riemannian metric $\langle\cdot,\cdot\rangle_W^{(\cdot)}$ via an embedding $\iota : M \to W$. For every $p \in M$, the linear map $\mathrm{R}_{\iota(p)} : W \to W$ satisfying*

$$\mathrm{R}_{\iota(p)}|_{T_{\iota(p)}\iota(M)} = -\operatorname{id}_{T_{\iota(p)}\iota(M)} \quad \textit{and} \quad \mathrm{R}_{\iota(p)}|_{N_{\iota(p)}\iota(M)} = \operatorname{id}_{N_{\iota(p)}\iota(M)}$$

is called the ***linear normal space involution at*** *$\iota(p)$. Furthermore, the* ***affine normal space involution*** *$\mathcal{R}_p : W \to W$* ***at*** *$\iota(p)$ is defined by*

$$\mathcal{R}_{\iota(p)}(X) := \mathrm{R}_{\iota(p)}\big(X - \iota(p)\big) + \iota(p).$$

Definition 2.6 (Extrinsic Symmetric Space)**.** *Let $M \hookrightarrow W$ be a connected Riemannian embedded submanifold of a vector space W with Riemannian metric $\langle\cdot,\cdot\rangle_W^{(\cdot)}$ via an embedding $\iota : M \to W$. Then, $M \hookrightarrow W$ is called* ***extrinsic symmetric with respect to the embedding*** *ι if $\mathcal{R}_{\iota(p)}\big(\iota(M)\big) = \iota(M)$ for all $p \in M$.*

As it turns out, both SE_3 and $\mathrm{SO}_3 \times \mathbb{R}^3$ are extrinsic symmetric spaces with respect to their embeddings into the corresponding vector spaces that we will consider in Sections 3.1 and 4.1.

3 Endpoint Geodesic Formula for SE_3

3.1 The Embedding

To simplify notations, in this section we use the letter M to denote SE_3, i.e. $M := \mathrm{SE}_3$. Consider the real, 16-dimensional vector space

$$W := \left\{ \begin{bmatrix} 0 & B_x \\ B_y & 0 \end{bmatrix} := \begin{bmatrix} 0 & \begin{bmatrix} B & x \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -B^\top & y \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \;\middle|\; B \in \mathbb{R}^{3\times 3}, x, y \in \mathbb{R}^3, a \in \mathbb{R} \right\},$$

where the big matrices should be understood as (2×2)-block matrices with each block consisting of a (4×4)-matrix. Also consider the map

$$\iota : M \to W\,,\; X = \begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix} \mapsto \frac{1}{2}\begin{bmatrix} 0 & X \\ -X^{-1} & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -R^\top & R^\top v \\ 0 & -1 \end{bmatrix} & 0 \end{bmatrix}. \tag{3.1}$$

Clearly, ι is an embedding of M into the vector space W. We also write $\overline{M} := \iota(M)$ and $\overline{X} := \iota(X)$ for any $X \in M$. Later, we will need the following property of ι that relates the embedding ι with the actions of the Lie group M on itself and on the embedded $\overline{M}$.

Lemma 3.1. *Consider the following two group actions*

$$l : M \times M \to M \ , \ (X_0, X) \mapsto X_0 \cdot X$$

of M on M and

$$\lambda : M \times \overline{M} \to \overline{M} \ , \ \left(X_0, \overline{X}\right) \mapsto \mathrm{Ad}_{\left[\begin{smallmatrix} X_0 & 0 \\ 0 & I_4 \end{smallmatrix}\right]} \left(\overline{X}\right)$$

of M on $\overline{M}$. For any $X_0 \in M$, define the maps $l_{X_0} : M \to M$, $X \mapsto l(X_0, X)$ and $\lambda_{X_0} : \overline{M} \to \overline{M}$, $\overline{X} \mapsto \lambda(X_0, \overline{X})$. Then, the embedding $\iota : M \to \overline{M}$ is equivariant with respect to the actions l and λ, i.e.

$$\iota \circ l_{X_0} = \lambda_{X_0} \circ \iota$$

for all $X_0 \in M$.

Proof. Given any $X, X_0 \in M$, we have

$$\iota \left(l_{X_0} (X) \right) = \iota \left(X_0 \cdot X \right) = \frac{1}{2} \begin{bmatrix} 0 & X_0 \cdot X \\ -X^{-1} \cdot X_0^{-1} & 0 \end{bmatrix}$$

and

$$\begin{aligned} \lambda_{X_0} \left(\iota \left(X \right) \right) = \lambda_{X_0} \left(\frac{1}{2} \begin{bmatrix} 0 & X \\ -X^{-1} & 0 \end{bmatrix} \right) &= \frac{1}{2} \begin{bmatrix} X_0 & 0 \\ 0 & I_4 \end{bmatrix} \cdot \begin{bmatrix} 0 & X \\ -X^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} X_0^{-1} & 0 \\ 0 & I_4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & X_0 \cdot X \\ -X^{-1} \cdot X_0^{-1} & 0 \end{bmatrix}. \end{aligned}$$

This proves the lemma. ∎

3.2 A Riemannian Metric on W

The next step is to equip W with a Riemannian metric such that we can ultimately calculate normal spaces and normal space involutions.

Lemma 3.2. *For every $Z = \left[\begin{smallmatrix} 0 & B_x \\ B_y & 0 \end{smallmatrix}\right] \in W$, let $\langle \cdot, \cdot \rangle_W^{(\cdot)}$ be defined point-wise by*

$$\langle \cdot, \cdot \rangle_W^Z : W \times W \to \mathbb{R} \ , \ \langle Z_1, Z_2 \rangle_W^Z := \\ 8 \operatorname{tr} \left(\left(\begin{bmatrix} -B_y & 0 \\ 0 & I_4 \end{bmatrix} Z_1 \begin{bmatrix} B_x & 0 \\ 0 & I_4 \end{bmatrix} \right)^\top \cdot \begin{bmatrix} -B_y & 0 \\ 0 & I_4 \end{bmatrix} Z_2 \begin{bmatrix} B_x & 0 \\ 0 & I_4 \end{bmatrix} \right).$$

Then, $\langle \cdot, \cdot \rangle_W^{(\cdot)}$ is a smooth symmetric bilinear form on W. Furthermore, for every

$X = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right] \in M$*, the smooth symmetric bilinear form*

$$\langle \cdot, \cdot \rangle_W^X : W \times W \to \mathbb{R}\,,\; \langle Z_1, Z_2 \rangle_W^X := 8\,\mathrm{tr}\left(\left(\mathrm{Ad}_{\left[\begin{smallmatrix} X^{-1} & 0 \\ 0 & I_4 \end{smallmatrix}\right]}(Z_1)\right)^\top \cdot \mathrm{Ad}_{\left[\begin{smallmatrix} X^{-1} & 0 \\ 0 & I_4 \end{smallmatrix}\right]}(Z_2)\right) \tag{3.2}$$

defines a scalar product on W*.*

Proof. The first part of the lemma is clear since $\langle \cdot, \cdot \rangle_W^{(\cdot)}$ is just a combination of the usual Frobenius scalar product and matrix multiplications and thus a symmetric bilinear form on W. Clearly, it is smooth, too. For the second part fix an $X = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right] \in M$. All that is left to show is non-degeneracy. To see this, an explicit computation shows that

$$\begin{aligned} \langle Z_1, Z_2 \rangle_W^X = 8\Big(&2\mathrm{tr}\left(B_1^\top B_2\right) + \left\langle R^\top x_1 - a_1 R^\top v, R^\top x_2 - a_2 R^\top v \right\rangle \\ &+ \left\langle -B_1^\top v + y_1, -B_2^\top v + y_2 \right\rangle + 2a_1 a_2\Big). \end{aligned}$$

From this calculation the non-degeneracy follows. ■

The next lemma is critical for our construction. It shows that $\overline{M}$ can be viewed as a submanifold of an open neighbourhood of $\overline{M}$ in W. The arguments are analogous to those in [17], Lemma 2.

Lemma 3.3. *Consider* $M \hookrightarrow W$ *embedded as in (3.1). There exists an open neighbourhood* $U \subset W$ *of* $\overline{M}$ *such that* $\tau_U^* \langle \cdot, \cdot \rangle_W^{(\cdot)}$ *defines a Riemannian metric on* U*, where* $\tau_U : U \to W$ *is the canonical inclusion. Furthermore,* $(M, \iota^* \langle \cdot, \cdot \rangle_W^{(\cdot)})$ *is a Riemannian submanifold of* $(U, \tau_U^* \langle \cdot, \cdot \rangle_W^{(\cdot)})$*.*

Proof. Define the continuous map

$$\varphi : W \to \mathrm{Sym}^2(W^*)\;,\; Z \mapsto \langle \cdot, \cdot \rangle_W^Z\,.$$

Here, the notation $\mathrm{Sym}^2(W^*)$ is used for the space of all symmetric bilinear forms on W. Lemma 3.2 implies that $\varphi(X) \in \mathrm{Sym}^2(W^*)$ is non-degenerate for all $X \in M$. Since φ is continuous there is an open neighbourhood U_X of $\overline{X}$ in W such that $\varphi(\widetilde{X})$ is non-degenerate for all $\widetilde{X} \in U_X$. We construct the open neighbourhood U by setting

$$U := \bigcup_{X \in M} U_X.$$

Consequently, U is an open subset of W as a union of open subsets $U_X \subset W$ and it fulfills $\overline{M} \subset U$ since every $\overline{X}$ lies in its neighbourhood U_X. Moreover, the symmetric bilinear form $\varphi(X) = \langle \cdot, \cdot \rangle_W^X$ is non-degenerate for every $\overline{X} \in U$ by construction of U. This is equivalent to the statement that $\tau_U^* \langle \cdot, \cdot \rangle_W^{(\cdot)}$ defines a Riemannian metric on U with τ_U^* being the pull-back of the canonical inclusion $\tau_U : U \to W$. Since M is embedded into U by ι, the last part of the lemma is clear, too. ■

For fixed $X \in M$, we can now define normal spaces with respect to the scalar product $\langle \cdot, \cdot \rangle_W^X$ in the next section. Because of the above lemma, we do not have to worry about what happens outside of an open neighbourhood of $\overline{M}$. This makes computations much easier since we know that the symmetric bilinear form $\langle \cdot, \cdot \rangle_W^{(\cdot)}$ is a scalar product there and thus non-degenerate.

3.3 SE_3 is an Extrinsic Symmetric Space

Lemma 3.4. *Consider the embedding $\iota : M \hookrightarrow W$ as given in (3.1). For every $X = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right] \in M$, we have the tangent space*

$$T_{\overline{X}}\overline{M} = \left\{ \begin{bmatrix} 0 & \begin{bmatrix} R\Omega & R\omega \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Omega R^\top & -\Omega R^\top v + \omega \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \,\middle|\, \Omega \in \mathfrak{so}_3, \omega \in \mathbb{R}^3 \right\}$$

and the normal space

$$N_{\overline{X}}\overline{M} = \left\{ \begin{bmatrix} 0 & \begin{bmatrix} RS & R\widetilde{\omega} + sv \\ 0 & s \end{bmatrix} \\ \begin{bmatrix} -SR^\top & SR^\top v - \widetilde{\omega} \\ 0 & -s \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \,\middle|\, S \in \mathrm{Sym}_3, \widetilde{\omega} \in \mathbb{R}^3, s \in \mathbb{R} \right\}.$$

with respect to the scalar product (3.2).

Proof. The tangent space at $X = I_4$ is

$$T_{\overline{I_4}}\overline{M} = \left\{ \begin{bmatrix} 0 & \begin{bmatrix} \Omega & \omega \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Omega & \omega \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \,\middle|\, \Omega \in \mathfrak{so}_3, \omega \in \mathbb{R}^3 \right\}.$$

Since ι is an equivariant map with respect to the two actions l and λ as defined in Lemma 3.1, the tangent space at an arbitrary $X \in M$ is given by

$$\begin{aligned} T_{\overline{X}}\overline{M} &= \mathrm{Ad}_{\left[\begin{smallmatrix} X & 0 \\ 0 & I_4 \end{smallmatrix}\right]} \left(T_{\overline{I_4}}\overline{M}\right) \\ &= \left\{ \begin{bmatrix} 0 & \begin{bmatrix} R\Omega & R\omega \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Omega R^\top & -\Omega R^\top v + \omega \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \,\middle|\, \Omega \in \mathfrak{so}_3, \omega \in \mathbb{R}^3 \right\} \end{aligned}$$

for any $X = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right] \in M$. For the normal spaces one calculates

$$\left\langle \begin{bmatrix} 0 & \begin{bmatrix} R\Omega & R\omega \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Omega R^\top & -\Omega R^\top v + \omega \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \begin{bmatrix} RS & R\widetilde{\omega} + sv \\ 0 & s \end{bmatrix} \\ \begin{bmatrix} -SR^\top & SR^\top v - \widetilde{\omega} \\ 0 & -s \end{bmatrix} & 0 \end{bmatrix} \right\rangle_W^X$$

$$= 8\mathrm{tr}\left(\begin{bmatrix} 0 & \begin{bmatrix} -\Omega & 0 \\ \omega^\top & 0 \end{bmatrix} \\ \begin{bmatrix} -\Omega & 0 \\ \omega^\top & 0 \end{bmatrix} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \begin{bmatrix} S & \widetilde{\omega} \\ 0 & s \end{bmatrix} \\ \begin{bmatrix} -S & -\widetilde{\omega} \\ 0 & -s \end{bmatrix} & 0 \end{bmatrix} \right) = 0.$$

Due to dimensional reasons, the normal space at $\overline{X}$ cannot be bigger. This proves the lemma. ■

Lemma 3.5. *The embedding $\iota : M \to W$, as defined in (3.1), is isometric with respect to the Riemannian metric on M induced by $\langle \cdot, \cdot \rangle_F$, as defined in (2.2), and the Riemannian metric $\langle \cdot, \cdot \rangle_W^{(\cdot)}$ on W, as defined in (3.2).*

Proof. We have to show that

$$\left\langle X\zeta_1, X\zeta_2 \right\rangle_X = \left\langle \mathrm{d}_X f(X\zeta_1), \mathrm{d}_X f(X\zeta_2) \right\rangle_W^X$$

for all $X \in M = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right]$ and $X\zeta_1, X\zeta_2 \in T_X M$. For the left-hand side we get

$$\begin{aligned}
\left\langle X\zeta_1, X\zeta_2 \right\rangle_X &= \left\langle \begin{bmatrix} R\Omega_1 & R\omega_1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} R\Omega_2 & R\omega_2 \\ 0 & 0 \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right]} \\
&= \left\langle \begin{bmatrix} R^\top & -R^\top v \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R\Omega_1 & R\omega_1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} R^\top & -R^\top v \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R\Omega_2 & R\omega_2 \\ 0 & 0 \end{bmatrix} \right\rangle_F \\
&= \left\langle \begin{bmatrix} \Omega_1 & \omega_1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \Omega_2 & \omega_2 \\ 0 & 0 \end{bmatrix} \right\rangle_F = 4\left(\mathrm{tr}\left(\Omega_1^\top \Omega_2\right) + \omega_1^\top \omega_2\right).
\end{aligned}$$

Accordingly, for the right-hand side we get

$$\begin{aligned}
&\left\langle \mathrm{d}_X f(X\zeta_1), \mathrm{d}_X f(X\zeta_2) \right\rangle_W^X \\
&= 8\,\mathrm{tr}\left(\left(\mathrm{Ad}_{\left[\begin{smallmatrix} X^{-1} & 0 \\ 0 & I_4 \end{smallmatrix}\right]} \left(\frac{1}{2} \begin{bmatrix} 0 & X\zeta_1 \\ \zeta_1 X^{-1} & 0 \end{bmatrix} \right) \right)^\top \cdot \mathrm{Ad}_{\left[\begin{smallmatrix} X^{-1} & 0 \\ 0 & I_4 \end{smallmatrix}\right]} \left(\frac{1}{2} \begin{bmatrix} 0 & X\zeta_2 \\ \zeta_2 X^{-1} & 0 \end{bmatrix} \right) \right) \\
&= 2\,\mathrm{tr}\left(\begin{bmatrix} 0 & \zeta_1 \\ \zeta_1 & 0 \end{bmatrix}^\top \cdot \begin{bmatrix} 0 & \zeta_2 \\ \zeta_2 & 0 \end{bmatrix} \right) = 4\left(\mathrm{tr}\left(\Omega_1^\top \Omega_2\right) + \omega_1^\top \omega_2\right).
\end{aligned}$$

■

Lemma 3.6. *Consider the embedding $\iota : M \hookrightarrow W$ as given in (3.1). For every $X = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right] \in M$, the map*

$$\mathrm{R}_{\overline{X}} : W \to W\,, \quad \begin{bmatrix} 0 & \begin{bmatrix} B & x \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -B^\top & y \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & \begin{bmatrix} RB^\top R & \widetilde{y} \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -(RB^\top R)^\top & \widetilde{x} \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix}$$

with

$$\widetilde{y} = av + R\left(B^\top v - y\right) \quad \text{and} \quad \widetilde{x} = -R^\top(x - av) + R^\top B R^\top v$$

is the linear normal space involution at $\overline{X}$.

Proof. Fix an $X = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right] \in M$. For the condition on the tangent spaces we calculate

$$\mathrm{R}_{\overline{X}}\left(\begin{bmatrix} 0 & \begin{bmatrix} R\Omega & R\omega \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Omega R^\top & -\Omega R^\top v + \omega \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \begin{bmatrix} R(R\Omega)^\top R & \widetilde{y} \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} -R^\top R\Omega R^\top & \widetilde{x} \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix}$$

for all $\Omega \in \mathfrak{so}_3$, $\omega \in \mathbb{R}^3$ and with

$$\widetilde{y} = R\left((R\Omega)^\top v + \Omega R^\top v - \omega\right) = -R\omega \quad \text{and} \quad \widetilde{x} = -R^\top R\omega + R^\top R\Omega R^\top v = \Omega R^\top v - \omega.$$

This shows $\mathrm{R}_{\overline{X}}|_{T_{\overline{X}}\overline{M}} = -\mathrm{id}_{T_{\overline{X}}\overline{M}}$. For the condition on the normal spaces we calculate

$$\mathrm{R}_{\overline{X}}\left(\begin{bmatrix} 0 & \begin{bmatrix} RS & R\widetilde{\omega} + sv \\ 0 & s \end{bmatrix} \\ \begin{bmatrix} -SR^\top & SR^\top v - \widetilde{\omega} \\ 0 & -s \end{bmatrix} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \begin{bmatrix} R(RS)^\top R & \widetilde{y} \\ 0 & s \end{bmatrix} \\ \begin{bmatrix} -R^\top RSR^\top & \widetilde{x} \\ 0 & -s \end{bmatrix} & 0 \end{bmatrix}$$

for all $S \in \mathrm{Sym}_3$, $\widetilde{\omega} \in \mathbb{R}^3$, $s \in \mathbb{R}$ and with

$$\widetilde{y} = sv + R\left((RS)^\top v - SR^\top v + \widetilde{\omega}\right) = R\widetilde{\omega} + sv \quad \text{and} \quad \widetilde{x} = -R^\top R\widetilde{\omega} + R^\top RSR^\top v = SR^\top v - \widetilde{\omega}.$$

This shows $\mathrm{R}_{\overline{X}}|_{N_{\overline{X}}\overline{M}} = \mathrm{id}_{N_{\overline{X}}\overline{M}}$. ■

Corollary 3.7. *$M \hookrightarrow W$ is an extrinsic symmetric space with respect to the embedding $\iota : M \to W$ as defined in (3.1).*

Proof. One can check that $\mathrm{R}_{\overline{X}} \equiv \mathcal{R}_{\overline{X}}$ for all $X = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right] \in M$, so we have to verify that $\mathrm{R}_{\overline{X}}(\overline{M}) = \overline{M}$ for all $X \in M$ to show that $M \hookrightarrow W$ is extrinsic symmetric with respect to the embedding $\iota : M \to W$ as defined in (3.1). Given an $X_0 = \left[\begin{smallmatrix} R_0 & v_0 \\ 0 & 1 \end{smallmatrix}\right] \in M$, we have

$$\mathrm{R}_{\overline{X}}\left(\overline{X_0}\right) = \begin{bmatrix} 0 & \begin{bmatrix} RR_0^\top R & \widetilde{y} \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -\left(RR_0^\top R\right)^\top & \widetilde{x} \\ 0 & -1 \end{bmatrix} & 0 \end{bmatrix}$$

with

$$\widetilde{y} = v + RR_0^\top (v - v_0) \quad \text{and} \quad \widetilde{x} = R^\top (v - v_0) + R^\top R_0 R^\top v = \left(RR_0^\top R\right)^\top \widetilde{y},$$

so $\mathrm{R}_{\overline{X}}(M) \subset M$. Also, given any $X_2 = \left[\begin{smallmatrix} R_2 & v_2 \\ 0 & 1 \end{smallmatrix}\right] \in M$, we find an $X_1 = \left[\begin{smallmatrix} R_1 & v_1 \\ 0 & 1 \end{smallmatrix}\right] \in M$ with $\mathrm{R}_{\overline{X}}\left(\overline{X_1}\right) = \overline{X_2}$ by choosing $R_1 = RR_2^\top R$ and $v_1 = v - RR_2^\top (v_2 - v)$. ■

3.4 Deriving the Endpoint Geodesic Formula on SE_3

Theorem 3.8 (Endpoint Geodesic Formula on SE_3)**.** *Consider the embedding $\iota : M \hookrightarrow W$ as given in (3.1). Let $X = \left[\begin{smallmatrix} R & v \\ 0 & 1 \end{smallmatrix}\right] \in M$ and $Z \in W$ be arbitrary. Then the endpoint geodesic formula*

$$\mathrm{R}_Q\left(\mathrm{R}_P(Z)\right) = \mathrm{Ad}_{\mathrm{e}^{2\Theta}}(Z) \tag{3.3}$$

holds, where

$$P := \overline{X} = \frac{1}{2}\begin{bmatrix} 0 & X \\ -X^{-1} & 0 \end{bmatrix}, \; Q := \mathrm{Ad}_{k\mathrm{e}^{\xi}k^{-1}}(P), \; \Theta := \mathrm{Ad}_k(\xi)$$

and

$$k = \begin{bmatrix} \begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} I_3 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}, \; \xi = \begin{bmatrix} \zeta & 0 \\ 0 & -\zeta \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \Omega & \omega \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} -\Omega & -\omega \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

with $\Omega \in \mathfrak{so}_3$ and $\omega \in \mathbb{R}^3$.

Proof. Fix an arbitrary $Z = \left[\begin{smallmatrix} 0 & B_x \\ B_y & 0 \end{smallmatrix}\right] \in W$. We start with the left-hand side of (3.3). Firstly, we compute Q more explicitly. We get

$$Q = \mathrm{Ad}_{k\mathrm{e}^{\xi}k^{-1}}(P) = \frac{1}{2}\begin{bmatrix} X\mathrm{e}^{\zeta}X^{-1} & 0 \\ 0 & \mathrm{e}^{-\zeta} \end{bmatrix}\begin{bmatrix} 0 & X \\ X^{-1} & 0 \end{bmatrix}\begin{bmatrix} X\mathrm{e}^{-\zeta}X^{-1} & 0 \\ 0 & \mathrm{e}^{\zeta} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & X\mathrm{e}^{2\zeta} \\ -\mathrm{e}^{-2\zeta}X^{-1} & 0 \end{bmatrix}.$$

Applying Lemma 2.2 to $\mathrm{e}^{2\zeta}$ resp. $\mathrm{e}^{-2\zeta}$ leads to

$$\mathrm{e}^{2\zeta} = \mathrm{e}^{\left[\begin{smallmatrix} 2\Omega & 2\omega \\ 0 & 0 \end{smallmatrix}\right]} = \begin{bmatrix} \mathrm{e}^{2\Omega} & \omega^+ \\ 0 & 1 \end{bmatrix}$$
$$\mathrm{e}^{-2\zeta} = \mathrm{e}^{\left[\begin{smallmatrix} -2\Omega & -2\omega \\ 0 & 0 \end{smallmatrix}\right]} = \begin{bmatrix} -\mathrm{e}^{2\Omega} & \omega^- \\ 0 & 1 \end{bmatrix},$$

where $\omega^+ := 2\int_0^1 \mathrm{e}^{2t\Omega}\omega \,\mathrm{d}t$ and $\omega^- := -2\int_0^1 \mathrm{e}^{-2t\Omega}\omega \,\mathrm{d}t$. and thus we have

$$Q = \frac{1}{2}\begin{bmatrix} 0 & \begin{bmatrix} R\mathrm{e}^{2\Omega} & R\omega^+ + v \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -\mathrm{e}^{-2\Omega}R^\top & \mathrm{e}^{-2\Omega}R^\top v + \omega^- \\ 0 & -1 \end{bmatrix} & 0 \end{bmatrix}.$$

Now we can compute

$$\begin{aligned} \mathrm{R}_Q\left(\mathrm{R}_P(Z)\right) &= \mathrm{R}_Q\left(\begin{bmatrix} 0 & \begin{bmatrix} RB^\top R & av + R\left(B^\top v - y\right) \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -(RB^\top R)^\top & -R^\top(x-av) + R^\top BR^\top v \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & \begin{bmatrix} R\mathrm{e}^{2\Omega}R^\top B\mathrm{e}^{2\Omega} & u_1 \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -\mathrm{e}^{-2\Omega}B^\top R\mathrm{e}^{-2\Omega}R^\top & u_2 \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix} \end{aligned}$$

with

$$\begin{aligned} u_1 &= a(R\omega^+ + v) + R\mathrm{e}^{2\Omega}\left(R^\top BR^\top(R\omega^+ + v) + R^\top(x-av) - R^\top BR^\top v\right) \\ &= aR\omega^+ + av + R\mathrm{e}^{2\Omega}R^\top B\omega^+ + R\mathrm{e}^{2\Omega}R^\top x - aR\mathrm{e}^{2\Omega}R^\top v, \\ u_2 &= -\mathrm{e}^{-2\Omega}R^\top\left(av + R(B^\top v - y) - a(R\omega^+ + v)\right) + \mathrm{e}^{-2\Omega}R^\top RB^\top R\mathrm{e}^{-2\Omega}R^\top(R\omega^+ + v) \\ &= -\mathrm{e}^{-2\Omega}B^\top v + \mathrm{e}^{2\Omega}y + a\mathrm{e}^{-2\Omega}\omega^+ + \mathrm{e}^{-2\Omega}B^\top R\mathrm{e}^{-2\Omega}\omega^+ + \mathrm{e}^{-2\Omega}B^\top R\mathrm{e}^{-2\Omega}R^\top v. \end{aligned}$$

For the right-hand side of (3.3) we first compute

$$\mathrm{e}^{2\Theta} = k\mathrm{e}^{2\xi}k^{-1} = \begin{bmatrix} X\mathrm{e}^{2\zeta}X^{-1} & 0 \\ 0 & e^{-2\zeta} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} R\mathrm{e}^{2\Omega}R^\top & -R\mathrm{e}^{2\Omega}R^\top v + R\omega^+ + v \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \mathrm{e}^{-2\Omega} & \omega^- \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

and then we get

$$\begin{aligned} \mathrm{Ad}_{\mathrm{e}^{2\Theta}}(Z) &= \mathrm{e}^{2\Theta}\begin{bmatrix} 0 & \begin{bmatrix} B & x \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -B^\top & y \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix}\mathrm{e}^{-2\Theta} \\ &= \begin{bmatrix} 0 & \begin{bmatrix} R\mathrm{e}^{2\Omega}R^\top B\mathrm{e}^{2\Omega} & \widetilde{u}_1 \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -\mathrm{e}^{-2\Omega}B^\top R\mathrm{e}^{-2\Omega}R^\top & \widetilde{u}_2 \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix} \end{aligned}$$

with

$$\widetilde{u}_1 = R\mathrm{e}^{2\Omega}R^\top B\omega^+ + R\mathrm{e}^{2\Omega}R^\top x - a\left(R\mathrm{e}^{2\Omega}R^\top v - R\omega^+ - v\right),$$
$$\widetilde{u}_2 = -\mathrm{e}^{-2\Omega}B^\top\left(-R\mathrm{e}^{-2\Omega}R^\top v + R\omega^- + v\right) + \mathrm{e}^{-2\Omega}y - a\omega^-.$$

Since $\mathrm{e}^{-2\Omega}\omega^+ = -\omega^-$, we have $u_1 = \widetilde{u}_1$ and $u_2 = \widetilde{u}_2$ and this proves the theorem. ■

Remark 3.9. The above theorem is still true if we use the Riemannian exponential map on $\overline{M}$, which is easily defined from (2.3), instead of the matrix exponential. To see that, one can show that the normal space involutions in Lemma 3.6 are of the same form as those described in [19]. Thus, the proof of Theorem 3.14, [19], for the endpoint geodesic formula using the Riemannian exponential map, also applies here.
Using the Riemannian exponential map instead of the matrix exponential is helpful since the above endpoint geodesic formula can then be used to recover the endpoint geodesic $\gamma : \mathbb{R} \to \overline{M}$ that satisfies $\gamma(0) = P$ and $\gamma(1) = Q$, for $P, Q \in \overline{M}$. This procedure works for all examples of extrinsic symmetric spaces that are mentioned in [19], too. Thus, this construction gives a systematic way of finding endpoint geodesics.

4 Endpoint Geodesic Formula for $\mathrm{SO}_3 \times \mathbb{R}^3$

4.1 The Embedding

In this section we consider $M := \mathrm{SO}_3 \times \mathbb{R}^3$ in its matrix representation as defined in (2.5). We also consider the real, 16-dimensional vector space

$$W := \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D+U \end{array}\right] \in \mathbb{R}^{7\times 7} \;\middle|\; A \in \mathbb{R}^{3\times 3}, D = \operatorname{diag}(d_1, \dots, d_4), U = \begin{bmatrix} 0_3 & u \\ 0 & 0 \end{bmatrix} \right\},$$

with zero blocks of suitable sizes. Then, the map

$$\iota : M \to W\ ,\ X \mapsto \iota(X) := X \tag{4.1}$$

clearly is an embedding of M into W with $\iota(M) = M$. As a Lie group, M acts on itself by matrix multiplication and the embedding ι is trivially equivariant with respect to that action. There is also a transitive group action $\phi : \big(M \times M\big) \times M \to M$, given by

$$\Bigg(\Bigg(\underbrace{\left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & I_4 + V_1 \end{array}\right]}_{X_1}, \underbrace{\left[\begin{array}{c|c} R_2 & 0 \\ \hline 0 & I_4 + V_2 \end{array}\right]}_{X_2}\Bigg), \underbrace{\left[\begin{array}{c|c} R & 0 \\ \hline 0 & I_4 + V \end{array}\right]}_{X}\Bigg)$$
$$\mapsto X_1 X X_2^{-1} = \left[\begin{array}{c|c} R_1 R R_2^\top & 0 \\ \hline 0 & I_4 + (V_1 + V - V_2) \end{array}\right].$$

We will need that action for the formulation of the endpoint geodesic formula in the next subsection. This action extends to W in an obvious way.

4.2 $\mathrm{SO}_3 \times \mathbb{R}^3$ is an Extrinsic Symmetric Space

Consider the vector space W equipped with the usual Frobenius scalar product, i.e.

$$\left\langle \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & D_1 + U_1 \end{array}\right], \left[\begin{array}{c|c} A_2 & 0 \\ \hline 0 & D_2 + U_2 \end{array}\right] \right\rangle_W := \operatorname{tr}\left(A_1^\top A_2 + D_1 D_2 + U_1^\top U_2\right). \tag{4.2}$$

Lemma 4.1. *Consider the embedding $\iota : M \to W$ as given in (4.1). Given any $X \in M$, we have the tangent space*

$$T_X M = \left\{ \left[\begin{array}{c|c} R\Omega & 0 \\ \hline 0 & U \end{array}\right] \in \mathbb{R}^{7\times 7} \,\middle|\, \Omega \in \mathfrak{so}_3, U = \begin{bmatrix} 0_3 & u \\ 0 & 0 \end{bmatrix} \right\}$$

and the normal space

$$N_X M = \left\{ \left[\begin{array}{c|c} RS & 0 \\ \hline 0 & D \end{array}\right] \in \mathbb{R}^{7\times 7} \,\middle|\, S \in \mathrm{Sym}_3, D = \operatorname{diag}(d_1, \ldots, d_4) \right\}$$

with respect to the scalar product (4.2).

Lemma 4.2. *Consider the embedding $\iota : M \to W$ as given in (4.1). Given any $X \in M$, the linear normal space involution at X is given by*

$$\mathrm{R}_X : W \to W\,, \quad \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D + U \end{array}\right] \mapsto \left[\begin{array}{c|c} RA^\top R & 0 \\ \hline 0 & D - U \end{array}\right]$$

and the affine normal space involution at X is given by

$$\mathcal{R}_X : W \to W\,, \quad \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D + U \end{array}\right] \mapsto \left[\begin{array}{c|c} RA^\top R & 0 \\ \hline 0 & D + (2V - U) \end{array}\right].$$

Proof. One immediately sees that

$$\mathrm{R}_X|_{T_X M}(Z) = -Z \quad \text{and} \quad \mathrm{R}_X|_{N_X M}(Z) = Z$$

for all $X \in M$ and all $Z \in W$. The affine normal space involutions are

$$\begin{aligned}\mathcal{R}_X(Z) &= \mathrm{R}_X(Z - X) + X \\ &= \mathrm{R}_X\left(\left[\begin{array}{c|c} A - R & 0 \\ \hline 0 & (D - I_4) + (U - V) \end{array}\right]\right) + \left[\begin{array}{c|c} R & 0 \\ \hline 0 & I_4 + V \end{array}\right] \\ &= \left[\begin{array}{c|c} RA^\top R & 0 \\ \hline 0 & D + (2V - U) \end{array}\right].\end{aligned}$$

This proves the lemma. ■

Corollary 4.3. *$M \hookrightarrow W$ is an extrinsic symmetric space.*

Proof. We have to check that $\mathcal{R}_X(M) = M$ for all $X \in M$. Given any $X_0 \in M$, we have

$$\mathcal{R}_X(X_0) = \left[\begin{array}{c|c} RR_0^\top R & 0 \\ \hline 0 & I_4 + (2V - V_0) \end{array}\right] \in M.$$

This shows that $\mathcal{R}_X(M) \subset M$. Furthermore, for any $X_2 \in M$ we find an $X_1 \in M$ such that $\mathcal{R}_X(X_1) = X_2$ by choosing $R_1 = RR_2^\top R$ and $v_1 = 2v - v_2$. ■

4.3 Deriving the Endpoint Geodesic Formula on $\mathrm{SO}_3 \times \mathbb{R}^3$

Theorem 4.4 (Endpoint Geodesic Formula on $\mathrm{SO}_3 \times \mathbb{R}^3$)**.** *Consider the embedding $\iota : M \to W$ as given in (4.1). Let $X \in M$ be arbitrary and define*

$$\begin{aligned} P &:= \left[\begin{array}{c|c} R & 0 \\ \hline 0 & I_4 + V \end{array}\right], \\ Q &:= \phi\left(\left(\left[\begin{array}{c|c} R\mathrm{e}^{\Omega} R^\top & 0 \\ \hline 0 & I_4 + \Sigma \end{array}\right], \left[\begin{array}{c|c} \mathrm{e}^{-\Omega} & 0 \\ \hline 0 & I_4 - \Sigma \end{array}\right]\right), \left[\begin{array}{c|c} R & 0 \\ \hline 0 & I_4 + V \end{array}\right]\right), \\ \Theta &:= \left(\left[\begin{array}{c|c} R\Omega R^\top & 0 \\ \hline 0 & \Sigma \end{array}\right], \left[\begin{array}{c|c} -\Omega & 0 \\ \hline 0 & -\Sigma \end{array}\right]\right) \end{aligned}$$

for any $\Omega \in \mathfrak{so}_3$ and $\sigma \in \mathbb{R}^3$. Then the endpoint geodesic formula

$$\mathcal{R}_Q \circ \mathcal{R}_P = \phi_{\mathrm{e}^{2\Theta}} \tag{4.3}$$

holds on M.

Proof. Given any $\left[\begin{array}{c|c} S & 0 \\ \hline 0 & I_4 + U \end{array}\right] \in M$, we start with the left-hand side of (4.3). It gives

$$\mathcal{R}_Q\left(\mathcal{R}_P\left(\left[\begin{array}{c|c} S & 0 \\ \hline 0 & I_4 + U \end{array}\right]\right)\right) = \mathcal{R}_Q\left(\left[\begin{array}{c|c} RS^\top R & 0 \\ \hline 0 & I_4 + (2V - U) \end{array}\right]\right).$$

We calculate Q more explicitly. It is

$$\begin{aligned}
Q &= \phi\left(\left(\left[\begin{array}{c|c} R\mathrm{e}^{\Omega}R^{\top} & 0 \\ \hline 0 & I_4+\Sigma \end{array}\right], \left[\begin{array}{c|c} \mathrm{e}^{-\Omega} & 0 \\ \hline 0 & I_4-\Sigma \end{array}\right]\right), \left[\begin{array}{c|c} R & 0 \\ \hline 0 & I_4+V \end{array}\right]\right) \\
&= \left[\begin{array}{c|c} R\mathrm{e}^{2\Omega} & 0 \\ \hline 0 & I_4+(V+2\Sigma) \end{array}\right].
\end{aligned}$$

Using this, we get

$$\mathcal{R}_Q\left(\left[\begin{array}{c|c} RS^{\top}R & 0 \\ \hline 0 & I_4+(2V-U) \end{array}\right]\right) = \left[\begin{array}{c|c} R\mathrm{e}^{2\Omega}R^{\top}S\mathrm{e}^{2\Omega} & 0 \\ \hline 0 & I_4+(4\Sigma+U) \end{array}\right].$$

Now we calculate the right-hand side of (4.3). We get

$$\begin{aligned}
&\phi_{\mathrm{e}^{2\Theta}}\left(\left[\begin{array}{c|c} S & 0 \\ \hline 0 & I_4+U \end{array}\right]\right) \\
&= \phi\left(\left(\left[\begin{array}{c|c} R\mathrm{e}^{2\Omega}R^{\top} & 0 \\ \hline 0 & I_4+2\Sigma \end{array}\right], \left[\begin{array}{c|c} \mathrm{e}^{-2\Omega} & 0 \\ \hline 0 & I_4-2\Sigma \end{array}\right]\right), \left[\begin{array}{c|c} S & 0 \\ \hline 0 & I_4+U \end{array}\right]\right) \\
&= \left[\begin{array}{c|c} R\mathrm{e}^{2\Omega}R^{\top}S\mathrm{e}^{2\Omega} & 0 \\ \hline 0 & I_4+(4\Sigma+U) \end{array}\right]
\end{aligned}$$

which proves the theorem. ■

Remark 4.5. Similar to the situation described in Remark 3.9, the above endpoint geodesic formula can be used to solve for explicit endpoint geodesics in this case, too. But now we do not have to be so careful about distinguishing matrix exponential and Riemannian exponential map since they are the same in this case due to the bi-invariance of the Riemannian metric on M.

Remark 4.6. All results of Section 3 and 4 can be generalized to the case of SE_n resp. $\mathrm{SO}_n \times \mathbb{R}^n$ for arbitrary n with minor adjustments.

5 Endpoint Geodesic Formulas for Subspaces of SE_3

For this section, we need some more background on the notions of symmetric subspaces and Lie subtriples. For more details we refer to [1] and [5].

Definition 5.1 (Lie Triple System)**.** *A vector space* $\mathfrak{m}$ *together with a trilinear map* $[\cdot,\cdot,\cdot] : \mathfrak{m}\times\mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$ *is called a* ***Lie triple system*** *if*

(i) $[u,v,w] = -[v,u,w]$,

(ii) $[u,v,w] + [w,u,v] + [v,w,u] = 0$,

(iii) $[u,v,[w,x,y]] = [[u,v,w],x,y] + [w,[u,v,x],y] + [w,x,[u,v,y]]$

hold for all $u, v, w, x, y \in \mathfrak{m}$.

As an important example, suppose that we have a Lie algebra $\mathfrak{k}$ with Cartan decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$. Then, $\mathfrak{m}$ becomes a Lie triple system with the trilinear map $[\cdot, [\cdot, \cdot]]$, where $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{k}$ restricted to $\mathfrak{m}$. Furthermore, simply connected Riemannian symmetric spaces and Lie triple systems are in one-to-one correspondence (e.g. [5]).

Definition 5.2 (Symmetric Subspace)**.** *Let M be a Riemannian symmetric space and $S \subset M$ a submanifold. Then, S is called a* ***Riemannian symmetric subspace*** *if for every $q \in S$ there exists an isometry $s_q : M \to M$ that fixes q and satisfies $s_q(S) = S$ as well as $\mathrm{d}_q s_q|_{T_qS} = -\mathrm{id}_{T_qS}$ and $\mathrm{d}_q s_q|_{N_qS} = \mathrm{id}_{N_qS}$.*

Definition 5.3 (Lie Subtriple)**.** *Given a vector space $\mathfrak{m}$ with trilinear map $[\cdot, [\cdot, \cdot]]$. A linear subspace $\mathfrak{s} \subset \mathfrak{m}$ is called a* ***Lie subtriple*** *if it is invariant under the trilinear map on $\mathfrak{m}$, i.e. $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$.*

Similar to the case of simply connected Riemannian symmetric spaces and Lie triple systems, there is a connection between Lie subtriples and Riemannian symmetric subspaces, too.

Theorem 5.4. *Let $M \cong G/K$ be a Riemannian symmetric space with Lie group G and a compact subgroup $K \subset G$ and let $S \subset M$ be a geodesically complete submanifold, i.e. every maximal geodesic in S is defined for all $t \in \mathbb{R}$. Then*

(i) S is a Riemannian symmetric subspace

(ii) $S = \exp_e(\mathfrak{s})$ where $\mathfrak{s} \subset \mathfrak{m} = T_eM$ is a Lie subtriple

are equivalent.

Note that in the situation of the above theorem, the Lie subtriple $\mathfrak{s} \subset \mathfrak{m}$ in general is not a subalgebra but only a subspace. In fact, the subalgebras of $\mathfrak{m}$ correspond to the subgroups of $M \cong G/K$ which of course are also symmetric subspaces of M.

5.1 Classification of Subspaces

Our first goal is to classify all of the symmetric subspaces of SE_3 in the sense of Definition 5.2. They fall into two different classes. Firstly, all of the Lie subgroups of SE_3 are symmetric subspaces in a trivial way. They are listed in [18], Table 3.1. In this paper, we focus on the second class of symmetric subspaces, which are no Lie subgroups of SE_3. Because of Theorem 5.4, a classification of the non-trivial symmetric subspaces of SE_3 is equivalent to a classification of the non-trivial Lie subtriples of $\mathfrak{se}_3$. To do so, we define the following basis of $\mathfrak{se}_3$:

$$t_x := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad t_y := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad t_z := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$r_x := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad r_y := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad r_z := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This leads to the following classification, which is up to conjugation.

Theorem 5.5 ([22], Table 1)**.** *There are exactly seven conjugacy classes of non-trivial Lie subtriples of $\mathfrak{se}_3$. With the basis defined above they are*

(1) $\mathfrak{m}_{2A} := \langle t_z, r_x \rangle$

(2) $\mathfrak{m}_{2A}^p := \langle t_z, r_x + pt_x \rangle$, $p \in \mathbb{R}$

(3) $\mathfrak{m}_{2B} := \langle r_x, r_y \rangle$

(4) $\mathfrak{m}_{3A} := \langle t_x, t_z, r_x \rangle$

(5) $\mathfrak{m}_{3B} := \langle t_z, r_x, r_y \rangle$

(6) $\mathfrak{m}_4 := \langle t_x, t_y, r_x, r_y \rangle$

(7) $\mathfrak{m}_5 := \langle t_x, t_y, t_z, r_x, r_y \rangle$.

Proof. The idea of the proof is rather simple: we have to check the double Lie bracket property of Definition 5.3 for arbitrary combinations of basis elements of $\mathfrak{se}_3$ under its Lie bracket, which is the usual matrix commutator. A brute force calculation then leads to the seven triples of the theorem, we omit the details. ■

For most of the above triples (only $\mathfrak{m}_{2A}^p$ is an exception), one can show that the matrix exponential is surjective and thus for any of these subtriples $\mathfrak{m}$, the corresponding subspace is given by $M = \mathrm{e}^{\mathfrak{m}}$.

5.2 Deriving the Endpoint Geodesic Formula on Certain Subspaces

In this section, we will take two of the symmetric subspaces of SE_3 classified in Theorem 5.5 and derive endpoint geodesic formulas for them. This works for the other symmetric subspaces in an analogous way, too. Since the computations are very similar to those of Section 3, we will not give all the details and leave out the proofs.
The first example we consider is the symmetric subspace $M_{3A} \subset \mathrm{SE}_3$ corresponding to

the Lie subtriple $\mathfrak{m}_{3A} = \langle t_x, t_z, r_x \rangle$, i.e.

$$M_{3A} = \mathrm{e}^{\mathfrak{m}_{3A}} = \left\{ \begin{bmatrix} 1 & 0 & \overline{v} \\ 0 & R & v \\ 0 & 0 & 1 \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & \cos a & -\sin a & z\left(\frac{\cos a}{a} - \frac{1}{a}\right) \\ 0 & \sin a & \cos a & z\frac{\sin a}{a} \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, x, z \in \mathbb{R} \right\}.$$

It has two degrees of freedom in the translational part and one in the rotational part. The vector space in which we will embed M_{3A} is

$$W_{3A} := \left\{ \begin{bmatrix} 0 & \begin{bmatrix} b & 0 & X \\ 0 & B & x \\ 0 & 0 & a \end{bmatrix} \\ \begin{bmatrix} -b & 0 & \overline{y} \\ 0 & -B^\top & y \\ 0 & 0 & -a \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \;\middle|\; B \in \mathbb{R}^{2\times 2}, x, y \in \mathbb{R}^2, a, b, X, \overline{y} \in \mathbb{R} \right\}$$

via the embedding

$$\iota : M_{3A} \to W_{3A}\,, \quad \begin{bmatrix} 1 & 0 & \overline{v} \\ 0 & R & v \\ 0 & 0 & 1 \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} 0 & \begin{bmatrix} 1 & 0 & \overline{v} \\ 0 & R & v \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 & \overline{v} \\ 0 & -R^\top & R^\top v \\ 0 & 0 & -1 \end{bmatrix} & 0 \end{bmatrix}.$$

Lemma 5.6. *Consider the symmetric subspace $M_{3A} \hookrightarrow W_{3A}$ embedded as above and let $X \in M_{3A}$. Then, the following statements hold.*

(i) The symmetric bilinear form

$$\langle \cdot, \cdot \rangle_{W_{3A}}^X : W_{3A} \times W_{3A} \to \mathbb{R}\,, \; \langle Z_1, Z_2 \rangle_{W_{3A}}^X := 8\operatorname{tr}\left(\left(\operatorname{Ad}_{\left[\begin{smallmatrix} X^{-1} & 0 \\ 0 & I_4 \end{smallmatrix}\right]}(Z_1) \right)^\top \cdot \operatorname{Ad}_{\left[\begin{smallmatrix} X^{-1} & 0 \\ 0 & I_4 \end{smallmatrix}\right]}(Z_2) \right)$$
$$= 8\Big(2\operatorname{tr}\left(B_1^\top B_2\right) + 2b_1 b_2 + \left\langle R^\top x_1 - a_1 R^\top v, R^\top x_2 - a_2 R^\top v \right\rangle + \langle X_1 - a_1 \overline{v}, X_2 - a_2 \overline{v} \rangle$$
$$+ \left\langle -B_1^\top v + y_1, -B_2^\top v + y_2 \right\rangle + \left\langle -b_1 \overline{v} + \overline{y}_1, -b_2 \overline{v} + \overline{y}_2 \right\rangle + 2a_1 a_2 \Big)$$

defines a scalar product on W_{3A}.

(ii) We have the tangent spaces

$$T_{\iota(X)}\iota(M_{3A}) = \left\{ \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 & \overline{\omega} \\ 0 & R\Omega & R\omega \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \overline{\omega} \\ 0 & \Omega R^\top & -\Omega R^\top v + \omega \\ 0 & 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \;\middle|\; \begin{bmatrix} 0 & 0 & \overline{\omega} \\ 0 & \Omega & \omega \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{m}_{3A} \right\}$$

and the normal spaces

$$N_{\iota(X)}\iota(M_{3A}) = \left\{ \begin{bmatrix} 0 & \begin{bmatrix} r & 0 & \widetilde{\overline{\omega}} \\ 0 & RS & R\widetilde{\omega} + sv \\ 0 & 0 & s \end{bmatrix} \\ \begin{bmatrix} -r & 0 & -\widetilde{\overline{\omega}} \\ 0 & -SR^\top & SR^\top v - \widehat{\omega} \\ 0 & 0 & -s \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \;\middle|\; S \in \mathrm{Sym}_2, \widetilde{\omega} = \begin{bmatrix} \widetilde{t} \\ t \end{bmatrix}, \widehat{\omega} = \begin{bmatrix} \widehat{t} \\ t \end{bmatrix} \in \mathbb{R}^2, r, s, \widetilde{\overline{\omega}} \in \mathbb{R} \right\}.$$

(iii) The map $\mathrm{R}_{\iota(X)} : W_{3A} \to W_{3A}$ *given by*

$$\begin{bmatrix} 0 & \begin{bmatrix} b & 0 & X \\ 0 & B & x \\ 0 & 0 & a \end{bmatrix} \\ \begin{bmatrix} -b & 0 & \overline{y} \\ 0 & -B^\top & y \\ 0 & 0 & -a \end{bmatrix} & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & \begin{bmatrix} b & 0 & -\overline{y} \\ 0 & RB^\top R & \widetilde{y} \\ 0 & 0 & a \end{bmatrix} \\ \begin{bmatrix} -b & 0 & -X \\ 0 & -(RB^\top R)^\top & \widetilde{x} \\ 0 & 0 & -a \end{bmatrix} & 0 \end{bmatrix}$$

with

$$\widetilde{y} = av + R\left(B^\top v - y\right) \quad \text{and} \quad \widetilde{x} = -R^\top(x - av) + R^\top B R^\top v$$

is the normal space involution at $\iota(X)$. *Furthermore,* $M_{3A} \to W_{3A}$ *embedded via* $\iota : M_{3A} \to W_{3A}$ *is an extrinsic symmetric space.*

Theorem 5.7 (Endpoint Geodesic Formula on M_{3A})**.** *Consider the symmetric subspace* $M_{3A} \hookrightarrow W_{3A}$ *embedded as above and let* $X \in M_{3A}$ *be arbitrary. Then the following endpoint geodesic formula holds:*

$$\mathrm{R}_Q\left(\mathrm{R}_P\left(Z\right)\right) = \mathrm{Ad}_{e^{2\Theta}}(Z)$$

where

$$P := \iota(X) = \frac{1}{2}\begin{bmatrix} 0 & X \\ -X^{-1} & 0 \end{bmatrix},\; Q := \mathrm{Ad}_{ke^{\xi}k^{-1}}\left(P\right),\; \Theta := \mathrm{Ad}_k\left(\xi\right)$$

and

$$k = \begin{bmatrix} \begin{bmatrix} 1 & 0 & \overline{v} \\ 0 & R & v \\ 0 & 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}, \ \xi = \begin{bmatrix} \begin{bmatrix} 0 & 0 & \overline{\omega} \\ 0 & \Omega & \omega \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 0 & -\overline{\omega} \\ 0 & -\Omega & -\omega \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

with $\begin{bmatrix} 0 & 0 & \overline{\omega} \\ 0 & \Omega & \omega \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{m}_{3A}$.

The other example we will discuss is the symmetric subspace $M_4 \subset \mathrm{SE}_3$ corresponding to the Lie subtriple $\mathfrak{m}_4 = \langle t_x, t_y, r_x, r_y \rangle$. This time the vector space in which we embed will be

$$W_4 := \left\{ \begin{bmatrix} 0 & \begin{bmatrix} B & x \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -B^\top & y \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \;\middle|\; B \in \mathbb{R}^{3\times 3}, x, y \in \mathbb{R}^3, a \in \mathbb{R} \right\}$$

which is the same as the vector space W in the case of SE_3. The embedding is again

$$\iota : M_4 \to W_4 \ , \ X = \begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} 0 & X \\ -X^{-1} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -R^\top & R^\top v \\ 0 & -1 \end{bmatrix} & 0 \end{bmatrix}.$$

Lemma 5.8. *Consider the symmetric subspace $M_4 \hookrightarrow W_4$ embedded via ι and let $X \in M_4$. Then the following statements hold.*

(i) The symmetric bilinear form

$$\langle \cdot, \cdot \rangle_{W_4}^X : W_4 \times W_4 \to \mathbb{R} \ , \ \langle Z_1, Z_2 \rangle_{W_4}^X := 8\mathrm{tr} \left(\left(\mathrm{Ad}_{\left[\begin{smallmatrix} X^{-1} & 0 \\ 0 & I_4 \end{smallmatrix}\right]} (Z_1) \right)^\top \cdot \mathrm{Ad}_{\left[\begin{smallmatrix} X^{-1} & 0 \\ 0 & I_4 \end{smallmatrix}\right]} (Z_2) \right)$$

defines a scalar product on W_4.

(ii) We have the tangent spaces

$$T_{\iota(X)}\iota(M_4) = \left\{ \begin{bmatrix} 0 & \begin{bmatrix} R\Omega & R\omega \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Omega R^\top & -\Omega R^\top v + \omega \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \;\middle|\; \begin{bmatrix} \Omega & \omega \\ 0 & 0 \end{bmatrix} \in \mathfrak{m}_4 \right\}$$

and the normal spaces

$$N_{\iota(X)}\iota(M_4) = \left\{ \begin{bmatrix} 0 & \begin{bmatrix} RS & R\widetilde{\omega} + sv \\ 0 & s \end{bmatrix} \\ \begin{bmatrix} -SR^\top & SR^\top v - \widehat{\omega} \\ 0 & -s \end{bmatrix} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \ \middle| \ S = \begin{bmatrix} d_1 & c_1 & b \\ c_2 & d_2 & a \\ b & a & d_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}, \widetilde{\omega} = \begin{bmatrix} t_1 \\ t_2 \\ \widetilde{t} \end{bmatrix}, \widehat{\omega} = \begin{bmatrix} -t_1 \\ -t_2 \\ \widehat{t} \end{bmatrix} \in \mathbb{R}^{3\times 3}, s \in \mathbb{R} \right\}.$$

(iii) *The map*

$$\mathrm{R}_{\iota(X)} : W_4 \to W_4, \begin{bmatrix} 0 & \begin{bmatrix} B & x \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -B^\top & y \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & \begin{bmatrix} RB^\top R & \widetilde{y} \\ 0 & a \end{bmatrix} \\ \begin{bmatrix} -(RB^\top R)^\top & \widetilde{x} \\ 0 & -a \end{bmatrix} & 0 \end{bmatrix}$$

with

$$\widetilde{y} = av + R\left(B^\top v - y\right) \quad \text{and} \quad \widetilde{x} = -R^\top(x - av) + R^\top B R^\top v$$

is the normal space involution at $\iota(X)$. Furthermore, $M_4 \to W_4$ embedded via $\iota : M_4 \to W_4$ is an extrinsic symmetric space.

Theorem 5.9 (Endpoint Geodesic Formula on M_4)**.** *Consider the symmetric subspace $M_4 \hookrightarrow W_4$ embedded via ι and let $X \in M_4$ be arbitrary. Then the following endpoint geodesic formula holds:*

$$\mathrm{R}_Q\left(\mathrm{R}_P\left(Z\right)\right) = \mathrm{Ad}_{e^{2\Theta}}(Z)$$

where

$$P := \iota(X) = \frac{1}{2}\begin{bmatrix} 0 & X \\ -X^{-1} & 0 \end{bmatrix}, \ Q := \mathrm{Ad}_{ke^{\xi}k^{-1}}\left(P\right), \ \Theta := \mathrm{Ad}_k\left(\xi\right)$$

and

$$k = \begin{bmatrix} \begin{bmatrix} R & v \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} I_3 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}, \ \xi = \begin{bmatrix} \zeta & 0 \\ 0 & -\zeta \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \Omega & \omega \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} -\Omega & -\omega \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

with $\begin{bmatrix} \Omega & \omega \\ 0 & 0 \end{bmatrix} \in \mathfrak{m}_4$.

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