Stochastic orders and shape properties for a new distorted proportional odds model

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Abstract

Building on recent developments in models focused on the shape properties of odds ratios, this paper introduces two new models that expand the class of available distributions while preserving specific shape characteristics of an underlying baseline distribution. The first model offers enhanced control over odds and log-odds functions, facilitating adjustments to skewness, tail behavior, and hazard rates. The second model, with even greater flexibility, describes odds ratios as quantile distortions. This approach leads to an enlarged log-logistic family capable of capturing these quantile transformations and diverse hazard behaviors, including non-monotonic and bathtub-shaped rates. Central to our study are the shape relations described through stochastic orders; we establish conditions that ensure stochastic ordering both within each family and across models under various ordering concepts, such as hazard rate, likelihood ratio, and convex transform orders.

Keywords: Stochastic order, Odds ratio, Log-logistic, Distortion

MSC Classification: 60E15 , 60E05 , 62E10

1 Introduction

The development of flexible distribution models is a key pursuit in statistical research, particularly in applications that require detailed control over distributional shape properties, such as survival analysis, reliability engineering, and actuarial science. Traditional methods for constructing such models often involve adding parameters to established families or transforming specific functional components, such as survival or hazard functions. Some recent examples of this approach are Alzaatreh et al. (2013), Kharazmi et al. (2021) or Vasconcelos et al. (2024). However, unlike our main interest that concentrates on stochastic order relations and shape properties, these authors focus on the basic characterisations of the distributions and practical statistical aspects. Recent advancements have emphasized extending these frameworks to odds functions, providing a broader and potentially more versatile approach for representing complex real-world data. The proportional hazards rate (PHR) model, for example, redefines the survival function by raising a baseline survival function to a power, resulting in a distribution with hazard rates proportional to the original, as studied in Marshall and Olkin (1997). While the PHR model preserves hazard rate monotonicity, it is limited in capturing distributions with heavy tails or non-standard failure rates. Expanding on this, Lando et al. (2022) proposed the increasing odds rate (IOR) class, which benchmarks distributions against the log-logistic distribution rather than the exponential. This shift enables the IOR class to model distributions with bathtub-shaped hazard rates or heavy tails, crucial for applications where standard assumptions about hazard monotonicity are too restrictive. The present work builds on these foundations by introducing a modified proportional odds model that leverages transformations of the baseline distribution's odds function, creating a new family of distributions that inherit specific shape properties. The proportional odds model has been extensively studied from multiple perspectives and has become a mainstay in survival analysis and reliability theory in recent years. Bennett (1983) introduced the model and developed the maximum likelihood estimation for the semi-parametric version. Collett (2023) applied the proportional odds model to analyze survival data for women with breast tumors. Additionally, Dinse and Lagakos (1983, 1984) and Rossini and Tsiatis (1996) used this model for the analysis of interval-censored data. Applications in reliability analysis are detailed in Crowder et al. (1991), while Kirmani and Gupta (2001) examined its structural properties, deriving results on stochastic comparisons and aging properties. Notably, they established connections between the proportional odds model and the behaviour of geometric minima and maxima (see Theorem 4, Kirmani and Gupta (2001)). More recently, Sankaran and Jayakumar (2008) proposed new distribution families inspired by the proportional odds model. The model suggested in this work is grounded in two main transformations: (1) defining proportionality for both odds and log-odds functions, thus allowing for flexible skewness and tail control, and (2) embedding the structure of an enlarged log-logistic distribution, a more generalized form that subsumes the log-logistic model and includes parameters that govern asymmetry, spread, and tail behaviour. By transforming the baseline distribution F through a suitable quantile function, the new model achieves adaptable odds behaviours, preserving key properties such as convexity, concavity, or monotonicity of odds rates depending on parameter settings. In survival and reliability studies, the ability to manipulate odds and hazard rates simultaneously is crucial for accurate modeling. The enlarged log-logistic (ELL) distribution, introduced here, offers an innovative quantile-based transformation that extends beyond the conventional proportional hazards models. This model accommodates distributions with varying shapes, including those with increasing, decreasing, or even non-monotonic hazard rates, which is particularly useful in capturing lifetime data with complex failure patterns. By parameterizing the model to allow for different convexity behaviours, we create a flexible toolkit for modeling distributions that meet practical shape requirements without sacrificing theoretical rigor. In applications where risk management and reliability assessments are paramount – such as in actuarial science, insurance, and public health – these new models offer greater adaptability. Their use of quantile-based transformations enables tailored fitting to specific data characteristics, including tail behaviour and asymmetry, which are often critical for accurate risk assessment. Moreover, by defining the odds functions as quantile transformations of the baseline distribution, the model provides a straightforward pathway for comparing reliability characteristics across different parameter choices, making it easier for practitioners to interpret and apply.

The paper is structured as follows. Section 2 provides preliminary definitions and background essential for understanding the proposed models, including a review of stochastic orders and shape properties in distribution families. In Section 3, we introduce the first modified proportional odds model, with a focus on its properties and stochastic comparisons with the baseline distribution. Section 4 extends the model by presenting a distorted odds ratio model, leveraging additional parameters to further control distributional shape. Finally, in Section 5, we explore the enlarged log-logistic distribution, detailing its implications for odds and hazard rate behaviors.

2 Preliminaries and basic definitions

We shall represent by X, F and f the baseline random variable, its cumulative and density functions (that we will be assuming to exist), respectively. Analogously, Y, G and g, possibly with some subscripts to denote parameters, will represent the new models to be studied. Moreover, survival functions are represented as $\overline{F}(x) = 1 - F(x)$ or $\overline{G}(x) = 1 - G(x)$. We shall refer to the random variables or to their distribution functions as is more convenient. In fact, the characterisations we will be discussing depend only on the distribution, so the random variables will appear only as a convenience. We recall the usual notions which were briefly mentioned in the Introduction. Given a distribution function F, its hazard rate and reversed hazard rate are denoted with $h_F(x) = \frac{f(x)}{\overline{F}(x)}$ and $\widetilde{h}_F(x) = \frac{f(x)}{F(x)}$ respectively, its odds function with $\Lambda_F(x) = \frac{F(x)}{\overline{F}(x)}$, and its odds rate with $\lambda_F(x) = \Lambda'_F(x) = \frac{f(x)}{\overline{F}^2(x)}$. While the monotonicity of the hazard rate function has been extensively studied in the literature, for the odds function, which is always increasing, the interest relies on its growth rate, characterised by monotonicity of λ_F. These functions may be used to define some classes of distributions.

Definition 2.1. *We say that*

1. *X or F have increasing (decreasing) hazard rate, represented by $F \in \mathrm{IHR}$ ($F \in \mathrm{DHR}$), if h_F is increasing (decreasing);*
2. *X or F have increasing (decreasing) odds rate, represented by $F \in \mathrm{IOR}$ ($F \in \mathrm{DOR}$), if λ_F is increasing (decreasing);*
3. *X or F have convex (concave) log-odds if $\log \Lambda_F(x)$ is convex (concave).*

The IHR and DHR families are well-known in the literature, while the IOR family has been receiving less attention. Some properties of the IOR class are studied in a systematic way in Lando et al. (2022). The DOR family is only briefly mentioned in Arab et al. (2024), and, also recently discussed in Chen et al. (2024), although with a different terminology. Note that $F \in \mathrm{IOR}$ ($F \in \mathrm{DOR}$) is equivalent to the odds ratio Λ_F being convex (concave), so these odds ratio classes describe a shape property of the corresponding distributions.

We now recall some common stochastic order notions that will be considered later.

Definition 2.2. *Consider two distribution functions F_1 and F_2, with densities f_1 and f_2, respectively. We say that,*

1. *F_1 is smaller than F_2 in the usual stochastic order, denoted as $F_1 \leq_{st} F_2$, if $\overline{F}_1(x) \leq \overline{F}_2(x)$, for every $x \in \mathbb{R}$;*
2. *F_1 is smaller than F_2 in the hazard rate order, denoted as $F_1 \leq_{hr} F_2$, if $h_{F_1}(x) \geq h_{F_2}(x)$, for every $x \in \mathbb{R}$;*
3. *F_1 is smaller than F_2 in the reversed hazard rate order, denoted as $F_1 \leq_{rh} F_2$, if $\widetilde{h}_{F_1}(x) \geq \widetilde{h}_{F_2}(x)$, for every $x \in \mathbb{R}$;*
4. *F_1 is smaller than F_2 in the likelihood rate order, denoted as $F_1 \leq_{lr} F_2$, if $\frac{f_2(x)}{f_1(x)}$, is increasing.*
5. *F_1 is smaller that F_2 in the dispersive order, denoted as $F_1 \leq_{disp} F_2$, if $F_2^{-1} \circ F_1(x) - x$ increases in x.*

Some of the classes of distributions mentioned in Definition 2.1 may be characterised via a different type of stochastic order, namely the convex transform order, defined by a shape restriction on the transformation that maps one distribution to the one being compared.

Definition 2.3 (van Zwet (1964))**.** *Given two distribution functions F_1 and F_2, we say that F_1 is smaller than F_2 in the convex transform order, represented by $F_1 \leq_c F_2$, if $F_2^{-1} \circ F_1$ is convex.*

Let us now fix, for the sequel, two reference distributions: the standard exponential, with distribution function $\mathcal{E}(x) = 1 - e^{-x}$, and the log-logistic, with distribution function $\mathcal{L}(x) = 1 - \frac{1}{x+1} = \frac{x}{x+1}$. It is well-known that $F \in$ IHR ($F \in$ DHR) if and only if $F \leq_c \mathcal{E}$ ($F \geq_c \mathcal{E}$). Analogously, as referred in Lando et al. (2022), it is easily seen that $F \in$ IOR ($F \in$ DOR) if and only if $F \leq_c \mathcal{L}$ ($F \geq_c \mathcal{L}$).

For a systematic study of properties of the stochastic orders defined above, and a number of other interesting stochastic order relations, and relations among them, we refer the interested reader to the monographs Shaked and Shanthikumar (2007) or Marshall and Olkin (2007).

3 A first modified proportional odds model

The study of the growth rate of the odds function is fundamental in the characterization of distribution families that maintain specific shape properties such as the IOR, which is crucial for modeling various real-world phenomena, particularly in reliability and survival analysis. In this context, we introduce a modified proportional odds model that leverages the properties of the IOR and log-odds convexity to create new distribution families. This approach aligns with and extends findings from Marshall and Olkin (1997), who explored models with proportional log-odds ratios but focused more on hazard rate properties than the broader proportional odds framework. This section will define and explore the model, demonstrating its applicability and properties that respond to these shape requirements, and showing how it provides a generalization of the IOR class while maintaining mathematical tractability.

Definition 3.1. *Let $\beta, \theta > 0$. Given a baseline distribution function F, we define the $G_{\beta,\theta}$ distribution function by*

$$\Lambda_{G_{\beta,\theta}}(x) = \beta \Lambda_F^\theta(x) = \beta \left(\frac{F(x)}{\overline{F}(x)} \right)^\theta . \tag{1}$$

It is obvious that $G_{\beta,1}$ has odds ratio $\Lambda_{G_{\beta,1}}$ proportional to Λ_F, while for $G_{1,\theta}$ we have that $\log \Lambda_{G_{1,\theta}}(x) = \theta \log \Lambda_F(x)$, that is, (1) covers the case of a model with proportional log-odds ratio. Note that the model corresponding to $\theta = 1$ has been studied in Marshall and Olkin (1997), although the authors did not refer to it as the proportional odds model, and were mainly interested in hazard rate properties.

Taking into account that $\Lambda_{G_{\beta,\theta}}(x) = \frac{G_{\beta,\theta}(x)}{\overline{G}_{\beta,\theta}(x)} = \frac{1}{\overline{G}_{\beta,\theta}(x)} - 1$, it follows easily that, for each $x \in \mathbb{R}$,

$$G_{\beta,\theta}(x) = \frac{\beta F^\theta(x)}{\beta F^\theta(x) + \overline{F}^\theta(x)}, \qquad \text{and} \qquad \overline{G}_{\beta,\theta}(x) = \frac{\overline{F}^\theta(x)}{\beta F^\theta(x) + \overline{F}^\theta(x)}. \tag{2}$$

Note that when taking $\theta = 1$, the following shape characterisation is a straightforward consequence of the convexity properties of the function $\frac{\beta x}{1+(1-\beta)x}$ when $x \in [0,1]$.

Proposition 3.2. *If F is concave, then $G_{\beta,1}$ for $\beta \geq 1$ is also concave. If F is convex, then $G_{\beta,1}$ for $\beta \leq 1$ is also convex.*

Note that the corresponding transformation for $\theta \neq 1$ and general $\beta > 0$ is easily seen to be neither convex nor concave, so no conclusion about the convexity of $G_{\beta,\theta}$ can be drawn.

From (2), the density and hazard rate functions for $G_{\beta,\theta}$ are easily obtained:

$$g_{\beta,\theta}(x) = \beta\theta f(x) \frac{F^{\theta-1}(x)\overline{F}^{\theta-1}(x)}{(\beta F(x)^\theta + \overline{F}^\theta(x))^2}, \tag{3}$$

and

$$h_{G_{\beta,\theta}}(x) = \beta\theta h_F(x) \frac{F^{\theta-1}(x)}{\beta F(x)^\theta + \overline{F}^\theta(x)} = \beta\theta h_F(x) \left(\frac{F(x)}{\overline{F}(x)} \right)^{\theta-1} \frac{\overline{G}_{\beta,\theta}(x)}{\overline{F}(x)}. \tag{4}$$

Defining $T_{\beta,\theta}(x) = \beta\theta \frac{x^{\theta-1}}{\beta x^\theta + (1-x)^\theta}$, the first equality in (4) may be rewritten as $h_{G_{\beta,\theta}}(x) = h_F(x) T_{\beta,\theta}(F(x))$, providing an immediate characterisation for some of the classes of distributions mentioned in Definition 2.1.

Theorem 3.3.

1. *If $F \in \mathrm{IHR}$ and $\beta \leq 1$, then $G_{\beta,1} \in \mathrm{IHR}$.*
2. *If $F \in \mathrm{DHR}$ and $\beta \geq 1$, then $G_{\beta,1} \in \mathrm{DHR}$.*
3. *If $F \in \mathrm{IHR}$ or $F \in \mathrm{IOR}$, then, for $\theta \geq 1$, $G_{\beta,\theta} \in \mathrm{IOR}$.*

Proof. The result is immediate once we verify the monotonicity properties of $T_{\beta,\theta}$. It is easily verified that $T_{\beta,\theta}$ is monotone only for $\theta = 1$, that $T_{\beta,1}$ is increasing for $\beta \leq 1$, and $T_{\beta,1}$ is decreasing for $\beta \geq 1$. With regard to the preservation of the IOR property, it follows directly from (1) taking into account that $\theta \geq 1$. □

Given that $G_{\beta,\theta}$ is a transformation of F, it is natural to compare the baseline distribution F with the transformed distribution $G_{\beta,\theta}$. Specifically, we are interested in understanding how the transformation applied to F affects key reliability properties and relationships.

Theorem 3.4.

1. *If* $\beta \leq 1$*, then* $F \leq_{lr} G_{\beta,1}$*.*
2. *If* $\beta \geq 1$*, then* $F \geq_{lr} G_{\beta,1}$*.*

Proof. It is easily verified that $\frac{g_{\beta,1}(x)}{f(x)} = \frac{\beta}{(1+(\beta-1)F(x))^2}$, so the result follows immediately. □

Corollary 3.5.

1. *If* $\beta \leq 1$*, then* $F \leq_{rh} G_{\beta,1}$*,* $F \leq_{hr} G_{\beta,1}$ *and* $F \leq_{st} G_{\beta,1}$*.*
2. *If* $\beta \geq 1$*, then* $F \geq_{rh} G_{\beta,1}$*,* $F \geq_{hr} G_{\beta,1}$ *and* $F \geq_{st} G_{\beta,1}$*.*
3. *If* $\theta \neq 1$*,* F *and* $G_{\beta,\theta}$ *are not comparable with respect to the standard order (hence, also not comparable with respect to* $\leq_{hr}$*,* $\leq_{rh}$ *or* $\leq_{lr}$*).*

Proof. Parts 1. and 2. are an immediate consequence of Theorem 3.4 and Theorem 1.C.1 in Shaked and Shanthikumar (2007). For part 3., we need to look at

$$\overline{G}_{\beta,\theta}(x) - \overline{F}(x) \overset{\text{sgn}}{=} \frac{\overline{F}(x)^{\theta-1}}{\beta F(x)^{\theta} + \overline{F}(x)^{\theta}} - 1,$$

so, the conclusion follows by analysing the sign of

$$S_{\beta,\theta}(x) = \frac{(1-x)^{\theta-1}}{\beta x^{\theta} + (1-x)^{\theta}} - 1, \quad x \in [0,1].$$

Differentiating, one finds $S'_{\beta,\theta}(x) \overset{\text{sgn}}{=} (1-x)^{\theta} - \beta x^{\theta-1}(\theta - x)$. For $\theta \neq 1$, one has $S_{\beta,\theta}(0) = 0$, $S'_{\beta,\theta}$ is positive for x near 0 if $\theta > 1$, and is negative if $\theta < 1$. Finally, noting that $S_{\beta,\theta}(1) = -1$ if $\theta > 1$, and $S_{\beta,\theta}(1) = +\infty$ if $\theta < 1$, the result is proved. □

Note that when $\theta = 1$ the following explicit bounds for $h_{G_{\beta,1}}$ are immediate:

1. For $\beta \leq 1$, $\beta h_F(x) \leq h_{G_{\beta,1}}(x) \leq h_F(x)$,
2. For $\beta \geq 1$, $h_F(x) \leq h_{G_{\beta,1}}(x) \leq \beta h_F(x)$.

The stochastic comparisons outlined in the two latter results, provide valuable insights into how transformations of a baseline distribution F using $G_{\beta,\theta}$ influence its comparative properties under various stochastic orderings. These findings highlight the impact of parameters like β and θ on the tail behaviour, risk profiles, and reliability characteristics of transformed distributions. This knowledge is crucial for applications in reliability engineering, risk management, survival analysis, and operations research, where understanding the effects of distributional changes on failure rates, risk assessments, and system longevity is essential. Moreover, recognizing when direct comparisons between distributions are not feasible underscores the complexity of certain transformation models and prompts further exploration into customized analytical approaches for these scenarios.

4 A distorted odds ratio model

The previous section studied stochastic ordering relations between distributions defined by a specific transformation of the odds function, having in mind the possibility of mixing the proportionality of the odds ration and of the log odds ratio, each controlled by an appropriate parameter. Observe that the odds function of the $G_{\beta,\theta}$ distribution appears as a distortion of the odds ratio Λ_F of the baseline distribution function F, adding a powerful layer of flexibility in shaping distributional properties. This approach generalizes the structure of the popular proportional hazards rate (PHR) model, where the survival function is expressed as $\overline{F}^{\theta}(x)$ for some $\theta > 0$, may also be viewed as a model where the odds function is distorted in a similar way. In fact, the odds function if the PHR model is of the form $(1 + \Lambda_F(x))^{\theta} - 1$. It is worth noting that an odds ratio of this latter form corresponds to transforming the underlying distribution F by the quantile function of a Pareto distribution with survival function $(x + 1)^{-\frac{1}{\theta}}$. This observation will be explored later in Section 5 in more generality. Such transformations not only introduce new shape parameters but also allow for targeted control over properties like tail behaviour and hazard rate. Building on this odds function framework, we can generalize the concept of distortion models to encompass a unified family of distributions. By setting specific parameter values, we can transition seamlessly between models that exhibit proportional odds, proportional hazards, or log-odds proportionality. This unified approach aligns with recent developments in survival analysis and reliability theory, where the need for flexible, interpretable models is paramount.

Definition 4.1. *Let $\alpha \geq 0$, β, $\theta > 0$. Given a baseline distribution function F, we define the $G_{\alpha,\beta,\theta}$ distribution function by*

$$\Lambda_{G_{\alpha,\beta,\theta}}(x) = \beta\left((\alpha + \Lambda_F(x))^{\theta} - \alpha^{\theta}\right). \tag{5}$$

It is obvious that the model introduced in Definition 3.1 is a particular case of (5), taking $\alpha = 0$, while the PHR model is obtained by choosing $(\alpha, \beta, \theta) = (1, 1, \theta)$.

Taking into account that $\Lambda_{G_{\alpha,\beta,\theta}}(x) = \frac{1}{\overline{G}_{\alpha,\beta,\theta}(x)} - 1$, the following explicit representations for the distributions $G_{\alpha,\beta,\theta}$ introduced in Definition 4.1 are immediate:

$$\overline{G}_{\alpha,\beta,\theta}(x) = \frac{1}{1 + \beta((\alpha + \Lambda_F(x))^{\theta} - \alpha^{\theta})}, \tag{6}$$

and

$$G_{\alpha,\beta,\theta}(x) = 1 - \frac{1}{1 + \beta((\alpha + \Lambda_F(x))^{\theta} - \alpha^{\theta})}, \tag{7}$$

while the density function is represented as

$$g_{\alpha,\beta,\theta}(x) = \frac{\beta\theta(\alpha + \Lambda_F(x))^{\theta}}{\overline{F}(x)(\alpha\overline{F}(x) + F(x))}\overline{G}^2_{\alpha,\beta,\theta}(x)f(x) = \beta\theta(\alpha + \Lambda_F(x))^{\theta-1}\frac{\overline{G}^2_{\alpha,\beta,\theta}(x)}{\overline{F}(x)}h_F(x).$$

Remark 4.2. *The function $G_{\alpha,\beta,\theta}$ can be equivalently written as*

$$G_{\alpha,\beta,\theta}(x) = \frac{\beta\left((\alpha + (1-\alpha)F(x))^{\theta} - (\alpha\overline{F}(x))^{\theta}\right)}{\overline{F}^{\theta}(x) + \beta\left((\alpha + (1-\alpha)F(x))^{\theta} - (\alpha\overline{F}(x))^{\theta}\right)}. \tag{8}$$

In the special case where $\alpha = 1$, $G_{1,\beta,\theta}$ is the recently defined MPHR model introduced in Balakrishnan et al. (2018). Das and Kayal (2021) later extended this model by incorporating a scale parameter, calling it MPHRS. Similarly, our models can be generalised by introducing a scale parameter in the same way.

Remark 4.3. *The family of distributions $G_{\alpha,\beta,\theta}$ depends on three parameters. Here is a brief description of the effect of each one of them. The parameter β influences the spread and tail weight, as large values push representation of $G_{\alpha,\beta,\theta}$ to the right. Thus, β acts as a scale parameter. The effect of θ reflects mainly on the tail growth rate, thus affecting the extremal behaviour, with large values of θ pushing the distribution towards its upper limit. Finally, the parameter α shifts the distribution along the xx-axis, introducing asymmetry and potentially skewed, heavy-tailed as α decreases. Both α and θ behave like shape parameters.*

Although it seems that the $G_{\alpha,\beta,\theta}$ family is not closed under formation of maximums, that is, in general the distribution of the form $G^n_{\alpha,\beta,\theta}(x)$, $n \geq 2$, no longer belongs to the $G_{\alpha,\beta,\theta}$ family, we may still find an *extreme geometrical stability property* (see Marshall and Olkin (1997)).

Theorem 4.4. *Let $X_1, X_2, \ldots$ be independent and with distribution function $G_{\alpha,\beta,\theta}$, for some fixed values of $\alpha \geq 0$, $\beta, \theta > 0$, and consider N, independent from the X_n, with geometric distribution, $P(N = n) = p(1-p)^{n-1}$, $n \geq 1$, for some $p \in [0,1]$. Define $U = \min\{X_1, \ldots, X_N\}$ and $V = \max\{X_1, \ldots, X_N\}$. Then, the distribution function of U and V are $G_{\alpha,\frac{\beta}{p},\theta}$ and $G_{\alpha,\beta p,\theta}$, respectively. Or, equivalently, the family of distributions $G_{\alpha,\beta,\theta}$ has geometric extreme stability.*

Proof. Proceeding by conditioning, the distribution function of U is

$$\overline{F}_U(x) = \sum_{n=1}^{\infty} \overline{G}^n_{\alpha,\beta,\theta}(x) p(1-p)^{n-1} = \frac{p\overline{G}_{\alpha,\beta,\theta}(x)}{1-(1-p)\overline{G}_{\alpha,\beta,\theta}(x)}.$$

Using now the representation for $\overline{G}_{\alpha,\beta,\theta}$ that follows from (8), the result is immediate. The case of V is treated analogously. □

4.1 Preservation of monotonicity properties

Given the expressions above, we have the following representation for the hazard rate function:

$$h_{G_{\alpha,\beta,\theta}}(x) = \frac{g_{\alpha,\beta,\theta}(x)}{\overline{G}_{\alpha,\beta,\theta}(x)} = \frac{\beta\theta(\alpha+\Lambda_F(x))^{\theta-1}}{1+\beta\left((\alpha+\Lambda_F(x))^{\theta}-\alpha^{\theta}\right)} \cdot \frac{h_F(x)}{\overline{F}(x)} = \beta\theta h_F(x) T_{\alpha,\beta,\theta}(\Lambda_F(x)),$$

where

$$T_{\alpha,\beta,\theta}(x) = \frac{(\alpha+x)^{\theta-1}(x+1)}{1+\beta\left((\alpha+x)^{\theta}-\alpha^{\theta}\right)}, \quad x \in (0,1). \tag{9}$$

Hence, we may prove monotonicity properties for $h_{G_{\alpha,\beta,\theta}}$ looking at the monotonicity of $T_{\alpha,\beta,\theta}$, which will be addressed via $U_{\alpha,\beta,\theta}(x) = \frac{1}{T_{\alpha,\beta,\theta}(x)}$, for simplicity. After differentiation and some simple algebraic manipulation, one gets

$$U'_{\alpha,\beta,\theta}(x) = \frac{D_{\alpha,\beta,\theta}(x)}{(x+1)^2(\alpha+x)^{\theta}},$$

where $D_{\alpha,\beta,\theta}(x) = \beta(1-\alpha)(\alpha+x)^{\theta} + (\beta\alpha^{\theta}-1)(\theta x+\alpha+\theta-1)$, and the sign of $U'_{\alpha,\beta,\theta}$ coincides with the sign of $D_{\alpha,\beta,\theta}$. We have that $D'_{\alpha,\beta,\theta}(x) = \theta\beta(1-\alpha)(\alpha+x)^{\theta-1}+\theta(\beta\alpha^{\theta}-1)$ and $D''_{\alpha,\beta,\theta}(x) = (1-\alpha)\beta\theta(\theta-1)(\alpha+x)^{\theta-2}$. Therefore, $D''_{\alpha,\beta,\theta}(x) \overset{\text{sgn}}{=} \text{sgn}((1-\alpha)(\theta-1))$, so $D'_{\alpha,\beta,\theta}$ is either increasing or decreasing. Now, the sign of $D_{\alpha,\beta,\theta}(0) = \alpha^\theta\beta\theta - \alpha - (\theta - 1)$ will play a significant role.

Theorem 4.5. *Let $G_{\alpha,\beta,\theta}$ be given by (5) and $D_{\alpha,\beta,\theta}$ be the polynomial defined above.*

1. *If $D_{\alpha,\beta,\theta}(0) < 0$, $(1-\alpha)(\theta-1) < 0$, and $F \in \mathrm{IHR}$, then $G_{\alpha,\beta,\theta} \in \mathrm{IHR}$.*
2. *If $D_{\alpha,\beta,\theta}(0) > 0$, $(1-\alpha)(\theta-1) > 0$, and $F \in \mathrm{DHR}$, then $G_{\alpha,\beta,\theta} \in \mathrm{DHR}$.*
3. *If $\alpha = 1$ or $\theta = 1$, $\beta > 1$, $D_{\alpha,\beta,\theta}(0) < 0$ and $F \in \mathrm{IHR}$, then $G_{\alpha,\beta,\theta} \in \mathrm{IHR}$.*
4. *If $\alpha = 1$ or $\theta = 1$, $\beta \leq 1$, $D_{\alpha,\beta,\theta}(0) > 0$ and $F \in \mathrm{DHR}$, then $G_{\alpha,\beta,\theta} \in \mathrm{DHR}$.*

Proof. In the first case, $D(x) < 0$ for every $x > 0$. Hence $U'_{\alpha,\beta,\theta}$ is always negative, so $U_{\alpha,\beta,\theta}$ is decreasing and, therefore, $T_{\alpha,\beta,\theta} = \frac{1}{U_{\alpha,\beta,\theta}}$ is increasing, so the conclusion is straightforward. The remaining cases are analogous, reversing signs and monotony directions for cases 2. and 4. $\square$

Note that this result extends Theorem 3.3, allowing now for an interplay of the different parameters.

The preservation of the monotonicity of the odds ratio is easily described in analogous terms, extending the final part of Theorem 3.3.

Theorem 4.6.

1. *If $\theta \geq 1$ and $F \in \mathrm{IOR}$, then $G_{\alpha,\beta,\theta} \in \mathrm{IOR}$.*
2. *If $\theta \leq 1$ and $F \in \mathrm{DOR}$, then $G_{\alpha,\beta,\theta} \in \mathrm{DOR}$.*

Proof. Just note that $\lambda'_{G_{\alpha,\beta,\theta}}(x) \overset{\text{sgn}}{=} \lambda'_F(x)(\alpha + \Lambda_F(x)) + (\theta-1)\lambda_F^2(x)$, and the conclusion is immediate. $\square$

4.2 Stochastic comparisons between $G_{\alpha,\beta,\theta}$ and F

We now address some stochastic ordering relations between the baseline distribution F and the family of transformed distributions $G_{\alpha,\beta,\theta}$ introduced in Definition 4.1.

Theorem 4.7.

1. *If $\theta > 1$ and $\alpha^{\theta-1}\beta\theta > 1$, then $G_{\alpha,\beta,\theta} \leq_{st} F$.*
2. *If $\theta < 1$ and $\alpha^{\theta-1}\beta\theta < 1$, then $G_{\alpha,\beta,\theta} \geq_{st} F$.*

Proof. We need to characterise the sign of

$$\overline{G}_{\alpha,\beta,\theta}(x) - \overline{F}(x) = \frac{1}{1 + \beta((\alpha + \Lambda_F(x))^\theta - \alpha^\theta)} - \frac{1}{\Lambda_F(x) + 1} = H(\Lambda_F(x)),$$

where $H(x) = \frac{1}{1+\beta((\alpha+x)^\theta - \alpha^\theta)} - \frac{1}{x+1}$, which has the same sign variation as $P(x) = x - \beta(\alpha + x)^\theta + \alpha^\theta\beta$. After differentiation, we have $P'(x) = 1 - \theta\beta(\alpha + x)^{\theta-1}$ and $P''(x) = -\beta\theta(\theta - 1)(\alpha + x)^{\theta-1}$. When $\theta > 1$, $P''(x) < 0$, so $P'(x)$ is decreasing. If $P'(0) = 1 - \beta\theta\alpha^{\theta-1} < 0$ it follows that $P'(x) < 0$, for every $x > 0$, hence $P(x)$ is decreasing. Since that $P(0) = 0$ we have the negativeness of $P(x)$. The case $\theta < 1$ is handled analogously. $\square$

Sufficient conditions for the hazard rate order follow immediately by remarking that

$$H^*(x) = \frac{h_{G_{\alpha,\beta,\theta}}(x)}{h_F(x)} = \beta\theta T_{\alpha,\beta,\theta}(\Lambda_F(x)),$$

where $T_{\alpha,\beta.\theta}$ is defined by (9). Noting that $H^*(0) = \alpha^{\theta-1}$, taking into account the properties of $T_{\alpha,\beta.\theta}$ mentioned above, the following statement is obvious.

Theorem 4.8.

1. *If $\alpha^{\theta-1} > 1$ and $T_{\alpha,\beta,\theta}$ is increasing, then $G_{\alpha,\beta,\theta} \leq_{hr} F$.*
2. *If $\alpha^{\theta-1} < 1$ and $T_{\alpha,\beta,\theta}$ is decreasing, then $G_{\alpha,\beta,\theta} \geq_{hr} F$.*

For a characterisation of the monotony of $T_{\alpha,\beta,\theta}$, please see the discussion about Theorem 4.5.

Now, the likelihood order follows easily.

Theorem 4.9.

1. *Assume that $\theta > 1$ and $F \leq_{hr} G_{\alpha,\beta,\theta}$. Then $G_{\alpha,\beta,\theta} \leq_{lr} F$.*
2. *Assume that $\theta < 1$ and $F \geq_{hr} G_{\alpha,\beta,\theta}$. Then $G_{\alpha,\beta,\theta} \geq_{lr} F$.*

Proof. Note that

$$\frac{g_{\alpha,\beta,\theta}(x)}{f(x)} = \frac{\beta\theta(\alpha+\Lambda_F(x))^\theta \overline{G}^2(x)}{\overline{F}(x)(\alpha\overline{F}(x)+F(x))} = \beta\theta(\alpha+\Lambda_F(x))^{\theta-1}\left(\frac{\overline{G}(x)}{\overline{F}(x)}\right)^2.$$

The monotonicity of the first parenthesis of the final expression on the right is fully defined by the sign of the exponent, while the monotonicity of the second term depends on monotonicity the hazard rate order between the distributions F and $G_{\alpha,\beta,\theta}$. □

5 An enlarged log-logistic family of distributions

We have treated the models introduced in Definitions 3.1 and 4.1 by defining distributions through their odds functions, taken as distortions of some given underlying odds function Λ_F. Naturally, we may instead consider the new odds function as distortion of the initial distribution function F. We mentioned, just before Definition 4.1, that the odds function for the PHR model, if interpreted as a distortion of the baseline distribution, corresponds to transforming F by a suitable quantile function. This approach may be extended to the full class of models considered in Definition 4.1, leading to the introduction of a new family of distributions.

Definition 5.1. *The enlarged log-logistic distribution with parameters $\alpha \geq 0$, β, $\theta > 0$, denoted with* $\mathrm{ELL}(\alpha,\beta,\theta)$ *has distribution function*

$$K_{\alpha,\beta,\theta}(x) = 1 - \frac{1}{\left(\frac{x}{\beta}+\alpha^\theta\right)^{\frac{1}{\theta}} + 1 - \alpha}, \quad x \geq 0. \tag{10}$$

The parameters β and $\frac{1}{\theta}$ are obviously scale and shape parameters, respectively, and α is a second shape parameter, having an effect on the asymmetry, skewness, and tail weight of the distribution. Moreover, it is straightforward to verify that $K_{0,1,1}$ is the standard log-logistic, already introduced before and denoted with $\mathcal{L}$, while $K_{0,\beta,\theta}$ represents the log-logistic with distribution function $\mathcal{L}_{\beta,\frac{1}{\theta}}(x) = 1 - \left((\frac{x}{\beta})^{\frac{1}{\theta}} + 1\right)^{-1}$.

Explicit expressions for the density $k_{\alpha,\beta,\theta}$, hazard rate $h_{\alpha,\beta,\theta}$, and quantile function $K^{-1}_{\alpha,\beta,\theta}$ for the distribution function $K_{\alpha,\beta,\theta}$ are given below:

$$k_{\alpha,\beta,\theta}(x)=\frac{\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}-1}}{\beta\theta\left(\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}}+1-\alpha\right)^{2}},\quad h_{\alpha,\beta,\theta}(x)=\frac{\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}-1}}{\beta\theta\left(\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}}+1-\alpha\right)},\quad x\geq 0.$$

and

$$K^{-1}_{\alpha,\beta,\theta}(u)=\beta\left(\left(\frac{1}{1-u}+\alpha-1\right)^{\theta}-\alpha^{\theta}\right),\quad u\in[0,1].$$

It is now obvious that $\Lambda_{G_{\alpha,\beta,\theta}}(x)=K^{-1}_{\alpha,\beta,\theta}\circ F(x)$. Therefore, ordering properties within the $K_{\alpha,\beta,\theta}$ family translate easily to the $G_{\alpha,\beta,\theta}$ class of distributions, the convexity of the odds of $G_{\alpha,\beta,\theta}$ being the most obvious, corresponding to the convex transform order between $K_{\alpha,\beta,\theta}$ and F. In other words, the convexity properties of the baseline distribution F with respect to $K_{\alpha,\beta,\theta}$ are inherited by $G_{\alpha,\beta,\theta}$. Recall that the convexity of the odds defines interesting classes, namely the IOR and DOR families of distributions (see Lando et al. (2022)). This naturally leads to an interest in exploring stochastic ordering relationships within the family of distributions defined by (10).

The increasingness of the hazard rate or the odds rate is simple to characterise, as described next.

Theorem 5.2.

1. *If $\alpha+\theta>1$ then $K_{\alpha,\beta,\theta}\in$ DHR, for every $\beta>0$.*
2. *If $\theta\leq 1$ then $K_{\alpha,\beta,\theta}\in$ IOR, while if $\theta\geq 1$ then $K_{\alpha,\beta,\theta}\in$ DOR.*

Proof. For part 1., note that

$$h'_{\alpha,\beta,\theta}(x)=-\left(\theta\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}}+(1-\alpha)(\theta-1)\right).$$

Since $h'_{\alpha,\beta,\theta}(0)=-(\alpha+\theta-1)$, it follows that $h'_{\alpha,\beta,\theta}(x)<0$ for every $x\geq 0$, given that h' is decreasing. For part 2., the odds rate of $K_{\alpha,\beta,\theta}$ is given by $\lambda_{K_{\alpha,\beta,\theta}}(x)=\frac{1}{\beta\theta}\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}-1}$, so the conclusion is obvious. □

The following result characterises the $\leq_{st}$-order comparability within the ELL family.

Theorem 5.3. *Assume the parameters $\alpha\geq 0$, β, $\theta>0$ and $\alpha_1\geq 0$, β_1, $\theta_1>0$ of the enlarged log-logistic distribution functions (10) satisfy one of the following assumptions:*

(ST1) $\theta<\theta_1$, $\alpha^{\theta-1}\beta\theta<\alpha_1^{\theta_1-1}\beta_1\theta_1$ and $\alpha_1(1-\theta)-\alpha(1-\theta_1)\geq 0$;
(ST2) $\theta=\theta_1$, $\beta<\beta_1$, $\alpha^{\theta-1}\beta<\alpha_1^{\theta_1-1}\beta_1$ and $(1-\theta)\left(\alpha_1^{\theta}\beta_1-\alpha^{\theta}\beta\right)>0$.

Then $K_{\alpha,\beta,\theta}\leq_{st}K_{\alpha_1,\beta_1,\theta_1}$.

Proof. For the general set of parameters (α,β,θ) denote $\overline{K}_{\alpha,\beta,\theta}(x)=1-K_{\alpha,\beta,\theta}(x)$, and define

$$V(x)=\frac{1}{\overline{K}_{\alpha,\beta,\theta}(x)}-\frac{1}{\overline{K}_{\alpha_1,\beta_1,\theta_1}(x)}=\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{\frac{1}{\theta}}-\left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right)^{\frac{1}{\theta_1}}+\alpha_1-\alpha.$$

Noting that $\overline{K}_{\alpha_1,\beta_1,\theta_1}(x) - \overline{K}_{\alpha,\beta,\theta}(x) \overset{\text{sgn}}{=} V(x)$, the proof is concluded if we prove that $V(x) \geq 0$, for every $x \geq 0$. It is obvious that $V(0) = 0$. We separate the two cases, according to which assumption is satisfied.

(ST1): We have $V(+\infty) = \infty \times \text{sgn}\left(\frac{1}{\theta} - \frac{1}{\theta_1}\right) = +\infty$. Differentiating, we find

$$V'(x) = \frac{1}{\beta\theta}\left(\frac{x}{\beta} + \alpha^\theta\right)^{\frac{1}{\theta}-1} - \frac{1}{\beta_1\theta_1}\left(\frac{x}{\beta_1} + \alpha_1^{\theta_1}\right)^{\frac{1}{\theta_1}-1},$$

so $V'(0) = \frac{\alpha^{1-\theta}}{\beta\theta} - \frac{\alpha_1^{1-\theta_1}}{\beta_1\theta_1} > 0$. Now, if we prove that $V'(x) \geq 0$, for every $x \geq 0$, it follows that V is increasing, hence $V(x) \geq 0$, and the conclusion follows. Therefore, we need to prove that

$$V'(x) \geq 0 \quad \Leftrightarrow \quad P(x) = \frac{\left(\frac{x}{\beta} + \alpha^\theta\right)^{\frac{1}{\theta}-1}}{\left(\frac{x}{\beta_1} + \alpha_1^{\theta_1}\right)^{\frac{1}{\theta_1}-1}} \geq \frac{\beta\theta}{\beta_1\theta_1}.$$

Noting that $P(0) = \frac{\alpha^{1-\theta}}{\alpha_1^{1-\theta_1}} > \frac{\beta\theta}{\beta_1\theta_1}$ and $P(+\infty) = +\infty$, we now look at the monotony of P. Differentiating, one observes that $P'(x) \overset{\text{sgn}}{=} L(x)$, where $L(x) = \frac{1}{\beta\beta_1}\left(\frac{1}{\theta} - \frac{1}{\theta_1}\right)x + \frac{1-\theta}{\theta}\frac{\alpha_1^{\theta_1}}{\beta} - \frac{1-\theta_1}{\theta_1}\frac{\alpha^\theta}{\beta_1}$. The assumptions imply that both the slope and intercept of $L(x)$ are positive, hence P is increasing, implying that $P(x) \geq \frac{\beta\theta}{\beta_1\theta_1}$, thus $V'(x) = 0$ has no solution.

(ST2): This case is treated analogously, so we just highlight the relevant differences. We now have $V(+\infty) = \infty \times \text{sgn}\left(\frac{1}{\beta} - \frac{1}{\beta_1}\right) = +\infty$, and $P'(x) \overset{\text{sgn}}{=} (1-\theta)\left(\alpha_1^\theta\beta_1 - \alpha^\theta\beta\right)$, assumed to be positive.

□

The previous result allows for an immediate pointwise comparison of the odds ratio of the $G_{\alpha,\beta,\theta}$ family.

Corollary 5.4. *Let $G_{\alpha,\beta,\theta}$ be given by (5). Under either of the assumptions of Theorem 5.3, it holds that $\Lambda_{G_{\alpha,\beta,\theta}}(x) \leq \Lambda_{G_{\alpha_1,\beta_1,\theta_1}}(x)$ for every $x \geq 0$.*

Proof. Remember that $\Lambda_{G_{\alpha,\beta,\theta}}(x) = K^{-1}_{\alpha,\beta,\theta}(F(x))$. Under the assumptions of Theorem 5.3, we have that $K_{\alpha,\beta,\theta}(x) \geq K_{\alpha_1,\beta_1,\theta_1}(x)$, for every $x \geq 0$. But this is equivalent to $K^{-1}_{\alpha,\beta,\theta}(x) \leq K^{-1}_{\alpha_1,\beta_1,\theta_1}(x)$ for every $x \geq 0$, so the result follows immediately. □

Conditions for the particular case of one parameter comparison, corresponding to the ELL that characterise the PHR, the proportional odds or the proportional log-odds models, are immediate from Theorem 5.3. We state the result, for sake of completeness.

Corollary 5.5. *For the enlarged log-logistic distribution functions (10) we have that:*

1. *If $\alpha \geq \alpha_1 \geq 0$ and $\theta \leq 1$ then $K_{\alpha,\beta,\theta} \leq_{st} K_{\alpha_1,\beta,\theta}$ for every $\beta > 0$.*
2. *If $\beta \leq \beta_1$ and $\theta <\leq 1$ then $K_{\alpha,\beta,\theta} \leq_{st} K_{\alpha,\beta_1,\theta}$ for every $\alpha \geq 0$.*
3. *If $\theta < \theta_1 \leq 1$, $\alpha^{\theta_1-\theta} > \frac{\theta}{\theta_1}$ and $\frac{1-\theta}{\alpha^\theta\theta} > \frac{1-\theta_1}{\alpha^{\theta_1}\theta_1}$ then $K_{\alpha,\beta,\theta} \leq_{st} K_{\alpha,\beta,\theta_1}$ for every $\beta > 0$.*

We now prove a general set of conditions providing the $\leq_{hr}$-comparability within the ELL family.

Theorem 5.6. *Assume the parameters $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $\alpha_1 > 0$, $\beta_1 > 0$, $\theta_1 > 0$ satisfy the following assumptions:*

(HR1) (i) $\beta\theta\alpha^{\theta-1} \le \beta_1\theta_1\alpha_1^{\theta_1-1}$, *and (ii)* $\beta\theta\alpha^{\theta} \le \beta_1\theta_1\alpha_1^{\theta_1}$,
(HR2) $\theta < \theta_1$,
(HR3) $(1-\alpha_1)(\theta_1-1) \ge 0$,
(HR4) $\left(\frac{1}{\alpha}-1\right)(\theta-1) \le \left(\frac{1}{\alpha_1}-1\right)(\theta_1-1)$.

Then $K_{\alpha,\beta,\theta} \le_{hr} K_{\alpha_1,\beta_1,\theta_1}$.

Proof. We shall prove that

$$\begin{aligned} V(x) &= \frac{1}{h_{\alpha_1,\beta_1,\theta_1}(x)} - \frac{1}{h_{\alpha,\beta,\theta}(x)} \\ &= \beta_1\theta_1\left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right) + (1-\alpha_1)\beta_1\theta_1\left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right)^{1-\frac{1}{\theta_1}} \\ &\qquad -\beta\theta\left(\frac{x}{\beta}+\alpha^{\theta}\right) - (1-\alpha)\beta\theta\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{1-\frac{1}{\theta}} \ge 0, \end{aligned}$$

which clearly implies the conclusion. We start by noting that, taking into account *(HR1-i)* and *(HR2)*, $V(0) = \beta_1\theta_1\alpha_1^{\theta_1-1} - \beta\theta\alpha^{\theta-1} \ge 0$ and $V(+\infty) = \infty \times \mathrm{sgn}(\theta_1-\theta) = +\infty$, hence the conclusion follows if we prove that V is increasing. Direct differentiation gives

$$V'(x) = \theta_1 - \theta + (1-\alpha_1)(\theta_1-1)\left(\frac{x}{\beta_1}+\alpha_1^{\theta_1}\right)^{-\frac{1}{\theta_1}} - (1-\alpha)(\theta-1)\left(\frac{x}{\beta}+\alpha^{\theta}\right)^{-\frac{1}{\theta}}, \quad (11)$$

so, given *(HR2)* and *(HR3)*, the nonnegativity of V' follows if we prove that

$$Q(x) = \frac{\beta_1^{\frac{1}{\theta_1}}}{\beta^{\frac{1}{\theta}}}\frac{\left(x+\beta\alpha^{\theta}\right)^{\frac{1}{\theta}}}{\left(x+\beta_1\alpha_1^{\theta_1}\right)^{\frac{1}{\theta_1}}} \ge \frac{(1-\alpha)(\theta-1)}{(1-\alpha_1)(\theta_1-1)}.$$

It easily seen that *(HR3)* and *(HR1-ii)* imply that $Q'(x) \ge 0$ for every $x \ge 0$, hence Q is increasing. Finally, *(HR4)* means that $Q(0) \ge \frac{(1-\alpha)(\theta-1)}{(1-\alpha_1)(\theta_1-1)}$, so the theorem is proved. □

Theorem 5.6 does not allow to choose $\alpha = 0$, therefore leaving out of the comparisons the important case of the log-logistic distribution $\mathcal{L}$, as the expression (11) means, when taking $x = 0$, that α appears as a denominator. The way out of this can be sorted adapting the expressions above by continuity when $\alpha \longrightarrow 0$.

Corollary 5.7. *Assume the parameters $\beta > 0$, $0 < \theta \le 1$ and $\alpha_1 > 0$, $\beta_1 > 0$, $\theta_1 > 0$ satisfy (HR2) and (HR3). Then $K_{0,\beta,\theta} \le_{hr} K_{\alpha_1,\beta_1,\theta_1}$.*

Proof. With respect to the proof of Theorem 5.6 note that, after allowing $\alpha \longrightarrow 0$, we need that $\theta \le 1$ to fulfill the appropriate version of $Q(0) = 0 \ge \frac{(1-\alpha)(\theta-1)}{(1-\alpha_1)(\theta_1-1)} = \frac{\theta-1}{(1-\alpha_1)(\theta_1-1)}$. □

Moreover, note that Theorem 5.6 proof's argument depends crucially on $\theta < \theta_1$, and breaks down if we assume equality of these two parameters.

Corollary 5.8. *Assume the parameters $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $\alpha_1 > 0$, $\beta_1 > 0$ are such that $\alpha > \alpha_1$, $\beta\alpha^{\theta-1} \le \beta_1\alpha_1^{\theta-1}$ and*

$$\begin{cases} \text{if } \theta \ge 1, & (1-\alpha)\beta^{\frac{1}{\theta}} \le (1-\alpha_1)\beta_1^{\frac{1}{\theta}}, \\ \\ \text{if } \theta \le 1, & \beta\alpha^{\theta} < \beta_1\alpha_1^{\theta}. \end{cases}$$

Then $K_{\alpha,\beta,\theta} \le_{hr} K_{\alpha_1,\beta_1,\theta}$.

Proof. Rewrite the function

$$\begin{aligned} V(x) &= \frac{1}{h_{\alpha_1,\beta_1,\theta}(x)} - \frac{1}{h_{\alpha,\beta,\theta}(x)} \\ &= \left(\beta_1\alpha_1^{\theta} - \beta\alpha^{\theta}\right)\theta + (1-\alpha_1)\beta_1\theta\left(\frac{x}{\beta_1} + \alpha_1^{\theta}\right)^{1-\frac{1}{\theta}} - (1-\alpha)\beta\theta\left(\frac{x}{\beta} + \alpha^{\theta}\right)^{1-\frac{1}{\theta}}. \end{aligned}$$

The assumptions imply that $V(0) \ge 0$ and $V(+\infty) \ge 0$, possibly equal to $+\infty$. The equation $V'(x) = 0$ translates into

$$1 + \frac{\beta\alpha^{\theta} - \beta_1\alpha_1^{\theta}}{x + \beta_1\alpha_1^{\theta}} = \frac{\beta}{\beta_1}\left(\frac{1-\alpha}{1-\alpha_1}\right)^{\theta},$$

which may have at most one root for $x \ge 0$. Moreover, $V'(0) = \frac{\alpha-\alpha_1}{\alpha\alpha_1} > 0$, therefore V starts increasing at $x = 0$. Hence, $V(x) > 0$ for every $x \ge 0$, and the conclusion follows. □

Again, as for the general result, Corollary 5.8 does not include the case $\alpha = 0$, but this can be handled in exactly the same way as in Corollary 5.7. We state, without proof, the corresponding result.

Corollary 5.9. *Assume the parameters $\beta > 0$, $\theta > 0$ and $\alpha_1 > 0$, $\beta_1 > 0$ are such that $\beta^{\frac{1}{\theta}} \le (1-\alpha_1)\beta_1^{\frac{1}{\theta}}$. Assume, further, than one of the following conditions is satisfied:*

(HR5) $(1-\alpha_1)(\theta-1) \ge 0$;

(HR6) $(1-\alpha_1)(\theta-1) < 0$ *and* $\alpha_1 + (1-\alpha_1)\beta_1^{1-\frac{1}{\theta}} \frac{(1-\alpha_1^{\theta})^{1-\frac{1}{\theta}} - \beta}{\left(\beta_1(1-\alpha_1^{\theta})-\beta\right)^{1-\frac{1}{\theta}}} \ge 0$.

Then $K_{0,\beta,\theta} \le_{hr} K_{\alpha_1,\beta_1,\theta}$.

Proof. We need to look now at he sign of

$$V(x) = \frac{1}{h_{\alpha_1,\beta_1,\theta}(x)} - \frac{1}{h_{0,\beta,\theta}(x)} = \beta_1\alpha_1^{\theta}\theta + (1-\alpha_1)\beta_1\theta\left(\frac{x}{\beta_1} + \alpha_1^{\theta}\right)^{1-\frac{1}{\theta}} - \beta\theta\left(\frac{x}{\beta}\right)^{1-\frac{1}{\theta}}.$$

We have $V(0) = \beta_1\alpha_1^{\theta-1}\theta > 0$. Moreover,

$$V(+\infty) = \begin{cases} \infty \times \operatorname{sgn}\left((1-\alpha_1)\beta_1^{\frac{1}{\theta}} - \beta^{\frac{1}{\theta}}\right) & \text{if } 1 - \dfrac{1}{\theta} > 0, \\ \\ \beta_1\alpha_1^{\theta} & \text{if } 1 - \dfrac{1}{\theta} < 0. \end{cases}$$

Therefore, under our assumptions, $V(+\infty) = +\infty$ for every $\theta > 0$. Seeking for extreme points of V, we need to solve $V'(x) = 0$, which translates into

$$P(x) = \frac{x}{x + \beta_1 \alpha_1^\theta} = \frac{\beta}{\beta_1} \frac{1}{(1-\alpha_1)^\theta}.$$

It is easy to verify that P is increasing, $P(0) = 0$, $P(+\infty) = 1$, and the right hand side of he equation is less or equal than 1, so this equation has exactly one solution, equal to $x_0 = \frac{\beta\beta_a\alpha_1^\theta}{\beta_1(1-\alpha_1^\theta)-\beta}$. Assuming *(HR5)*, it follows that $V'(x) \geq 0$, for every $x \geq 0$, hence V remains positive. If assuming *(HR6)*, V has a minimum at x_0, and our assumptions mean that $V(x_0) \geq 0$ so, again, we conclude that V stays positive, thus concluding the proof. □

Finally, a characterisation of convex transform order relationships.

Theorem 5.10. *For the enlarged log-logistic distribution functions (10) we have that:*

1. *If $\theta \leq \theta_1$ and $\alpha(\theta_1 - 1) + \alpha_1(1-\theta) > 0$, then for every β, $\beta_1 > 0$, $K_{\alpha,\beta,\theta} \leq_c K_{\alpha_1,\beta_1,\theta_1}$.*
2. *If $\theta \geq \theta_1$ and $\alpha(\theta_1 - 1) + \alpha_1(1-\theta) < 0$, then for every β, $\beta_1 > 0$, $K_{\alpha_1,\beta_1,\theta_1} \leq_c K_{\alpha,\beta,\theta}$.*

Proof. First note that as the β is a scale parameter and the convex transform order is invariant with respect to scale parameters, we may assume that $\beta = \beta_1 = 1$. We need to look at the convexity/concavity of

$$\psi(x) = K^{-1}_{\alpha_1,1,\beta_1} \circ K_{\alpha,1,\theta}(x) = \left(\left(x + \alpha^\theta\right)^{\frac{1}{\theta}} + \alpha_1 - \alpha\right)^{\frac{1}{\theta_1}} - \alpha_1^{\theta_1}.$$

Simple differentiation and simplification show that $\psi''(x) \stackrel{\text{sgn}}{=} (\theta_1 - \theta)\left(x + \alpha^\theta\right)^{\frac{1}{\theta}} + (1 - \theta)(\alpha_1 - \alpha)$. Therefore, ψ is convex if $\theta_1 - \theta \geq 0$ and $\psi''(0) = \alpha(\theta_1 - 1) + \alpha_1(1-\theta) > 0$, and it is concave if both these two inequalities are reversed. □

The following particular cases are now obvious.

Corollary 5.11. *For the enlarged log-logistic distribution functions (10) we have that:*

1. *If $\theta \geq 1$, then for every $\alpha \geq 0$, $\mathcal{L} = K_{0,\beta,1} \leq_c K_{\alpha,\beta,\theta} \leq_c K_{0,\beta,\theta}$.*
2. *If $\theta \leq 1$, then for every $\alpha \geq 0$, $K_{0,\beta,\theta} \leq_c K_{\alpha,\beta,\theta} \leq_c K_{0,\beta,1} = \mathcal{L}$.*

Remark 5.12. *As mentioned above, the* IOR *family may be characterised as the class of distributions that are dominated, with respect to the convex transform order, by the standard log-logistic $K_{0,1,1}$ (which is equivalent, for this purpose, to $K_{0,\beta,1}$, for every $\beta > 0$). Denote with $D_{\alpha,\beta,\theta}$ the family of distributions that are dominated, with respect to the convex transform order, by the $K_{\alpha,\beta,\theta}$ distribution. We have then that* IOR $= D_{0,\beta,1}$*, for every $\beta > 0$. Moreover, the transitivity of the $\leq_c$-ordering implies that, for $\theta \geq 1$ and $\alpha \geq 0$,* IOR $= D_{0,\beta,1} \subset D_{\alpha,\beta,\theta} \subset D_{0,\beta,\theta}$*. This inclusion implies that, for this choice of parameters, the IOR class remains nested within this more general family, hence meaning that the requirement that $G \in D_{\alpha,\beta,\theta}$ is less stringent that $G \in$* IOR*. As emphasized in Lando et al. (2022), the* IOR *already encompasses several well-known distributions with interesting shape properties, namely, allows heavy tailed distributions or for bathtub shaped hazard rates.*

In Theorem 4.6 we described conditions implying the monotonicity of the odds rate $\lambda_{G_{\alpha,\beta,\theta}}$. This monotonicity, following the Lando et al. (2022), translates into either

$G_{\alpha,\beta,\theta} \leq_c \mathcal{L} = K_{0,1,1}$, equivalent to $G_{\alpha,\beta,\theta} \in$ IOR, or $\mathcal{L} = K_{0,1,1} \leq_c G_{\alpha,\beta,\theta}$, equivalent to $G_{\alpha,\beta,\theta} \in$ DOR. We may now describe a more general form of the convex transform relations between the $G_{\alpha,\beta,\theta}$ and $K_{\alpha,\beta,\theta}$ families of distributions.

Theorem 5.13. *Let $G_{\alpha,\beta,\theta}$ be described by (5) (or (7) for a more explicit expression) and $K_{\alpha_1,\beta_1,\theta_1}$ as in (10). If $F \in$ IOR and θ, $\theta_1 \geq 1$, then $G_{\alpha,\beta,\theta} \leq_c K_{\alpha_1,\beta_1,\theta_1}$. On the other hand, if $F \in$ DOR and θ, $\theta_1 \leq 1$, then $K_{\alpha_1,\beta_1,\theta_1} \leq_c G_{\alpha,\beta,\theta}$.*

Proof. Assume that $F \in$ IOR and θ, $\theta_1 \geq 1$. Due to the invariance of the convex transform order with respect to scale parameters, we may assume the $\beta_1 = 1$. Hence, we want to prove the convexity of

$$\psi(x) = K^{-1}_{\alpha_1,1,\theta_1} \circ G_{\alpha,\beta,\theta}(x) = \left(\beta\left((\alpha + \Lambda_F(x))^\theta - \alpha^\theta\right) + \alpha_1\right)^{\theta_1} - \alpha_1^{\theta_1}.$$

Differentiation shows that

$$\psi'(x) = \beta\theta\theta_1\left(\beta\left((\alpha + \Lambda_F(x))^\theta - \alpha^\theta\right) + \alpha_1\right)^{\theta_1 - 1} \lambda_F(x)\,(\alpha + \Lambda_F(x))^{\theta-1},$$

which, under our assumptions, is clearly increasing, so ψ is convex. The second statement is proved analogously. □

Theorem 5.14. *Let $G_{\alpha,\beta,\theta}$ be described by (5) (or (7) for a more explicit expression) and $K_{\alpha_1,\beta_1,\theta_1}$ as in (10). If $F \in$ IOR, θ, $\theta_1 \geq 1$ and $\beta\beta_1\theta\theta_1 f(0)\alpha^{\theta-1}\alpha^{\theta_1-1} \geq 1$, then $G_{\alpha,\beta,\theta} \leq_{disp} K_{\alpha_1,\beta_1,\theta_1}$. On the other hand, if $F \in$ DOR, θ, $\theta_1 \leq 1$ and $\beta\beta_1\theta\theta_1 f(0)\alpha^{\theta-1}\alpha^{\theta_1-1} \leq 1$, then $K_{\alpha_1,\beta_1,\theta_1} \leq_{disp} G_{\alpha,\beta,\theta}$.*

Proof. The result follows by studying the monotonicity of the function $\phi(x) = K^{-1}_{\alpha_1,\beta_1,\theta_1} \circ G_{\alpha,\beta,\theta}(x) - x$. Observe that if $F \in$ IOR and $\theta, \theta_1 \geq 1$, ϕ' is increasing while the additional assumption ensures that $\phi'(0) \geq 0$, establishing the nonnegativeness of ϕ'. The second part of the theorem follows in a similar manner. □

Remark 5.15. *Notice that $K_{\alpha_1,\beta_1,\theta_1}(0) = G_{\alpha,\beta,\theta}(0) = 0$. Thus, under the same conditions as in Theorem 5.14 we can easily get the respective results for the usual stochastic order by applying Theorem 3.B.13(a) of Shaked and Shanthikumar (2007).*

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