Convex combinations of random variables stochastically dominate the parent for a large class of heavy tailed distributions

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Abstract

Stochastic dominance of a random variable by a convex combination of its independent copies has recently been shown to hold within the relatively narrow class of distributions with concave odds function. We show that a key property for this stochastic dominance result to hold is the subadditivity of the cumulative distribution function of the reciprocal of the random variable of interest, referred to as the inverted distribution. This enlarges significantly the family of distributions for which the dominance is verified. Moreover, we study the relation between the class of distributions with concave odds function and the class we introduce showing conditions under which the concavity of the odds function implies the subadditivity of inverted distribution.

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1 Introduction

Stochastic dominance is a widely-used tool in probability, which expresses some notion of one random variable being larger than another from a distributional point of view (see Shaked and Shanthikumar (2007)). The applications of this concept are numerous within different fields, such as statistics, economics, and finance, as is easily seen by the enormous references in the literature dealing with these concepts. Although the topic has been studied extensively, recent results, discussed below, have outlined some “surprising” behaviours of stochastic dominance, especially when we consider sums of random variables. While it may be intuitive, in a non-random setting, that summing the same quantity on both sides of some expression should not affect inequalities, and that a convex combination of points belongs to the convex hull, these basic principles are not generally true when random elements are involved. For example, Pomatto et al. (2020) have shown that, under some conditions, the ordering between a pair of random variables can be obtained by summing an independent “noise” to both, while Chen et al. (2024) proved that, within a given family of probability distributions, a random variable can be dominated by convex combinations of independent copies from it. In both cases, the results are related to the variability, or the tail-heaviness, of the random variables involved.

In this paper, we show that the dominance result recently obtained by Chen et al. (2024) holds under broader conditions. To be more specific, we search for conditions under which, given n i.i.d. copies of X, say $X_1, \dots, X_n$, and weights $\theta_1, \dots, \theta_n \geq 0$ such that $\theta_1 + \cdots + \theta_n = 1$, we have

$$X \leq_{st} \theta_1 X_1 + \cdots + \theta_n X_n. \tag{1}$$

where $\leq_{st}$ represents the standard stochastic dominance (see Definition 1 below to recall the formal definition). The implications of this result in terms of decision making under uncertainty, with meaningful applications in insurance and economic models, are quite remarkable, as it has been already explained by Chen et al. (2024). Using the same terminology of the referenced paper, the relation in (1) represents an "unexpected" stochastic dominance result. Indeed, it is maybe intuitive to think that a convex combination is somewhere in between its components, which is actually the case for random variables with finite mean. In particular, if $\mathrm{E}\, X < \infty$, (1) holds trivially, with equality in distribution, if and only if only one coefficient is strictly positive. Differently, if $\mathrm{E}\, X < \infty$ and at least two coefficients are strictly positive, (1) does not hold, as follows from the fact that the random variables X and $\theta_1 X_1 + \cdots + \theta_n X_n$ have the same expectation and they are comparable in terms of variability, where X is more variable that $\theta_1 X_1 + \cdots + \theta_n X_n$ in terms of the convex order (Shaked and Shanthikumar 2007), as we will discuss later. Chen et al. (2024) proved that (1) holds if X is an increasing convex transformation of a standard Pareto random variable, that these authors call a *super-Pareto* random variable. This property, as described later (see Proposition 12 below), is equivalent to the concavity of the odds function (the class of distributions defined through shape properties of the odds functions has recently been studied in Lando et al. (2023)). Moreover, the super-Pareto assumption implies that the expectation of X is infinite (Proposition 2 in Chen et al. (2024)), which indeed is necessary for (1) to hold. However, the assumptions in Chen et al. (2024) rule out many important models for which stochastic dominance is still verified. For example, we shall prove that the Fréchet distribution (with parameter 1), which is strictly sub-Pareto, using the terminology of Chen et al. (2024), satisfies (1). Furthermore, the super-Pareto assumption implies that X is absolutely continuous, while it is reasonable to expect that (1) may hold even for some discrete models, as it can be seen in some special cases. The above examples motivate the interest in finding weaker conditions for (1). In our main result we show that the stochastic domination still holds for a class of distributions characterised by a suitable subadditivity assumption that, while extending the result obtained in Chen et al. (2024), allows for discrete distributions as well, as we show by providing one such example. Moreover, we also address the relationship between the class of distributions we obtained and the super-Pareto class introduced by Chen et al. (2024) and discuss some relevant characterisation properties of the family of distributions we found.

2 Main result

Given a random variable X, we shall represent by F_X and $\overline{F}_X = 1 - F_X$ its cumulative distribution and survival functions, possibly using other subscripts if different such objects are under consideration. In general, we shall not be assuming the existence of densities. We shall also be referring to the odds function $\Lambda_X(x) = \frac{F_X(x)}{\overline{F}_X(x)}$, again possibly with different subscripts. Throughout this paper, "increasing" and "decreasing" are taken as "non-decreasing" and "non-increasing", respectively, and the generalised inverse of an increasing function v is denoted as $v^{-1}(u) = \sup\{x \in \mathbb{R} : v(x) \leq u\}$. Moreover, a function v is said to be subadditive if $v(x+y) \leq v(x) + v(y)$, for every x, y. We recall the definition of stochastic dominance.

Definition 1 *Given two random variables X and Y, we say that Y stochastically dominates X, denoted as $X \leq_{st} Y$, if $\overline{F}_X(x) \leq \overline{F}_Y(x)$, for every $x \in \mathbb{R}$.*

Note that we shall refer to the random variables or to their distribution functions, with the same notations, as is more convenient. In fact, the stochastic orders and the characterisations we will be discussing depend only on the distribution functions.

Bearing in mind that $1 - F_X(\frac{1}{x})$ is generally referred to as the *inverted distribution* of X, as this is the cumulative distribution function of $\frac{1}{X}$ in the continuous case, we introduce a new class of distributions that will be central to our main result.

Definition 2 *We say that a nonnegative random variable X is* InvSub *(for "inverted subadditive") if $1 - F_X(\frac{1}{x})$ is subadditive.*

We first present a simple characterisation of this class.

Lemma 3 *A random variable X is* InvSub *if and only if*

$$F_X(\tfrac{x}{\theta}) + F_X(\tfrac{x}{1-\theta}) \leq F_X(x) + 1, \quad \forall x \geq 0,\, \theta \in (0,1). \tag{2}$$

Proof. The subadditivity of $1 - F_X(\frac{1}{x})$ is obviously equivalent to $1 - F_X(\frac{x}{\theta}) + 1 - F_X(\frac{x}{1-\theta}) \geq 1 - F_X(x)$, for every ≥ 0 and $\theta \in (0,1)$, which is clearly a rewriting of (2). □

Example 4 *A simple example of a class of distributions that are* InvSub *is obtained by considereing random variables X_a with Fréchet distribution, that is, with cumulative distribution function $H_a(x) = a^{1/x}$, for $x > 0$, for some $a \in (0,1)$. In fact, given $\theta \in [0,1]$, $1 + H_a(x) - H_a(\frac{x}{\theta}) - H_a(\frac{x}{1-\theta}) = (1 - a^{\theta/x})(1 - a^{(1-\theta)/x}) \geq 0$, so H_a satisfies (2). Moreover, note that it is easily seen that $\mathrm{E}\, X_a$ is infinite and the odds function is $\Lambda_{H_a}(x) = \frac{a^{1/x}}{1-a^{1/x}}$, which is convex.*

It is well-known that, for nonnegative functions, concavity implies subadditivity. Hence, when densities exist, a rather simple sufficient condition is available.

Proposition 5 *Assume the nonnegative random variable X has density f_X such that $V_F(x) = x^2 f_X(x)$ is increasing. Then X is* InvSub.

Proof. Just note that $(1-F_X(\frac{1}{x}))' = V_F(\frac{1}{x})$ that is, according to the assumption, decreasing. Hence, $1-F_X(\frac{1}{x})$ is concave, vanishes at zero, and is therefore subadditive. □

This very simple characterisation allows to show that a second family of distributions is InvSub.

Example 6 *Consider nonnegative random variables X_b, where $b \geq 0$, with survival function $\overline{F}_b(x) = \frac{1}{1+x^b \operatorname{Log}(x+1)}$, for $x > 0$. It is easily seen that these distributions satisfy the monotonicity assumption of Proposition 5 whenever $b < 1$. Therefore X_b, for $b < 1$ are* InvSub. *Moreover, note that the odds function is $\Lambda_b(x) = x^b \operatorname{Log}(x+1)$, that can be checked to not be concave nor convex.*

The existence of a density is not necessary. Indeed, the InvSub condition is also compatible with discrete models, as we illustrate in the next example.

Example 7 *Let $F_X(x) = (1-p)^{\lceil \frac{1}{x} \rceil}$, be defined in the completed half line $(0, +\infty]$, where $p \in (0,1)$ and $\lceil \cdot \rceil$ denotes the ceiling function. This cumulative distribution function is a right-continuous step function, with jumps at points $\frac{1}{k}$, $k = 1, 2, 3, \ldots, +\infty$, and in particular, it assigns positive mass p to $+\infty$. This clearly implies that $\mathrm{E}\, X = +\infty$. Now, $1 - F_X(\frac{1}{x}) = 1-(1-p)^{\lceil x \rceil}$ is the left-continuous version of the geometric cumulative distribution function, which can be seen to be subadditive.*

We present our main result stating a general condition for the stochastic order dominance between a random variable and a convex linear combination of its independent copies.

Theorem 8 *Let $X_1, \ldots, X_n$ be independent random variables with the same cumulative distribution function F_X as the nonnegative random variable X that is* InvSub. *Given any $\theta_1, \ldots, \theta_n > 0$ such that $\theta_1 + \cdots + \theta_n = 1$, the stochastic dominance (1) holds.*

Proof. We proceed by induction on the number of random variables. Conditioning and using the independence of the random variables, it is easily seen that, for every $x \geq 0$ and $\theta \in (0, 1)$,

$$\mathrm{P}\left(\theta X_1 + (1-\theta) X_2 > x\right) = 1 - \int_0^{x/\theta} F_X(\tfrac{x-\theta t}{1-\theta})\, F_X(dt).$$

To find an upper bound for the integral, we consider the decomposition described in Figure 1, from which follows easily that

$$\int_0^{x/\theta} F_X(\tfrac{x-\theta t}{1-\theta})\, F_X(dt) \leq F_X(\tfrac{x}{1-\theta}) F_X(x) + F_X(x)\left(F_X(\tfrac{x}{\theta}) - F_X(x)\right) \leq F_X(x),$$

using (2) for the last inequality. So, it follows that $\mathrm{P}\left(\theta X_1 + (1-\theta) X_2 > x\right) \geq 1 - F_X(x) = \mathrm{P}(X > x)$, so (1) holds for $n = 2$.

Assume now that (1) holds whenever considering $n-1$ random variables. Given $\theta_1, \ldots, \theta_n > 0$

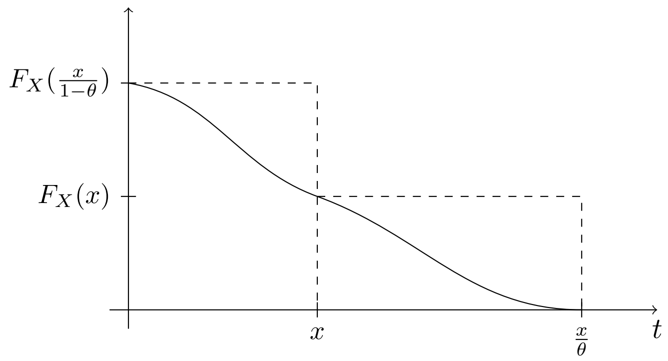

Figure 1: Upper bound for the integral in the initial induction step.

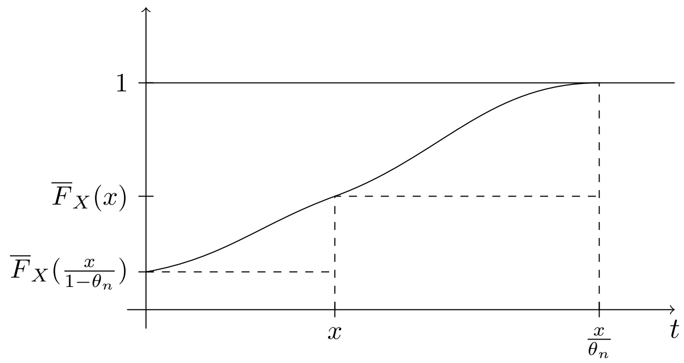

Figure 2: Lower bound for the integral in the induction step.

satisfying $\theta_1 + \cdots + \theta_n = 1$, we have that

$$\begin{aligned}
&\mathrm{P}\left(\theta_1 X_1 + \cdots + \theta_n X_n > x\right) \\
&\quad = \mathrm{P}\left(X_n \geq \tfrac{x}{\theta_n}\right) + \mathrm{P}\left(\theta_1 X_1 + \cdots + \theta_n X_n \geq x, X_n \leq \tfrac{x}{\theta_n}\right) \\
&\quad = \overline{F}_X(\tfrac{x}{\theta_n}) + \int_0^{x/\theta_n} \mathrm{P}\left(\tfrac{\theta_1 X_1 + \cdots + \theta_{n-1} X_{n-1}}{1-\theta_n} \geq \tfrac{x-\theta_n t}{1-\theta_n}\right) F_X(dt) \\
&\quad \geq \overline{F}_X(\tfrac{x}{\theta_n}) + \int_0^{x/\theta_n} \overline{F}_X(\tfrac{x-\theta_n t}{1-\theta_n})\, F_X(dt),
\end{aligned}$$

using the induction hypothesis. We need now to find a lower bound for this integral, which may achieved using the decomposition depicted in Figure 2, from which follows that

$$\begin{aligned}
&\mathrm{P}\left(\theta_1 X_1 + \cdots + \theta_n X_n > x\right) \\
&\quad \geq \overline{F}_X(\tfrac{x}{\theta_n}) + \int_0^{x/\theta_n} \overline{F}_X(\tfrac{x-\theta_n t}{1-\theta_n})\, F_X(dt) \\
&\quad \geq \overline{F}_X(\tfrac{x}{\theta_n}) + \overline{F}_X(\tfrac{x}{1-\theta_n}) F_X(x) + \overline{F}_X(x)\left(F_X(\tfrac{x}{\theta_n}) - F_X(x)\right) \\
&\quad \geq \overline{F}_X(\tfrac{x}{\theta_n}) + \overline{F}_X(\tfrac{x}{1-\theta_n}) + \overline{F}_X(x)\left(\overline{F}_X(x) - \overline{F}_X(\tfrac{x}{\theta_n}) - \overline{F}_X(\tfrac{x}{1-\theta_n})\right).
\end{aligned}$$

Finally, noting that the subadditivity assumption implies that the large parenthesis is negative, it follows that $\mathrm{P}\left(\theta_1 X_1 + \cdots + \theta_n X_n > x\right) \geq \overline{F}_X(\frac{x}{\theta_n}) + \overline{F}_X(\frac{x}{1-\theta_n}) \geq \overline{F}_X(x)$, using (2) written in terms of the survival function, thus concluding the proof. □

It is easy to see that, if X has finite mean, then X is more variable then $\theta_1 X_1 + \cdots + \theta X_n$ in terms of the convex order, that is, for every convex function ϕ, $\mathrm{E}\,\phi(\theta_1 X_1 + \cdots + \theta X_n) \leq \mathrm{E}\,X$ (this follows by using repeatedly the definition of convexity), meaning, for instance, that X has larger variance (when it is finite) than the convex combination. Differently, as already remarked in Chen et al. (2024), the stochastic dominance stated in (1) is crucially linked to the fact that we are dealing with random variables with infinite means (see Proposition 2 in Chen et al. (2024)). We present here another proof, using more elementary arguments.

Proposition 9 *Let $X_1, \ldots, X_n$ be independent random variables such that $\mathrm{P}(X_1 = \cdots = X_n) < 1$, and $\theta_1, \ldots, \theta_n > 0$ such that $\theta_1 + \cdots + \theta_n = 1$. If (1) holds, then X has infinite mean.*

Proof. It is enough to prove the case $n = 2$. Denote $Y = \theta X_1 + (1-\theta) X_2$, where X_1 and X_2 are independent and have the same distribution as X and are such that $\mathrm{P}(X_1 \neq X_2) > 0$. Moreover, assume $\mathrm{E}(X)$ is finite. As then follows that $\mathrm{E}(Y) = \mathrm{E}(X)$, both finite, this, together with (1), implies that X and Y have the same distribution. Therefore, $\mathrm{Var}\left(\sqrt{X}\right) = \mathrm{Var}\left(\sqrt{Y}\right)$, implying that $\mathrm{E}\left(\sqrt{X}\right) = \mathrm{E}\left(\sqrt{Y}\right)$. But this is not possible, as Jensen's inequality implies that $\mathrm{E}\left(\sqrt{Y}\right) = \mathrm{E}\left(\sqrt{\theta X_1 + (1-\theta) X_2}\right) > \theta\,\mathrm{E}\left(\sqrt{X}\right) + (1-\theta)\,\mathrm{E}\left(\sqrt{X}\right) = \mathrm{E}\left(\sqrt{X}\right)$, the inequality being strict because $\mathrm{P}(X_1 \neq X_2) > 0$. □

3 Comparing with earlier results

Our Theorem 8 states the same stochastic domination as proved in the first part of Theorem 1 in Chen et al. (2024), assuming independence instead of the variant of negative dependence these authors considered. However, the result in Chen et al. (2024) is more specific on the behaviour of the distribution of the random variables. Let us quote the notions and results relevant for comparing the assumptions in Theorem 1 in Chen et al. (2024) and the subadditivity assumption in our Theorem 8. We shall represent the standard Pareto distribution function by $\mathcal{P}(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$, for $x \geq 1$. Note that it is straightforward to verify that $\mathcal{P}$ satisfies the subadditivity assumption in Theorem 8.

Definition 10 (Chen et al. (2024)) *A random variable Y is super-Pareto if $Y \stackrel{d}{=} h(Z)$, where $F_Z = \mathcal{P}$ and h is an increasing, convex and nonconstant function.*

As mentioned in Chen et al. (2024), the super-Pareto family includes the generalised Pareto distributions, the Burr distributions and the log-logistic distribution. Therefore, it includes some common models either in economical applications or in extreme value theory. With this notion, an adapted version of the main result in Chen et al. (2024) is quoted below.

Theorem 11 (adapted version of Theorem 1 in Chen et al. (2024)) *Let $X_1, \ldots, X_n$ be independent random variables with the same cumulative distribution function F_X as the non-negative super-Pareto random variable X, and $\theta_1, \ldots, \theta_n > 0$ such that $\theta_1 + \cdots + \theta_n = 1$. Then (1) is satisfied.*

An equivalent characterisation of the super-Pareto family is given in the following result.

Proposition 12 (Proposition 1 in Chen et al. (2024)) *A random variable X, with cumulative distribution function F_X, is super-Pareto if and only if $\frac{1}{\overline{F}_X(x)}$ is concave.*

Note that the concavity of $\Lambda_X(x) = \frac{F_X(x)}{\overline{F}_X(x)} = \frac{1}{\overline{F}_X(x)} - 1$ implies the differentiability of F_X at almost every point of the support, hence the concavity of Λ_X is equivalent to the decreasingness of the odds rate $\lambda_X(x) = \Lambda'_X(x) = \frac{f_X(x)}{\overline{F}^2_X(x)}$, considering side derivatives at the points of nondifferentiability. This family of distributions has been addressed in Lando et al. (2023) or, more recently, in Arab et al. (2024), being referred as the DOR family, that can be characterised using an appropriate stochastic dominance relation. We need some additional definitions to describe these relations more precisely.

Definition 13 *We say that a random variable X with cumulative distribution function F_X is* DOR *if its odds function $\Lambda_X(x)$ is concave.*

Definition 14 *Given two distribution functions F_1 and F_2, we say that, F_1 is smaller than F_2 in the convex transform order, represented by $F_1 \leq_c F_2$, if $F_2^{-1} \circ F_1$ is convex.*

As before, we refer indifferently to random variables or distribution functions.

The following result relates the shape of the odds function with the Pareto distribution through the convex transform order.

Proposition 15 *F_X is* DOR *(or, equivalently, X is super-Pareto) if and only if $\mathcal{P} \leq_c F_X$, where $\mathcal{P}$ is the standard Pareto distribution. Analogously F_X has increasing odds rate if and only if $F_X \leq_c \mathcal{P}$.*

Proof. Noting that the quantile of the standard Pareto is $\mathcal{P}^{-1}(u) = \frac{1}{1-u}$, we have $\mathcal{P}^{-1} \circ F_X = \frac{1}{\overline{F}_X} = \Lambda_X + 1$, so the result follows immediately. $\square$

The DOR (or, the super-Pareto) class seems a relatively narrow family of distributions, thus the interest in enlarging the scope of applicability of the standard stochastic dominance (1) to a wider family of distributions. As mentioned in Example 4, the Fréchet class, with cumulative distribution functions $H_a(x) = a^{1/x}$, for some $a \in (0, 1)$, satisfies the assumption on Theorem 8. On other hand, Λ_{H_a} is convex, hence $H_a \leq_c \mathcal{P}$, so random variables with cumulative distribution H_a are not super-Pareto. This provides an example where our main result Theorem 8 implies (1), while Theorem 11 is not applicable, as its assumptions are not satisfied. Further, note that, as the convex transform order is transitive, it follows that if X is super-Pareto, we have $H_a \leq_c \mathcal{P} \leq_c F_X$.

We present next two examples showing that shape conditions about the odds function seem not to be an appropriate way to characterise the stochastic dominance (1).

Example 16 *The random variables X_b introduced in Example 6 have odds function $\Lambda_b(x) = x^b \operatorname{Log}(x + 1)$ that, as mentioned before, are not concave nor convex for $b < 1$. Nevertheless, as referred in Example 6, X_b is* InvSub.

Example 17 *Consider cumulative distribution functions of the form* $H_a^*(x) = H_a(x)\left(1 + \frac{1}{e^{2x}-1}\right)$. *Choosing* $a \in (0,1)$ *sufficiently small (say, for example,* $a = 0.5$*), it can be verified that (2) is satisfied, while the corresponding odds function is not concave nor convex.*

Before explicitly relating the super-Pareto class of distributions with the family of distributions satisfying the subaddivity assumption in Theorem 8, we present a more general transformation result.

Theorem 18 *Let* X *be* InvSub *and* h *a continuous star-shaped function. Then* $h(X)$ *is* InvSub*.*

Proof. Note that, as h is continuous, $F_{h(X)}(x) = F_X(h^{-1}(x))$. Since h is star-shaped, its inverse, h^{-1}, is anti-star-shaped (see Lemma 4.1 in Arab et al. (2024)), hence, given $\theta \in (0,1)$, it follows that $h^{-1}(\frac{x}{\theta}) \leq \frac{h^{-1}(x)}{\theta}$ and $h^{-1}(\frac{x}{1-\theta}) \leq \frac{h^{-1}(x)}{1-\theta}$. As every function considered is increasing, we have

$$\begin{aligned} & F_{h(X)}(\tfrac{x}{\theta}) + F_{h(X)}(\tfrac{x}{1-\theta}) \\ = \ & F_X\left(h^{-1}(\tfrac{x}{\theta})\right) + F_X\left(h^{-1}(\tfrac{x}{1-\theta})\right) \\ \leq \ & F_X(h^{-1}(x)) + 1 = F_{h(X)}(x) + 1 \end{aligned}$$

using (2) for the last inequality, thus, taking into account Lemma 3, the proof is concluded. □

Finally, we relate the super-Pareto class of distributions with the family satisfying the subaddivity assumption in Theorem 8.

Corollary 19 *Let* X *be super-Pareto (or,* X *be* DOR*) such that the increasing convex transformation* h *in Definition 10 satisfies* $h(0) \leq 0$*. Then* X *is* InvSub*.*

Proof. According to Definition 10, $X \overset{d}{=} h(Z)$, where $F_Z = \mathcal{P}$. As mentioned above F_Z is subadditive. On the other hand, as h is convex and such that $h(0) \leq 0$, it holds that h is star-shaped. Hence, the conclusion follows applying Theorem 18. □

The assumption, in Corollary 19, that $h(0) \leq 0$ means that the super-Pareto variable has support of the form $[s_x, \infty)$ where $s_x \leq 0$, possibly being $-\infty$. In particular, super-Pareto variables whose support is not strictly contained in $[0, +\infty)$ are within the scope of applicability Theorem 18.

4 Some further classes of distributions

The assumptions considered in Theorem 8 suggest considering some further classes of distributions, some of which, to the best knowledge of the authors, have not been considered in the literature. Therefore, a brief discussion about their characterisations is now addressed. We present some definitions, to start with.

Definition 20 *We say that a nonnegative random variable* X *with cumulative distribution function* F_X *is*

1. NBU *if* $\overline{F}_X(x)\overline{F}_X(x) \geq \overline{F}_X(x+y)$, *for every* $x \geq 0$*;*

2. NWU *if* $\overline{F}_X(x)\overline{F}_X(x) \leq \overline{F}_X(x+y)$, *for every* $x \geq 0$.

Note that the characterisation of X being NBU or NWU, that are well known classes of distributions, may be rewritten as $\operatorname{Log} \overline{F}_X$ being subadditve or superadditive, respectively (see the initial notes in Shaked and Shanthikumar (2007)), together with being nonnegative.

Noting that $1 - F_X(\frac{1}{x}) = \mathrm{P}(\frac{1}{X} < x)$, Theorem 8 sets an assumption about the distribution of $\frac{1}{X}$. We consider next some other well known classes of distributions that we shall relate with the InvSub family.

Definition 21 *The nonnegative random variable* X *is said to be inverted-*NBU *or inverted-*NWU*, if* $\frac{1}{X}$ *is of class* NBU *or* NWU*, respectively.*

Remark that Theorem 18 sets conditions for X being DOR (the assumption in Theorem 1 in Chen et al. (2024)) implying that X is InvSub.

Proposition 22 *Assume* X *has absolutely continuous distribution. If* X *is inverted-*NWU*, then* X *is* InvSub.

Proof. As noted after Definition 20, $\frac{1}{X}$ being NWU means that $\operatorname{Log} \overline{F}_{\frac{1}{X}}(x) = \operatorname{Log} F_X(\frac{1}{x})$ is superadditive, that is $\operatorname{Log} F_X(\frac{1}{x+y}) \geq \operatorname{Log} F_X(\frac{1}{x}) + \operatorname{Log} F_X(\frac{1}{y})$, which implies that $F_X(\frac{1}{x+y}) \geq \exp\left(\operatorname{Log} F_X(\frac{1}{x}) + \operatorname{Log} F_X(\frac{1}{y})\right) \geq F_X(\frac{1}{x}) + F_X(\frac{1}{y}) - 1$, as the exponential is superadditive. Finally, going to the complementary sets, this last inequality is just $\overline{F}_X(\frac{1}{x+y}) \leq \overline{F}_X(\frac{1}{x}) + \overline{F}_X(\frac{1}{y})$, that is, $\overline{F}_X(\frac{1}{x})$ is subadditive, that is, X is InvSub. □

Note that the distribution function H_a^* given in Example 17 shows that the InvSub class is strictly larger that the inverted-NWU family.

We now present some general characterisations of the classes just introduced. For this purpose, some more stochastic order notions are needed.

Definition 23 *Given two distribution functions* F_1 *and* F_2 *we say that*

1. F_1 *is smaller than* F_2 *in the superadditive order, represented by* $F_1 \leq_{su} F_2$ *if* $F_2^{-1} \circ F_1$ *is superadditive;*

2. F_1 *is smaller than* F_2 *in the subadditive order, represented by* $F_1 \leq_{sb} F_2$ *if* $F_2^{-1} \circ F_1$ *is subadditive.*

We first note that, apparently, there is no general description of the NWU class, apart from its definition, so a similar situation should be expected for the new inverted-NWU family. There is, however, a simple characterisation of the closely related NBU class.

Theorem 24 (Theorem 4.B.11 in Shaked and Shanthikumar (2007)) *A random variable is* NBU *if and only if* $X \leq_{su} Z$*, where* Z *is exponentially distributed.*

A simple similar characterisation would follow by inversion, but unfortunately, in general, the inverse of a superaddditive function may not be subadditive. According to Proposition 1 in Østerdal (2006) the extra assumptions of continuity and strict monotonicity are required, providing the following simple, but partial, characterisation.

Proposition 25 *A random variable X with strictly positive density along its support is* NWU *if and only if $Z \leq_{su} X$, where Z is exponentially distributed.*

We note that the NWU includes distributions whose cumulative distribution function does not even has a density, as it follows easily from the examples mentioned in Cai and Kalashnikov (2000).

The following results gives a complete description for the inverted-NWU class, avoiding the strict continuity difficulties.

Theorem 26 *A nonnegative random variable X is* NWU *if and only if $X \leq_{sb} H^*$, where H^* has an inverted-Fréchet distribution.*

Proof. Recall that, H^* having inverted-Fréchet distribution means that $\frac{1}{H^*}$ has distribution function H_a, for some $a \in (0,1)$. Therefore, the cumulative distribution function $F_{H^*}(x) = 1 - a^x$, for $x \geq 0$, so $F_{H^*}^{-1} \circ F_X(x) = \frac{\operatorname{Log} \overline{F}_X(x)}{\operatorname{Log} a}$. Taking into account that $a \in (0,1)$, the subadditivity of $F_{H^*}^{-1} \circ F_X(x)$ translates into $\overline{F}_X(x)\overline{F}_X(x) \leq \overline{F}_X(x+y)$, so the result is proved. □

The following characterisation is now immediate.

Corollary 27 *A nonnegative random variable X is inverted-*NWU *if and only if $\frac{1}{X} \leq_{sb} H^*$, where H^* has an inverted-Fréchet distribution.*

We may obtain a similar characterisation for the InvSub class using the subadditive order and choosing the appropriate benchmarking distribution. Remembering that the Pareto distribution is InvSub, the following is straightforward by computing the quantile function of the Pareto.

Theorem 28 *A nonnegative random variable X with absolutely continuous distribution is* InvSub *if and only if $X \leq_{sb} Z$, where Z is an inverted-Pareto random variable.*

Finally, we show that InvSub distributions have infinite means.

Proposition 29 *Assume X is* InvSub*. Then* $\mathrm{E}\, X$ *is infinite.*

Proof. As X is InvSub, the stochastic dominance (1) holds. But, then Proposition 9 implies that $\mathrm{E}\, X$ is infinite. □

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References

Arab, I., Lando, T., Oliveira, P.E., 2024. Inequalities and bounds for expected order statistics from transform-ordered families. arXiv 2403.03802 [stat.ME]. doi:10.48550/arXiv.2403.03802.

Cai, J., Kalashnikov, V., 2000. NWU property of a class of random sums. J. Appl. Probab. 37, 283–289. doi:10.1017/s0021900200015436.

Chen, Y., Embrechts, P., Wang, R., 2024. Technical note—an unexpected stochastic dominance: Pareto distributions, dependence, and diversification. Oper. Res. Articles in Advance. doi:10.1287/opre.2022.0505.

Lando, T., Arab, I., Eduardo Oliveira, P., 2023. Nonparametric inference about increasing odds rate distributions. J. Nonparametr. Stat. , 1–20doi:10.1080/10485252.2023.2220050.

Østerdal, L.P., 2006. Subadditive functions and their (pseudo-)inverses. J. Math. Anal. Appl. 317, 724–731. doi:10.1016/j.jmaa.2005.05.039.

Pomatto, L., Strack, P., Tamuz, O., 2020. Stochastic dominance under independent noise. Journal of Political Economy 128, 1877–1900.

Shaked, M., Shanthikumar, J.G., 2007. Stochastic orders. Springer Series in Statistics.