On the number of shellable arrangements of pseudolines

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Abstract

An arrangement of pseudolines is a finite collection of bi-infinite simple curves where each pair intersects exactly once. Using the correspondence between rhombic tilings of 2-dimensional zonotopes and pseudoline arrangements, we enumerate the subset of arrangements of n pseudolines that include extreme pseudolines. Additionally, we derive a recursive formula for counting shellable arrangements of pseudolines and establish bounds along with asymptotic results.

1 Introduction

Pseudoline arrangements were introduced by Friedrich Levi in the 1920s as a natural generalization of line arrangements, allowing more flexibility as the curves are not restricted to being straight. The set $A(n)$ of non-isomorphic arrangements of n pseudolines is a subject of interest in both Discrete Geometry and Combinatorics due to its connections to various combinatorial structures, such as commutation classes of reduced words for the longest permutation in the symmetric group, oriented matroids of rank 3 or rhombic tilings of 2-dimensional zonotopes.

Although pseudoline arrangements can be interpreted in various combinatorial ways, no known formula exists for the cardinality of $A(n)$. The sequence A006245 of the Online Encyclopedia of Integer Sequences (OEIS) lists all currently known values of $|A(n)|$, which are available only up to $n = 16$. Knuth [10] was the first to establish bounds for $|A(n)|$, showing that $|A(n)|$ grows as $2^{\Theta(n^2)}$. Since then, several authors have refined Knuth's bounds [3, 5, 6, 11], but there is still much to be discovered.

However, for some subsets of $A(n)$, enumeration results exist. For instance, the number of pseudoline arrangements that correspond to commutation classes with only one reduced word is 4 for every $n \geq 4$ [9]. There is also an enumeration for the commutation classes that are related to Gelfand-Cetlin polytopes [2].

Recently, a new family of pseudoline arragements was introduced in [13] to compute the degree of connectivity for the flip graph of $A(n)$. These arrangements are known as *shellable* and their definition relies on the notion of *extreme* pseudolines. In this paper we will explore the arrangements that contain these special pseudolines, as well as the shellable arrangements. We start Section 2 with some background information on pseudolines arrangements and their relation with rhombic tilings of 2-dimensional zonotopes, which will be crucial for the rest of the paper. Section 3 is dedicated to the extreme pseudolines and to their interpretation on the tilings, where we give a formula for the number of arrangements containg extreme pseudolines in terms of the cardinality of $A(n-1)$ and $A(n-2)$. In Section 4, using the methods from Section 3, we present a recursive formula to compute the number of shellable arrangements, along with some bounds and asymptotic results.

2 Definitions and background

A *pseudoline* is an unbounded simple curve that divides the Euclidean plane into two connected regions. A finite collection of pseudolines is called an *arrangement of pseudolines* if every pair of pseudolines intersects exactly once. The *crossings* of an arrangement are the points in which two pseudolines intersect. Throughout this paper, every arrangement is assumed to be *simple*, that is, no three of its curves meet at the same point. A simple arrangement of n pseudolines partition the plane into several connected components, called *cells*, where exactly $2n$ of them are unbounded.

Following the terminology used in [13], it will be assumed that every arrangement is *marked*, meaning that one of the unbounded cells is designated as *north-cell* (labeled N) and pseudolines are oriented in such a way that the north-cell is in their left half-plane. The *south-cell* (labeled S) of the arrangement is the unique unbounded cell that is separated from the north-cell by all pseudolines. The *cannonical labeling* of a marked arrangement is the labeling of the pseudolines such that an oriented curve from the north-cell to the south-cell that has all crossings on the right intersects the pseudolines in increasing order (see Figure 1). We denote by l_i the pseudoline with label i.

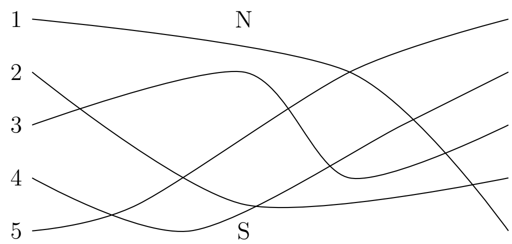

Figure 1: A marked arrangement of 5 pseudolines.

Two arrangements are *isomorphic* if there is an isomorphism of the induced cell decompositions respecting the labellings of the pseudolines. One can check the previous condition by examining the orders in which every pseudoline intersects the others when traversed from left to right. For instance, the arrangements depicted in Figure 2 are not isomorphic since in the arrangement on the left, l_3 intersects l_4 before l_1, whereas in the arrangement on the right, l_3 intersects l_1 before l_4. Given a positive integer n, we denote

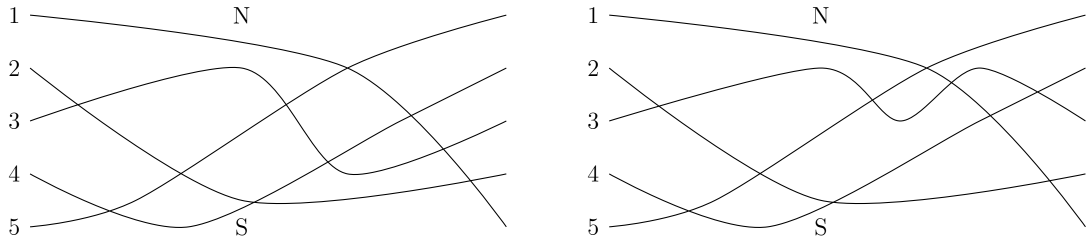

Figure 2: Non-isomorphic arrangements of 5 pseudolines.

by $A(n)$ the set of non-isomorphic arrangements of n pseudolines. For $n \leq 8$, every arrangement of pseudolines is isomorphic to an arrangement of lines, meaning that we can "stretch" the pseudolines into lines without changing the combinatorial properties of the arrangement. In fact, it is a NP-hard problem to decide if an arrangement of pseudolines is stretchable [14].

The set $A(n)$ is in bijection with a variety of interesting combinatorial objects such as commutation classes of reduced words for the longest permutation in the symmetric group, oriented matroids of rank 3 and rhombic tilings of 2-dimensional zonotopes [4, 1, 7]. This final family of objects will be used as an alternative approach obtain and interpret the results of this paper.

Definition 1. A 2-dimensional *zonotope* $Z(V)$ is a centrally symmetric $2n$-gon defined to be the Minkowski sum of a set of n vectors $V = \{v_1, \ldots, v_n\}$ in $\mathbb{R}^2$, *i.e*,

$$Z(V) = \left\{ \sum_{i=1}^{n} c_i v_i : -1 \leq c_i \leq 1, \text{ for all } 1 \leq i \leq n \right\}.$$

A *rhombic tiling* of $Z(V)$ is a tiling of $Z(V)$ made of rhombus with all edges congruent and parallel to the edges of $Z(V)$. A rhombic tiling together with a distinguished vertex of the boundary of $Z(V)$ is a *marked zonotopal tiling*.

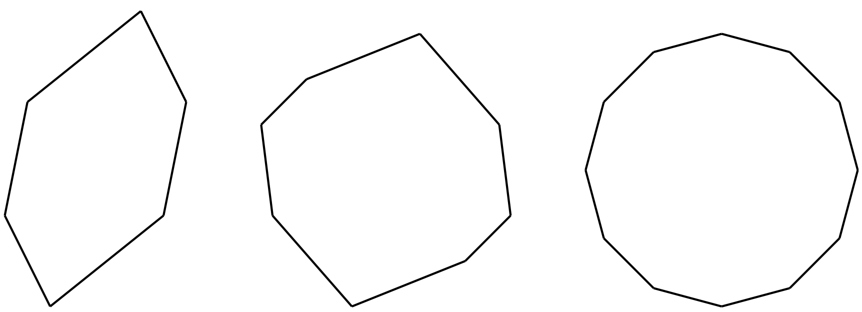

Figure 3: Some examples of zonotopes.

Each edge of a zonotopal tiling can be labeled with an integer such that parallel edges have the same label. For simplicity, we write only the labels of edges that are to the left

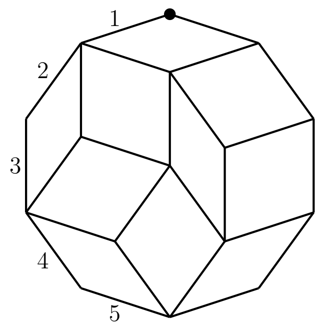

Figure 4: A marked zonotopal tiling.

of the distinguished vertex from top to bottom, with integers from 1 to n in increasing order (see Figure 4).

The following result states the connection between pseudoline arrangements and zonotopal tilings.

Theorem 2.1 ([7]). *Let V be a set of n pairwise non-collinear vectors in $\mathbb{R}^2$. There is a bijection between marked zonotopal tilings of $Z(V)$ and marked arrangements of n pseudolines.*

An intuitive way to understand the relationship between these two types of objects is to consider an arrangement as a planar graph, consisting of its crossings and the arcs connecting them. The corresponding zonotopal tiling represents the dual graph of the arrangement, where a point is placed in each cell, and two points are connected if they are separated by a single pseudoline. The distinguished vertex of the tiling is the one that is placed in the north-cell of the arrangement. Thereby, we will call it the *north-vertex* of the tiling. Conversely, if we have a marked zonotopal tiling, we can construct an arrangement where pseudoline l_i intersects only the tilling's edges that are labelled with i. Figure 5 shows an example which describes the previous construction, and relates the pseudoline arrangement depicted in Figure 1 with the zonotopal tiling of Figure 4.

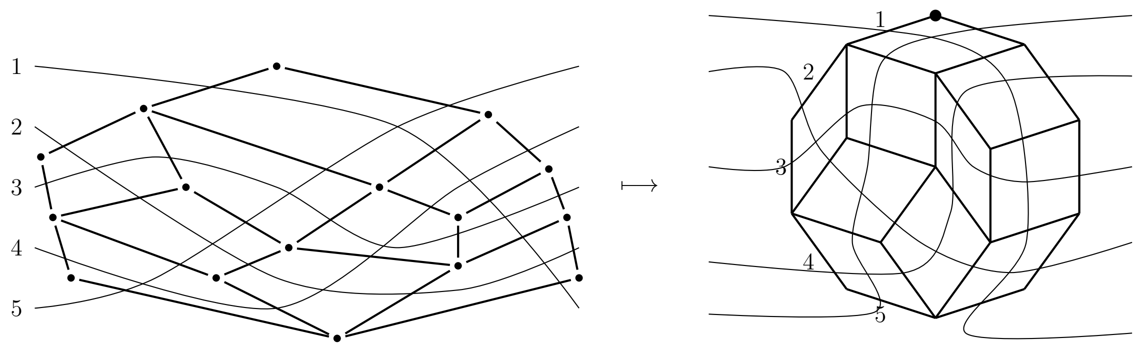

Figure 5: A pseudoline arrangement and its associated tiling.

Since Theorem 2.1 is independent of the choice of V, we can assume that $Z(V)$ has all edges with equal length and all interior angles with $\frac{\pi}{n}$ radians, *i.e*, $Z(V)$ is a regular polygon. The set of all marked zonotopal tilings of $Z(V)$ will be denoted by $T(n)$ and it will be assumed that the north-vertex will be the top most one of the polygon, unless otherwise stated.

The main advantage of working with a regular polygon is the use of its symmetries to generate new tilings from previous ones. For instance, all the tilings of an octagon are related by rotations (see Figure 6). In the following sections, we will these rotacional symmetries to simplify the study of some subsets of $T(n)$. It is worth noting that this technique was already used by Tenner in [15] to compute some statistics on $T(n)$.

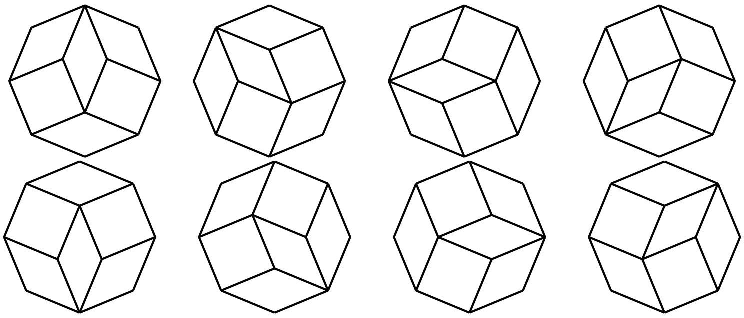

Figure 6: All rhombic tilings of an octagon.

3 Extreme pseudolines

As we said in the beginning, a pseudoline divides the Euclidean plane into two half-planes. Thus, given an arrangement $\mathcal{A} \in A(n)$ and fixing a pseudoline l, the crossings of $\mathcal{A}$ not lying on l must be distributed between the two half-planes of l. However, in certain arrangements, there is an extreme behavior defined as follows.

Definition 2. [13] Let $\mathcal{A} \in A(n)$. We say that a pseudoline l in $\mathcal{A}$ is *extreme* if all crossings that do not lie on l belong to the same half-plane induced by l.

In the arrangement ilustrated in Figure 1, l_1 is an extreme pseudoline since all the crossings not lying in l_1 are below it. In fact, it is the only extreme pseudoline of the arrangement. Figure 7 depicts an arrangement with two extreme pseudolines: l_2 and l_3.

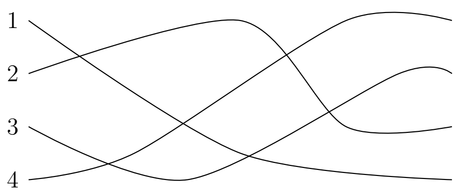

Figure 7: A pseudoline arrangement with two extremes.

The interpertation of an extreme pseudoline in the language of tilings is as follows.

Definition 3. Given $T \in T(n)$, denote by T_i the *strip* formed by the rhombus that contain some edge with label i. Then, the strip T_i is *extreme* if all rhombus lie above T_i or all rhombus lie below T_i.

Note that the crossings of an arrangement are encoded by the rhombus on its corresponding tiling. Thus, l_i is an extreme pseudoline of $\mathcal{A} \in A(n)$ if and only if T_i is an extreme strip of the tiling $T \in T(n)$ associated to the dual graph of $\mathcal{A}$. In Figure 8 is depicted the arrangement of Figure 1 and its corresponding tiling with the extremes emphasized on both objects. In this section we will focus on the arrangements that contain at least one extreme pseudoline. Due to the symmetries mencioned in the previous section, we will choose to do this study using the tilings.

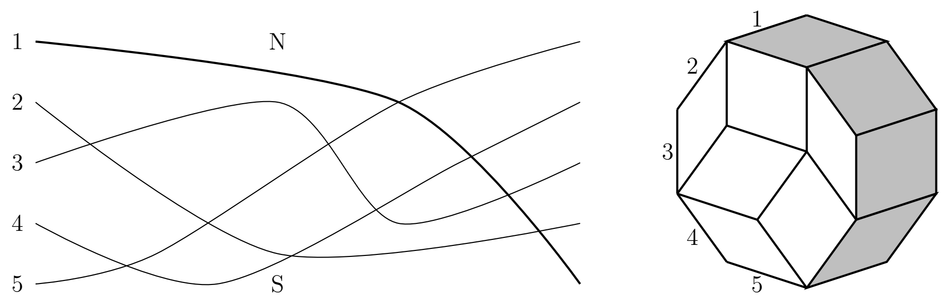

Figure 8: An arrangement and its corresponding tiling with their extremes emphasized.

Let $ET(n) := \{T \in T(n) : T \text{ contains extreme strips}\}$. This set can be writen as

$$ET(n) = \bigcup_{i=1}^{n} ET_i(n), \tag{1}$$

where $ET_i(n) := \{T \in T(n) : T_i \text{ is extreme in } T\}$, with $i \in [n] := \{1, \ldots, n\}$. We have the following.

Lemma 3.1. *Given $n \geq 1$, there is a bijection between $ET_1(n)$ and $ET_i(n)$ for all $i \in [n]$.*

Proof. Define the following map:

$$\begin{aligned} f : ET_1(n) &\longrightarrow ET_i(n) \\ T &\longmapsto f(T) = T', \end{aligned}$$

where T' is obtained from T by a counter-clockwise rotation of $(i-1)\frac{\pi}{n}$ radians with respect to the center of the polygon. To prove that f is well-defined, just see that the strip T_1 in T is identified with the strip T'_i in T'. Since T_1 was an extreme strip in T, we have that T'_i is an extreme strip in T', and so $T' \in ET_i(n)$. The inverse of f can be given by

$$\begin{aligned} g : ET_i(n) &\longrightarrow ET_1(n) \\ T &\longmapsto g(T) = T'', \end{aligned}$$

where T'' is obtained from T by a clockwise rotation of $(i-1)\frac{\pi}{n}$ radians, proving that f is bijective. □

The previous argument could not be used for every zonotope because we are not guarantee to have rotational symmetry. Hidden in a rotation is a shift of the north-vertex. More precisely, rotating a tiling by $i\frac{\pi}{n}$ radians counterclockwise has the same effect as shifting the north-vertex i vertices to the right. Figure 9 ilustrates the previous situation: T' is obtained from T by changing the north-vertex one vertex to the right, and T'' is obtained from T by a rotation of $\frac{\pi}{5}$ radians. Although T' and T'' are diferent rhombic tilings of a 10-gon, they are equivalent as marked zonotopal tilings. If we were working with arrangements, this procedure would be equivalent to shift the inicial choice for the north-cell.

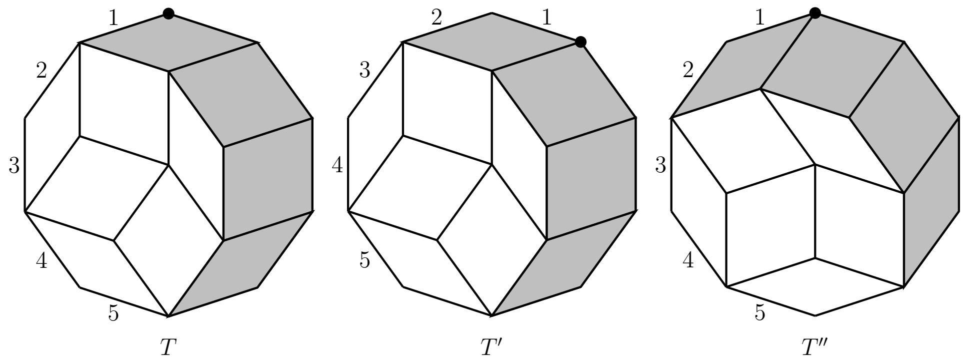

Figure 9: Rotation a tiling and shifting the north-vertex

As we saw in Figure 7, an arrangement can have more than one extreme pseudoline and so, a tiling can also have more than one extreme strip. In the following Lemma, we will see that these objects cannot have more than two extremes.

Proposition 3.2. *Let $T \in T(n)$ with $n > 3$. Then, T contains at most 2 extreme strips.*

Proof. Every tiling that contains an extreme strip can be obtained by rotating a tiling $T' \in ET_1(n)$. Since rotations do not change the number of extreme strips, we can assume that $T \in ET_1(n)$. Thus, T must correspond to one of the cases shown in Figure 10. If T_i is another extreme strip of T, then $i = 2$ or $i = n$, otherwise the rhombus with labels $\{1, i-1\}$ and $\{1, i+1\}$ would lie on different sides of the strip T_i. Since each extreme strip must contain $n+1$ edges of the polygon's border and two distinct strips share exactly one rhombus, it is impossible for T_1, T_2 and T_n to be extreme strips of T simultaneously. □

From the proof of the previous lemma, we conclude that if $T \in ET_1(n)$ contains another extreme strip T_i besides T_1, then $i = 2$ or $i = n$. The fact that every tiling containing extreme strips can be obtained from a tiling $T' \in ET_1(n)$ using rotations implies the following.

Corollary 3.3. *Let $i, j \in [n]$ and $T \in T(n)$. If T_i and T_j are extreme strips, then $|i - j| = 1$ or $i, j \in \{1, n\}$.*

Using the principle of inclusion-exclusion,

$$|ET(n)| = \sum_{\phi \neq A \subseteq [n]} (-1)^{|A|+1} \left| \bigcap_{i \in A} ET_i(n) \right|. \tag{2}$$

From Proposition 3.2 and Corollary 3.3, equation (2) can be simplified to

$$|ET(n)| = \sum_{i=1}^{n} |ET_i(n)| - |ET_1(n) \cap ET_n(n)| - \sum_{i=1}^{n-1} |ET_i(n) \cap ET_{i+1}(n)|. \tag{3}$$

Similiar to Lemma 3.1, we have a relation between the sets $ET_1(n) \cap ET_n(n)$ and $ET_i(n) \cap ET_{i+1}(n)$, for $i \in [n-1]$.

Lemma 3.4. *There is a bijection between $ET_1(n) \cap ET_n(n)$ and $ET_i(n) \cap ET_{i+1}(n)$, for all $i \in [n-1]$.*

Proof. A tiling $T \in ET_1(n) \cap ET_n(n)$ has one of the forms shown in Figure 11. Using similiar arguments as in Lemma 3.1, we have the result. □

Before stating the main result of this section, we need the following enumeration results.

Proposition 3.5. *Let $n \geq 3$. Then, $|ET_1(n)| = 2|T(n-1)|$.*

Proof. Suppose that $T \in ET_1(n)$. We have two possibilities for T: either the strip contains the entire left border of the polygon, or it contains the entire right border (see Figure 10). In both cases, we have a centrally symmetric $2(n-1)$-gon contained in T which can be

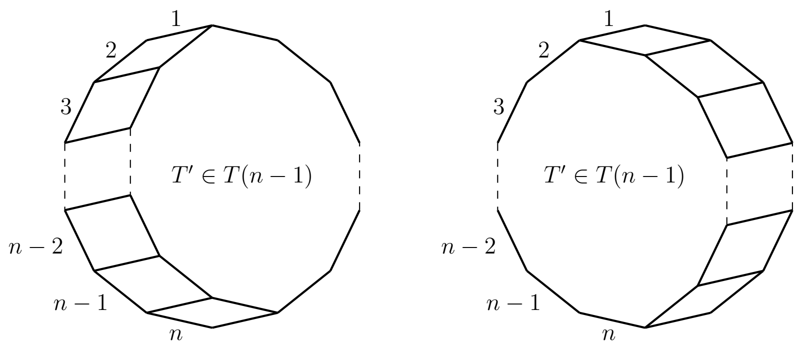

Figure 10: Possible forms for a tiling in $ET_1(n)$.

tilled in $|T(n-1)|$ ways. Hence $|ET_1(n)| = 2|T(n-1)|$. □

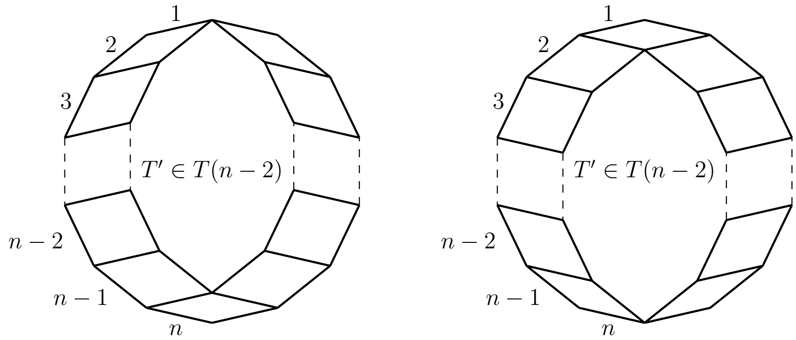

Figure 11: Possible forms for a tiling in $ET_1(n) \cap ET_n(n)$.

Proposition 3.6. *Let* $n \geq 4$. *Then,* $|ET_1(n) \cap ET_n(n)| = 2|T(n-2)|$.

Proof. Suppose that $T \in ET_1(n) \cap ET_n(n)$. Since each extreme strip contains $n+1$ edges of the border of the tiling, then T must correspond to one of the cases shown in Figure 11. In both cases, we have a centrally symmetric $2(n-2)$-gon contained in T which can be tiled in $|T(n-2)|$ ways. Hence $|ET_1(n) \cap ET_n(n)| = 2|T(n-2)|$. □

Joining all the information, we have the following enumeration for $ET(n)$.

Theorem 3.7. *For* $n \geq 4$, *we have* $|ET(n)| = 2n(|T(n-1)| - |T(n-2)|)$.

Proof. From Lemmas 3.1 and 3.5,

$$|ET_i(n)| = |ET_1(n)| = 2|T(n-1)|,$$

for all $n \geq 3$, and from Lemmas 3.4 and 3.6,

$$|ET_i(n) \cap ET_{i+1}(n)| = |ET_1(n) \cap ET_n(n)| = 2|T(n-2)|,$$

for all $n \geq 4$. Hence, from (3) we have the result. □

To convert the previous result into the language of arrangements, just note that $ET(n)$ is in bijection with the set $EA(n) := \{\mathcal{A} \in A(n) : \mathcal{A}$ contains extreme pseudolines$\}$. Since $|T(n)| = |A(n)|$, we have

$$|EA(n)| = 2n(|A(n-1)| - |A(n-2)|),$$

for all $n \geq 4$. As consequence of Theorem 3.7, we get a new way to express the cardinality of $T(n)$.

Corollary 3.8. *For* $n \geq 4$, $|T(n)| = 1 + \displaystyle\sum_{i=4}^{n+1} \frac{|ET(i)|}{2i}$.

Proof. From the previous theorem, $|ET(n+1)| = 2(n+1)(|T(n)| - |T(n-1)|)$, that is

$$\frac{|ET(n+1)|}{2(n+1)} = |T(n)| - |T(n-1)|.$$

But then,

$$\sum_{i=4}^{n+1} \frac{|ET(i)|}{2i} = \sum_{i=3}^{n} (|T(i)| - |T(i-1)|) = T(n) - T(2).$$

Since $|T(2)| = 1$, we have the result. □

Not every tiling contains extremes strips. Figure 12 illustrates the only examples in $T(5)$, out of the 62 possible tilings, that do not contain extremes. Using the formula obtained for $|ET(n)|$ and the available values for $|T(n)|$, we have computed $|ET(n)|$ up to $n = 17$. The conclusion that we took is that $ET(n)$ is a very small part of $T(n)$ (for $n = 16$, the set $ET(n)$ corresponds to aproximately 0.6% of $T(n)$). This is not a surprise because of the rapidly grow of $T(n)$ and the fact that $|ET(n)| \leq 2n|T(n-1)|$. Based on these observations, we end this section with the following conjecture.

Conjecture 3.9. *We have* $|ET(n)| = o(|T(n)|)$*, that is,* $\lim_{n\to\infty} \frac{|ET(n)|}{|T(n)|} = 0$.

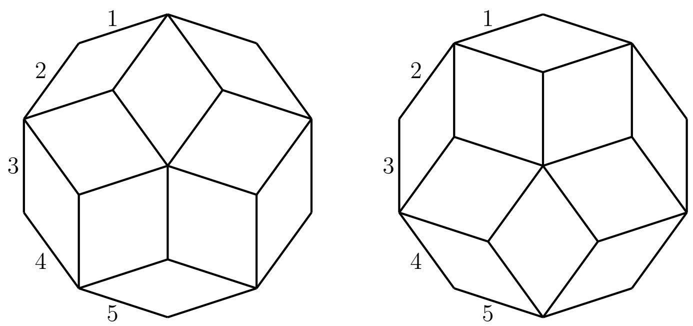

Figure 12: The only rhombic tilings of a 10-gon that do not contain extreme strips.

4 Counting shellable arrangements

A common method for generating new arrangements from existing ones is by adding pseudolines. According to Levi's Extension Lemma [12], if two points do not lie on the same pseudoline, it is always possible to add a new pseudoline to the arrangement that passes through those points. The inverse operation can also be performed by removing a pseudoline from an arrangement to form a new one. For $\mathcal{A} \in A(n)$ and a pseudoline l, we denote the arrangement obtained by removing l from $\mathcal{A}$ as $\mathcal{A} - l \in A(n-1)$.

Definition 4. [13] Let $\mathcal{A} \in A(n)$. We say that $\mathcal{A}$ is *shellable* if it contains an extreme pseudoline l and $\mathcal{A} - l$ is shellable or empty.

A shellable arrangement $\mathcal{A} \in A(n)$ can be interpreted as a chain of arrangements $\mathcal{A} = \mathcal{A}_n \supseteq \mathcal{A}_{n-1} \supseteq \cdots \supseteq \mathcal{A}_0 = \phi$, where $\mathcal{A}_j \in A(j)$ for all $j \in [n]$, and we transition from $\mathcal{A}_j$ to $\mathcal{A}_{j-1}$ by removing an extreme pseudoline. The arrangement depicted in Figure 1 is shellable, which we can see by removing l_1, l_3, l_4, l_5 and l_2 in this order (see Figure 13). The sequence obtained by iteratively removing extreme pseudolines from a shellable arrangement is called a *shellable sequence* [13], and we will denote it as a word $s = i_1 \cdots i_n$, where l_{i_j} is the j-th pseudoline to be removed. Shellable arrangements were introduced in [13] to compute the degree of connectivity of the flip graph, a graph structure defined on $A(n)$ where two arrangements are connected by an edge if they differ by a single flip (see [8, 7] for more details on this topic).

This section is dedicated to enumerate the shellable arrangements. Given the fact that they always contain some extreme pseudoline, the methods we use will be analogous to those in the previous section. We start by giving an interpertation of shellable arrangements in the context of tilings.

Definition 5. Let $T \in T(n)$. A tiling T is *shellable* if it contains an extreme strip T_i and $T - T_i$ is shellable or empty, where $T - T_i$ is the tiling obtained from T by shrinking T_i down to a path. The set of shellable tilings is denoted by $ST(n)$.

Figure 13 illustrates the shrinking process of a shellable tiling, where strips are sequentially shrunk. Removing a pseudoline l_i from an arrangement $\mathcal{A} \in A(n)$ is equivalent

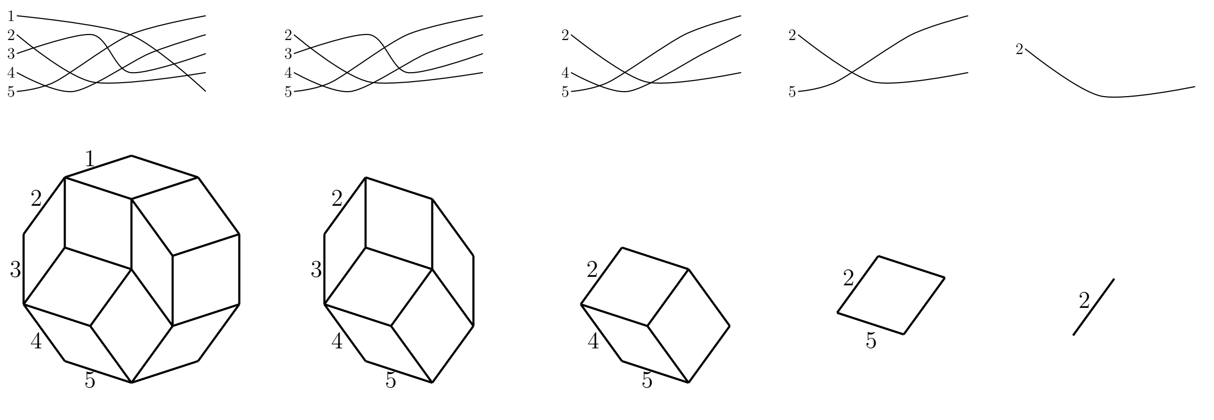

Figure 13: Shrinking process of a shellable tiling.

to shrink the strip T_i of the tiling $T \in T(n)$ corresponding to the dual of $\mathcal{A}$. Thus, $\mathcal{A}$ is shellable if and only if T is shellable. Additionally, it makes sense for a tiling to have a shellable sequence, defined by the order in which the strips are iteratively shrunk.

Analogous to $ET(n)$, the set $ST(n)$ can be writen as

$$ST(n) = \bigcup_{i=1}^{n} ST_i(n),$$

where $ST_i(n) = \{T \in ST(n) : T_i \text{ is an extreme strip}\}$. We also have the following.

Lemma 4.1. *Let $n \geq 1$.*

1. *There is a bijection between $ST_1(n)$ and $ST_i(n)$ for all $i \in [n]$.*
2. *There is a bijection between $ST_1(n) \cap ST_n(n)$ and $ST_i(n) \cap ST_{i+1}(n)$ for all $i \in [n-1]$.*

Proof. Just note that rotating a tiling does not affect the shellable property. Using the same aproach as in Lemmas 3.1 and 3.4 we have the result. □

Proposition 4.2.

1. $|ST_1(n)| = 2|ST(n-1)|$, *for all $n \geq 3$,*
2. $|ST_1(n) \cap ST_n(n)| = 2|ST(n-2)|$, *for all $n \geq 4$.*

Proof. Suppose that $T \in ST_1(n)$. Then, T must correspond to one of the cases of Figure 10. In both cases we have a centrally symmetric $2(n-1)$-gon contained in T that will correspond to a tiling $T' \in T(n-1)$. Since T is shellable, it admits a shellable sequence $s = i_1 \cdots i_n$. If $i_j = 1$ for some $j \in [n]$, then $i_1 \cdots \widehat{i_j} \cdots i_n$ (i_j omitted) is a shellable sequence for T', and so $T' \in ST(n-1)$. Conversely, if $T' \in ST(n-1)$ and $s' = i_1 \cdots i_{n-1}$ is a shellable sequence of T', then the concatenation $1 \cdot s'$ is a shellable sequence for T. Therefore, $T \in ST_1(n)$ if and only if $T' \in ST(n-1)$, implying that $|ST_1(n)| = 2|ST(n-1)|$.

For statement 2, if $T \in ST_1(n) \cap ST_n(n)$ then T must correspond to one of the cases of Figure 11. In both cases we have a centrally symmetric $2(n-2)$-gon contained in T that will correspond to a tiling $T' \in T(n-2)$. Using similiar arguments as in statement 1, $T \in ST_1(n) \cap ST_n(n)$ if and only if $T' \in ST(n-2)$, and we obtain that $|ST_1(n) \cap ST_n(n)| = 2|ST(n-2)|$. □

The previous two results give us a recursion formula to compute the number of shellable tilings.

Theorem 4.3. *For $n \geq 4$, we have that $|ST(n)| = 2n(|ST(n-1)| - |ST(n-2)|)$.*

Proof. Using the principle of inclusion-exclusion,

$$|ST(n)| = \sum_{\phi \neq A \subseteq [n]} (-1)^{|A|+1} \left| \bigcap_{i \in A} ST_i(n) \right|. \tag{4}$$

Using Proposition 3.2 and Corollary 3.3, we can simplify (4) to

$$|ST(n)| = \sum_{i=1}^{n} |ST_i(n)| - |ST_1(n) - ST_n(n)| - \sum_{i=1}^{n-1} |ST_i(n) \cap ST_{i+1}(n)|.$$

Then, from Lemma 4.1 and Proposition 4.2 the result follows. □

Since shellable tilings are in bijection with shellable arrangements, the number of shellable arrangements follows the same recursive formula as stated in the previous theorem.

The sequence $s_n = |ST(n)|$ is not listed in the OEIS. It satisfies a second order linear recurrence with polynomial coefficients with initial values $s_2 = 1$ and $s_3 = 2$, for which a closed formula seems dificult to obtain. Using the sequence of ratios of consecutive terms, one can express s_n in another way.

Proposition 4.4. *For $n \geq 4$,*

$$s_n = \frac{2^n n!}{24} \prod_{i=4}^{n} \left(1 - \frac{1}{r_{i-1}}\right),$$

where $r_n = \frac{s_n}{s_{n-1}}$ for all $n \geq 4$.

Proof. Dividing both sides of the recursion obtained in Theorem 4.3 by s_{n-1},

$$\frac{s_n}{s_{n-1}} = 2n\left(1 - \frac{s_{n-2}}{s_{n-1}}\right) \Leftrightarrow r_n = 2n\left(1 - \frac{1}{r_{n-1}}\right).$$

We have that r_n satisfies a first order non-linear recursion with initial value $r_3 = \frac{s_3}{s_2} = 2$. Since

$$s_n = s_2 \frac{s_3}{s_2} \frac{s_4}{s_3} \cdots \frac{s_n}{s_{n-1}} = s_2 r_3 \prod_{i=4}^{n} r_i,$$

using the recurrence relation for r_n and values for s_2 and r_3,

$$s_n = 2 \prod_{i=4}^{n} 2i \left(1 - \frac{1}{r_{i-1}}\right) = \frac{2^n n!}{24} \prod_{i=4}^{n} \left(1 - \frac{1}{r_{i-1}}\right). \qquad \square$$

Being $s_n = |ST(n)|$, we have that $s_n > 0$ for all n. But then, r_n must also be positive, implying that $0 < 1 - \frac{1}{r_{n-1}} < 1$. Therefore, $r_n < 2n$ and $s_n < \frac{2^n n!}{24}$ for all n. However, these upper bounds are not very acurate, as we can see in Table 1 for the case of r_n. Notice that r_n seems to be between $2(n-1)$ and $2n-1$ for $n \geq 6$. The next result confirms this claim.

n	3	4	5	6	7	8	16	32	64	128
r_n	2	4	7.5	10.4	≈12.65	≈14.74	≈30.89	≈62.95	≈126.98	≈254.99

Table 1: Some values of r_n

Proposition 4.5. *Let r_n be as in the Proposition 4.4. Then, $2(n-1) < r_n < 2n-1$ for all $n \geq 6$.*

Proof. We use induction on n to prove both inequalities. For $n = 6$, we have $r_n = 10.4$ which satisfies the claim. Suppose that $2(n-1) < r_n < 2n-1$ for some $n \geq 6$. We want to prove that $2n < r_{n+1} < 2(n+1)-1$. For the first inequality, using the recurence relation and the induction hypothesis,

$$r_{n+1} = 2(n+1) - \frac{2(n+1)}{r_n} > 2(n+1) - \frac{2(n+1)}{2(n-1)} = 2(n+1) - \frac{n+1}{n-1}.$$

Since $n \geq 6$ and the sequence $\frac{n+1}{n-1}$ is decreasing, we have $\frac{n+1}{n-1} \leq \frac{6+1}{6-1} = \frac{7}{5}$ for all $n \geq 6$. Hence,

$$r_{n+1} \geq 2(n+1) - \frac{7}{5} = 2n + \frac{3}{5} > 2n.$$

For the second inequality, using again the recurrence relation and the induction hypothesis,

$$r_{n+1} = 2(n+1) - \frac{2(n+1)}{r_n} < 2(n+1) - \frac{2(n+1)}{2n-1}.$$

Since $2(n+1) > 2n-1$, we have $\frac{2(n+1)}{2n-1} > 1$, and so $r_{n+1} < 2(n+1) - 1$. □

As a consequence, we obtain better bounds for s_n.

Corollary 4.6. *For all n we have* $\frac{5}{64}\frac{2^n n!}{n} < s_n < \frac{4}{63}\frac{(2n)!}{2^n n!}$.

Proof. Since $s_n = 2\prod_{i=4}^{n} r_n$, from the previous result and the values for r_4 and r_5,

$$60\prod_{i=5}^{n-1} 2i < s_n < 60\prod_{i=6}^{n}(2i-1).$$

Then,

$$\frac{60}{2\cdot 4\cdot 6\cdot 8\cdot 2n}\prod_{i=1}^{n} 2i < s_n < \frac{60}{1\cdot 3\cdot 5\cdot 7\cdot 9}\prod_{i=1}^{n}(2i-1),$$

which simply to

$$\frac{5}{64n}\prod_{i=1}^{n} 2i < s_n < \frac{4}{63}\prod_{i=1}^{n}(2i-1).$$

Recall that $\prod_{i=1}^{n} 2i = 2^n n!$ and $\prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{2^n n!}$. Aplying these identities to our inequalities, we have the result. □

To give some motivation before stating our last result, we need to return to the definition of a shellable sequence. A tiling is shellable if the empty tiling can be obtained by iteratively shrinking extreme strips. In each step of this process we have two cases: or all rhombus are above the strip that will be shrunk, or all are below it. All this information can be recorded in a signed permutation, that is, a permutation $w = (w(1), \ldots, w(n))$ such that $w(i) \in \{\pm 1, \ldots, \pm n\}$ for all $i \in [n]$ and $|w(i)| \neq |w(j)|$ for all $i \neq j$. Given a shellable sequence $s = i_1 \cdots i_n$ of T, associate to s the signed permutation $w_s = (w_s(1), \ldots, w_s(n))$ where $w_s(j) = i_j$ if all rhombus were above T_{i_j} before shrinking it, or $w_s(j) = -i_j$ otherwise. We have the following.

Proposition 4.7. *Given $n \geq 2$ and a signed permutation $w = (w(1), w(2), \ldots, w(n))$, there is at most one shellable tiling with a shellable sequence associated to w.*

Proof. We proceed by induction on n. For $n = 2$ the result is true since $|T(2)| = 1$. Let $w = (w(1), w(2), \ldots, w(n))$ be a signed permutation and suppose that there is $T, T' \in ST(n)$ where both have shellable sequences associated to w. Without lost of generality assume that $w(1) = 1$. Since $w(1)$ is related to the first strip to be shrunk, we have that T_1 is an extreme strip on both tilings. The fact that $w(1) > 0$ implies that T and T' are of the form of the right tiling of Figure 10. If we shrink the strip T_1 from both tilings we obtain two tilings $T^1, T^2 \in T(n-1)$ which will have shellable sequences associated to the signed permutation $(w(2), w(3), \ldots, w(n))$. Using the induction hypothesis, $T^1 = T^2$ implying that $T = T'$. □

A direct consequence of this result is that the number of shellable tilings is at most the number of signed permutations, that is, $2^n n!$. Our final result shows how s_n is related to $2^n n!$ in the limit.

Theorem 4.8. *There is a constant $c > 0$ such that $s_n \sim \frac{2^n n!}{c\sqrt{n}}$.*

Proof. We just need to prove that $\lim_{n\to\infty} \frac{s_n}{\frac{2^n n!}{\sqrt{n}}} = L$ for some $L \neq 0$. The first step is to show that the previous limit exist. From Proposition 4.4,

$$\lim_{n\to\infty} \frac{s_n}{\frac{2^n n!}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{24}\sqrt{n} \prod_{i=4}^{n} \left(1 - \frac{1}{r_{i-1}}\right). \tag{5}$$

Let $x_n = \frac{1}{24}\sqrt{n} \prod_{i=4}^{n} \left(1 - \frac{1}{r_{i-1}}\right)$ for $n \geq 4$. Then,

$$\frac{x_n}{x_{n+1}} = \sqrt{\frac{n}{n+1}} \frac{1}{\left(1 - \frac{1}{r_n}\right)}. \tag{6}$$

Since $r_n < 2n$ for all $n \geq 3$, from (6) we get

$$\frac{x_n}{x_{n+1}} > \sqrt{\frac{n}{n+1}} \frac{1}{(1 - \frac{1}{2n})} = \sqrt{\frac{n}{n+1}} \frac{2n}{2n-1}. \tag{7}$$

Squaring both sides of (7) yields

$$\left(\frac{x_n}{x_{n+1}}\right)^2 > \frac{n}{n+1}\frac{4n^2}{4n^2-4n+1} = \frac{4n^3}{4n^3-3n+1},$$

and we have $\dfrac{4n^3}{4n^3-3n+1} > 1$ for all $n \geq 4$. But then, $\dfrac{x_n}{x_{n+1}} > 1$ implying that (x_n) is a decreasing sequence. Since $x_n > 0$, the limit (5) exists.

To prove that $L \neq 0$, from Proposition 4.5 we have $r_n > 2(n-1)$ for all $n \geq 6$. Using the values of Table 1, $\displaystyle\prod_{i=4}^{6}\left(1-\frac{1}{r_{i-1}}\right) = \frac{13}{40}$, and from (5),

$$\lim_{n\to\infty} \frac{s_n}{\frac{2^n n!}{\sqrt{n}}} > \lim_{n\to\infty} \frac{13}{960}\sqrt{n}\prod_{i=7}^{n}\left(1-\frac{1}{2(i-2)}\right) = \lim_{n\to\infty} \frac{13}{960}\sqrt{n}\prod_{i=7}^{n}\frac{2(i-2)-1}{2(i-2)}.$$

The previous product can be written as $\displaystyle\prod_{i=5}^{n-2}\frac{2i-1}{2i}$. Multiplying and dividing by the missing terms for the product to go from 1 to n, we get

$$\lim_{n\to\infty} \frac{13}{960}\sqrt{n}\prod_{i=7}^{n}\frac{2(i-2)-1}{2(i-2)} = \lim_{n\to\infty} \frac{26}{525}\frac{4n(n-1)}{(2n-1)(2(n-1)-1)}\sqrt{n}\prod_{i=1}^{n}\frac{2i-1}{2i} \tag{8}$$

$$= \lim_{n\to\infty} \frac{26}{525}\frac{4n^2-4n}{4n^2-8n+3}\frac{\sqrt{n}(2n)!}{2^{2n}(n!)^2}. \tag{9}$$

Using Stirling's aproximation $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ and the fact that $\displaystyle\lim_{n\to\infty}\frac{4n^2-4n}{4n^2-8n+3} = 1$, we have

$$(9) = \lim_{n\to\infty} \frac{26}{525}\frac{\sqrt{n}\sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n}}{2^{2n}2\pi n\left(\frac{n}{e}\right)^{2n}} = \lim_{n\to\infty} \frac{26}{525}\frac{2\sqrt{\pi}n2^{2n}\left(\frac{n}{e}\right)^{2n}}{2^{2n}2\pi n\left(\frac{n}{e}\right)^{2n}} = \frac{26}{525}\frac{\sqrt{\pi}}{\pi}. \tag{10}$$

Therefore, $L \geq \frac{26}{525}\frac{\sqrt{\pi}}{\pi} > 0$ and $c = \frac{1}{L}$. □

Noticing that $\frac{26}{525}\frac{\sqrt{\pi}}{\pi} \approx 0.028$, the constant c in the previous theorem satisfy $c < 36$. To obtain a lower bound for c, we may useu Proposition 4.6 to get

$$L = \lim_{n\to\infty} \frac{s_n}{\frac{2^n n!}{\sqrt{n}}} < \lim_{n\to\infty} \frac{4}{63}\frac{\frac{(2n)!}{2^n n!}}{\frac{2^n n!}{\sqrt{n}}} = \lim_{n\to\infty} \frac{4}{63}\frac{\sqrt{n}(2n)!}{2^{2n}(n!)^2} = \frac{4}{63}\frac{\sqrt{\pi}}{\pi} \approx 0.036,$$

and so $c > 27$. Based on computacional data, we found that $c \approx 34.3$.

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References

[1] A. Björner, M. Vergnas, B. Sturmfels, N. Whitel, and M. Ziegler. *Oriented Matroids*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2 edition, 1999.

[2] Y. Cho, J. Kim, and E. Lee. Enumeration of gelfand-cetlin type reduced words. *Electronic Journal of Combinatorics*, 29, 2020.

[3] A. Dumitrescu and R. Mandal. New lower bounds for the number of pseudoline arrangements. *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 410–425, 2019.

[4] S. Elnitsky. Rhombic tilings of polygons and classes of reduced words in coxeter groups. *Journal of Combinatorial Theory, Series A*, 77(2):193–221, 1997.

[5] S. Felsner. On the number of arrangements of pseudolines. *Discrete & Computational Geometry*, 18:257–267, 1997.

[6] S. Felsner and P. Valtr. Coding and counting arrangements of pseudolines. *Discrete & Computational Geometry*, 46:405–416, 2011.

[7] S. Felsner and H. Weil. Sweeps, arrangements and signotopes. *Discrete Applied Mathematics*, 109(1):67–94, 2001.

[8] S. Felsner and G. Ziegler. Zonotopes associated with higher bruhat orders. *Discrete Mathematics*, 241(1):301–312, 2001.

[9] G. Gutierres, R. Mamede, and J. Santos. Commutation classes of the reduced words for the longest element of $\mathfrak{S}_n$. *The Electronic Journal of Combinatorics*, 27(2), 2020.

[10] D. Knuth. *Axioms and Hulls*. Lecture Notes in Computer Science, 1992.

[11] F. Kühnast, J. Dallant, S. Felsner, and M. Scheucher. An improved lower bound on the number of pseudoline arrangements. *arXiv2402.13107*, 2024.

[12] F. Levi. Die teilung der projektiven ebene durch gerade oder pseudogerade. *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse*, 78:256–267, 1926.

[13] Y. Radtke, S. Felsner, J. Obenaus, S. Roch, M. Scheucher, and B. Vogtenhuber. Flip graph connectivity for arrangements of pseudolines and pseudocircles. *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 4849–4871, 2024.

[14] P. Shor. Stretchability of pseudolines is np-hard. *Applied Geometry And Discrete Mathematics*, pages 531–554, 1990.

[15] B. Tenner. Tiling-based models of perimeter and area. *Advances in Applied Mathematics*, 119:102059, 2020.