MULTIPLIERS, W-ALGEBRAS AND THE GROWTH OF GENERALIZED POLYNOMIAL IDENTITIES

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ABSTRACT. Let A be a W-algebra over a field F of characteristic zero, where W is any F-algebra. We develop a comprehensive theory of generalized identities independently on the algebraic structure of W, using the multiplier algebra of A. We also characterize the generalized varieties of almost polynomial growth generated by finite dimensional W-algebras. Finally, we provide a counter-example to the Specht property of generalized T_W-ideals in characteristic zero.

1. INTRODUCTION

Let A and W be associative algebras over a field F of characteristic zero and let $F\langle X\rangle$ be the free algebra generated by the countable set $X = \{x_1, x_2, \ldots\}$. A is called W-algebra if it is a W-bimodule such that

$$w(a_1a_2) = (wa_1)a_2, \quad (a_1a_2)w = a_1(a_2w), \quad (a_1w)a_2 = a_1(wa_2).$$

for all $a_1, a_2 \in A$ and for all $w \in W$. For instance, A is an A-algebra if the action of A on itself is given by left and right multiplication and, if $W \cong F$, then such definition coincides with the usual one of F-algebra.

The free W-algebra is denoted by $W\langle X\rangle$ and its elements are called W-polynomials or generalized polynomials. In particular, we may consider these elements as polynomials containing variables of X and elements of W as some sort of coefficients that may appear also between two or more variables. A W-polynomial is an identity for the W-algebra A if it vanishes under all substitutions of its variables by elements of A. The set $\mathrm{Id}^W(A)$ of all generalized polynomial identities of A is a T_W-ideal of $W\langle X\rangle$, i.e. a stable ideal by endomorphisms of $W\langle X\rangle$ and, as in the usual case of F-algebras, one of the main problems is to find a set of generators of $\mathrm{Id}^W(A)$. Nowadays, generators of T_W-ideals are known only in a few cases. The interested reader can find in [7] the generalized identities of the full matrix algebra $M_n(F)$ for all $n \geq 1$ in case $W = M_n(F)$ acts by left and right multiplication, and in [25] the T_W-ideal of $UT_2(F)$, the 2×2-upper triangular matrix algebra, in case $W = UT_2$ acts either as the subalgebra D of diagonal matrices or the full algebra UT_2 by left and right multiplication. These are the only known examples of generalized ideals at present.

Generalized identities first appeared in [3] in the setting of primitive rings and later on they were studied by several authors from the perspective of the algebraic structure and properties of the rings satisfying them. We refer to [5] and its bibliography for an exhaustive dissertation of such topic from this point of view. In more recent years a combinatorial approach arose, focusing on the study of generalized codimensions, which are essentially a quantitative tool for roughly measuring how many generalized identities are satisfied by an algebra. The codimension sequence $c_n(A)$ of a F-algebra A was first introduced by Regev in [26] as the dimension of the space of multilinear polynomials of degree n reduced modulo the polynomial identities of A. In the same paper, he also proved that such sequence is exponentially bounded if A satisfies a non-trivial identity. Later, in 1999, Giambruno and Zaicev in [15] captured such exponential growth, providing an explicit method to compute it and solving the Amitsur conjecture, stating that the limit $\lim_{n\to+\infty} \sqrt[n]{c_n(A)}$ exists and is a non-negative integer called the PI-exponent of A.

Following this approach, Gordienko in [17] defines the generalized codimension sequence and proves the Amitsur conjecture for any finite dimensional algebra A nonetheless only in the case $W = A$ and the action is given by left and right multiplication. Furthermore, in [25] the authors study the generalized variety generated by UT_2 and prove that it has almost polynomial growth of the generalized codimensions if $W = UT_2$ acts either as a subalgebra isomorphic to F or as the subalgebra D by left and right multiplication. All these results were established by selecting a priori who W and its action are.

The present paper consists of three parts. The first one aims to develop a comprehensive, self-contained and as general as possible theory of W-algebras that allows us to disregard the choice of W. In order to reach this goal, we will introduce the notion of multiplier algebra $\mathcal{M}(A)$ and we establish a duality between W-actions and $\mathcal{M}(A)$-actions with an additional condition. It is worth mentioning that multipliers are an extremely versatile

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tool that appears in various and different areas of mathematics, such as non commutative analysis in the context of C^*-algebras (see for instance [10–12]) or category theory (see [8, 9, 20, 22]).

Then, after a brief dissertation about the consequences of the theory that has just been developed on algebras with polynomial identities, the second part is devoted to the classification of the W-varieties of almost polynomial growth generated by finite dimensional algebras. Such characterization was made also in other settings, including group graded algebras [30], algebras with involution [13], special Jordan algebras [24] and algebras with derivation with some mild restrictions [4, 27].

Finally, the last part of the paper concerns some possible further directions. In particular, a counter-example to the Specht property in characteristic zero will be given in the last section, proving that the T_W-ideal of generalized identities of the unital Grassmann algebra E is not finitely generated, if W is not finitely generated and acts either as the finite dimensional Grassmann algebra E_k or the full algebra E by left and right multiplication.

2. Multipliers and W-algebras

Throughout this paper, F will denote a field of characteristic 0 and all algebras will be associative F-algebras. Let A be an algebra and $\mathrm{End}_F(A)$ be the algebra of the endomorphisms of A acting on the left of A.

Definition 2.1. Given an algebra W, we say that A is a W*-algebra* if it is a W-bimodule, i.e., A is both a left and a right W-module with the compatibility relation, for all $w_1, w_2 \in W$, $a \in A$,

$$(w_1 a)w_2 = w_1(a w_2) \tag{2.1}$$

and also satisfies the following conditions, for all $w \in W$, $a_1, a_2 \in A$

$$w(a_1 a_2) = (w a_1)a_2, \quad (a_1 a_2)w = a_1(a_2 w), \quad (a_1 w)a_2 = a_1(w a_2). \tag{2.2}$$

For instance, if $W = F$, then A is just a F-algebra. Moreover, W has itself the structure of W-algebra by taking the left and right W-actions as the usual left and right multiplication of W.

To formally describe the action of W over an F-algebra A, it is useful to introduce the notion of multiplier of A (see [22], where they are called bimultiplications). Multipliers often appear in the context of C^*-algebras and in particular they are used to study partial group actions and the generalization of C^*-crossed products (see for instance [10, 11]).

Definition 2.2. Let $L, R \in \mathrm{End}_F(A)$. We say that (R, L) is a *multiplier* of A if for all $a, b \in A$, one has that

$$R(ab) = aR(b), \quad L(ab) = L(a)b, \quad R(a)b = aL(b). \tag{2.3}$$

We say that L is a *left multiplier* and R is a *right multiplier* of A.

We denote by $\mathcal{M}(A)$ the set of all multipliers and we notice that $\mathcal{M}(A)$ is a F-algebra with unit by setting

$$(R_1, L_1) + (R_2, L_2) = (R_1 + R_2, L_1 + L_2)$$
$$\alpha(R, L) = (\alpha R, \alpha L)$$
$$(R_1, L_1) \cdot (R_2, L_2) = (R_2 R_1, L_1 L_2)$$

for all $(R_1, L_1), (R_2, L_2), (R, L) \in \mathcal{M}(A)$ and for all $\alpha \in F$. The unit element is (id_A, id_A).

Definition 2.3. The algebra $\mathcal{M}(A)$ is called the *multiplier algebra* of A.

Notice that for a fixed $m \in A$, if one considers the endomorphism of A of right and left multiplications by m:

$$R_m : a \mapsto am; \qquad L_m : a \mapsto ma,$$

for all $a \in A$, then (R_m, L_m) is a multiplier of A called *inner multiplier* of A, and

$$\begin{array}{rccc} \mu: & A & \to & \mathcal{M}(A) \\ & m & \mapsto & (R_m, L_m) \end{array} \tag{2.4}$$

is a homomorphism of F-algebras. Also, $\mathcal{IM}(A) := \mu(A)$ is a two-sided ideal of $\mathcal{M}(A)$, called *inner multiplier ideal* of A. Conversely, if A has unit element 1_A, then for any $(R, L) \in \mathcal{M}(A)$

$$R(1_A) = R(1_A)1_A = 1_A L(1_A) = L(1_A),$$

thus letting $m = R(1_A) = L(1_A)$, we get

$$R(a) = R(a 1_A) = aR(1_A) = am$$
$$L(a) = L(1_A a) = L(1_A)a = ma$$

for all $a \in A$. Hence $(R, L) = (R_m, L_m) \in \mathcal{IM}(A)$. This leads us to the following result.

Proposition 2.4. *μ is an isomorphism if and only if A has unit element.*

To establish a correspondence between multipliers and W-actions, it is necessary to ensure that the multipliers are permutable. In other words, given two multipliers (R_1, L_1) and (R_2, L_2), we are wondering whether

$$R_2L_1 = L_1R_2 \tag{2.5}$$

holds or not.

In general this is not true; in fact, one can consider an algebra A with trivial multiplication, i.e., $ab = 0$ for all $a, b \in A$. In this case, any couple of linear transformations (f, g) is a multiplier of A, but of course we cannot expect that (2.5) may hold in such general case, since this should imply that in particular $fg = gf$ for all $f, g \in \mathrm{End}_F A$. Hence, we need an extra condition on A.

Clearly, any multiplier (R, L) permutes with any inner multiplier. Consequently, by Propositions 2.4, it follows that in the case of an algebra with a unit, we have the following result.

Corollary 2.5. *Let A be a unital F-algebra. Then $A \cong \mathcal{IM}(A) = \mathcal{M}(A)$ and $R_aL_b = L_aR_b$, for all $a, b \in A$.*

Unital algebras are not the only type of algebras for which the multiplier algebra satisfies the permutability property, as highlighted in the following proposition.

Recall that A is said *non-degenerate algebra* if there is no nonzero element $a \in A$ such that $ab = ba = 0$, for all $b \in A$, or equivalently if the homomorphism μ defined in (2.4) is injective. Clearly, every unital algebra is non-degenerate.

Proposition 2.6 [12, Proposition 7.9]**.** *Let A be either non-degenerate or idempotent, i.e. $A^2 = A$. Then $R_2L_1 = L_1R_2$, for all $(R_1, L_1), (R_2, L_2) \in \mathcal{M}(A)$.*

We are now in a position to establish a connection between multiplier algebras and W-algebras. To this end, recall that if A is both a right and a left W-module, then the right and left actions of W on A induce the following F-algebras homomorphisms:

$$\rho : W^{\mathrm{op}} \to \mathrm{End}_F(A) \quad \text{and} \quad \lambda : W \to \mathrm{End}_F(A)$$

where $\rho(w)(a) = aw$ and $\lambda(w)(a) = wa$, for all $w \in W$ and $a \in A$, called respectively *right* and *left representation* of W on A, and vice versa. Here W^{op} denotes *opposite algebra* of W, which is the underlying vector space of W endowed with the opposite product $\cdot^{\mathrm{op}}$ defined by $w_1 \cdot^{\mathrm{op}} w_2 := w_2w_1$ for all $w_1, w_2 \in W$.

Now, let A be a W-algebra. Since A is both a left and right W-module, one can consider the corresponding right and left representation ρ and λ of W on A. From (2.2), it directly follows that for all $w \in W$,

$$\big(\rho(w), \lambda(w)\big) \in \mathcal{M}(A).$$

Then, if one sets

$$\begin{array}{rccc} \Phi : & W & \to & \mathcal{M}(A) \\ & w & \mapsto & \big(\rho(w), \lambda(w)\big) \end{array} \tag{2.6}$$

we get a homomorphism of F-algebras which, by (2.1), satisfies the additional condition, for all $w_1, w_2 \in W$,

$$\rho(w_2)\lambda(w_1) = \lambda(w_1)\rho(w_2).$$

Definition 2.7. The homomorphism Φ defined in (2.6) is called *acting homomorphism* of W on A.

As a result, the action of W on A induces an action of the subalgebra $\overline{W} = \Phi(W)$ of $\mathcal{M}(A)$ on A. In other words, A can be regarded as a $\overline{W}$-algebra. In this sense, we will say that W acts on A as the algebra $\overline{W}$. Furthermore, notice that $\overline{W}$ is a subalgebra of $\mathcal{M}(A)$ that satisfies the condition (2.5) of permutability. Additionally, Φ is injective if A is a faithful W-bimodule, i.e., the left and right actions of each $w \neq 0$ in W on A are nontrivial.

The converse is not generally true because not all multipliers of A satisfy the condition (2.5) of permutability. However, if A is either non-degenerate or idempotent, then the converse is also true by Proposition 2.6. Indeed, if W is a subalgebra of $\mathcal{M}(A)$, then A becomes a W-algebra by setting, for all $(R, L) \in W$, $a \in A$,

$$(R, L) \cdot a = L(a), \quad a \cdot (R, L) = R(a).$$

Note that the above characterization aligns with the fact that the category of associative F-algebras is weakly action representable, as proved by Janelidze in [20]. For an overview of the basic constructions, we also refer the reader to Section 2 of [9]. This observation further justifies the term 'acting homomorphism' used to refer to Φ (see [9, Proposition 2.3]). Furthermore, it is worth highlighting that [8, Proposition 1.11], closely related to [20, Proposition 4.5], establishes that for any W-algebra A there exists a unique acting homomorphism Φ such that A is a $\overline{W}$-algebra, where $\overline{W} = \Phi(W)$.

Now, we can state the following theorem which proof comes directly from the previous arguments and Corollary 2.5.

Theorem 2.8. *Let W be an F-algebra. If A is a unital W-algebra, then the action of W on A is equivalent to the action of a suitable subalgebra B of A by left and right multiplication. Moreover, if W is a unital algebra, then B has unit element.*

Proof. Let Φ the acting homomorphism of W on A and consider $\overline{W} = \Phi(W) \subseteq \mathcal{M}(A)$. Since A has unit element, by Corollary 2.5, $\overline{W}$ is isomorphic to a subalgebra B of A and its action on A is given by left and right multiplication. Clearly, if W has unit element 1_W, then $1_{\overline{W}} = \Phi(1_W)$ is the unit of $\overline{W}$ and, as a consequence, B is also a unital algebra. □

3. The Wedderburn-Malcev decomposition

In this section, we describe the structure of finite-dimensional W-algebras. To this end, we first describe the action of multipliers on finite-dimensional algebras.

Let A be a finite dimensional algebra and denote by $J = J(A)$ the Jacobson radical of A. Recall that A is called *semisimple* if and only if $J = 0$, and A is called *simple* if it has no nontrivial ideals and $A^2 = 0$.

Since the base field F is of characteristic zero, by Wedderburn-Malcev Theorem for associative algebras (see [16, Theorem 3.4.3]), there exists a unique, up to isomorphism, maximal semisimple subalgebra $B \subseteq A$ such that we can write A as a direct sum of vector spaces

$$A = B + J. \tag{3.1}$$

Then we have the following results.

Proposition 3.1. *Let A be a finite dimensional algebra over a field F of characteristic zero and J be its Jacobson radical. Then $R(J), L(J) \subseteq J$ for all $(R, L) \in \mathcal{M}(A)$.*

Proof. Let $A = B + J$ as in (3.1), and take $(R, L) \in \mathcal{M}(A)$ and $j \in J$. Then there exist $b \in B$ and $j' \in J$ such that $R(j) = b + j'$. Hence, for all $\bar{b} \in B$, $R(j)\bar{b} = jL(\bar{b}) \in J$ since J is an ideal of A and by (2.3). As a consequence, $R(j)\bar{b} = b\bar{b} + j'\bar{b} \in J$. Thus, given that $b \in B$, $b\bar{b} = 0$ for all $\bar{b} \in B$. Since B is semisimple, b must be equal to 0, and $R(j) = j' \in J$. Analogously, we can prove that $L(j) \in J$. □

Similarly, we can prove the following.

Proposition 3.2. *If A is a finite dimensional algebra such that $A = B_1 \oplus \cdots \oplus B_k + J$, where $B_1, \ldots, B_k$ are simple algebras and J is the Jacobson radical of A, then $R(B_i), L(B_i) \subseteq B_i + J$, for all $(R, L) \in \mathcal{M}(A)$ and $1 \le i \le k$.*

Now, let W be an F-algebra and suppose that A is a finite-dimensional W-algebra. For $I \subseteq A$, we consider

$$WI = \{wa \ : \ w \in W, \ a \in I\} \quad \text{and} \quad IW = \{aw \ : \ a \in I, \ w \in W\}.$$

We say that I is W-invariant if $WI, IW \subseteq I$. Thus, an ideal (subalgebra) of I of A is called *W-ideal* (*W-subalgebra*) if it is W-invariant.

Notice that the Wedderburn-Malcev decomposition may not be W-invariant. In fact, in general B in (3.1) is not W-subalgebra of A. For example, if $J \neq 0$ and W acts on A as the subalgebra J by left and right multiplication, then $WB + BW \subseteq J$, that is, B is not W-invariant. However, we have the following result.

Theorem 3.3. *Let W be an algebra over a field F of characteristic zero and A be a finite dimensional W-algebra. Then the Jacobson radical J of A is a W-ideal of A. Moreover, if F is algebraically closed, then $A = B_1 \oplus \cdots \oplus B_k + J$, where B_i is a simple algebra such that $WB_i, B_iW \subseteq B_i + J$ for all $1 \le i \le k$.*

Proof. Let Φ be the acting homomorphism of W on A. Then $\Phi(w) = (\rho(w), \lambda(w)) \in \mathcal{M}(A)$ for all $w \in W$. By Proposition 3.1 $\rho(w)(J), \lambda(w)(J) \subseteq J$ for all $w \in W$. Hence, by the definition of ρ and λ, $jw, wj \in J$ for all $j \in J$ and $w \in W$. Thus, J is a W-ideal of A.

Now suppose that F is algebraically closed of characteristic zero. Let $A = B + J$ as in (3.1). Since F is algebraically closed, by the Wedderburn-Artin Theorem (see [19, Theorem 2.1.6]) we can write $B = B_1 \oplus \cdots \oplus B_k$, where $B_1, \ldots, B_k$ are simple algebras. Then $A = B_1 \oplus \cdots \oplus B_k + J$. As a consequence of Proposition 3.2, $\rho(w)(B_i), \lambda(w)(B_i) \subseteq B_i + J$ for all $w \in W$. Thus, by the definition of ρ and λ, $bw, wb \in B_i + J$ for all $b \in B_i$ and $w \in W$, and we are done. □

We say that a W-algebra A is a *W-simple* if $A^2 \neq 0$ and A has no non-zero W-ideals.

Corollary 3.4. *Let A be a finite dimensional W-algebra over an algebraically closed field F of characteristic zero. Then A is W-simple if and only if it is simple.*

Proof. First, assume that A is simple. Then clearly A is also W-simple. So, suppose that A is W-simple. By Theorem 3.3 J is a W-ideal of A, then $J = 0$. Moreover, since F is algebraically closed, by the second part of Theorem 3.3 we have that $A = B_1 \oplus \cdots \oplus B_k$, where $B_1, \ldots, B_k$ are simple algebras and $WB_i, B_iW \subseteq B_i$ for all $1 \le i \le k$. Since $B_1, \ldots, B_k$ are also ideals of A, it follows that $k = 1$ and A is simple, as required. □

Corollary 3.5. *If A is a finite dimensional W-simple algebra over an algebraically closed field F of characteristic zero, then A is isomorphic to the W-algebra $M_n(F)$ of matrices of order n over F for some $n \geq 1$ with the W-action given by left and right multiplication of an appropriate unital subalgebra.*

Proof. By Corollary 3.4 it follows that A is a simple algebra, and by Wedderburn Theorem (see [19, Theorem 1.4.4]) it is isomorphic to $M_n(F)$ for some $n \geq 1$. Moreover, since $M_n(F)$ has unit, by Theorem 2.8 we are done. □

4. Free W-algebra and generalized identities

Let W be an F-algebra. It is not restrictive to assume that W is a unital algebra since if not, we can consider the unital algebra $W^+ = W + F1_W$ obtained from W by adding the unit element 1_W. So, from now on W will be a unital F-algebra. Recall that a homomorphisms $\varphi : A \to B$ between W-algebras A, B must satisfy $\varphi(wav) = w\varphi(a)v$ for $a \in A$, $w, v \in W$.

Since the class of W-algebras is a non-trivial variety in the sense of universal algebra, it contains the free W-algebra $W\langle X\rangle$ freely generated by the countably infinite set of variables $X := \{x_1, x_2, \ldots\}$, i.e., $W\langle X\rangle$ is uniquely determined up to an isomorphism by the following universal property: given a W-algebra A, any map $\varphi : X \to A$ can be uniquely extended to a homomorphism of W-algebras $\bar{\varphi} : W\langle X\rangle \to A$, which we call the *evaluation* of $W\langle X\rangle$ at elements $\varphi(x_1), \varphi(x_2), \ldots$ from A. Notice that the evaluation $\bar{\varphi}$ of $W\langle X\rangle$ in A depends not only on the map φ but also on the right and left actions of W on A, and consequently on the corresponding right and left representations of W on A. Hence, $\bar{\varphi}$ depends on the acting homomorphism Φ of W on A. Since A can be a W-algebra with respect to different W-actions, we will denote the evaluation $\bar{\varphi}$ by $\bar{\varphi}_\Phi$, or simply φ_Φ, when necessary to clarify which action we are considering.

We can describe $W\langle X\rangle$ as follows: $W\langle X\rangle$ is generated as an algebra by the set $\{vx_iw \mid i \geq 1, v, w \in W\}$ subject to the relations

$$1_W x_i 1_W = x_i,\ (v_1 + v_2)x_iw = v_1x_iw + v_2x_iw,\ vx_i(w_1 + w_2) = vx_iw_1 + vx_iw_2,\ v(x_i + x_j)w = vx_iw + vx_jw$$

for all $v, v_1, v_2, w, w_1, w_2 \in W$ and $i, j \geq 1$, and the product given first by juxtaposition and then by multiplication in W. Moreover, given a basis $\mathcal{B}_W := \{w_i\}_{i\in\mathcal{I}}$ of W, then a basis of $W\langle X\rangle$ is the following

$$\mathcal{B}_{W\langle X\rangle} := \left\{w_{i_0}x_{j_1}w_{i_1}x_{j_2}\cdots w_{i_{n-1}}x_{j_n}w_{i_n} \mid j_1, \ldots, j_n \geq 1,\ w_{i_0}, \ldots, w_{i_n} \in \mathcal{B}_W,\ n \geq 1\right\}.$$

Clearly, $W\langle X\rangle$ has a natural structure of W-algebra by using the multiplication of W. The elements of $W\langle X\rangle$ are called *W-polynomials* or *generalized polynomials* when there is no ambiguity about the role of W. A *T_W-ideal* of the free W-algebra is an W-ideal which in addition is invariant under all W-algebra endomorphisms of $W\langle X\rangle$, i.e., under the endomorphisms that we call *substitutions*, which send variables of $x_i \in X$ to elements of $W\langle X\rangle$.

Given a W-algebra A, a generalized polynomial $f(x_1, \ldots, x_n) \in W\langle X\rangle$ is a *W-identity* of A, or *generalized identity* when the role of W is clear, and we write $f \equiv 0$, if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. We denote by $\mathrm{Id}^W(A)$ the set of generalized identities of A, which is a T_W-ideal of the free W-algebra $W\langle X\rangle$. Note that $\mathrm{Id}^W(A)$ is the intersection of all kernels of evaluations of $W\langle X\rangle$ from A.

For $n \geq 1$, we denote by P_n^W the vector space of *multilinear W-polynomials* in the variables $x_1, \ldots, x_n$, that is

$$P_n^W := \mathrm{span}_F\{w_{i_0}x_{\sigma(1)}w_{i_1}x_{\sigma(2)}\cdots w_{i_{n-1}}x_{\sigma(n)}w_{i_n} \mid \sigma \in S_n, w_{i_0}, \ldots, w_{i_n} \in \mathcal{B}_W\},$$

where S_n denotes the symmetric group acting on $\{1, \ldots, n\}$. Since F has characteristic zero, a standard argument shows that the T_W-ideal $\mathrm{Id}^W(A)$ is completely determined by its multilinear generalized polynomials. Thus it is reasonable to consider the quotient space

$$P_n^W(A) := \frac{P_n^W}{P_n^W \cap \mathrm{Id}^W(A)},$$

and so one can define the nth *W-codimension*, or *generalized codimension*, of A as

$$c_n^W(A) := \dim_F P_n^W(A), \quad n \geq 1.$$

Note that $c_n^W(A)$ is not necessarily finite, indeed we will show an example of infinite generalized codimensions in the last section. However, if at least one between W and A is finite dimensional, then $c_n^W(A)$ is finite for all $n \geq 1$, and it is interesting to ask if the limit

$$\exp^W(A) := \lim_{n\to\infty} \sqrt[n]{c_n^W(A)}$$

exists.

In case $W = F$, we are dealing with ordinary polynomial identities, and it is well known that the limit $\exp(A) := \exp^W(A)$ exists and is a nonnegative integer, called (ordinary) *exponent* of A (see [14,15]). In particular, if A is a finite dimensional F-algebra, then

$$\exp(A) = \max\{\dim_F\left(B_{i_1} \oplus B_{i_2} \oplus \cdots \oplus B_{i_r}\right) \mid B_{i_1}JB_{i_2}J\cdots JB_{i_r} \neq 0,\ 1 \leq r \leq k,\ i_p \neq i_s,\ 1 \leq p, s \leq r\},$$

where $A = B_1 \oplus \cdots \oplus B_k + J$ with $B_1, \ldots, B_k$ simple algebras and $J = J(A)$ is the Jacobson radical of A.

In [17] Gordienko captured the exponential growth of the generalized codimension sequence of a finite dimensional unital algebra A that acts on itself by left and right multiplication. More precisely, he proved that if A is a finite dimensional unital A-algebra, then $\exp^A(A) = \exp(A)$. Using the relationship between W-algebras and multiplier algebras, we can generalize this result as follows.

Theorem 4.1. *Let W be a unital algebra over a field F of characteristic zero. If A is a finite dimensional unital W-algebra, then* $\exp^W(A) = \exp(A)$.

Proof. Since A is a unital algebra, by Theorem 2.8 it follows that the action of W on A is equivalent to the action of a suitable unital subalgebra B of A by left and right multiplication. Then, since A is finite dimensional, following step by step the proof of Theorem 3 in [17] we get the desired conclusion. □

Corollary 4.2. *If A is a finite dimensional unital W-algebra, then the sequence of generalized codimensions $c_n^W(A)$, $n \geq 1$, grows exponentially or is polynomially bounded.*

A variety of W-algebras generated by a W-algebra A is denoted by $\mathrm{var}^W(A)$ and is called *W-variety*, or *generalized variety*, and $\mathrm{Id}^W(\mathcal{V}) := \mathrm{Id}^W(A)$. The *growth* of $\mathcal{V} = \mathrm{var}^W(A)$ is the growth of the sequence $c_n^W(\mathcal{V}) := c_n^W(A)$, $n \geq 1$. We say that the generalized variety $\mathcal{V}$ has *polynomial growth* if $c_n^W(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has *almost polynomial growth* if $c_n^W(\mathcal{V})$ is not polynomially bounded but every proper W-subvariety $\mathcal{U}$ of $\mathcal{V}$ has polynomial growth.

Let A be a finite dimesional W-algebra and consider the acting homomorphism $\Phi : W \to \mathcal{M}(A)$ of W on A. Since $\overline{W} = \Phi(W)$ is finite dimensional and F is of characteristic zero, by the Wedderburn-Malcev Theorem we can decompose

$$\overline{W} = \overline{W}_{ss} + \bar{J},$$

where $\overline{W}_{ss}$ is a semisimple subalgebra of $\overline{W}$ (isomorphic to $\overline{W}/\bar{J}$) and $\bar{J} = J(\overline{W})$ is the Jacobson radical of $\overline{W}$.

Now, denotes by $\pi : \overline{W} \to \overline{W}/\bar{J}$ the canonical epimorphism. Since $\overline{W}_{ss}$ isomorphic to $\overline{W}/\bar{J}$, we can assume that $\pi : \overline{W} \to \overline{W}_{ss}$. Hence, if one defines the map $\Phi_{ss} : W \to \mathcal{M}(A)$ such that $\Phi_{ss} = \pi \circ \Phi$, then Φ_{ss} is a homomorphism of F-algebra and determines another W-algebra structure on A, which means that Φ_{ss} is another acting homomorphism of W on A. Thus, we denote by $A_{\overline{W}_{ss}}$ the W-algebra A under this new action, where W acts on A as the algebra $\overline{W}_{ss} = \Phi_{ss}(W)$. Using this notation, we have the following result.

Theorem 4.3. *Let W be a unital algebra over a field F of characteristic zero. If A is a unital finite dimensional W-algebra, then $A_{\overline{W}_{ss}} \in \mathrm{var}^W(A)$.*

Proof. Let $f \in \mathrm{Id}^W(A)$ with $\deg f = n$ and assume, as we may, that f is a multilinear. Decompose f as $f = f_{ss} + f_J$, where f_{ss} is a multilinear W-polynomial in which in each monomial appears only elements of W that are sent through Φ into $\overline{W}_{ss}$, and f_J is a multilinear W-polynomial in which in each monomial appears at least an element of W that is sent through Φ into $\bar{J}$. As a consequence, for all maps $\varphi : X \to A$, we have that

$$\varphi_{\Phi_{ss}}(f) = \varphi_{\Phi_{ss}}(f_{ss}) = \varphi_{\Phi}(f_{ss}). \tag{4.1}$$

Moreover, since A is a unital algebra, by Theorem 2.8 it follows that $\overline{W}$ is isomorphic to a subalgebra of A that acts on A by left and right multiplication. Then, since A is finite dimensional and f is a W-identity of A, it follows that

$$\varphi_{\Phi}(f_{ss}) = -\varphi_{\Phi}(f_J) \in J(A) \tag{4.2}$$

for all maps $\varphi : X \to A$, where $J(A)$ is the Jacobson radical of A.

Now, for any fixed map $\varphi : X \to A$, if $\varphi_{\Phi}(f_{ss}) \in J^q$ for some $q \geq 1$, then by the definition of f_{ss} and f_J we have $\varphi_{\Phi}(f_J) \in J^{q+1}$. Thus, if q is the biggest integer such that $\varphi_{\Phi}(f_{ss}) \in J^q$ and $\varphi_{\Phi}(f_{ss}) \notin J^{q+1}$, from (4.2) it follows that $\varphi_{\Phi}(f_{ss}) = -\varphi_{\Phi}(f_J) = 0$. By using (4.1) we can then conclude that $\varphi_{\Phi_{ss}}(f) = 0$. As this holds for any map $\varphi : X \to A$, we deduce that $f \in \mathrm{Id}^W(A_{\overline{W}_{ss}})$, as required. □

5. W-identities of 2×2-upper triangular matrix algebra

In this section we present the results in [25] about generalized identities of the algebra UT_2 of 2×2 upper triangular matrices. Notice that although such results were obtained in the case of UT_2-algebras, the present work seeks to generalize and extend the findings to the case where W is an arbitrary unital algebra.

To this end, from now on, let $\mathcal{B}_W = \{w_i\}_{i \in I}$ be a fixed ordered basis of W over F, chosen so that $w_0 = 1_W$. For simplicity, when considering a finite-dimensional unital W-algebra A with the W-action defined by left and right multiplication of a unital subalgebra $B \subseteq A$, we fix an ordered basis $\mathcal{B}_B = \{b_0 = 1_B, b_1, \ldots, b_n\}$ of B over F. Then, under these assumptions, we may assume without loss of generality that $\Phi(w_i) = (R_{b_i}, L_{b_i}) \in \mathcal{M}(A)$ for all $0 \leq i \leq n$ and $w_i \in \ker \Phi$ for all $i \geq n + 1$, where Φ is the acting homomorphism of W on A. We will use this notation throughout.

We start with the following simple lemma.

Lemma 5.1. *If B is a unital subalgebra of UT_2 of dimension 2, then either $B \cong D$ or $B \cong C$, where $D = Fe_{11} \oplus Fe_{22}$ and $C = F \oplus Fe_{12}$.*

Proof. If B is semisimple, then it is clear that it must be $B \cong F \oplus F$, that is $B \cong D$. If B is not semisimple, then a basis can be chosen so that it contains an element of the radical, say $j = \alpha e_{11} + \beta e_{22} + \gamma e_{12}$, for some $\alpha, \beta, \gamma \in F$. If $\alpha \neq 0$ or $\beta \neq 0$ then j can not lie in the radical since it would be non-nilpotent. Hence $j = \gamma e_{12}$ and $B \cong C$. □

As a direct consequence of the previous Lemma plus Theorem 2.8, we get that on UT_2 one can define four non-equivalent structures of W-algebra according to which subalgebra of UT_2 is acting by left and right multiplication. We will denote them by UT_2^F, UT_2^D, UT_2^C and simply UT_2 depending on whether $\overline{W} = \Phi(W)$ is isomorphic to F, D, C or the full algebra UT_2, respectively.

Notice that in the first case we are dealing with UT_2 with the ordinary structure of F-algebra and the ideal of identities was computed in [23], whereas UT_2^D and UT_2 were studied in [25]. In particular, using the following ordered bases for UT_2 and D respectively:

$$\mathcal{B}_{UT_2} = \{1_{UT_2} := e_{11} + e_{22}, e_{22}, e_{12}\} \quad \text{and} \quad \mathcal{B}_D = \{1_{UT_2}, e_{22}\},$$

the following theorems hold.

Theorem 5.2. *Let UT_2^F be the W-algebra UT_2 where W acts on it as the algebra F by left and right multiplication. Then $\mathrm{Id}^W(UT_2^F)$ is generated as T_W-ideal by the polynomials:*

$$w_i x, \quad x w_i, \quad [x_1, x_2][x_3, x_4],$$

for all $i \geq 1$. Moreover, $c_n^W(UT_2^F) = (n-2)2^{n-1} + 2$.

Theorem 5.3. *Let UT_2^D be the W-algebra UT_2 where W acts on it as the algebra D by left and right multiplication. Then $\mathrm{Id}^W(UT_2^D)$ is generated as T_W-ideal by the polynomials:*

$$w_i x, \quad x w_i, \quad [x_1, x_2] - [x_1, x_2, w_1],$$

for all $i \geq 2$. Moreover, $c_n^W(UT_2^D) = n2^{n-1} + 2$.

Theorem 5.4. *Let UT_2 be the W-algebra UT_2 where W acts on it as UT_2 itself by left and right multiplication. Then $\mathrm{Id}^W(UT_2)$ is generated as T_W-ideal by the polynomials:*

$$w_i x, \quad x w_i, \quad [x_1, x_2] - [x_1, x_2, w_1],$$

for all $i \geq 3$. Moreover, $c_n^W(UT_2) = (n+2)2^{n-1} + 2$.

Furthermore, in [25] it was also proved that while $\mathrm{var}^W(UT_2^F)$ and $\mathrm{var}^W(UT_2^D)$ generate two distinct varieties of W-algebras of almost polynomial growth, the same does not hold true for $\mathrm{var}^W(UT_2)$ since it contains UT_2^D. We can also remark that by Theorem 4.3, $UT_2^F \in \mathrm{var}^W(UT_2^C)$ thus $\mathrm{var}^W(UT_2^C)$ is not also of almost polynomial growth.

Just for the sake of completeness, here we state the result concerning T_W-ideal and W-codimensions of UT_2^C. This can be proven following step-by-step the proof in [25, Theorem 3.2] with the necessary yet intuitive modifications. Here, we take $\mathcal{B}_C = \{1_{UT_2}, e_{12}\}$ as the ordered basis for UT_2^C.

Theorem 5.5. *Let UT_2^C be the W-algebra UT_2 where W acts on it as the algebra C by left and right multiplication. Then $\mathrm{Id}^W(UT_2^C)$ is generated as T_W-ideal by the polynomials:*

$$w_i x, \quad x w_i, \quad [x_1, x_2][x_3, x_4], \quad [x_1, x_2] w_2, \quad w_2 [x_1, x_2],$$

for all $i \geq 2$. Moreover, $c_n^W(UT_2^C) = n2^{n-1} + 2$.

6. Classifying almost polynomial growth W-varieties

In this section, we characterize the generalized varieties of almost polynomial growth.

Let $M_n(F)$ be the algebra of matrices of order n over F for some $n > 1$, and consider the ordered basis $\mathcal{B}_{M_n(F)} = \{1_{M_n(F)}, e_{11}, \dots, e_{n-1,n-1}\} \cup \{e_{i,j} \mid i \neq j\}$ where the e_{ij}'s outside the diagonal are lexicographically ordered. Then, by using the same notation of the previous section, we can generalize [7, Proposition 2.1] as follows.

Proposition 6.1. *Let $M_n(F)$, $n > 1$, be the W-algebra with the action of $M_n(F)$ on itself by left and right multiplication. Then $\mathrm{Id}^W(M_n(F))$ is generated as T_W-ideal by the polynomials:*

$$w_i x, \quad x w_i, \quad [w_1 x_1 w_1, w_1 x_2 w_1],$$

for all $i \geq n^2 + 1$.

Lemma 6.2. *Let F be an algebraically closed field of characteristic zero and let consider the W-algebra $M_n(F)$. Then $\mathrm{var}^W(M_n(F))$ contains at least one of the algebras in $\{UT_2^F, UT_2^D\}$.*

Proof. Since $M_n(F)$ is a unital algebra, Theorem 2.8 ensures that the action of W is equivalent to the action of a subalgebra $B \subseteq M_n(F)$ by left and right multiplication. Moreover, by Theorem 4.3, we may assume without loss of generality that B is semisimple. Then, since F is algebraically closed, it follows from Wedderburn-Artin Theorem that

$$B \cong M_{t_1}(F) \oplus \cdots \oplus M_{t_k}(F)$$

for some $1 \leq k \leq n$ and $1 \leq t_i \leq n$ with $1 \leq i \leq k$.

Suppose first that $t_i = 1$ for all $1 \leq i \leq k$. If $k = 1$, then $B \cong F$ and we are dealing with the ordinary case. Consequently, we have $UT_2^F \in \mathrm{var}^W(M_n(F))$. Assume then that $k \geq 2$. In this case, we can express B as

$Fe_{j_1j_1} \oplus \cdots \oplus Fe_{j_kj_k}$. Clearly, the vector space $V = \text{span}_F\{e_{j_1j_1}, e_{j_2j_2}, e_{j_1j_2}\}$ forms a W-subalgebra of $M_n(F)$ isomorphic, as W-algebra, to UT_2^D. Therefore, we conclude that $UT_2^D \in \text{var}^W(M_n(F))$, completing this case.

Now, consider the case where $t_i \geq 2$ for some $1 \leq i \leq k$. We endow B with a natural W-algebra structure by defining the action of W on B as the action of B on itself via left and right multiplication. Since $M_{t_i}(F)$ is a minimal ideal of B, it follows that $M_{t_i}(F)$ is a W-subalgebra of B, and the W-action is determined by left and right multiplication of $M_{t_i}(F)$ on itself. Consequently, we have $\text{var}^W(M_{t_i}(F)) \subseteq \text{var}^W(B)$.

Through straightforward calculations, we obtain that $w_jx,\ xw_j,\ [w_1x_1w_1, w_1x_2w_1] \in \text{Id}_W(UT_2^F) \cap \text{Id}_W(UT_2^D)$ for all $j \geq t_i^2$. Therefore, by Proposition 6.1, it follows that $UT_2^F, UT_2^D \in \text{var}^W(M_{t_i}(F)) \subseteq \text{var}^W(B)$. Furthermore, since the W-action on $M_n(F)$ is given by left and right multiplication by B, it follows that B is a W-subalgebra of $M_n(F)$, and thus $\mathcal{V}^W(B) \subseteq \mathcal{V}^W(M_n(F))$. As a result, we conclude that $UT_2^F, UT_2^D \in \mathcal{V}^W(M_n(F))$, as required. □

Lemma 6.3. *Let $A = A_1 \oplus \cdots \oplus A_k + J$ be a finite dimensional unital algebra, where $A_1 \cong \cdots \cong A_k \cong F$, and suppose that A is a W-algebra. If there exist $1 \leq i, r \leq k$, $i \neq r$, such that $A_iJA_r \neq 0$, then $\text{var}^W(A)$ contains at least one of the algebras in $\left\{UT_2^F, UT_2^D\right\}$.*

Proof. Since A is unital, by Theorem 2.4 the action of W on A is equivalent to the action of a suitable unital subalgebra B of A by left and right multiplication. Additionally, by Theorem 4.3, we can assume that B is semisimple, i.e., $B \cong A_{p_1} \oplus \cdots \oplus A_{p_r}$ with $p_1, \ldots, p_r$ distinct elements in $\{1, \ldots, k\}$.

Now, since $A_iJA_r \neq 0$, there exist an element $j \in J$ such that $e_ije_r \neq 0$, where e_i and e_r denote the unit elements of A_i and A_r, respectively. For all $b \in B$, let

$$be_i = \alpha_b^{(i)}e_i, \quad e_ib = \beta_b^{(i)}e_i, \quad be_r = \alpha_b^{(r)}e_r, \quad e_rb = \beta_b^{(r)}e_r$$

for some $\alpha_b^{(i)}, \beta_b^{(i)}, \alpha_b^{(r)}, \beta_b^{(r)} \in F$. As a consequence, for all $b \in B$, we obtain

$$be_ije_r = (be_i)e_ije_r = \alpha_b^{(i)}e_ije_r \quad \text{and} \quad e_ije_rb = e_ije_r(e_rb) = \beta_b^{(r)}e_ije_r.$$

Now, consider the vector space

$$A' = \text{span}_F\{e_i,\ e_r,\ e_ije_r\}.$$

Clearly, A' is a unital W-subalgebra of A. Again by Theorems 2.4 and 4.3, the action of W on A' is equivalent to the action of a semisimple unital subalgebra B' of A' by left and right multiplication. Consequently, we have $\dim_F B' = 1$ or 2. If $\dim_F B' = 1$, then $B' \cong F$ and A' is isomorphic to UT_2^F as W-algebras. So, suppose instead that $\dim_F B' = 2$. Then $1_{B'} = 1_{A'}$, and since B' is semisimple, we can take $\{1_{B'}, e_r\}$ as a basis of B'. As a result, A' is isomorphic to UT_2^D as W-algebras. Thus, we conclude that either $UT_2^F \in \text{var}^W(A')$ or $UT_2^D \in \text{var}^W(A')$. Since $\text{var}^W(A') \subseteq \text{var}^W(A)$, the lemma is proved. □

Lemma 6.4. *Let A be a finite dimensional unital W-algebra. If $UT_2^F, UT_2^D \notin \text{var}^W(A)$, then the generalized codimension sequence $c_n^W(A)$ is polynomially bounded.*

Proof. Using an argument analogous to that used in the ordinary case (see [16, Theorem 4.1.9]), we can prove that generalized codimensions do not change upon extension of the base field. Therefore, we may assume F is algebraically closed.

Since the Jacobson radical $J = J(A)$ of A is a W-ideal, $\bar{A} = A/J$ is a W-algebra. Moreover, since $\bar{A}$ is semisimple and by Corollary 3.4, we have that $\bar{A} = \bar{A}_1 \oplus \cdots \oplus \bar{A}_k$ where $\bar{A}_1, \ldots, \bar{A}_k$ are W-simple algebras. By Corollary 3.5, for all $1 \leq i \leq k$, $\bar{A}_i$ is isomorphic to $M_{n_i}(F)$ as W-algebras, where $M_{n_i}(F)$ is the algebra of matrices of order $n_i \geq 1$ over F with the W-action given by left and right multiplication of a suitable subalgebra. If $n_i > 1$ for some i, then by Lemma 6.2 at least one among UT_2^F and UT_2^D belongs to $\text{var}^W(\bar{A}) \subseteq \text{var}^W(A)$, a contradiction. Thus, we can assume that $\bar{A}_i$ is isomorphic to F for all $1 \leq i \leq k$. As a result, by Wedderburn-Malcev Theorem for associative algebras (see [16, Theorem 3.4.3]) we conclude that

$$A = A_1 \oplus \cdots \oplus A_k + J,$$

where $A_1 \cong \cdots \cong A_k \cong F$ (as ordinary algebras). Thus, by Theorem 4.1 to complete the proof, it is enough to show that $A_iJA_r = 0$ for all $1 \leq i, r \leq m$, $i \neq r$.

Suppose to the contrary that there exist $1 \leq i, r \leq k$, $i \neq r$, such that $A_iJA_r \neq 0$. Then, by Lemma 6.3 it follows that either $UT_2^F \in \text{var}^W(A)$ or $UT_2^D \in \text{var}^W(A)$, a contradiction and the lemma is proved. □

Let A be a W-algebra (not necessarily finite-dimensional) and let Φ denote the acting homomorphism of W on A. Notice that the subalgebra $\overline{W} = \Phi(W) \subseteq \mathcal{M}(A)$ has a natural structure of W-algebra by left and right multiplication, more precisely, by defining $w\bar{w} := \Phi(w)\bar{w}$ and $\bar{w}w := \bar{w}\Phi(w)$, for all $w \in W$ and $\bar{w} \in \overline{W}$.

Since A is also $\overline{W}$-algebra, we can define a multiplication on the direct sum of vector spaces $\overline{W} \oplus A$ as follows:

$$(\bar{w}_1, a_1)(\bar{w}_2, a_2) := (\bar{w}_1\bar{w}_2, \bar{w}_1a_2 + a_1\bar{w}_2 + a_1a_2)$$

where $\bar{w}_1a_2 = L_1(a_2)$ and $a_1\bar{w}_2 = R_2(a_1)$, with $\bar{w}_1 = (L_1, R_1), \bar{w}_2 = (L_2, R_2) \in \overline{W}, a_1, a_2 \in A$. We refer to $\overline{W} \oplus A$ equipped with this multiplication as the *semi-direct product* of $\overline{W}$ and A, and we denote it by $\overline{W} \ltimes A$. Clearly, $\overline{W} \ltimes A$ is a unital F-algebra, with the unit given by $(1_{\overline{W}}, 0)$.

Remark that this construction naturally leads to the following diagram:

$$0 \longrightarrow A \xrightarrow{i_2} \overline{W} \ltimes A \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} \overline{W} \longrightarrow 0$$

with $\pi_1(\bar{w}, a) = \bar{w}$, $i_1(\bar{w}) = (\bar{w}, 0)$ and $i_2(a) = (0, a)$, respectively. Since the above diagram satisfies that $\pi \circ i_1 = id_{\overline{W}}$ and $i_2(A)$ is in the kernel of π_1, it is a split extension of F-algebras. For further details on split extensions of algebras and their relation with the action of F-algebras, we refer the reader to [20].

Notice that the algebra $\overline{W} \ltimes A$ naturally inherits the structure of W-algebra from the W-algebra structures of $\overline{W}$ and A. Specifically, for all $w \in W$, $\bar{w} \in \overline{W}$, and $a \in A$, the actions are defined as: $w(\bar{w} + a) = w\bar{w} + wa$ and $(\bar{w} + a)w = \bar{w}w + aw$. Thus, the semi-direct product $\overline{W} \ltimes A$ is a unital W-algebra.

Now, we are in a position to characterize the generalized varieties of polynomial growth.

Theorem 6.5. *Let W be a unital algebra over a field F of characteristic zero and A be a finite dimensional W-algebra. Then the sequence $c_n^W(A)$, $n \geq 1$, is polynomially bounded if and only if $UT_2^F, UT_2^D \notin \mathrm{var}^W(A)$.*

Proof. It is clear that if $c_n^W(A)$ is polynomially bounded, then $UT_2^F, UT_2^D \notin \mathrm{var}^W(A)$ since, by Theorems 5.2 and 5.3, UT_2^F and UT_2^D generate generalized varieties of exponential growth.

Now, assume that $UT_2^F, UT_2^D \notin \mathrm{var}^W(A)$. Let Φ denote the acting homomorphism of W in A, and let $\overline{W}$ denote the subalgebra $\Phi(W)$ of $\mathcal{M}(A)$. We can then consider the unital W-algebra $\overline{W} \ltimes A$. Since both $\overline{W}$ and A are finite-dimensional, it follows that $\overline{W} \ltimes A$ is also finite-dimensional. Thus, by Lemma 6.4 we have that $c_n^W(\overline{W} \ltimes A)$ is polynomially bounded. Additionally, since we can identify A with a suitable W-subalgebra of $\overline{W} \ltimes A$, it follows that $\mathrm{var}^W(A) \subseteq \mathrm{var}^W(\overline{W} \ltimes A)$. Therefore, $c_n^W(A) \leq c_n^W(\overline{W} \ltimes A)$ and consequently, $c_n^W(A)$ is also polynomially bounded, as required. □

As a consequence, we have the following characterization of generalized varieties of almost polynomial growth.

Corollary 6.6. *UT_2^F and UT_2^D are the only finite dimensional W-algebras generating generalized varieties of almost polynomial growth.*

7. On the Specht property

This final section deals with the Specht property of generalized T-ideals showing a conjecture and some future directions.

We say that a class of algebras has the Specht property if any T-ideal of that class is finitely generated. For example, a celebrated theorem of Kemer states that the class of associative algebras in characteristic zero has such property (see [21]). This result represents one of the milestones in the theory of algebras with polynomial identities, so that many mathematicians have continued to follow this line of research, attempting to solve the Specht problem in other contexts as well, such as algebras with additional structures, non-associative algebras and so on (see for instance [1,2,29]). All the affirmative answers are given in characteristic zero, since in positive characteristic there are several examples, even for associative algebras, of infinitely generated T-ideals (see [6,18,28]).

Throughout the paper, we have always considered finite dimensional W-algebras, although W had no restrictions on the dimension or on the number of generators. Regardless, a keen observer may notice that Theorems 5.2, 5.3, 5.4 and Proposition 6.1 raise an interesting matter regarding the number of generators of a T_W-ideal. In general, is it or is it not finitely generated?

Furthermore, working with infinite dimensional algebras can lead to other unusual situations. For instance, if W has infinite dimension, then P_n^W is also infinite dimensional and so can be $P_n^W(A)$. In other terms, this means that the generalized codimensions can be infinite, as already anticipated in Section 4.

It is clear that such "pathological" behavior of the codimensions may appear provided that both W and A are infinite dimensional, since if A has finite dimension, then the action of W, i.e. $\Phi(W)$, must be finite, as in the case of UT_2 for example.

Concerning the Specht property, it seems that the crucial aspect is not the dimension of W, but rather its being finitely generated. More precisely, we state the following conjecture.

Conjecture 7.1. *Let A be a W-algebra over a field of characteristic zero. If W is finitely generated, then $\mathrm{Id}^W(A)$ is finitely generated.*

To support the previous conjecture and simultaneously provide an example of infinite codimensions, we now compute the T_W-ideal of generalized identities of the unital Grassmann algebra E, assuming that either $\Phi(W) \cong E$ or $\Phi(W) \cong E_k$, the finite dimensional Grassmann algebra. To achieve this, we assume that W is not finitely generated.

Recall that the Grassmann algebra E with unit is the algebra generated over F by the elements $1, e_1, e_2, \ldots$ such that $e_i e_j = -e_j e_i$. Moreover, a basis of E consists of all the elements $g = e_{i_1} e_{i_2} \cdots e_{i_n}$ such that $i_1 < i_2 < \cdots < i_n$ and $n \geq 0$. Here, we are assuming that if $n = 0$ then $g = 1$. It is clear that by construction E is not finitely generated.

One can also consider a finite set of generators and construct the finite dimensional Grassmann algebra which is a subalgebra of E. More precisely, E_k is the algebra generated by $1, e_1, e_2, \dots, e_k$ and $\dim_F E_k = 2^k$.

Since W is not finitely generated and there is no loss of generality in considering E, we set $W = E$ for simplicity of notation.

We start computing the T_E-ideal of $E^{(k)}$, i.e., E with the action of E_k by left and right multiplication, for all $k \geq 1$. A simple computation shows that the polynomials

$$[x_1, x_2, x_3], \quad [e_i, x_1, x_2], \quad e_j x, \quad x e_j \tag{7.1}$$

for all $1 \leq i \leq k$ and for all $j \geq k+1$, belong to $\mathrm{Id}^E(E^{(k)})$. Moreover, notice that the polynomial $[x, g] = 0$, where $g = e_{i_1} \cdots e_{i_{2r}}$ lies in the center of E, and $[e_i, x, e_j] = 0$ are E-trivial polynomials in the sense of [25, Corollary 2.3]. Also, by straightforward calculations we have the equality

$$[e_i, x e_j] = [e_i, x] e_j + 2 x e_i e_j. \tag{7.2}$$

Lemma 7.2. *The following generalized polynomials are consequences of polynomials in* (7.1)*:*

$$[x_1, x_2][x_3, x_4] + [x_1, x_4][x_3, x_2]$$
$$[e_i, x_2][x_3, x_4] + [e_i, x_4][x_3, x_2]$$
$$[x_1, x_2, e_i]$$
$$[e_i, x_1][e_j, x_2] + 2[x_1, x_2] e_i e_j$$

for all $i, j \geq 1$.

Proof. Following the lines of [16, Theorem 4.1.8], one can prove that $[x_1, x_2, x_3]$ implies $[x_1, x_2][x_3, x_4] + [x_1, x_4][x_3, x_2]$ and $[e_i, x_1, x_2]$ implies $[e_i, x_2][x_3, x_4] + [e_i, x_4][x_3, x_2]$. Moreover, using Jacobi identity, it is clear that $[e_i, x_1, x_2]$ implies also $[x_1, x_2, e_i]$.

Now, take $[e_i, x_1, x_2]$ and substitute $x_1 e_j$ instead of x_1. Using $[x, g] = 0$, (7.2) and $[e_i, x_1, x_2] \in \mathrm{Id}^E(E^{(k)})$, we get that

$$\begin{aligned} [e_i, x_1 e_j, x_2] &= [e_i, x_1 e_j] x_2 - x_2 [e_i, x_1 e_j] = [e_i, x_1] e_j x_2 + 2 x_1 e_i e_j x_2 - x_2 [e_i, x_1] e_j - 2 x_2 x_1 e_i e_j \\ &= [e_i, x_1] e_j x_2 - x_2 [e_i, x_1] e_j + 2 x_1 x_2 e_i e_j - 2 x_2 x_1 e_i e_j = [e_i, x_1] e_j x_2 - x_2 [e_i, x_1] e_j + 2 [x_1, x_2] e_i e_j \\ &\equiv [e_i, x_1] e_j x_2 - [e_i, x_1] x_2 e_j + 2 [x_1, x_2] e_i e_j = [e_i, x_1][e_j, x_2] + 2 [x_1, x_2] e_i e_j. \end{aligned}$$

Thus $[e_i, x_1, x_2]$ imply $[e_i, x_1][e_j, x_2] + 2[x_1, x_2] e_i e_j$ and we are done. □

We are now in a position to compute a basis of $\mathrm{Id}^E(E^{(k)})$. In the next theorem, recall that the *support* of $g \in E$, denoted by $\mathrm{supp}(g)$, is the set of all the e_i's appearing in g. Moreover, we denote by E_1 the span over F of all the words in the generators of E of odd length.

Theorem 7.3. *Let* $E^{(k)}$, $k \geq 1$ *be the unital Grassmann algebra* E *over a field* F *of characteristic zero regarded as an* E*-algebra, where* E *acts as the subalgebra* E_k *by left and right multiplication. Then* $\mathrm{Id}^E(E^{(k)})$ *is generated, as* T_E*-ideal, by the polynomials*

$$[x_1, x_2, x_3], \quad [e_i, x_1, x_2], \quad e_j x, \quad x e_j$$

for all $1 \leq i \leq k$ *and for all* $j \geq k+1$. *Moreover,* $c_n^E(E^{(k)}) = 2^{n-1}(2^{k+1} - 1)$, *for all* $n \geq 1$.

Proof. Let I be the T_E-ideal generated by the above generalized polynomials. A straightforward computation shows that $I \subseteq \mathrm{Id}^E(E^{(k)})$. Thus in order to prove the opposite inclusion, let $f \in \mathrm{Id}^E(E^{(k)})$ be a multilinear generalized polynomial of degree n and suppose, as we may, that in f does not appear any e_j for all $j \geq k+1$.

By the Poincaré-Birkhoff-Witt Theorem and by $[x_1, x_2, e_i] \equiv 0$, f can be written as a linear combination of polynomials of the type

$$x_{c_1} \cdots x_{c_t} v_1 v_2 \cdots v_m e_{l_1} \cdots e_{l_s},$$

where $c_1 < \cdots < c_t$, $l_1 < \cdots < l_s$, $0 \leq s \leq k$ and $v_1, \dots, v_m$ are left-normed commutators containing variables and e_i's. Moreover, since E has unit element, without loss of generality in what follows we can disregard the tail $x_{c_1} \cdots x_{c_t}$ and suppose that the variables $x_1, \dots, x_n$ in f appear exclusively within the commutators.

Now, using Lemma 7.3 and the W-trivial polynomials mentioned above, we can write f as linear combination of the polynomials

$$[x_1, x_2][x_3, x_4] \cdots [x_{2m-1}, x_{2m}] e_{l_1} \cdots e_{l_s}, \tag{7.3}$$

if $n = 2m$ is even or

$$[e_{l_1}, x_1][x_2, x_3] \cdots [x_{2m}, x_{2m+1}] e_{l_2} \cdots e_{l_s}, \tag{7.4}$$

if $n = 2m+1$ is odd. In both cases $l_1 < \cdots < l_s$ and $0 \leq s \leq k$. This implies that such polynomials generate P_n^W modulo the generalized identities of E.

We claim that they are also linearly independent modulo $\mathrm{Id}^E(E^{(k)})$. To this end, first suppose that $\deg f = 2m$ and write

$$f = \sum_L \alpha_L [x_1, x_2][x_3, x_4] \cdots [x_{2m-1}, x_{2m}] e_{l_1} \cdots e_{l_s} \pmod{\mathrm{Id}^E(E^{(k)})},$$

where $L = \{l_1, \dots, l_s\}$. Let s be the smallest integer for which by contradiction there exists L such that $\alpha_L \neq 0$. Then, by making the evaluation $x_1 \mapsto g_1 = e_{j_1} \cdots e_{j_t} g'$ and $x_i \mapsto g_i$ for all $2 \leq i \leq 2m$, where $\{j_1, \dots, j_t\} \uplus L = \{1, \dots, k\}$, $g' \in E$ such that $g_1 \in E_1$, $\mathrm{supp}(g') \cap \{j_1, \dots, j_t\} = \emptyset$, $g_2, \dots, g_{2m} \in E_1$ and $\mathrm{supp}(g_i) \cap \mathrm{supp}(g_j) = \emptyset$ for all $i \neq j$, we get $2^m \alpha_L g_1 g_2 \cdots g_{2m} e_{l_1} \cdots e_{l_s} = 0$ that is $\alpha_L = 0$, a contradiction. Therefore, $f = 0$ and we are done in this case.

Now suppose that $\deg f = 2m + 1$ and write

$$f = \sum_L \alpha_L [e_{l_1}, x_1][x_2, x_3] \cdots [x_{2m}, x_{2m+1}] e_{l_2} \cdots e_{l_s} \pmod{\mathrm{Id}^E(E^{(k)})},$$

where $L = \{l_1, \dots, l_s\}$. With the same evaluation as before, one can get $f = 0$ also in this case.

Thus these polynomials are linearly independent modulo $\mathrm{Id}^E(E^{(k)})$ and since $P_n^E \cap I \subseteq P_n^E \cap \mathrm{Id}^E(E^{(k)})$, this proves that $I = \mathrm{Id}^E(E^{(k)})$ and the polynomials in (7.3) and (7.4) form a basis of P_n^E modulo $P_n^E \cap \mathrm{Id}^E(E^{(k)})$.

If we set $\gamma_{2m}(E)$ and $\gamma_{2m+1}(E)$ to be the number of polynomials in (7.3) and (7.4), respectively, then it is clear that $\gamma_{2m}(E) = 2^k$, $\gamma_{2m+1}(E) = 2^k - 1$ for all $m \geq 1$ and

$$c_n^E(E^{(k)}) = \sum_{t=0}^{n} \binom{n}{t} \gamma_t(E).$$

Therefore by counting

$$\begin{aligned} c_n^E(E^{(k)}) &= \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2h} 2^k + \sum_{h=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2h+1} (2^k - 1) \\ &= 2^k \left[\sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2h} + \sum_{h=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2h+1} \right] - \sum_{h=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2h+1} = 2^k 2^n - 2^{n-1} = 2^{n-1}(2^{k+1} - 1), \end{aligned}$$

as claimed. □

With similar techniques we can compute also the T_W-ideal of generalized identities of E when W acts by left and right multiplication as the full algebra E.

Theorem 7.4. *Let E be the unital Grassmann algebra over a field F of characteristic zero regarded as an E-algebra, where E acts on itself by left and right multiplication. Then $\mathrm{Id}^E(E)$ is generated, as T_E-ideal, by the polynomials*

$$[x_1, x_2, x_3], \quad [e_i, x_1, x_2], \tag{7.5}$$

for all $i \geq 1$. Moreover, $c_n^E(E) = +\infty$, for all $n \geq 1$.

Proof. Following the lines of the previous Theorem, one can prove that modulo $\mathrm{Id}^E(E)$, any multilinear polynomial f can be written as linear combination of

$$[x_1, x_2][x_3, x_4] \cdots [x_{2m-1}, x_{2m}] e_{l_1} \cdots e_{l_s},$$

if $n = 2m$ is even or

$$[e_{l_1}, x_1][x_2, x_3] \cdots [x_{2m}, x_{2m+1}] e_{l_2} \cdots e_{l_s},$$

if $n = 2m+1$ is odd. In both cases $l_1 < \cdots < l_s$ but here we have no limit on the value of s. It is also easily proven that these polynomials are linearly independent modulo $\mathrm{Id}^E(E)$, thus the polynomials in (7.5) generate $\mathrm{Id}^E(E)$ and consequently $c_n^E(E) = +\infty$ for all $n \geq 1$. □

Corollary 7.5. *The T_E-ideal of generalized polynomial identities of the Grassmann algebra E with either the action by left and right multiplication of E_k, for any k, or the action of E, has not the Specht property.*

Proof. The claim follows directly from Theorems 7.3 and 7.4. In fact, concerning $\mathrm{Id}^E(E^{(k)})$, it is clear that for instance the identity $e_j x$ for any fixed $j \geq k + 1$ can not follow from the remaining ones. Similarly, for $\mathrm{Id}^E(E)$, any identity $[e_i, x_1, x_2]$ can not follow from $[x_1, x_2, x_3]$ since the latter one has degree 3 and, by the multiplication rules in E, it can not follow also by any other identity of the type $[e_j, x_1, x_2]$ with $j \neq i$. □

Notice that the previous corollary shows that the Specht property may fail whenever W is not finitely generated, regardless its action on A, strengthening Conjecture 7.1.

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