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UNCERTAINTY PRINCIPLES FOR THE q-HANKEL TRANSFORM

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ABSTRACT: We prove two propositions related to the support of functions and their q-Hankel transform. The first says that if a function f and its q-Hankel transform both vanish at the points q^{-n}, $n = 1, 2, ...$ then f must vanish identically. The second asserts that if f is supported at $[0, T]$ and its q-Hankel transform at $[0, \Omega]$ then $\Omega T \geq (q; q)_{\infty}^{2}$.

KEYWORDS: uncertainty principles; q-Bessel functions; q-Hankel transform.
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1. Introduction

The Fourier transform of a $L^1(\mathbf{R})$ function supported on a finite interval (a, b)

$$f^{\wedge}(\omega) = \int_a^b f(t) e^{-\omega i t} dt \tag{1}$$

defines an entire function. Therefore, if $f^{\wedge}$ itself has compact support, then it must vanish identically since it vanishes on a set with an accumulation point. By Fourier inversion f itself must vanish identically. This is the most simple manifestation of the uncertainty principle of Fourier analysis which says, in general, that a function and its transform cannot be simultaneously small. The present note pretends to address the question of how to prove such a statement if, instead of the Lebesgue measure, one is working with a measure without an accumulation point outside the interval (a, b).

Consider a number q in the real interval $(0, 1)$. The prototype of the situation just described is the discrete Jackson q-integral

$$\int_0^{\infty} f(t)\, d_q t = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n)\, q^n. \tag{2}$$

where the spectrum of the measure is $\{q^n\}_{n=-\infty}^{\infty}$ which has zero as the only accumulation point. Using the q-integral and a suitable chosen q-analogue of

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the Bessel function (which we will define in the next section), Koornwinder and Swarttouw defined in [5] a q-analogue of the Hankel transform, $H_q^\nu f$, setting

$$\left(H_q^\nu f\right)(x)=\int_0^\infty (xt)^{\frac{1}{2}} J_\nu^{(3)}\left(xt;q^2\right) f(t)\, d_q t \tag{3}$$

For the transform $H_q^\nu f$, we will prove that, in a convenient normalized space, if f and $H_q^\nu f$ vanish at all the points of the spectrum outside the interval $(0,1)$, then f must vanish in the equivalent classes of the normalized space considered. The presentation is organized as follows. In the next section we introduce the notions about q-calculus to be used in the remaining of the paper. In the third section we prove our main theorem and deduce from it a proposition about uniqueness sets in a certain Hilbert space of entire functions. In the last section we obtain some estimates on the kernel of the integral transform and use them to conclude, from a general proposition due to de Jeu [6], that the length of the support of f times the length of the support of $H_q^\nu f$ must be bigger than a certain positive quantity, paralleling a classical result about Fourier transforms.

2. Basic definitions and facts

The third Jackson q-Bessel function or the Hahn-Exton q-Bessel function is defined by

$$J_\nu(z;q)=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)/2}}{(q^{\nu+1};q)_n(q;q)_n}z^{2n+\nu} \tag{4}$$

The notation $J_\nu^{(3)}(z;q)$ is used to distinguish it from the other two known q-Bessel functions. Since this is the only Bessel function appearing on the text, we will drop the superscript for shortness of the notations and write $J_\nu(z;q)=J_\nu^{(3)}(z;q)$. The symbols in the above definitions are

$$(a;q)_n=(1-q)(1-aq)\ldots\left(1-aq^{n-1}\right) \tag{5}$$

with the zero and infinite cases as

$$(a;q)_0=1 \tag{6}$$

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n=\prod_{k=0}^{\infty}(1-aq^n) \tag{7}$$

The infinite product above can be written in series form by means of the the Euler formula:

$$(z;q)_\infty = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} x^n \tag{8}$$

The $q-$integral in the finite interval $(0,a)$ is

$$\int_0^a f(t)\, d_q t = (1-q)\, a \sum_{n=0}^{\infty} f(aq^n)\, q^n \tag{9}$$

and in the interval $(0,\infty)$

$$\int_0^\infty f(t)\, d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n)\, q^n \tag{10}$$

We will denote by $L_q^p(X)$ the Banach space induced by the norm

$$\|f\|_p = \left[\int_X |f(t)|^p\, d_q t\right]^{\frac{1}{p}}. \tag{11}$$

For entire indices, the functions $J_n(x;q)$ are generated by the relation, valid for $|xt| < 1$,

$$\frac{\left(qxt^{-1};q\right)_\infty}{(xt;q)_\infty} = \sum_{n=-\infty}^{\infty} J_n(x;q) t^n \tag{12}$$

It was shown in [5] that the q-Hankel transform satisfies the inversion formula

$$f(t) = \int_0^\infty (xt)^{\frac{1}{2}} \left(H_q^\nu f\right)(x)\, J_\nu\left(xt;q^2\right) d_q x = \left(H_q^\nu \left(H_q^\nu f\right)\right)(t) \tag{13}$$

where t takes the values $q^k, k \in Z$.

3. A vanishing theorem for the q-Hankel transform

The main tool in the proof of the main result in this section is the following completeness criterion, derived in [2] as a consequence of the Phragmén-Lindelöf principle for functions of order less than one.

Theorem A. *Let f and g be defined by their power series expansions as $f(z) = \sum_{n=0}^{\infty}(-1)^n a_n z^{2n}$ and $g(z) = \sum_{n=0}^{\infty}(-1)^n b_n z^{2n}$. Denote by λ_n the nth zero of g. If the order of f is less than one, then the sequence $\{f(\lambda_n x)\}$ is complete $L_q(0,1)$ if, as $n \to \infty$,*

$$\frac{a_n}{b_n} \to 0 \tag{14}$$

Now we formulate and proof our main result, which is an uncertainty principle of a qualitative nature. Essentially it says that a $L_q^1(\mathbf{R}^+)$ function and its q-Hankel transform cannot be both simultaneously supported inside the interval $(0,1)$.

Theorem 1. *Let $f \in L_q^1(\mathbf{R}^+)$ such that both f and its q-Hankel transform vanish at the points $q^{-n}, n \in \mathbf{N}_0$, then*

$$f(q^k) = 0, k \in \mathbf{Z}. \tag{15}$$

that is, $f \equiv 0$ almost everywhere in $L_q^1(\mathbf{R}^+)$. If f is analytic then f must vanish identically in the whole complex plane.

Proof. Let $f \in L_q^1(\mathbf{R}^+)$. If $f(q^{-n}) = 0, n \in \mathbf{N}_0$, then the q-Hankel transform of f is

$$H_q^\nu f(\omega) = \int_0^1 (\omega t)^{\frac{1}{2}} J_\nu \left(\omega t; q^2\right) f\left(t\right) d_q t. \tag{16}$$

Our second assumption says that

$$(H_q^\nu f)(q^{-n}) = 0, n \in \mathbf{N}_0 \tag{17}$$

therefore, setting $\omega = q^{-n}$ in (16) gives

$$\int_0^1 \left(q^{-n} t\right)^{\frac{1}{2}} J_\nu \left(q^{-n} t; q^2\right) f(t) d_q t = 0, n \in \mathbf{N}_0 \tag{18}$$

Now, in the set up of Theorem A take $f(z) = J_\nu\left(z; q^2\right)$ and $g(z) = (z^2; q^2)_\infty$. Using (4) and (8) together with the trivial observation that $\{q^{-n}\}$ is the sequence of zeros of g gives that, if $\nu > -1$, the sequence $\left\{J_\nu\left(q^{-n}x; q^2\right)\right\}$ is complete in $L_q^1(0,1)$. This, together with (18) implies that $f \equiv 0$ in $L_q^1(0,1)$, that is,

$$f(q^n) = 0, n \in \mathbf{N}_0 \tag{19}$$

Combining this with the assumption $f(q^{-n}) = 0, n \in \mathbf{N}_0$ gives

$$f(q^k) = 0, k \in \mathbf{Z} \tag{20}$$

This proves that $f \equiv 0$ almost everywhere in $L_q^1(\mathbf{R}^+)$. Since the set $\{q^k, k \in \mathbf{Z}\}$ has an accumulation point, if f is analytic then it must be the null function. $\square$

Following [1] we introduce the space

$$PW_q^\nu = \left\{ f \in L_q^2\left(\mathbf{R}^+\right) : f\left(x\right) = \int_0^1 \left(tx\right)^{\frac{1}{2}} J_\nu\left(xt;q^2\right) u\left(t\right) d_q t, u \in L_q^2\left(0,1\right) \right\} \tag{21}$$

This can be interpreted as a q-Bessel version of the Paley Wiener space of bandlimited functions. Clearly, PW_q^ν is a Hilbert space of analytic functions. Observe also that, if $\left(H_q^\nu f\right)\left(q^{-n}\right) = 0, n \in \mathbf{N}$, then taking into account definitions (9) and (2), $f = \left(H_q^\nu\left(H_q^\nu f\right)\right)$ is of the form required in (21). Using these concepts, we have the following consequence of the vanishing theorem:

Corollary 1. $\Gamma = \{q^{-n}, n \in \mathbf{N}\}$ *is a set of uniqueness for the space* PW_q^ν*.*

Proof. Take $f \in PW_q^\nu$ such that $f(q^{-n}) = 0, n \in \mathbf{N}$. If f is of the form required in (21) then $f = H_q^\nu u^*$ where $u^* \in L_q^2\left(\mathbf{R}^+\right)$ is obtained from $u \in L_q^2(0,1)$ by prescribing $u(q^{-n}) = 0, n \in \mathbf{N}$. By the inversion formula (13), $u^* = H_q^\nu f$. We conclude that $H_q^\nu f(q^{-n}) = 0, n \in \mathbf{N}$. By Theorem 1, $f \equiv 0$. $\square$

Remark 1. *Observe that we proved the following characterization of* PW_q^ν*:*

$$PW_q^\nu = \left\{f \in L_q^2\left(\mathbf{R}^+\right) : \left(H_q^\nu f\right)\left(q^{-n}\right) = 0, n \in \mathbf{N}\right\} \tag{22}$$

The property $\left(H_q^\nu f\right)\left(q^{-n}\right) = 0, n \in \mathbf{N}$ *can thus be seen as a sort of "q-Hankel-bandlimitedness". It was shown in* [1] *that there are many features in this space analogous to the classical Paley Wiener space, including a sampling theorem and a reproducing kernel.*

4. An uncertainty principle

With the purpose of extending the Donoho and Stark uncertainty principle [3] to an abstract setting, de Jeu [6] obtained a very general proposition, from which we just quote a special case.

Theorem B *If there is a Plancherel theorem for the integral transform in* $L^2(X)$ *whose kernel is* $K(x,t)$*, then, if the support of* f *is* T *and the support of* $(Kf)(x) = \int_X K(x,t)f(t)d\mu(t)$ *is* Ω*, the following inequality holds:*

$$\|\mathbf{1}_{T\times\Omega}K(x,t)\|_{L^2(\mu,X)\times L^2(\mu,X)} \geq 1 \tag{23}$$

In order to use Theorem B to extract more valuable information about the size of the supports in our study of the q-Hankel transform, we must first obtain bounds for its kernel.

Lemma 1. *If $\nu \geq 0$ and $|x| < q^{-\frac{1}{2}}$, the inequality holds:*

$$|J_\nu(x;q)| \leq \frac{1}{(q;q)_\infty} \tag{24}$$

Proof. If $\nu > 0$, $y > -\frac{1}{2}$ and $x \in \mathbf{R}$, the following q-analogue of the Sonine integral was proved in [1]:

$$\frac{(q;q)_\infty}{(q^\nu;q)_\infty} x^{-\nu} J_{y+\nu}(x;q) = \int_0^1 t^{\frac{y}{2}} \frac{(tq;q)_\infty}{(tq^\nu;q)_\infty} J_y(xt^{\frac{1}{2}};q) d_q t \tag{25}$$

Setting $y = 0$ in (25) and taking absolute values gives

$$|J_\nu(x;q)| \leq \left| x^\nu \frac{(q^\nu;q)_\infty}{(q;q)_\infty} \right| \int_0^1 \left| \frac{(tq;q)_\infty}{(tq^\nu;q)_\infty} J_0(xt^{\frac{1}{2}};q) \right| d_q t \tag{26}$$

We need to estimate the integrand in (25). For the infinite product, observe that if $0 < t < 1$, then

$$\frac{(tq;q)_\infty}{(tq^\nu;q)_\infty} < \frac{1}{(q^\nu;q)_\infty} \tag{27}$$

Now we will show that, if $t < 1$ and $|x| < q^{-\frac{1}{2}}$ then

$$\left| J_0(xt^{\frac{1}{2}};q) \right| \leq 1 \tag{28}$$

This can be seen using a generating function argument as follows. Substituting t by $t^{-1}q$ in (12) and multiplying the two resulting identities gives, if $|xq| < |t| < |x|^{-1}$ (which holds if $|x| < q^{-\frac{1}{2}}$ and $|xt| < 1$)

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} t^{n-m} q^m J_n(x;q) J_m(x;q) = 1 \tag{29}$$

Equating coefficients of t^0 in (29) reveals that, if $|x| < q^{-\frac{1}{2}}$, $\sum_{k=-\infty}^{\infty} q^k \left[J_k(x;q)\right]^2 = 1$. In particular,

$$|J_k(x;q)| \leq q^{-\frac{k}{2}},\ k = 0, 1, ... \tag{30}$$

Now, if $t < 1$ and $|x| < q^{-\frac{1}{2}}$ we also have $|xt| < q^{-\frac{1}{2}}$. Setting $k = 0$ in (30) gives(28). Using this estimates in (26) together with (27) gives

$$|J_\nu(x;q)| \leq \left| x^\nu \frac{1}{(q;q)_\infty} \right|. \tag{31}$$

This proves the lemma. □

We can now state a proposition providing information of a quantitative nature about the supports of f and $H_q^\nu f$.

Theorem 2. *Suppose that $\nu \geq 0$. If the support of f is contained in $[0,T]$ and the support of $H_q^{\nu} f$ is contained in $[0,\Omega]$, then*

$$\Omega T \geq (q;q)_{\infty}^{2} \tag{32}$$

Proof. First observe that if $\Omega T \geq 1$ then the proposition is trivial, since $(q;q)_{\infty} < 1$. Thus we can assume without loss of generalization that $\Omega T < 1$. In this case we have $|xt| < 1$ in the square $[0,T] \times [0,\Omega]$ and the use of (24) together with the definition of the q-integral gives

$$\left\| \mathbf{1}_{T\times\Omega}(x,t)\,(xt)^{\frac{1}{2}}\, J_{\nu}\left(xt;q^{2}\right)\right\|_{L_q^2(X)\times L_q^2(X)} = \int_0^{\Omega}\left[\int_0^{T}\left[(tx)^{\frac{1}{2}}\, J_{\nu}\left(xt;q^{2}\right)\right]^{2} d_qt\right] d_qx \tag{33}$$

$$\leq \int_0^{\Omega}\int_0^{T}\left[\frac{1}{(q;q)_{\infty}}\right]^{2} d_qt d_qx = \frac{\Omega T}{(q;q)_{\infty}^{2}} \tag{34}$$

now observe that applying Theorem B to the q-Hankel transform gives

$$1 \leq \left\| \mathbf{1}_{T\times\Omega}\,(xt)^{\frac{1}{2}}\, J_{\nu}\left(xt;q^{2}\right)\right\|_{L_q^2(X)\times L_q^2(X)} \tag{35}$$

and the result is proved. $\square$

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