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QUADRATIC LIE SUPERALGEBRAS WITH REDUCTIVE EVEN PART

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Abstract: The aim of this paper is to exibit some non trivial examples of quadratic Lie superalgebras such that the even part is a reductive Lie algebra and the action of the even part on the odd part is not completely reducible and to give an inductive classification of this class of quadratic Lie superalgebras. The notion of generalized double extension of quadratic Lie superalgebras proposed by I. Bajo, S. Benayadi and M. Bordemann [1] has a crucial importance in this work. In particular we will improve some results of [4], in the sense that we will not demand that the action of the even part on the odd part is completely reducible, which naturally makes the proofs of our results more difficult.

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Introduction

A Lie superalgebra $\mathfrak{g}$ is called *quadratic* if there is a bilinear form B on $\mathfrak{g}$ such that B is non-degenerate, supersymmetric, even, and $\mathfrak{g}$-invariant. In this case, B is called an invariant scalar product on $\mathfrak{g}$. For example, the semisimple Lie algebras and basic classical Lie superalgebras are quadratic Lie superalgebras [12, 13]. But, interestingly, some solvable Lie superalgebras are also quadratic Lie superalgebras [2, 14]. Moreover, the structure of quadratic Lie algebras has a significant role in conformal field theory, Sugawara construction exists precisely for quadratic Lie algebras [10]. The finite dimensional quadratic Lie algebras over a field $\mathbb{K}$ of characteristic zero are well known, as we can see in the following works: [3, 7, 9, 11, 14]. In [14] A. Medina and P. Revoy introduce the notion of double extension with the purpose to give an inductive classification of quadratic Lie algebras. H. Benamor and S. Benayadi in [2] extended the notion of double extension to quadratic

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Lie superalgebras. It was shown that it was more difficult to use double extension to classify quadratic Lie superalgebras than to classify quadratic Lie algebras. Among the main difficulties are that: a one-dimensional subspace of a Lie superalgebra $\mathfrak{g}$ is not necessarily a Lie subsuperalgebra of $\mathfrak{g}$; there is no analog to the Lie Theorem for solvable Lie superalgebras; a decomposition similar to the Levi-Malcev decomposition in Lie algebras is not verified in the case of Lie superalgebras. Despite the difficulties of the subject, in [2] they presented an inductive classification of quadratic Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\dim \mathfrak{g}_{\bar{1}} = 2$. In [4], S. Benayadi obtained an inductive classification of quadratic Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, and the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible. Moreover, he also gave an inductive classification of the solvable quadratic Lie superalgebras such that the odd part is a completely reducible module on the even part. More recently, I. Bajo, S. Benayadi, and M. Bordemann ([1]) generalized the notion of double extension of quadratic Lie superalgebras in order to present an inductive classification of solvable quadratic Lie superalgebras. The aim of this paper is to present some non trivial examples of quadratic Lie superalgebras such that the even part is a reductive Lie algebra and the action of the even part on the odd part is not completely reducible and to give an inductive classification of this class of quadratic Lie superalgebras improving some results of [4]. In fact, in the second section of [4] S. Benayadi has demanded for quadratic Lie superalgebras that the action of the even part on the odd part is completely reducible, which we will not impose in our paper. Although, our results are similar, most of the methods used to prove them are naturally more complicated than those employed in [4].

1. Generalized double extension of quadratic Lie superalgebras

I. Bajo, S. Benayadi, and M. Bordemann presented the notion of generalized double extension of quadratic Lie superalgebras [1]. In this work we will use the particular case of generalized double extension by the one-dimensional abelian Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ to give an inductive classification of quadratic Lie superalgebras with even part a reductive Lie algebra.

Proposition 1.1 ([1]). *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a quadratic Lie superalgebra,* D *an odd skew-supersymmetric superderivation of* $(\mathfrak{g}, B)$*, and* $X_0 \in \mathfrak{g}_{\bar{0}}$ *such* *that* $D(X_0)=0, B(X_0,X_0)=0, D^2=\dfrac{1}{2}[X_0,.]_{\mathfrak{g}}$. *Define a map* $\varphi:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathbb{K}$ *by* $\varphi(X,Y)=-B(D(X),Y),\forall_{X,Y\in\mathfrak{g}}$. *Then* φ *is an odd element of* $Z^2(\mathfrak{g},\mathbb{K})$. *Moreover, the graded vector space* $\mathfrak{g}\oplus\mathbb{K}e^*$ *endowed with the multiplication defined by*

$$[X+\alpha e^*,Y+\beta e^*]=[X,Y]_{\mathfrak{g}}+\varphi(X,Y)e^*,\qquad \forall_{(X+\alpha e^*),(Y+\beta e^*)\in(\mathfrak{g}\oplus\mathbb{K}e^*)},$$

is a Lie superalgebra. In this case, $\mathfrak{g}\oplus\mathbb{K}e^*$ *is the central extension of* $\mathbb{K}e^*$ *by* $\mathfrak{g}$ *(by means of* φ*).*

Proposition 1.2 ([1])**.** *Consider two Lie superalgebras* $\mathfrak{g}$ *and* $\mathfrak{h}$*, a linear map* $\Omega:\mathfrak{g}\longrightarrow Der(\mathfrak{h})$*, and an even skew-supersymmetric bilinear map* $\varphi:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{h}$ *such that*

$$\big[\Omega(X),\Omega(Y)\big]-\Omega\big([X,Y]_{\mathfrak{g}}\big)=ad\big(\varphi(X,Y)\big),\forall_{X,Y\in\mathfrak{g}},$$

and

$$\begin{aligned}&(-1)^{xz}\Big\{\Omega(X)\big(\varphi(Y,Z)\big)-\varphi\big([X,Y]_{\mathfrak{g}},Z\big)\Big\}\\&+(-1)^{xy}\Big\{\Omega(Y)\big(\varphi(Z,X)\big)-\varphi\big([Y,Z]_{\mathfrak{g}},X\big)\Big\}\\&+(-1)^{yz}\Big\{\Omega(Z)\big(\varphi(X,Y)\big)-\varphi\big([Z,X]_{\mathfrak{g}},Y\big)\Big\}=0,\end{aligned}$$

for every $X\in\mathfrak{g}_x$*,* $Y\in\mathfrak{g}_y$*, and* $Z\in\mathfrak{g}_z$*. Then the* $\mathbb{Z}_2$*-graded vector space* $\mathfrak{k}=\mathfrak{g}\oplus\mathfrak{h}$ *endowed with the multiplication* $[,]:\mathfrak{k}\times\mathfrak{k}\longrightarrow\mathfrak{k}$ *defined by*

$$[X+f,Y+h] \;=\; [X,Y]_{\mathfrak{g}}+[f,h]_{\mathfrak{h}}+\Omega(X)(h)-(-1)^{xy}\Omega(Y)(f)+\varphi(X,Y),$$

where $(X+f)\in\mathfrak{k}_x$ *and* $(Y+h)\in\mathfrak{k}_y$*, is a Lie superalgebra. It is called the generalized semi-direct product of* $\mathfrak{g}$ *and* $\mathfrak{h}$ *(by means of* Ω *and* φ*). In particular, if* $\varphi=0$ *then* $\mathfrak{k}$ *is the semi-direct product of* $\mathfrak{g}$ *and* $\mathfrak{h}$ *(by means of* Ω*)* [15].

Now we will present the notion of the generalized double extension of a quadratic Lie superalgebra due to I. Bajo *et al.* [1]

Theorem 1.3. *Let* $(\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1},B)$ *be a quadratic Lie superalgebra,* D *an odd skew-supersymmetric superderivation of* $(\mathfrak{g},B)$*, and* X_0 *a non-zero element of* $\mathfrak{g}_{\bar 0}$ *such that* $D(X_0)=0, B(X_0,X_0)=0, D^2=\dfrac{1}{2}[X_0,.]_{\mathfrak{g}}$*. Let us* *consider the linear map* $\Omega : \mathbb{K}e \longrightarrow Der(\mathfrak{g}\oplus\mathbb{K}e^*)$ *defined by* $\Omega(e) = \widetilde{D}$, *where* $\widetilde{D} : \mathfrak{g} \oplus \mathbb{K}e^* \longrightarrow \mathfrak{g} \oplus \mathbb{K}e^*$ *satisfies* $\widetilde{D}(e^*) = 0$ *and*

$$\widetilde{D}(X) \;=\; D(X) - (-1)^x B(X, X_0)e^*, \qquad \forall_{X\in\mathfrak{g}_x},$$

which is extended by linearity to all $\mathfrak{g}$. *Consider the bilinear map* $\varphi : \mathbb{K}e \times \mathbb{K}e \longrightarrow \mathfrak{g}\oplus\mathbb{K}e^*$ *defined by* $\varphi(e, e) = X_0$. *Then* $\mathfrak{k} = \mathbb{K}e\oplus\mathfrak{g}\oplus\mathbb{K}e^*$ *equipped with the even skew-symmetric bilinear map** $[,] : \mathfrak{k} \times \mathfrak{k} \longrightarrow \mathfrak{k}$ *defined by:* $[e, e] = X_0; [e, X] = D(X) - B(X, X_0)e^*,\ \forall_{X\in\mathfrak{g}_x}; [X, Y] = [X, Y]_{\mathfrak{g}} - B(D(X), Y)e^*,\ \forall_{X,Y\in\mathfrak{g}}; [e^*, \mathfrak{k}] = \{0\}$, *is the generalized semi-direct product of* $\mathfrak{g} \oplus \mathbb{K}e^*$ *by the one-dimensional Lie superalgebra* $(\mathbb{K}e)_{\bar{1}}$ *(by means of* Ω *and* φ*). Further, the supersymmetric bilinear form* $\widetilde{B} : \mathfrak{k} \times \mathfrak{k} \longrightarrow \mathbb{K}$ *defined by* $\widetilde{B}\mid_{\mathfrak{g}\times\mathfrak{g}}= B, \widetilde{B}(e^*, e) = 1, \widetilde{B}(\mathfrak{g}, e) = \widetilde{B}(\mathfrak{g}, e^*) = \{0\}, \widetilde{B}(e, e) = \widetilde{B}(e^*, e^*) = 0$, *is an invariant scalar product on* $\mathfrak{k}$. *The quadratic Lie superalgebra* $(\mathfrak{k}, \widetilde{B})$ *is called the generalized double extension of* $(\mathfrak{g}, B)$ *by the one-dimensional Lie superalgebra* $(\mathbb{K}e)_{\bar{1}}$ *(by means of* D *and* X_0*).*

Remark 1.4. If $X_0 = 0$ then $\mathfrak{k}$ is the semi-direct product of $\mathfrak{g} \oplus \mathbb{K}e^*$ by $\mathbb{K}e$ by means of the linear map $\Omega : \mathbb{K}e \longrightarrow Der(\mathfrak{g} \oplus \mathbb{K}e^*)$ defined by $\Omega(e) = \widetilde{D}$, where $\widetilde{D} : \mathfrak{g} \oplus \mathbb{K}e^* \longrightarrow \mathfrak{g} \oplus \mathbb{K}e^*$ satisfies $\widetilde{D}(e^*) = 0$ and $\widetilde{D}\mid_{\mathfrak{g}}= D$.

We will use the next results of [2] and [1] to formalize an inductive classification of our class of quadratic Lie superalgebras.

Proposition 1.5. *Suppose that* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *is a* B*-irreducible quadratic Lie superalgebra such that* $\dim\mathfrak{g} > 1$. *If* $\mathfrak{z}(\mathfrak{g})\cap\mathfrak{g}_{\bar{0}} \neq \{0\}$ *then* $(\mathfrak{g}, B)$ *is a double extension of a quadratic Lie superalgebra* $(\mathfrak{h}, \widetilde{B})$ *(*$\dim\mathfrak{h} = \dim\mathfrak{g} - 2$*) by a one-dimensional Lie algebra. If* $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} \neq \{0\}$ *then* $(\mathfrak{g}, B)$ *is a generalized double extension of a quadratic Lie superalgebra* $(\mathfrak{h}, \widetilde{B})$ *(*$\dim\mathfrak{h} = \dim\mathfrak{g} - 2$*) by a one-dimensional Lie superalgebra with even part equal to zero.*

2. Examples of Quadratic Lie superalgebras with reductive even part and the action of the even part on the odd part is not completely reducible

In what follows we shall consider finite dimensional Lie superalgebras over an algebraically closed commutative field $\mathbb{K}$ of characteristic zero and finite dimensional representations. For basic definitions and results about Lie superalgebras see [15]. This work intends to be a continuation of [4].

*Remark that since B is even and $X_0 \in \mathfrak{g}_{\bar{0}}$ then $[e, X] = D(X) - (-1)^x B(X, X_0)e^*$, $\forall_{X\in\mathfrak{g}_x}$.

We start by collecting some examples to illustrate the class of Lie superalgebras studied in this work. In the first examples we will present quadratic Lie superalgebras such that the even part is a reductive Lie algebra and the action of the even part on the odd part is completely reducible. It is well known ([12],[13]) that the classical simple Lie superalgebras $A(m,n)$, $B(m,n)$, $D(m,n)$, $C(n)$, $D(2,1,\alpha)$, $F(4)$, $G(3)$ are examples of quadratic Lie superalgebras such that the even part is a reductive Lie algebra and the action of the even part on the odd part is completely reducible. Another example is obtained considering the two-dimensional abelian Lie superalgebra such that the even part is zero. Let $\{e_1,e_2\}$ be a basis of $\mathfrak{M}$ and $B : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathbb{K}$ be the bilinear form on $\mathfrak{M}$ defined by $B(e_1,e_1) = B(e_2,e_2) = 0$ and $B(e_1,e_2) = -B(e_2,e_1) = 1$. Clearly, B is an invariant scalar product on $\mathfrak{M}$. And, if γ is another invariant scalar product on $\mathfrak{M}$, then there exists $k \in \mathbb{K}$ such that $\gamma = kB$. It is straightforward to ensure that $(\mathfrak{M}, B)$ is a quadratic Lie superalgebra such that $\mathfrak{M}_{\bar{0}}$ is a reductive Lie algebra and the action of $\mathfrak{M}_{\bar{0}}$ on $\mathfrak{M}_{\bar{1}}$ is completely reducible. Other examples and inductive classification of quadratic Lie superalgebras with reductive even part and the action of the even part on the odd part is completely reducible can be found in [4]. Next we will present some examples of quadratic Lie superalgebras with reductive even part and the action of the even part on the odd part is not completely reducible. We will divide the following cases in two classes begining by solvable examples.

Example 2.1. Let $(\mathfrak{M}, B)$ be the two-dimensional abelian quadratic Lie superalgebra with $\mathfrak{M}_{\bar{0}} = \{0\}$ introduced before. Consider the linear map $D : \mathfrak{M} \longrightarrow \mathfrak{M}$ defined by $D(e_1) = 0$ and $D(e_2) = e_1$. It is easy to prove that D is an even skew-supersymmetric superderivation of $(\mathfrak{M}, B)$. Let us consider the double extension of the quadratic Lie superalgebra $(\mathfrak{M}, B)$ by the one-dimensional Lie algebra $(\mathbb{K}e)_{\bar{0}}$ by means of $\psi : \mathbb{K}e \longrightarrow \mathrm{Der}_a(\mathfrak{M}, B)$ defined by $\psi(e) = D$. Computing the map $\varphi : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathbb{K}e^*$ we obtain that $\varphi(e_1,e_1) = \varphi(e_1,e_2) = \varphi(e_2,e_1) = 0$, $\varphi(e_2,e_2) = e^*$. Doing some calculations we obtain that the non-zero values of the multiplication on the Lie superalgebra $\mathfrak{k} = \mathbb{K}e \oplus \mathfrak{M} \oplus \mathbb{K}e^*$ are $[e,e_2] = -[e_2,e] = e_1, [e_2,e_2] = e^*$. We easily see that the double extension $\mathfrak{k} = \mathbb{K}e \oplus \mathfrak{M} \oplus \mathbb{K}e^*$ of $\mathfrak{M}$ by the one-dimensional Lie algebra $\mathbb{K}e$ by means of ψ verifies: $\mathfrak{k}_{\bar{0}} = \mathbb{K}e \oplus \mathbb{K}e^*$ is a reductive Lie algebra, $\mathfrak{k}_{\bar{1}} = \mathfrak{M}$, $[\mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{1}}] = \langle e_1 \rangle$, and $\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{k}_{\bar{1}} = \langle e_1 \rangle$, then $\mathfrak{k}_{\bar{1}}$ is not a completely reducible $\mathfrak{k}_{\bar{0}}$-module.

Example 2.2. Let V be a 4-dimensional vector space and $\{e_1, e_2, e_3, e_4\}$ a basis of V. Consider the antisymmetric bilinear form $B : V \times V \longrightarrow \mathbb{K}$ defined by $B(e_1, e_2) = B(e_3, e_4) = 1$, and the rest are zero. The symplectic Lie algebra $sp(V, B)$ determined by B is the Lie subalgebra of $gl(V)$ formed by $f \in End(V)$ with $M = Mat(f, \{e_1, e_2, e_3, e_4\})$ satisfying $M^tL = -LM$ where $L = Mat(B, \{e_1, e_2, e_3, e_4\})$. Consider the Lie subalgebra of $\mathrm{sp}\,(V, B)$ generated by $e \in End(V)$ such that

$$M = Mat(e, \{e_1, e_2, e_3, e_4\}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

supposing that V is an abelian Lie superalgebra with $V = V_{\bar{1}}$. Let be $\mathfrak{g} = \mathbb{K}e \oplus \mathbb{K}e^* \oplus V$ the double extension of the quadratic Lie superalgebra (V, B) by $\mathbb{K}e$ (by means of the identity $id : \mathbb{K}e \longrightarrow \mathfrak{g}l(V)$) with $\mathfrak{g}_{\bar{0}} = \mathbb{K}e \oplus \mathbb{K}e^*$ a reductive Lie algebra. More, the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}} = V$ is not completely reducible. In fact, $V_1 = \langle e_1, e_3\rangle$ is a $\mathfrak{g}_{\bar{0}}$-submodule of V (more precisely, $[\mathfrak{g}_{\bar{0}}, V_1] = \{0\}$). Let V_2 be a complement of V_1 in V, that is, V_2 is a $\mathfrak{g}_{\bar{0}}$-submodule of V such that $V = V_1 \oplus V_2$. Then $V_2 = \langle X, Y\rangle$, where

$$\begin{aligned} X &= \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4, \text{ with } \alpha, \beta, \gamma, \delta \in \mathbb{K}(\beta \neq 0 \text{ or } \delta \neq 0), \\ Y &= \varepsilon e_1 + \zeta e_2 + \eta e_3 + \theta e_4, \text{ with } \varepsilon, \zeta, \eta, \theta \in \mathbb{K}(\zeta \neq 0 \text{ or } \theta \neq 0). \end{aligned}$$

As $e(X) = \delta e_1 + \beta e_3$ and $e(Y) = \theta e_1 + \zeta e_3$ then $[\mathfrak{g}_{\bar{0}}, V_2] \subseteq V_1$. Since V_2 is a $\mathfrak{g}_{\bar{0}}$-submodule of V, then $[\mathfrak{g}_{\bar{0}}, V_2] \subseteq V_2$, so $[\mathfrak{g}_{\bar{0}}, V_2] \subseteq V_1 \cap V_2 = \{0\}$. Therefore $e(X) = e(Y) = 0$, and hence $\beta = \delta = \zeta = \theta = 0$, which is a contradiction. We conclude that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is not completely reducible.
More generally, let (V, B) be a *symplectic vector space* (this means that, V is a vector space endowed with an antisymmetric bilinear form $B : V \times V \longrightarrow \mathbb{K}$) with even dimension $n = 2p$ such that there exists a basis $\{e_1, \ldots, e_{2p}\}$ of V such that $B(e_{2s+1}, e_{2s+2}) \neq 0$, for all $s \in \{0, \ldots, p-1\}$. Suppose that $p > 2$. Consider $e \in End(V)$ such that $e(e_2) = e_3, e(e_4) = e_1, e(e_1) = e(e_3) = 0, e(e_i) = 0, \forall_{i \in \{5,\ldots,2p\}}$. It is easy to see that $e \in \mathrm{sp}\,(V, B)$. Consider $\mathfrak{g} = \mathbb{K}e \oplus \mathbb{K}e^* \oplus V$ the double extension of the quadratic Lie superalgebra (V, B) by $\mathbb{K}e$ (by means of the identity $id : \mathbb{K}e \longrightarrow \mathfrak{g}l(V)$). Then $\mathfrak{g}_{\bar{0}} = \mathbb{K}e \oplus \mathbb{K}e^*$ is a reductive Lie algebra. More, the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is not completely reducible. In fact, let V_1 be the vector subspace of V generated by $\{e_1, e_3\} \cup \{e_i : 5 \leq i \leq n\}$. As $e(V_1) = \{0\}$ then V_1 is a $\mathfrak{g}_{\bar{0}}$-submodule of

V. Since $e(V) = \langle e_1, e_3 \rangle$ then it is clear that does not exist any $\mathfrak{g}_{\bar{0}}$ - module of V which is a complement of V_1 in V.

To present an example of a non solvable quadratic Lie superalgebra with even part reductive and the action of the even part on the odd part is not completely reducible we need to study first the double extension of the simple complex Lie superalgebra $spl(2,2)/\mathbb{C}I_4$ that is an example of quadratic Lie superalgebra with reductive even part and odd part completely reducible. This example was studied without details in [5].

Example 2.3. Consider the complex Lie superalgebra $\mathfrak{t} =$ spl $(2,2)$ (see preliminaries for definition). Set E_{ij} the matrix with 1 in line i and column j and the rest are zero, for $i, j \in \{1,2,3,4\}$. Denote

$$
\begin{array}{lll}
a_0 = I_4 = \sum_{i=1}^{4} E_{ii} & b_1 = E_{31} & b_5 = E_{13} \\
a_1 = E_{11} + E_{44} & b_2 = E_{32} & b_6 = E_{14} \\
a_2 = E_{22} + E_{44} & b_3 = E_{41} & b_7 = E_{23} \\
a_3 = E_{12} & b_4 = E_{42} & b_8 = E_{24} \\
a_4 = E_{21} & & \\
a_5 = E_{43} & & \\
a_6 = E_{34} & &
\end{array}
$$

Therefore $\mathfrak{t}_{\bar{0}} = \langle a_0, a_1, a_2, a_3, a_4, a_5, a_6 \rangle$ and $\mathfrak{t}_{\bar{1}} = \langle b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 \rangle$. The graded skew-symmetric multiplication of $\mathfrak{t}$ is given for the elements of the basis presented above by

$$
\begin{array}{lll}
[a_1, a_3] = -[a_2, a_3] = a_3 & [a_3, b_1] = -b_2 & [a_6, b_5] = -b_6 \\
[a_1, a_4] = -[a_2, a_4] = -a_4 & [a_3, b_3] = -b_4 & [a_6, b_7] = -b_8 \\
[a_1, a_5] = [a_2, a_5] = a_5 & [a_3, b_7] = b_5 & [b_1, b_5] = a_0 - a_2 \\
[a_1, a_6] = [a_2, a_6] = -a_6 & [a_3, b_8] = b_6 & [b_1, b_6] = a_6 \\
[a_3, a_4] = a_1 - a_2 & [a_4, b_2] = -b_1 & [b_1, b_7] = a_4 \\
[a_5, a_6] = a_1 + a_2 - a_0 & [a_4, b_4] = -b_3 & [b_2, b_5] = a_3 \\
[a_1, b_1] = -b_1 & [a_4, b_5] = b_7 & [b_2, b_7] = a_0 - a_1 \\
[a_1, b_4] = b_4 & [a_4, b_6] = b_8 & [b_2, b_8] = a_6 \\
[a_1, b_5] = b_5 & [a_5, b_1] = b_3 & [b_3, b_5] = a_5 \\
[a_1, b_8] = -b_8 & [a_5, b_2] = b_4 & [b_3, b_6] = a_1 \\
[a_2, b_2] = -b_2 & [a_5, b_6] = -b_5 & [b_3, b_8] = a_4 \\
[a_2, b_3] = b_3 & [a_5, b_8] = -b_7 & [b_4, b_6] = a_3 \\
[a_2, b_6] = -b_6 & [a_6, b_3] = b_1 & [b_4, b_7] = a_5 \\
[a_2, b_7] = b_7 & [a_6, b_4] = b_2 & [b_4, b_8] = a_2
\end{array}
$$

and the others are zero. Remark that

$\mathfrak{t}_{\bar{0}} = \langle H_1 = a_1 - a_2, F_1 = a_3, G_1 = a_4, H_2 = a_1 + a_2 - a_0, F_2 = a_5, G_2 = a_6, I_4\rangle$

and the non-zero values of the multiplication are:

$$\begin{array}{ll} [H_1, F_1] = 2F_1 & [H_2, F_2] = 2F_2 \\ [H_1, G_1] = -2G_1 & [H_2, G_2] = -2G_2 \\ [F_1, G_1] = H_1 & [F_2, G_2] = H_2. \end{array}$$

Therefore $\mathfrak{t}_{\bar{0}} = I \oplus J \oplus L$, where $I = \langle H_1, F_1, G_1\rangle$, $J = \langle H_2, F_2, G_2\rangle$, and $L = \mathbb{C}I_4$ are ideals of $\mathfrak{t}_{\bar{0}}$ such that I and J are isomorphic to sl (2) (meaning that $\mathfrak{t}_{\bar{0}} \simeq$ sl (2) $\oplus$ sl (2) $\oplus\, \mathbb{C}$). It is easy to verify that $\mathfrak{t}_{\bar{1}} = \mathfrak{t}_1 \oplus \mathfrak{t}_{-1}$, where $\mathfrak{t}_1 = R_1 \oplus R_2$ and $\mathfrak{t}_{-1} = R_3 \oplus R_4$, with $R_1 = \langle -b_6, b_8\rangle$, $R_2 = \langle -b_5, b_7\rangle$, $R_3 = \langle b_1, b_2\rangle$, and $R_4 = \langle b_3, b_4\rangle$. We have that $\mathfrak{t}_{-1}$ and $\mathfrak{t}_1$ are $I \simeq$ sl (2)-modules (and $J \simeq$ sl (2)-modules) isomorphic to $D(\frac{1}{2}) \oplus D(\frac{1}{2})$ (where $D(\dfrac{1}{2})$ denotes the two-dimensional irreducible representation of sl (2)). The Lie superalgebra $\mathfrak{t}$ verify $\mathfrak{z}(\mathfrak{t}) = \langle I_4\rangle$ and $[\mathfrak{t}_1, \mathfrak{t}_1] = [\mathfrak{t}_{-1}, \mathfrak{t}_{-1}] = \{0\}$. If we consider the bilinear form $S : \mathfrak{t} \times \mathfrak{t} \longrightarrow \mathbb{C}$ defined by

$$S(a, b) \;=\; \text{str } (ab), \forall_{a,b \in \mathfrak{t}},$$

then it is easy to check that

$$S(E_{ij}, E_{kl}) = 0, \text{ if either } j \neq k \text{ or } i \neq l$$

$$S(E_{ij}, E_{ji}) = \begin{cases} 1 & \text{if } 1 \leq i \leq 2 \\ -1 & \text{if } 3 \leq i \leq 4 \end{cases}$$

$$S(a, b) = (-1)^{\alpha\beta} S(b, a), \forall_{a \in \mathfrak{t}_\alpha, b \in \mathfrak{t}_\beta}$$

$$S([a, b], c) = S(a, [b, c]), \forall_{a,b,c \in \mathfrak{t}}.$$

We can verify that $\mathfrak{t}^{\perp S} = \langle a_0\rangle$ and $S(\mathfrak{t}_1, \mathfrak{t}_1) = S(\mathfrak{t}_{-1}, \mathfrak{t}_{-1}) = \{0\}$. Consequently, $(\mathfrak{g} = \mathfrak{t}/\mathfrak{t}^{\perp S}, B)$ is a quadratic Lie superalgebra, where $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ is defined by

$$B(\overline{a}, \overline{b}) \;=\; S(a, b), \forall_{a,b \in \mathfrak{t}}.$$

Remark that $\mathfrak{g} = \mathfrak{t}/\mathfrak{t}^{\perp S} =$ spl $(2, 2)/\mathbb{C}I_4$ is a simple Lie superalgebra with the Killing form equal to zero. Let us analyse the simple complex quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$. It follows that $\mathfrak{g}_{\bar{0}} = s_1 \oplus s_2$, where s_1 and

s_2 are two simple ideals of $\mathfrak{g}_{\bar{0}}$ such that $s_i =$ sl (2) $(i = 1, 2)$. Further, a basis of $\mathfrak{g}_{\bar{0}}$ is $\{e_i = \overline{a_i}, 1 \leq i \leq 6\}$ and a basis of $\mathfrak{g}_{\bar{1}}$ is $\{f_i = \overline{b_i}, 1 \leq i \leq 8\}$. We have, $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1} = \langle f_1, f_2, f_3, f_4\rangle$ and $\mathfrak{g}_1 = \langle f_5, f_6, f_7, f_8\rangle$ are s_i-submodules of $\mathfrak{g}_{\bar{1}}$ isomorphic to $D(\frac{1}{2}) \oplus D(\frac{1}{2})$, for $i \in \{1, 2\}$. Moreover, $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \{0\}$ and $B(\mathfrak{g}_1, \mathfrak{g}_1) = B(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) = \{0\}$. The non-zero values of the graded skew-symmetric multiplication of $\mathfrak{g}$ for the elements of the basis $\{e_1, \ldots, e_6\ , f_1, \ldots, f_8\}$ are

$$\begin{array}{lll}
[e_1, e_3] = -[e_2, e_3] = e_3 & [e_3, f_1] = -f_2 & [e_6, f_5] = -f_6 \\
[e_1, e_4] = -[e_2, e_4] = -e_4 & [e_3, f_3] = -f_4 & [e_6, f_7] = -f_8 \\
[e_1, e_5] = [e_2, e_5] = e_5 & [e_3, f_7] = f_5 & [f_1, f_5] = -e_2 \\
[e_1, e_6] = [e_2, e_6] = -e_6 & [e_3, f_8] = f_6 & [f_1, f_6] = e_6 \\
[e_3, e_4] = e_1 - e_2 & [e_4, f_2] = -f_1 & [f_1, f_7] = e_4 \\
[e_5, e_6] = e_1 + e_2 & [e_4, f_4] = -f_3 & [f_2, f_5] = e_3 \\
[e_1, f_1] = -f_1 & [e_4, f_5] = f_7 & [f_2, f_7] = -e_1 \\
[e_1, f_4] = f_4 & [e_4, f_6] = f_8 & [f_2, f_8] = e_6 \\
[e_1, f_5] = f_5 & [e_5, f_1] = f_3 & [f_3, f_5] = e_5 \\
[e_1, f_8] = -f_8 & [e_5, f_2] = f_4 & [f_3, f_6] = e_1 \\
[e_2, f_2] = -f_2 & [e_5, f_6] = -f_5 & [f_3, f_8] = e_4 \\
[e_2, f_3] = f_3 & [e_5, f_8] = -f_7 & [f_4, f_6] = e_3 \\
[e_2, f_6] = -f_6 & [e_6, f_3] = f_1 & [f_4, f_7] = e_5 \\
[e_2, f_7] = f_7 & [e_6, f_4] = f_2 & [f_4, f_8] = e_2
\end{array}$$

On the basis of $\mathfrak{g}$, $\{e_1, \ldots, e_6\ , f_1, \ldots, f_8\}$, the invariant scalar product B is defined by

$$\begin{array}{ll}
B(e_1, e_2) = -1 & B(f_1, f_5) = -1 \\
B(e_3, e_4) = 1 & B(f_2, f_7) = -1 \\
B(e_5, e_6) = -1 & B(f_3, f_6) = -1 \\
 & B(f_4, f_8) = -1,
\end{array}$$

and the rest are zero. Let us denote $\mathbb{C}e$ the one-dimensional Lie algebra. Consider the homomorphism of Lie superalgebras $\psi : \mathbb{C}e \longrightarrow \mathrm{Der}_a(\mathfrak{g}, B) \subseteq \mathrm{Der}(\mathfrak{g})$ determined by $(\psi(e)) \mid_{\mathfrak{g}_{\bar{0}}} = 0$, $(\psi(e)) \mid_{\mathfrak{g}_1} = Id_{\mathfrak{g}_1}$, and $(\psi(e)) \mid_{\mathfrak{g}_{-1}} = -Id_{\mathfrak{g}_{-1}}$. It is elementary to see that the double extension $(\mathfrak{k} = \mathbb{C}e \oplus \mathfrak{g} \oplus \mathbb{C}e^*, \widetilde{B})$ of $(\mathfrak{g}, B)$ by $\mathbb{C}e$ (by means of ψ) verifies: $\mathfrak{z}(\mathfrak{k}) = \mathbb{C}e^*$, $[e, \mathfrak{k}_{\bar{0}}] = \{0\}$, $\mathfrak{k}_{\bar{0}} = \mathbb{C}e \oplus \mathbb{C}e^* \oplus \mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra (in fact, $\mathfrak{g}_{\bar{0}}$ is the greatest semisimple ideal of $\mathfrak{k}_{\bar{0}}$, $\mathfrak{z}(\mathfrak{k}_{\bar{0}}) = \mathbb{C}e \oplus \mathbb{C}e^*$), and $\mathfrak{k}_{\bar{1}} = \mathfrak{g}_{\bar{1}}$ is a semisimple $\mathfrak{k}_{\bar{0}}$-module.

We will now show that in the complex Lie superalgebra defined before every odd skew symmetric superderivation on $(\mathfrak{k}, \widetilde{B})$ is inner. More precisely, set $D(e) = \sum_{i=1}^{8} \alpha_i f_i$ ($\alpha_i \in \mathbb{C}$, $i \in \{1, \ldots, 8\}$). Then $D = \mathrm{ad}\, A$, which is $D(X) = [A, X], \forall_{X \in \mathfrak{k}}$, where $A = \sum_{i=1}^{4} \alpha_i f_i - \sum_{i=5}^{8} \alpha_i f_i$. In fact, since $\mathfrak{z}(\mathfrak{k}) = \mathbb{C}e^*$ and D is an odd superderivation on $\mathfrak{k}$, we have $[D(e^*), X] = 0, \forall_{X \in \mathfrak{k}}$. Therefore $D(e^*) \in \mathfrak{z}(\mathfrak{k})$. As D is odd we conclude that $D(e^*) = 0$. Since $[e, \mathfrak{k}_{\bar{0}}] = \{0\}$ and D is an odd superderivation on $\mathfrak{k}$, we infer that $D([e, f_i]) = [D(e), f_i], \forall_{i \in \{1,\ldots,8\}}$. In particular, if $i \in \{1, 2, 3, 4\}$ we have $D(f_i) = [-D(e), f_i]$, while, if $i \in \{5, 6, 7, 8\}$ we have $D(f_i) = [D(e), f_i]$. Taking in account that $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \{0\}$, we conclude that $D(f_i) = [A, f_i], \forall_{i \in \{1,\ldots,8\}}$, therefore $D(X) = [A, X], \forall_{X \in \mathfrak{k}_{\bar{1}}}$. It remains to show that the last expression is valid for elements in $\mathfrak{k}_{\bar{0}}$. As D is an odd skew-supersymmetric superderivation on the quadratic Lie superalgebra $(\mathfrak{k}, \widetilde{B})$ it comes that

$$\widetilde{B}(D(X), Y) = -\widetilde{B}(X, [A, Y]), \forall_{X \in \mathfrak{k}_{\bar{0}}, Y \in \mathfrak{k}_{\bar{1}}}.$$

From the $\widetilde{B}$-invariance we obtain that $\widetilde{B}(D(X) - [A, X], Y) = 0, \forall_{X \in \mathfrak{k}_{\bar{0}}, Y \in \mathfrak{k}_{\bar{1}}}$. Since $\widetilde{B} \mid_{\mathfrak{k}_{\bar{1}} \times \mathfrak{k}_{\bar{1}}}$ is non-degenerate, $D(X) = [A, X], \forall_{X \in \mathfrak{k}_{\bar{0}}}$, and consequently $D = \mathrm{ad}\, A$ as required.

Now we can present a non solvable example of irreductive quadratic Lie superalgebra with reductive even part and the action of the even part on the odd part is not completely reducible.

Example 2.4. Let $(\mathfrak{k} = \mathbb{C}e \oplus \mathfrak{g} \oplus \mathbb{C}e^*, \widetilde{B})$ be the preceeding complex quadratic Lie superalgebra. We consider the inner odd skew-supersymmetric superderivation D of $(\mathfrak{k}, \widetilde{B})$ defined by $D(X) = [-f_3 - \frac{1}{2} f_6 + f_7, X], \forall_{X \in \mathfrak{k}}$. Explicitly, D is given for elements of the basis $\{e, e^*, e_1, \ldots, e_6\ , f_1, \ldots, f_8\}$ by

$$\begin{array}{ll}
D(e) = -f_3 + \frac{1}{2} f_6 - f_7 & D(f_1) = e_4 - \frac{1}{2} e_6 \\
D(e_2) = f_3 - \frac{1}{2} f_6 - f_7 & D(f_2) = +e^* - e_1 \\
D(e_3) = -f_4 - f_5 & D(f_4) = -\frac{1}{2} e_3 + e_5 \\
D(e_4) = \frac{1}{2} f_8 & D(f_5) = -e_5 \\
D(e_5) = -\frac{1}{2} f_5 & D(f_6) = 2D(f_3) = -(e^* + e_1) \\
D(e_6) = f_1 + f_8 & D(f_8) = -e_4,
\end{array}$$

and the rest are zero. Doing straightforward calculations, it is elementary to prove that $\widetilde{B}(e_1,e_1) = 0, D^2 = \dfrac{1}{2}[e_1,.]_{\mathfrak{k}}$. Consider the one-dimensional abelian Lie superalgebra $\mathbb{C}i$ with even part zero. Now, we are interested in the generalized double extension $(\mathfrak{h} = \mathbb{C}i \oplus \mathfrak{k} \oplus \mathbb{C}i^*, \overline{B})$ of $(\mathfrak{k}, \widetilde{B})$ by $\mathbb{C}i$ (by means of D and e_1). The graded skew-symmetric multiplication is defined by

$$\begin{aligned}
[i,i] &= e_1 \\
[i,X] &= D(X) - \widetilde{B}(X,e_1)i^*, \forall_{X\in\mathfrak{k}}, \\
[X,Y] &= [X,Y]_{\mathfrak{k}} - \widetilde{B}(D(X),Y)i^*, \forall_{X,Y\in\mathfrak{k}}, \\
[i^*,\mathfrak{h}] &= \{0\}
\end{aligned}$$

Further, the invariant scalar product $\overline{B}$ on $\mathfrak{h}$ is defined by

$$\begin{array}{ll}
\overline{B}(e,e^*) = 1 & \overline{B}(f_1,f_5) = -1 \\
\overline{B}(e_1,e_2) = -1 & \overline{B}(f_2,f_7) = -1 \\
\overline{B}(e_3,e_4) = 1 & \overline{B}(f_3,f_6) = -1 \\
\overline{B}(e_5,e_6) = -1 & \overline{B}(f_4,f_8) = -1 \\
 & \overline{B}(i,i^*) = -1,
\end{array}$$

and the rest are zero. It is easy to see that the generalized double extension $(\mathfrak{h} = \mathbb{C}i \oplus \mathfrak{k} \oplus \mathbb{C}i^*, \overline{B})$ of $(\mathfrak{k}, \widetilde{B})$ by $\mathbb{C}i$ (by means of D and e_1) is a quadratic Lie superalgebra such that $\mathfrak{h}_{\bar{0}} = \mathfrak{k}_{\bar{0}}$ is a reductive Lie algebra. Now we will show that the $\mathfrak{h}_{\bar{0}}$-module $\mathfrak{h}_{\bar{1}} = \mathfrak{k}_{\bar{1}} \oplus \mathbb{C}i \oplus \mathbb{C}i^*$ is not semisimple. In fact, it is clear that $\mathfrak{k}_{\bar{1}} \oplus \mathbb{C}i^*$ is a $\mathfrak{h}_{\bar{0}}$-submodule of $\mathfrak{h}_{\bar{1}}$. If we suppose that the $\mathfrak{h}_{\bar{0}}$-module $\mathfrak{h}_{\bar{1}}$ is semisimple then there exists M a $\mathfrak{h}_{\bar{0}}$-submodule of $\mathfrak{h}_{\bar{1}}$ such that $\mathfrak{h}_{\bar{1}} = (\mathfrak{k}_{\bar{1}} \oplus \mathbb{C}i^*) \oplus M$. Then $M = \mathbb{C}Z$, where

$$Z = \Big(\sum_{j=1}^{8} \lambda_j f_j\Big) + \alpha i + \beta i^*,$$

with $\lambda_j \in \mathbb{C}$ (for all $j \in \{1,\ldots,8\}$), $\beta \in \mathbb{C}$, and $\alpha \in \mathbb{C} \setminus \{0\}$. Since $[\mathfrak{h}_{\bar{0}}, M] \subseteq M$ then there exists $\gamma \in \mathbb{C}$ such that $\gamma Z = [e,Z]$, which is

$$\gamma Z = \Big(\sum_{j=1}^{8} \lambda_j [e,f_j]\Big) + \alpha[e,i].$$

It follows that

$$\gamma\Big(\big(\sum_{j=1}^{8}\lambda_j f_j\big)+\alpha i+\beta i^*\Big) = -\lambda_1 f_1-\lambda_2 f_2+(\alpha-\lambda_3)f_3-\lambda_4 f_4+\lambda_5 f_5 +(\lambda_6-\frac{1}{2}\alpha)f_6+(\alpha+\lambda_7)f_7+ \lambda_8 f_8+(\lambda_2-\frac{1}{2}\lambda_3-\lambda_6)i^*.$$

As $\alpha \neq 0$ then $\gamma = 0$, which implies that

$$\begin{cases} \lambda_1=\lambda_2=\lambda_4=\lambda_5=\lambda_8=0 \\ \alpha=\lambda_3=2\lambda_6=-\lambda_7 \\ -\frac{1}{2}\lambda_3-\lambda_6=0 \end{cases}$$

From the second and third conditions we get that $\lambda_6 = 0$. Again, by the second condition we infer that $\alpha = 0$, which contradicts the fact that $\alpha \neq 0$. We conclude that $\mathfrak{h}_{\bar{0}}$-module $\mathfrak{h}_{\bar{1}}$ is not semisimple.

Remark 2.5. The preceeding Example 2.4 shows that a generalized double extension of a quadratic Lie superalgebra $(\mathfrak{g}, B)$ by a one-dimensional Lie superalgebra by means an inner odd superderivation is not always the orthogonal sum of $(\mathfrak{g}, B)$ and the two abelian dimensional Lie superalgebra $\mathfrak{M} = \mathfrak{M}_{\bar{1}}$. This is an important phenomenon in generalized double extension, since it is well kown that in the case of the double extension of a quadratic Lie superalgebra $(\mathfrak{g}, B)$ by the one-dimensional Lie algebra by means of an inner even superderivation, it is the orthogonal direct sum of $(\mathfrak{g}, B)$ and the abelian two-dimensional Lie algebra.

3. Structure and inductive classification of quadratic Lie superalgebras with reductive even part

In this section, we will present our results of the quadratic Lie superalgebras such that the even part is a reductive Lie algebra. We notice that our conclusions differ from those obtained in [4] by S. Benayadi. We will make precisely the differences, by recalling throughout the section some Benayadi's results. In [4], it is used the following result to get an inductive classification of quadratic Lie superalgebras such that the even part is a reductive

Lie algebra and the action of the even part on the odd part is completely reducible

Lemma 3.1. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a* B*-irreducible quadratic Lie superalgebra such that* $\mathfrak{g}_{\bar{0}} \neq \{0\}$ *and the action of* $\mathfrak{g}_{\bar{0}}$ *on* $\mathfrak{g}_{\bar{1}}$ *is completely reducible. Then* $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$ *and* $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} = \mathfrak{z}(\mathfrak{g})$.

However, there is no analog to the preceeding lemma in our context. Indeed, for quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is not necessarily completely reducible, we can not ensure neither that $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$ nor that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} = \mathfrak{z}(\mathfrak{g})$, as we see by the Example 2.1. Moreover, it is proved in [6] for B-irreducible quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible the following result

Theorem 3.2. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a* B*-irreducible quadratic Lie superalgebra such that the action of* $\mathfrak{g}_{\bar{0}}$ *on* $\mathfrak{g}_{\bar{1}}$ *is completely reducible then* $soc(\mathfrak{g}) \subseteq soc(\mathfrak{g}_{\bar{0}})$.

Recall that a graded ideal of a Lie superalgebra $\mathfrak{g}$ is *minimal* if is not trivial and doesn't contain any non trivial graded ideal of $\mathfrak{g}$. The *socle* of $\mathfrak{g}$ (denoted by $soc(\mathfrak{g})$) is the sum of all minimal graded ideals of $\mathfrak{g}$. If $\mathfrak{g}$ is simple or one dimensional, we have $soc(\mathfrak{g}) = \{0\}$. Instead, for B-irreducible quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is not necessarily completely reducible we have the following result

Theorem 3.3. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a* B*-irreducible quadratic Lie superalgebra that is neither the one-dimensional Lie algebra nor a simple Lie superalgebra. If the Lie algebra* $\mathfrak{g}_{\bar{0}}$ *is reductive then* $soc(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

Proof: Remark that the hypothesis of this theorem imply that $\mathfrak{g}_{\bar{1}} \neq \{0\}$. In the first part of the proof we show that $\mathfrak{z}(\mathfrak{g}) \subseteq soc(\mathfrak{g})$. Choose a basis $\{X_1, \ldots, X_n\}$ of the vector space $\mathfrak{z}(\mathfrak{g})$. This means, $\mathfrak{z}(\mathfrak{g}) = \oplus_{k=1}^{n} \mathbb{K}X_k$. Moreover, for all $k \in \{1, \ldots, n\}$, $\mathbb{K}X_k$ is a minimal graded ideal of $\mathfrak{g}$. Consequently $\mathfrak{z}(\mathfrak{g}) = \oplus_{k=1}^{n} \mathbb{K}X_k \subseteq soc(\mathfrak{g})$.

Concerning the opposite inclusion, we take I a minimal graded ideal of $\mathfrak{g}$. Consequently $\mathfrak{g}/I^{\perp}$ is a simple or a one-dimensional Lie superalgebra. Suppose that $\mathfrak{g}/I^{\perp}$ is a one-dimensional Lie superalgebra. Invoking Proposition 2.6 of [6] we observe that if $\mathfrak{g}/I^{\perp}$ is quadratic (respectively non-quadratic) one-dimensional Lie superalgebra then there exists $X \in (\mathfrak{z}(\mathfrak{g}))_{\bar{0}}$ (respectively $X \in (\mathfrak{z}(\mathfrak{g}))_{\bar{1}}$) such that $I = \mathbb{K}X$. Thus I is a one-dimensional graded ideal of $\mathfrak{g}$ contained in its centre, and in this case we can conclude that $I \subseteq \mathfrak{z}(\mathfrak{g})$. Now, let us assume that $\mathfrak{g}/I^\perp$ is simple Lie superalgebra. By hypothesis $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, which means that $\mathfrak{g}_{\bar{0}} = s \oplus \mathfrak{z}(\mathfrak{g}_{\bar{0}})$, where s is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$. The minimal graded ideal I of $\mathfrak{g}$ can be written as $I = I_{\bar{0}} \oplus I_{\bar{1}}$, where $I_{\bar{0}} = I \cap \mathfrak{g}_{\bar{0}}$ and $I_{\bar{1}} = I \cap \mathfrak{g}_{\bar{1}}$. We infer that $I_{\bar{0}}$ is a reductive ideal of $\mathfrak{g}_{\bar{0}}$ and we may write $I_{\bar{0}} = I \cap s \oplus I \cap \mathfrak{z}(\mathfrak{g}_{\bar{0}})$. From the B-irreducibility of $\mathfrak{g}$ we get that $I \cap I^\perp \neq \{0\}$. Consequently $I \subseteq I^\perp$ and $[I, I] = \{0\}$. In particular $[I_{\bar{0}}, I_{\bar{0}}] = \{0\}$, and $I_{\bar{0}} = I \cap \mathfrak{z}(\mathfrak{g}_{\bar{0}})$. Therefore $I = I \cap \mathfrak{z}(\mathfrak{g}_{\bar{0}}) \oplus I \cap \mathfrak{g}_{\bar{1}}$. Recall that if s is semisimple then $s = [s, s]$ and using invariance of B we infer that $B(s, \mathfrak{z}(\mathfrak{g}_{\bar{0}})) = \{0\}$. On the other hand, since B is even and $s \subseteq \mathfrak{g}_{\bar{0}}$ we conclude that $B(s, \mathfrak{g}_{\bar{1}}) = \{0\}$. Therefore $s \subseteq I^\perp$. Consequently $(\mathfrak{g}/I^\perp)_{\bar{0}} \simeq \mathfrak{z}(\mathfrak{g}_{\bar{0}})/(I^\perp \cap \mathfrak{z}(\mathfrak{g}_{\bar{0}}))$. As a quadratic Lie superalgebra is solvable iff the even part is a solvable Lie algebra ([15],[13]), we get that $\mathfrak{g}/I^\perp$ is solvable. Since by hypothesis $\mathfrak{g}/I^\perp$ is simple, then $\mathfrak{g}/I^\perp = \{0\}$, which contradicts the fact that $I \neq \{0\}$. Consequently $\mathfrak{g}/I^\perp$ is a one-dimensional Lie superalgebra, completing the proof. ■

Next we will characterize the socle of a quadratic Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. In fact, this is the analog to the result of S. Benayadi [6] in the framework of quadratic Lie superalgebras such that the even part is a reductive Lie algebra and the action of the even part on the odd part is completely reducible. In [6], it is used Theorem 3.2 to prove the result, in our more general context we will apply Theorem 3.3

Corollary 3.4. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a quadratic Lie superalgebra such that* $\mathfrak{g}_{\bar{0}}$ *is a reductive Lie algebra. If* $soc(\mathfrak{g}) \neq \{0\}$*, then* $soc(\mathfrak{g}) = s \oplus \mathfrak{z}(\mathfrak{g})$*, where* s *is the greatest semisimple graded ideal of* $\mathfrak{g}$*.*

Proof: By Theorem 1.1 of [6] we can write $\mathfrak{g} = \bigoplus_{k=1}^{l} \mathfrak{g}_k$, where $\{\mathfrak{g}_k | 1 \leq k \leq l\}$ is a set of B-irreducible graded ideals of $\mathfrak{g}$ with $B(\mathfrak{g}_k, \mathfrak{g}_{k'}) = \{0\}$, for every $k, k' \in \{1, \ldots, l\}$ and $k \neq k'$. We may assume that there exist $m, n \in \mathbb{N}$ such that $\mathfrak{g}_k$ is one-dimensional Lie algebra, for all $k \in \{1, \ldots, m\}$, $\mathfrak{g}_k$ is simple Lie superalgebra, when $k \in \{m+1, \ldots, n\}$ and $\mathfrak{g}_k$ is neither the one-dimensional Lie algebra nor a simple Lie superalgebra, whereby $k \in \{n+1, \ldots, l\}$. Let s be the greatest semisimple graded ideal of $\mathfrak{g}$. And since $\mathfrak{g}$ is a quadratic Lie superalgebra then $s = [s, s]$. Consequently $s = [s, \mathfrak{g}] = \mathfrak{g} = \bigoplus_{k=1}^{l} s \cap \mathfrak{g}_k$, with $s \cap \mathfrak{g}_k = \{0\}$, for all $k \in \{1, \ldots, m\}$, $s \cap \mathfrak{g}_k = \mathfrak{g}_k$, when $k \in \{m+1, \ldots, n\}$, and $s \cap \mathfrak{g}_k = \{0\}$, where $k \in \{n+1, \ldots, l\}$. The last part follows because $\mathfrak{g}_k$ is B-irreducible not simple. We conclude that $s = \bigoplus_{k=m+1}^{n} \mathfrak{g}_k$. On the other hand, it is clear that $\mathfrak{z}(\mathfrak{g}) = \left(\bigoplus_{k=1}^{m} \mathfrak{g}_k\right) \oplus \left(\bigoplus_{k=n+1}^{l} \mathfrak{z}(\mathfrak{g}_k)\right)$. In view of Lemma 3.1 of [6] and Theorem 3.3, we have $soc(\mathfrak{g}) = s \oplus \mathfrak{z}(\mathfrak{g})$. ■

Corollary 3.5. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if $\mathfrak{z}(\mathfrak{g}) = \{0\}$.*

Proof: If $\mathfrak{g}$ is semisimple it is immediate that $\mathfrak{z}(\mathfrak{g}) = \{0\}$. Now let us assume that $\mathfrak{z}(\mathfrak{g}) = \{0\}$.
First case: Suppose that $\mathfrak{g}$ is B-irreducible. If $\mathfrak{g}_{\bar{1}} = \{0\}$ then $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$ is a simple Lie algebra . Now suppose that $\mathfrak{g}_{\bar{1}} \neq \{0\}$. Consider that $\mathfrak{g}$ is not simple, which means that $\mathfrak{g}$ contains non trivial graded ideals. Then take I a minimal graded ideal of $\mathfrak{g}$. Since $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, we invoke Theorem 3.3 and hypothesis $\mathfrak{z}(\mathfrak{g}) = \{0\}$ to infer that $soc(\mathfrak{g}) = \{0\}$, which contradicts the fact that $I \neq \{0\}$. Therefore the quadratic Lie superalgebra $\mathfrak{g}$ is simple.

Second case: Suppose that $\mathfrak{g}$ is not B-irreducible. Then $\mathfrak{g} = \bigoplus_{k=1}^{l} \mathfrak{g}_k$, where $\mathfrak{g}_k$ is a B-irreducible graded ideal of $\mathfrak{g}$, with $k \in \{1, \ldots, l\}$, such that $B(\mathfrak{g}_k, \mathfrak{g}_{k'}) = \{0\}$, for all $k, k' \in \{1, \ldots, l\}$ and $k \neq k'$. It is clear that $(\mathfrak{g}_k)_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{z}(\mathfrak{g}_k) = \{0\}$, whenever $k \in \{1, \ldots, l\}$. For all $k \in \{1, \ldots, lq(\mathfrak{g})\}$, since $\mathfrak{g}_k$ is B-irreducible then it is also simple. We conclude that $\mathfrak{g}$ is semisimple, which completes the proof. ■

Remark 3.6. Amongst other things, S. Benayadi used explicitly in [4] the definition of completely reducibility and Lemma 3.1 (more precisely, the fact that $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$) to prove the last result in the particular case where $\mathfrak{g}$ is B-irreducible and the action of the even part on the odd part is completely reducible (Theorem 2.1 of [4]). As we have showed before we could not use the same arguments in our case, instead we have applyed Theorem 3.3.

Now let us recall Theorem 2.3 of [4] where an inductive classification of quadratic Lie superalgebras with even part reductive and the action of the even part on the odd part is completely reducible, is presented. Consider $\mathfrak{U}$ the set formed by $\{0\}$, the basic classical Lie superalgebras, the one-dimensional Lie algebra, and the abelian quadratic two-dimensional Lie superalgebra $\mathfrak{M}$. We can formalize the result of S. Benayadi as follows [4]

Theorem 3.7. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible. Then either $\mathfrak{g}$ is an element of $\mathfrak{U}$, or $\mathfrak{g}$ is obtained by a sequence of elementary double extensions by the one-dimensional Lie algebra, and/or by orthogonal direct sums of quadratic Lie superalgebras from a finite number of elements of $\mathfrak{U}$.*

Finally, we will present an inductive classification of quadratic Lie superalgebra such that the even part is a reductive Lie algebra. Clearly, we improve the S. Benayadi's result, since we work with a larger class of Lie superalgebras. In the proof of the last theorem, S. Benayadi applied the Lemma 3.1 to solve the unique case: $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \neq \{0\}$. However, in our framework of quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is not necessarily completely reducible, we will have to consider two cases. So we will use an entirely new approach, in fact, we will use the generalized double extension of quadratic Lie superalgebras. We denote by $\mathfrak{W}$ the set formed by $\{0\}$, basic classical Lie superalgebras and one-dimensional Lie algebra.

Theorem 3.8. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Then either $\mathfrak{g}$ is an element of $\mathfrak{W}$, or $\mathfrak{g}$ is obtained by a sequence of double extensions by the one-dimensional Lie algebra, or generalized double extensions by the one-dimensional Lie superalgebra, and/or by orthogonal direct sums of quadratic Lie superalgebras from a finite number of elements of $\mathfrak{W}$.*

Proof: Our proof is by induction on the dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 0$ then $\mathfrak{g} = \{0\} \in \mathfrak{W}$. If $\dim \mathfrak{g} = 1$. As dimension of $\mathfrak{g}_{\bar{1}}$ is even then $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$ is the one-dimensional Lie algebra. We assume that the theorem is true if $\dim \mathfrak{g} < n$, where $n \geq 2$. Suppose that $\dim \mathfrak{g} = n$.
First case: Suppose that $\mathfrak{g}$ is B-irreducible. If $\mathfrak{z}(\mathfrak{g}) = \{0\}$ then according to Corollary 3.5 we conclude that $\mathfrak{g}$ is simple, and so it is a basic classical Lie

superalgebra. While, if $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ then $\mathfrak{g}$ is a double extension of a quadratic Lie superalgebra $\mathfrak{h}$ (such that $\dim \mathfrak{h} = \dim \mathfrak{g} - 2 < n$) by the one-dimensional Lie algebra, or generalized double extension of a quadratic Lie superalgebra $\mathfrak{h}$ ($\dim \mathfrak{h} = \dim \mathfrak{g} - 2 < n$) by the one-dimensional Lie superalgebra. Now we apply the induction hypothesis to $\mathfrak{h}$ and we obtain the result for $\mathfrak{g}$.

Second case: Now we consider that $\mathfrak{g}$ is not B-irreducible. Then $\mathfrak{g} = \bigoplus_{k=1}^{l} \mathfrak{g}_k$, where $\{\mathfrak{g}_k | 1 \leq k \leq l\}$ is a set of B-irreducible graded ideals of $\mathfrak{g}$ such that $B(\mathfrak{g}_k, \mathfrak{g}_{k'}) = \{0\}$, for all $k, k' \in \{1, \ldots, l\}$ and $k \neq k'$. It is clear that $(\mathfrak{g}_k)_{\bar{0}}$ is a reductive Lie algebra and $\dim \mathfrak{g}_k < n$, for all $k \in \{1, \ldots, l\}$. We apply the hypothesis to each $\mathfrak{g}_k$ with $k \in \{1, \ldots, l\}$. The theorem is completely proved. ∎

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