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THE HANKEL PENCIL CONJECTURE

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ABSTRACT: The Toeplitz pencil conjecture stated in [SS1] and [SS2] is equivalent to a conjecture for $n \times n$ Hankel pencils of the form $H_n(x) = (c_{i+j-n+1})$, where $c_0 = x$ is an indeterminate, $c_l = 0$ for $l < 0$, and $c_l \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, for $l \geq 1$. In this paper it is shown to be implied by another conjecture, we call root conjecture. This latter claims for a certain pair $(m_{nn}, m_{n-1,n})$ of submaximal minors of certain special $H_n(x)$ that, viewed as elements of $\mathbb{C}[x]$, there holds that $\text{roots}(m_{nn}) \subseteq \text{roots}(m_{n-1,n})$ implies $\text{roots}(m_{n-1,n}) = \{1\}$. We give explicit formulae in the c_i for these minors and show the root conjecture for minors $m_{nn}, m_{n-1,n}$ of degree ≤ 6. This implies the Hankel Pencil conjecture for matrices up to size 8×8. Main tools involved are a partial parametrization of the set of solutions of systems of polynomial equations that are both homogeneous and index sum homogeneous, and use of the Sylvester identity for matrices.

KEYWORDS: Hankel matrices, Toeplitz matrices, systems of polynomial equations, Sylvester identity.
AMS SUBJECT CLASSIFICATION (2000): 15A22 Matrix Pencils, 15A15 Determinants.

1. Introduction

A 1981 conjecture by Bumby, Sontag, Sussmann, and Vasconcelos, [BSSV], says that the polynomial ring $\mathbb{C}[y]$ is a so called Feedback Cyclization (FC) ring. Two exceptional cases of that conjecture remained unsolved. More background on this material is found in a 2004 paper by Schmale and Sharma [SS2]. These authors showed that one of the cases referred would follow from the truth of a deceptively simple looking conjecture they formulated for Toeplitz matrices.

Recall that Toeplitz and Hankel matrices correspond to each other simply by arranging columns in reverse order; but Hankel matrices have the additional convenience of being symmetric.

Consider the specific Hankel matrices over $\mathbb{C}[x]$, defined by

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$$H_n(x) = H_n(x; c_1, \ldots, c_{n+1}) = \begin{bmatrix} & & & x & c_1 & c_2 \\ & & x & c_1 & c_2 & c_3 \\ & & & & \vdots & \vdots \\ x & c_1 & \ldots & \ldots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & \ldots & \ldots & c_{n-1} & c_n \\ c_2 & c_3 & \ldots & \ldots & c_n & c_{n+1} \end{bmatrix};$$

for example

$$H_5(x) = \begin{bmatrix} & & x & c_1 & c_2 \\ & x & c_1 & c_2 & c_3 \\ x & c_1 & c_2 & c_3 & c_4 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix}.$$

Conjecture 1.1. (Hankel Pencil Conjecture HPnC). *If* $\det H_n(x) = 0$, *and* $c_1, ..., c_n \in \mathbb{C}^*$, *then the last two columns are dependent.*

The authors of [SS2] proved this conjecture for the cases $n = 3, 4$, and by computational algebraic geometry also for $n = 5$; i.e. they proved HP3C, HP4C, HP5C. They posed HPnC as a problem also in [SS1]. In [BL] a solution was proposed, but it was shown to have a serious gap [W].

In this paper we report progress on the conjecture. In section 2 we show that it is sufficient to prove it for the subclass of matrices $H_n(x)$ for which $c_1 = c_2 = 1$. This is done via a general observation on polynomial systems which we call index sum homogeneous. In section 3 we give an equivalent formulation of the conjecture using the Sylvester identity. We formulate it as a conjecture for a certain class of polynomials for which in section 4 we give explicit formulae. In section 5 we formulate for this class of polynomials and their modified inverses, and assuming wlog $c_1 = c_2 = 1$, a more general conjecture which we shall call root conjecture. We abbreviate it as RnC if referring to monic polynomials of degree $n - 2$. We show that RnC $\Rightarrow$ HPnC. In section 6 we prove RnC for $n \leq 8$. Via the new insights, the case $n = 5$, previously testing the limits of technology, can now be done by hand, the case $n = 6$ with some patience as well. For $n = 7, 8$ we use a 1993 486-PC, and Mathematica v. 2.2, but the computations are rapid so that it is reasonable to expect that more modern models and specialized software versions (or more patience) could extend our results to $n \leq 10$, at least. In section 7 we report briefly on other lines of attack and delimit our results via counter examples to more general and related conjectures that may seem reasonable.

2. To show HPnC one can assume $c_2 = c_1 = 1$.

Here we show it is sufficient to restrict attention to the subclass of matrices $H_n(x)$ for which the rightmost two entries of the first row are equal to 1.

We use the following definitions.

Definition 2.1. Let $p = p(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial in n variables. Then p is *degree-homogeneous* (d-homogeneous) of *degree* m if each of the monomials $x_1^{i_1} x_2^{i_1} \dots x_n^{i_n}$ occurring in it satisfies $i_1 + ... + i_n = m$. Furthermore, for evident reasons, we call p *index sum homogeneous* (is-homogeneous) of *i-sum* k if each of the monomials $x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ occurring in it satisfies $i_1 + 2i_2 + \dots + ni_n = k$.

For example

$$p(x_1, x_2, x_3, x_4, x_5, x_6) = \\ x_2^5 - 4x_1x_2^3x_3 + 3x_1^2x_2x_3^2 + 3x_1^2x_2^2x_4 - 2x_1^3x_3x_4 - 2x_1^3x_2x_5 + x_1^4x_6,$$

is as well d-homogeneous of degree 5 as is-homogeneous of i-sum 10.

If p is a polynomial in $\mathbb{C}[x_1, \dots, x_n]$, then its variety $V(p) = \{\underline{c} \in \mathbb{C}^n : p(\underline{c}) = 0\}$. If p is d-homogeneous, we can describe the set of all solutions for which the first coordinate is $\neq 0$ as

$$V.(p) := V(p) \cap (\mathbb{C}^* \times \mathbb{C}^{n-1}) = \{(c, cc_2, ..., cc_n) : p(1, c_2, ..., c_n) = 0, c \in \mathbb{C}^*\}.$$

We now give a similar description for the solutions with nonzero first and second coordinate of polynomials that are both d- and is-homogeneous. So let $V_{..}(p) := V(p) \cap ((\mathbb{C}^*)^2 \times \mathbb{C}^{n-2})$.

Lemma 2.2. *Let $\underline{x} = (x_1, \dots, x_n)$, and let $p = p(\underline{x}) \in \mathbb{C}[\underline{x}]$ be a d- and is-homogeneous polynomial. Then*

$$V_{..}(p) = \{(c, ca, ca^2c_3, ..., ca^{n-1}c_n) : c, a \neq 0, p(1, 1, c_3, ..., c_n) = 0\}. \qquad (*)$$

Proof. Let d be the degree and k the index sum of p.

Claim. In $\mathbb{C}(\underline{x})$ we have the identity

$$\frac{p(x_1,\ldots,x_n)}{x_1^d x_2^{k-d}} = p\left(\frac{x_1}{x_1},\frac{x_2}{x_1x_2},\frac{x_3}{x_1x_2^2},\ldots,\frac{x_j}{x_1x_2^{j-1}},\ldots,\frac{x_n}{x_1x_2^{n-1}}\right).$$

$\triangleright$ Evidently it is sufficient to prove this claim for every monomial occurring. So consider a monomial occurring in p, say $x_1^{i_1}x_2^{i_2}...x_n^{i_n}$. Upon substituting in it x_j by $\frac{x_j}{x_1x_2^{j-1}}$, $j=1,2,...,n$, we obtain

$$\left(\frac{x_1}{x_1}\right)^{i_1}\left(\frac{x_2}{x_1x_2}\right)^{i_2}\cdots\left(\frac{x_j}{x_1x_2^{j-1}}\right)^{i_j}\cdots\left(\frac{x_n}{x_1x_2^{n-1}}\right)^{i_n}.$$

The denominator of this expression is $x_1^{i_1+\ldots+i_n}x_2^{i_2+2i_3+3i_4+\ldots+(n-1)i_n}$. Thus the exponent of x_1 in it is d and the exponent of x_2 is $\sum_{\nu=1}^n(\nu-1)i_\nu = k-d$. The claim follows. $\triangleleft$

Now consider any $\underline{u} = (u_1,\ldots,u_n) \in V_{\cdot\cdot}(p)$. Since $u_1,u_2 \neq 0$ we can define $c,a \neq 0, c_3,\ldots,c_n$ such that $u_1 = c, u_2 = ca$, and $u_j = ca^{j-1}c_j$, for $j=3,\ldots,n$. So $\underline{u} = (c,ca,ca^2c_3,...,ca^{n-1}c_n) \in V_{\cdot\cdot}(p) \subseteq V_{\cdot}(p)$ implies by the characterization of $V_{\cdot}(p)$ that $p(1,a,a^2c_3,\ldots,a^{n-1}c_n)=0$. But then the identity used from left to right implies $p(1,1,...,a^{j-1}c_j/(1a^{j-1}),...) = p(1,1,c_3,...,c_n) = 0$. So $V_{\cdot\cdot}(p)$ is a subset of rhs$(*)$. Now apply p to an element on rhs$(*)$. Then d- and is-homogeneties, and the identity yield the computation

$$\begin{aligned} p(c,ca,ca^2c_3,...,ca^{n-1}c_n) &= c^d p(1,a,a^2c_3,\ldots,a^{n-1}c_n) \\ &= c^d p(1,1,c_3,\ldots,c_n) \\ &= 0, \end{aligned}$$

so $(c,ca,ca^2c_3,...,ca^{n-1}c_n) \in V_{\cdot\cdot}(p)$. $\square$

Corollary 2.3. *Assume we are given a system of d- and is-homogeneous polynomials* $p_1,....,p_m \in \mathbb{C}[\underline{x}]$. *If the system of equations*

$$p_1(1,1,\underline{x}_{3:n}) = 0,....,p_m(1,1,\underline{x}_{3:n}) = 0$$

allows only the solution $x_{3:n} = (1,1,...,1) \in (\mathbb{C}^*)^{n-2}$, *then the set of all solutions in* $(\mathbb{C}^*)^2 \times \mathbb{C}^{n-2}$ *of the system*

$$p_1(\underline{x}) = 0,....,p_m(\underline{x}) = 0$$

is given by $\{c(1,a,a^2,...,a^{n-1}) : c,a \in \mathbb{C}^*\}$.

Proof. The set of all the sought-for solutions is $\bigcap_{i=1}^{m} V_{\cdot\cdot}(p_i)$. Using the description of the sets $V_{\cdot\cdot}(p_j)$ given in lemma 2.2, the claim is easily deduced. □

Now consider a matrix $H_n(x)$ as in section 1. Obviously $\det H_n(x)$ is a polynomial in x with coefficients that are polynomials in $c_1, \ldots, c_{n+1}$. As long as we treat the c_j as indeterminates, we have polynomials $cx_j \in \mathbb{C}[c_1, \ldots, c_{n+1}]$, so that

$$\det H_n(x) = cx_0 + cx_1 \cdot x + cx_2 \cdot x^2 + ... + cx_{n-2} \cdot x^{n-2}.$$

Lemma 2.4. *The polynomials cx_j, $j = 0, 1 \ldots, n-2$, associated to Hankel matrix $H_n(x)$ are d-homogenous of degree $n-j$ and is-homogeneous of index sum $2n$.*

Proof. Let $H_n(x) = (h_{ij})$, $i, j = 1, \ldots, n$. Then we have

$$h_{ij} = \begin{cases} 0 & \text{if } i+j \leq n-2 \\ x & \text{if } i+j = n-1 \\ c_{i+j-n+1} & \text{if } i+j \geq n. \end{cases}$$

or, writing c_0 for x,

$$h_{ij} = \begin{cases} 0 & \text{if } i+j \leq n-2 \\ c_{i+j-n+1} & \text{if } i+j \geq n-1. \end{cases}$$

A monomial in any cx_j originates in a diagonal product $h_{1,\sigma(1)} h_{2,\sigma(2)} \ldots h_{n,\sigma(n)}$, occuring in a term in the determinant $\det H_n(x)$; here σ is a permutation on $\{1, \ldots, n\}$. For j fixed, exactly j of the h_{**} are equal to $x = c_0$, and $n-j$ are equal to some c_i with $i \geq 1$. This shows that cx_j is homogeneous of degree $n-j$. The i-sum of the diagonal product is the sum of the indices of the c_*s in it. The i-sum of the diagonal product is $\sum_{i=1}^{n}(i + \sigma(i) - n + 1) = 2\sum_{i=1}^{n} i - n^2 + n = 2n$. Note that x has i-sum 0. Therefore the i-sum of any monomial in cx_j is $2n$. □

Corollary 2.5. *If* HPnC *is true for the subclass of admissible matrices for which $c_1 = c_2 = 1$, then HPnC is true in general.*

Proof. Admitting throughout only $i \in \{0, ..., n-2\}$, $j \in \{1, ..., n+1\}$, and $l \in \mathbb{C}^*$, the general HPnC can be written in the form

$$\forall i, j \; cx_i(\underline{c}) = 0 \;\&\; c_j \in \mathbb{C}^* \quad \Rightarrow \quad \exists l \, \forall j \; c_j = l^{j-1} c_1.$$

Since $c_2 = c_1 = 1$ implies $l = 1$, the restricted HPnC has the form

$$\forall i, j \; cx_i(1, 1, c_{3:n+1}) = 0 \;\&\; c_j \in \mathbb{C}^* \quad \Rightarrow \quad \forall j \; c_j = 1.$$

The polynomials $cx_j(c_1, \ldots, c_{n+1})$ are d- and is-homogeneous, by lemma 2.4. So if we assume correctness of restricted HPnC, then by corollary 2.3, the solution of a system satisfying the hypothesis of general HPnC is given by $c_j = ca^{j-1}$ for some $c, a \in \mathbb{C}^*$. But this is precisely the claim. □

3. An equivalent formulation of HPnC

Given a matrix partition $A = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$ with A $n \times n$, E $k \times k$, let $E[\&i|\&j] = A[1:k, i|1:k, j]$, for $i, j = k+1, ..., n$. From these $(k+1) \times (k+1)$ matrices one can form the $n-k \times n-k$ matrix of minors $(\det E[\&i|\&j])_{i,j=k+1,\ldots,n}$. The Sylvester-identity says that

$$\det((\det E[\&i|\&j])) = (\det E)^{n-k-1} \det A;$$

see Brualdi and Schneider [BS] for a lucid introduction to determinantal identities.

Now define polynomial $m_{ij}(x) = \det H_n(x)[i^c|j^c]$, where for $s = i, j \in \{n-1, n\}$, $s^c = \{1, \ldots, n\} \setminus \{s\}$.

Proposition 3.1. *The* HPnC *is equivalent to saying that*

$$m_{nn}(x) m_{n-1,n-1}(x) = m_{n-1,n}^2(x) \;\;\&\;\; c_j \in \mathbb{C}^* \;\; \Rightarrow \\ c_j = a^{j-1} c_1 \textit{ for some } a \in \mathbb{C}, \;\; \textit{and } j = 1, ..., n+1.$$

Proof. Applying the Sylvester identity with $A = H_n(x)$, $k = n-2$, we find that

$$\begin{aligned} m_{nn}(x) \cdot m_{n-1,n-1}(x) - m_{n-1,n}^2(x) &= \det \begin{bmatrix} m_{nn}(x) & m_{n-1\,n}(x) \\ m_{n-1\,n}(x) & m_{n-1\,n-1}(x) \end{bmatrix} \\ &= \delta_{n-1} x^{n-2} \cdot \det H_n(x), \end{aligned}$$

where $\delta_{n-1} = \operatorname{sgn}(n-2, \ldots, 1)$ is determined as given at the beginning of the next section. Thus the hypothesis of HPnC is equivalent to the right hand side of the above implication; the claim follows. □

4. Formulae for the polynomials $m_{ij}(x)$ and their modified inverses

What we shall call the *inverse* of a polynomial $p(x) = \sum_{j=0}^{n} p_j x^j \in \mathbb{C}[x]$ of degree n can alternatively be defined as $\sum_{j=0}^{n} p_{n-j} x^j$ or as $x^n p(1/x)$ (the latter expression lives in $\mathbb{C}(x)$, but not in $\mathbb{C}[x]$). Assuming $p_0 \neq 0$, the inverse of p has as roots precisely the inverses of the roots of p.

In this section we prove formulae for the polynomials

$$m_{ij}(x) = \det H_n(x)[i^c|j^c],$$

and their *modified* inverses $\hat{m}_{ij}(x) = \delta_n x^{n-2} m_{ij}(1/x)$, where for integers $n \geq 0$ define δ_n by

$$\delta_n = \begin{cases} -1 & \text{if } n \equiv_4 0, 3 \\ 1 & \text{if } n \equiv_4 1, 2 \end{cases}, \text{ or equivalently, by } \delta_n = (-1)^{\lfloor (n-1)/2 \rfloor}.$$

We avoid handwaving arguments in the proof of theorem 4.2b below by taking a detour over a purely combinatorial lemma of interest in its own right.

Let $\mathcal{O}(n)$ be the set of ordered positive integer partitions of n into an odd number of parts, and $\mathcal{P}(n)$ the family of all positive ordered partitions of the integer n. Examples of elements in $\mathcal{O}(8)$ include $134, 22211, 31211$, etc. Here 134 for example is a shorthand for (1,3,4), given that the digits involved are≤ 9. Elements in $\mathcal{P}(7)$ include $61, 241, 1114$, etc.

Now let $o = (n_1, \ldots, n_{2k+1})$ be a partition in $\mathcal{O}(n)$. Examine n_i, $i = 1, 2, 3, \ldots$, successively from left to right and write the following:
if i is odd, write a string of form 111...1 of length $n_i - 1$; if this value is 0, let it void.
if i is even write the integer $n_i + 1$.

This defines a map ϕ on $\mathcal{O}(n)$. For example, we have $\mathcal{O}(13) \ni 31531 \overset{\phi}{\mapsto} 11211114 \in \mathcal{P}(12)$. This is no coincidence.

Proposition 4.1. (a) *The map $\phi : \mathcal{O}(n) \to \mathcal{P}(n-1)$ is bijective.*
(b) *Under this bijection the set of all partitions in $\mathcal{O}(n)$ for which the sum*

of the entries at even positions is l corresponds to the elements in $\mathcal{P}(n-1)$ of length $n-l-1$.

Proof. Applying ϕ to an element of $\mathcal{O}(n)$, we obtain a string of positive integers whose sum is $(n_1-1)+(n_2+1)+\ldots+(n_{2k-1}-1)+(n_{2k}+1)+n_{2k+1}-1=n-1$. It is now evident that we have constructed an injective map $\mathcal{O}(n) \xrightarrow{\phi} \mathcal{P}(n-1)$.

Conversely, let be given any positive partition of $n-1$. One can find on it from left to right for certain integers $n_1', n_2', \ldots$: a n_1'-string of 1s, a number $n_2' \geq 2$, a n_3'-string of 1s, a number $n_4' \geq 2, \ldots$, a n_{k+1}' string of 1s, etc., where one needs to admit $n_1', n_3', \ldots.$ to be possibly of value 0. This reading is unique and defines integers $n_1', n_2', \ldots.$ From left to right now
if i is odd: write the integer $n_i'+1$.
if i is even: write the integer $n_i'-1$.

So for the example above, starting with $11211114 \in \mathcal{P}(12)$, we find $n_1' = 2, n_2' = 2, n_3' = 4, n_4' = 4, n_5' = 0$. Applying the construction process just outlined leads back to 31531.

It is easy to see that this is the inverse of the map ϕ.

(b) Now fix l. Apologizing for the incoherence of notation, let $\mathcal{O}(n,l)$ the partitions of $\mathcal{O}(n)$ for which the sum of the entries at even positions is l, and let $\mathcal{P}(n-1,n-l-1)$ denote the set of all partitions in $\mathcal{P}(n-1)$ of length $n-l-1$. Consider $o=(n_1,n_2,\ldots,n_{2k},n_{2k+1}) \in \mathcal{O}(n,l)$. Then

$$\begin{aligned} \mathrm{length}(\phi(o)) &= (n_1-1)+1+(n_3-1)+1+\cdots+(n_{2k-1}-1)+1+n_{2k+1}-1 \\ &= n_1+n_3+\cdots+n_{2k+1}-1 \\ &= (n-l)-1. \end{aligned}$$

So we have an injective map $\phi|\mathcal{O}(n,l) : \mathcal{O}(n,l) \to \mathcal{P}(n-1,n-l-1)$; hence $\#\mathcal{O}(n,l) \leq \#\mathcal{P}(n-1,n-l-1)$. Since $\mathcal{O}(n) = \biguplus_{l\geq 1} \mathcal{O}(n,l)$, and $\mathcal{P}(n-1) = \biguplus_{l\geq 1} \mathcal{P}(n-1,l)$, and $\#\mathcal{O}(n) = \#\mathcal{P}(n-1)$ by part (a), we find $\#\mathcal{O}(n,l) = \#\mathcal{P}(n-1,n-l-1)$, and so the map is bijective. $\square$

Theorem 4.2. *With the understanding that $i_\nu \in \mathbb{Z}_{\geq 1}$ for all indices occurring, there hold the following formulae.*

(a) $m_{nn}(x) =$
$\delta_n \sum_{j=0}^{n-2} \left(\sum\{c_{i_1}c_{i_2}\cdots c_{i_{n-j-1}} : i_1+i_2+\ldots+i_{n-j-1}=n-1\}\right)\cdot(-x)^j,$

(b) $m_{n-1,n}(x) =$
$\delta_n \sum_{j=0}^{n-2} \big(\sum\{c_{i_1}c_{i_2}\cdots c_{1+i_{n-j-1}} : i_1+i_2+\ldots+i_{n-j-1}=n-1\}\big)\cdot(-x)^j$.

(c) $m_{n-1,n-1}(x) =$
$\delta_n \sum_{j=0}^{n-3} \big(\sum\{c_{i_1}c_{i_2}\cdots c_{1+i_{n-j-2}}c_{1+i_{n-j-1}} : i_1+i_2+\ldots+i_{n-j-1}=n-1\}\big)\cdot$
$(-x)^j + \ \delta_n c_{n+1}(-x)^{n-2}$.

Proof. For small n these formulae are verified directly. We now show them to hold true by induction. By definition $m_{nn}(x)$ is the left upper $n-1\times n-1$ minor of $H_n(x)$. Thus, developing according to row 1, we find

$$\begin{aligned} m_{nn}(x) &= \begin{vmatrix} & & & x & c_1 \\ & & x & c_1 & c_2 \\ & & & & \vdots \\ x & c_1 & \ldots & \ldots & c_{n-2} \\ c_1 & c_2 & \ldots & \ldots & c_{n-1} \end{vmatrix} \\ &= (-1)^{n-3}x \underbrace{\begin{vmatrix} & & & x & c_2 \\ & & x & c_1 & c_3 \\ & & & & \vdots \\ x & c_1 & \ldots & \ldots & c_{n-2} \\ c_1 & c_2 & \ldots & \ldots & c_{n-1} \end{vmatrix}}_{=:|A|} + (-1)^{n-2}c_1 \underbrace{\begin{vmatrix} & & & x & c_1 \\ & & x & c_1 & c_2 \\ & & & & \vdots \\ x & c_1 & \ldots & \ldots & c_{n-3} \\ c_1 & c_2 & \ldots & \ldots & c_{n-2} \end{vmatrix}}_{=:|B|} \end{aligned}$$

Note that the matrices A, B are $n-2\times n-2$, and $|A| = m_{n-1,n-2} = m_{n-2,n-1}$, $|B| = m_{n-1,n-1}(x)$, *relatively to* $H_{n-1}(x)$, which is symmetric. So we have by induction assumption the formulae:

$$\begin{aligned} |B| &= \delta_{n-1}\sum_{j=0}^{n-3}\Big(\sum\{c_{i_1}c_{i_2}\cdots c_{i_{n-j-2}} : i_1+i_2+\ldots+i_{n-j-2}=n-2\}\Big)\cdot(-x)^j. \\ |A| &= \delta_{n-1}\sum_{j=0}^{n-3}\Big(\sum\{c_{i_1}c_{i_2}\cdots c_{1+i_{n-j-2}} : i_1+i_2+\ldots+i_{n-j-2}=n-2\}\Big)\cdot(-x)^j. \end{aligned}$$

Next, with $A' := |A|/\delta_{n-1}, B' := |B|/\delta_{n-1}$, we can write

$$m_{nn}(x) = -^{n-3}(x|A| - c_1|B|) = -^{n-3}\delta_{n-1}(xA' - c_1B'). \quad (*)$$

Using $\mathrm{coeff}(xA', x^l) = \mathrm{coeff}(A', x^{l-1})$, next, check for $l = 1, 2, \ldots, n-2$, that

$$\begin{aligned}\mathrm{coeff}(xA', x^l) &= -^{l+1}\sum\{c_{i_1}c_{i_2}\cdots c_{1+i_{n-l-1}} : i_1 + i_2 + \ldots + i_{n-l-1} = n-2\}\\ \mathrm{coeff}(-c_1B', x^l) &= -^{l+1}\sum\{c_{i_1}c_{i_2}\cdots c_{i_{n-l-2}}\cdot c_1 : i_1 + i_2 + \ldots + i_{n-l-2} = n-2\}.\end{aligned}$$

We can write this in an alternative way as

$$\mathrm{coeff}(xA', x^l) = -^{l+1}\sum a_{l_1 l_2 \ldots l_{n-1}} c_1^{l_1} c_2^{l_2} \ldots c_{n-1}^{l_{n-1}},$$

and

$$\mathrm{coeff}(-c_1B', x^l) = -^{l+1}\sum b_{l_1 l_2 \ldots l_{n-1}} c_1^{l_1} c_2^{l_2} \ldots c_{n-1}^{l_{n-1}},$$

where:

$a_{l_1 l_2 \ldots l_{n-1}}$ is equal to the number of $n-l-1$-tuples $(i_1, ..., i_{n-l-2}, 1 + i_{n-l-1})$ containing l_1 entries 1, l_2 entries 2, ..., l_{n-1} entries $n-1$, while $(i_1, ..., i_{n-l-1})$ ranges over all positive $n-l-1$-tuples of sum $n-2$;

$b_{l_1 l_2 \ldots l_{n-1}}$ is equal to the number of $n-l-1$-tuples $(i_1, ..., i_{n-l-2}, 1)$ containing l_1 entries 1, l_2 entries 2, ..., l_{n-1} entries $n-1$, while $(i_1, ..., i_{n-l-2})$ ranges over all positive $n-l-2$-tuples of sum $n-2$.

With these definitions, the coefficient of x^l of the right hand side of (*), in other words

$$\mathrm{coeff}(m_{nn}(x), x^l) = -^{n+l-2}\delta_{n-1}\sum(a_{l_1 l_2 \ldots l_{n-1}} + b_{l_1 l_2 \ldots l_{n-1}}).$$

At the other hand by similar considerations as above for the a_*s and b_*s, this coefficient is claimed to be

$$-^l\delta_n\sum\{c_{i_1}c_{i_2}\cdots c_{i_{n-l-1}} : i_1 + i_2 + \ldots + i_{n-l-1} = n-1\},$$

or equivalently

$$-^l\delta_n\sum w_{l_1 l_2 \ldots l_{n-1}} c_1^{l_1} c_2^{l_2} \ldots c_{n-1}^{l_{n-1}},$$

where $w_{l_1 l_2 \ldots l_{n-1}}$ is the number of positive $n-l-1$-tuples of sum $n-1$ containing l_1 entries 1, l_2 entries 2, ..., l_{n-1} entries $n-1$.

Now the set whose cardinality $w_{l_1 l_2 \ldots l_{n-1}}$ counts can be divided into two disjoint subsets: namely the subset of tuples whose last component is ≥ 2, and the subset of tuples whose whose last component is 1. It is now easy to see that these subsets have cardinalities $a_{l_1 l_2 \ldots l_{n-1}}$ and $b_{l_1 l_2 \ldots l_{n-1}}$ respectively. Hence $\sum w_{l_1 l_2 \ldots l_{n-1}} = \sum a_{l_1 l_2 \ldots l_{n-1}} + b_{l_1 l_2 \ldots l_{n-1}}$. Finally one checks that $\delta_n = -^n\delta_{n-1}$ so that we have proved our claim concerning $m_{nn}(x)$.

(b) Consider once more the determinant defining $m_{nn}(x)$. It has $x$es in columns $1, ..., n-2$ and no x in column or row $n-1$. Circle some, say l, x-es. In the length n sequence $0\ 1\ 2\ 3 ...\ n-2\ n-1$, underline those integers j that are column indices of circled x-es. Call a set of consecutive underlined integers an *u-interval*; a set of consecutive not underlined integers a *n-interval*. The treated sequence necessarily begins and ends in n-intervals. Now going from left to right write down the sequence of lengths (i.e. cardinalities) of these intervals. This sequence is of odd length and partitions the integer n. It is $o = (n_1, n_2, ..., n_{2k+1}) \in \mathcal{O}(n)$, say. It has at its even positions the lengths of the u-intervals. The sum of these lengths equals the number of circled x-es. A little reflection shows now the following. There is one and only one possibility of circling $(n-l)$ cs such that the $l + (n-l) = n$ circles lie all in different rows and columns, i.e. such that they form a permutation. Indeed the indices of the circled cs written down as appearing from left to right coincide precisely with $\phi(o) \in \mathcal{P}(n-1)$.

$$\begin{bmatrix}
 & & & & & & & x & 1 \\
 & & & & & & x & 1 & 2 \\
 & & & & & x & 1 & 2 & 3 \\
 & & & & x & 1 & 2 & 3 & 4 \\
 & & & x & 1 & 2 & 3 & 4 & 5 \\
 & & x & 1 & 2 & 3 & 4 & 5 & 6 \\
 & x & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
x & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{bmatrix}.$$

To illustrate this process, consider the 9×9 matrix at the left whose determinant defines $m_{10,10}(x)$. (For readability we suppressed the cs.) We circled three x-es; the associated underlined sequence is $0\,1\,2\,\underline{3\,4}\,5\,6\,\underline{7}\,8\,9$. The sequence of lengths of n- and u-intervals thus is $o = 32212 \in o(10)$. Thus $\phi(o) = 113121$. To the circles chosen corresponds the word $c_1c_1xxc_3c_1xc_2c_1$, or after eliminating the x-es, $c_1c_1c_3c_1c_2c_1$.

What is the bearing of this discussion for our problem? We have shown in part a that the coefficient of x^l of $m_{nn}(x)$ is apart from signing equal to

$$*_1 : \sum \{c_{i_1}c_{i_2} \cdots c_{i_{n-l-1}} : i_1 + i_2 + \ldots + i_{n-l-1} = n-1\}.$$

By proposition 4.1 we can understand this sum now as the sum over all products $c_{i_1}c_{i_2} \cdots c_{i_{n-l-1}}$ for which $(i_1, \ldots, i_{n-l-1}) = \phi(o)$, as o ranges over all partitions of n of odd length with sum of even entries$=l$. Now, by symmetry $m_{n-1,n}(x) = m_{n,n-1}(x)$, and this latter polynomial can be simply obtained by adding 1 to the indices of the last column of the determinantal expression for m_{nn}. Our combinatorial interpretation of the sum $*_1$ now yields the formula (b).

(c) The formula in (b) can also be written
$m_{n-1,n}(x) =$

$$\delta_n \sum_{j=0}^{n-2} \left(\sum\{c_{1+i_1}c_{i_2}\cdots c_{i_{n-j-1}} : i_1 + i_2 + \ldots + i_{n-j-1} = n-1\}\right) \cdot (-x)^j.$$

and the coefficients of x^l interpreted as the sum of the words in the cs obtained in the transposed of the determinantal expression considered in (b). The transposed has as last index row $[2, 3, ..., n]$, so all entries are ≥ 2. c_{1+i_1} would represent the c chosen in the last row (necessarily the leftmost) and $c_{i_{n-j-1}}$ represents in all cases the c chosen in the last column. Now to obtain from our transposed minor the minor $m_{n-1,n-1}$ we have to add 1 to each index in the last column, and thus can largely use the reasoning we used in part b. There is one point to observe: if the coefficient consists of only one letter, i.e. if $n-j-1 = 1$, so $j = n-2$, then the index has to be augmented by 2, for then the letter is found in the lower right corner and so has been increased by 1 as being in the last row, and once from augmenting as lying in the last column. Our formula given in part c reflects these facts. $\square$

Corollary 4.3. *There hold the formulae*

$$\text{(a) } \hat{m}_{nn}(x) = -^n \sum_{j=0}^{n-2} \left(\sum\{c_{i_1}...c_{i_{j+1}} : i_1 + \ldots + i_{j+1} = n-1\}\right)(-x)^j.$$

(a') $\hat{m}_{nn}(x) =$
$-^n \sum_{j=0}^{n-2} \left(\sum\{\binom{j+1}{l_1,l_2,\ldots,l_{n-1}} c_1^{l_1} c_2^{l_2} \ldots c_{n-1}^{l_{n-1}} : 1l_1 + 2l_2 + ... + (n-1)l_{n-1} = n-1\}\right)$
$(-x)^j$.

$$\text{(b) } \hat{m}_{n-1,n}(x) = -^n \sum_{j=0}^{n-2} \left(\sum\{c_{i_1}...c_{1+i_{j+1}} : i_1 + \ldots + i_{j+1} = n-1\}\right)(-x)^j.$$

(c) $\hat{m}_{n-1,n-1}(x) =$
$-^n c_{n+1} + -^n \sum_{j=1}^{n-2} \left(\sum\{c_{i_1}...c_{1+i_j}c_{1+i_{j+1}} : i_1 + \ldots + i_{j+1} = n-1\}\right)(-x)^j.$

(d) *If* $c_1 = c_2 = 1$, *then all the polynomials* $\hat{m}_{ij}$, $i, j \in \{n-1, n\}$ *are monic.*

Proof. (a),(b),(c). These formulae follow directly from the definitions of $\hat{m}_{nn}, \hat{m}_{n-1,n}$, and $\hat{m}_{n-1,n-1}$, respectively.
(a') We can write the inner sum in part a as $\sum a_{l_1 l_2 \ldots l_{n-1}} c_1^{l_1} c_2^{l_2} \ldots c_{n-1}^{l_{n-1}}$, where $a_{l_1 l_2 \ldots l_{n-1}}$ is the number of positive $j+1$-tuples of sum $n-1$ containing l_1 entries 1, l_2 entries 2, ..., l_{n-1} entries $n-1$.

Using the definition of the multinomial coefficient occurring in (a'), see [A, p77], the claim follows.
(d) The leading coefficient in the polynomials above is found considering only the terms corresponding to $j = n-2$. This choice forces the inner sums to collapse to $c_1^{n-1}, c_1^{n-2}c_2$, and $c_1^{n-3}c_2^2$ respectively. The claim follows. □

5. A more general conjecture: the root conjectures RnC

On the basis of sections 2,3,4 we can formulate HPnC as follows

Proposition 5.1. HPnC *is equivalent to the following assertion for modified inverse polynomials.*

$$\hat{m}_{nn}(x) \cdot \hat{m}_{n-1,n-1}(x) = \hat{m}_{n-1,n}^2(x) \ \& \ c_1 = c_2 = 1 \quad \Rightarrow \quad c_3 = \ldots = c_{n+1} = 1. \ (*)$$

Proof. Assume $(*)$ correct. Consider the formulation of HPnC as given in proposition 3.1. In view of the result of section 2, we can formulate it as
$$m_{nn}(x) \cdot m_{n-1,n-1}(x) = m_{n-1,n}^2(x) \ \& \ c_1 = c_2 = 1 \quad \Rightarrow$$
$$c_1 = c_2 = \ldots = c_n = c_{n+1} = 1.$$

Since passing to the modified inverse of a degree $n-2$ polynomial is an involutive process, one sees that the first of the hypothesis is equivalent to $\hat{m}_{nn}(x) \cdot \hat{m}_{n-1,n-1}(x) = \hat{m}_{n-1,n}^2(x)$. Thus $(*)$ implies HPnC. The discussion shows that the converse also holds true. □

For a polynomial $p \in \mathbb{C}[x]$ define $\mathrm{roots}(p) = \{c \in \mathbb{C} : p(c) = 0\}$.

Conjecture 5.2. (Root conjecture RnC).

If $\mathrm{roots}(\hat{m}_{n,n}) \subseteq \mathrm{roots}(\hat{m}_{n-1,n})$ & $c_1 = c_2 = 1$, *then* $\mathrm{roots}(\hat{m}_{n-1,n}) = \{1\}$.

Proposition 5.3. (a) *Suppose* $c_1 = c_2 = 1$. *Then the following are equivalent.*

(i) $\text{roots}(\hat{m}_{n-1,n}) = \{1\}$.
(ii) $c_3 = c_4 = ... = c_n = 1$.
(iii) $\hat{m}_{n-1,n} = \hat{m}_{nn}$.
(iv) $\exists a \in \mathbb{C}^* \ \text{roots}(\hat{m}_{n,n}) = \{a\}$.

(b) *In particular the root conjecture can be formulated as*

RnC: $\text{roots}(\hat{m}_{n,n}) \subseteq \text{roots}(\hat{m}_{n-1,n})$ & $c_1 = c_2 = 1 \ \Rightarrow \ $ (i) $\vee$ (ii) $\vee$ (iii) $\vee$ (iv).

Proof. (a) (i)$\Leftarrow$(ii): If $c_i = 1$ for all $i = 1, ..., n$, then from the simple combinatorial fact [A, p.80] that

$$\#\{i \in \mathbb{Z}_{\geq 1}^{j+1} : i_1 + \ldots + i_{j+1} = n-1\} = \binom{n-2}{j},$$

for $j = 0, ..., n-2$, and the formula in corollary 4.3b, we get that $\hat{m}_{n-1,n} = (x-1)^{n-2}$.
(i)$\Rightarrow$(ii): By corollary 4.3d, $\hat{m}_{n-1,n}$ is monic. The hypothesis implies that

$$\sum\{c_{i_1}c_{i_2}\ldots c_{i_j}c_{1+i_{j+1}} : i_1+\ldots+i_{j+1} = n-1\} = \binom{n-2}{j}, \quad j = 0, 1, 2, ..., n-2.$$

For a fixed j, consider $(i_1, \ldots, i_{j+1})$ as ranging over the set $\mathcal{P} = \mathcal{P}(n-1, j+1)$ of all positive integer $j+1$-tuples of sum $n-1$. Then

$$\max\{\max(i_1, \ldots, i_j, 1+i_{j+1}) : (i_1, \ldots, i_{j+1}) \in \mathcal{P}\} = n-j,$$

and this value is achieved exactly once namely when $(i_1, \ldots, i_j, 1+i_{j+1}) = (1, 1, ..., 1, n-j)$. Writing above equation for $j = n-2, n-3, ..., 1, 0$ successively, and using $c_1 = c_2 = 1$, one finds $c_3 = 1, c_4 = 1, ..., c_n = 1$.
(ii)$\Leftrightarrow$(iii). We use similar ideas. Part iii is equivalent to saying that

$$\sum\{c_{i_1}...c_{i_{j+1}} : i_1+\ldots+i_{j+1} = n-1\} = \sum\{c_{i_1}...c_{1+i_{j+1}} : i_1+\ldots+i_{j+1} = n-1\},$$

for $j = n-2, n-3, ..., 0$. All indices occurring in either side are$\leq 1+(n-1) = n$. So if (ii) is satisfied, then so is (iii). Conversely, suppose (iii). We know $c_1 = c_2 = 1$. Assume $c_1 = c_2 = ... = c_{n-k-1} = 1$ already established. Write the equation for $j = k$. Then the left hand side is a sum of 1s, while the right hand side is also a sum of 1s except for one term that is c_{n-k}. Since both sides have the same number of terms, we find $c_{n-k} = 1$. So induction yields (ii).

(ii)⇐(iv). Suppose $\hat{m}_{nn}(x) = (x-a)^{n-2} = \sum_{j=0}^{n-2} \binom{n-2}{j} x^j (-a)^{n-2-j}$. So, using the formulae of corollary 4.3, we have $\binom{n-2}{j} a^{n-2-j} = \text{coeff}(\hat{m}_{nn}, -^{n-2-j} x^j) = \sum\{c_{i_1} c_{i_2} \dots c_{i_j} c_{i_{j+1}} : i_1 + \dots + i_{j+1} = n-1\}, \quad j = 0, 1, 2, ..., n-2$. Choosing $j = n-3$, this specializes to $(n-2)a = n-2$. Hence $a = 1$. With this then going into the comparison of coefficients above, and choosing $j = n-3, ..., 0$ successively, one finds (ii) proceeding similarly as in the proof of implication '(i)⇒(ii)' above.
(ii)⇒(iv). Supposing (ii), we find by almost exactly the same reasoning as in '(i)⇐(ii)' before, that $\hat{m}_{nn}(x) = (x-1)^{n-2}$. So roots $\hat{m}_{nn}(x) = \{1\}$, hence (iv) holds.

Part (b) is now clear. □

The relevance of the root conjecture for our problem follows from the following proposition.

Proposition 5.4. *For every $n \geq 3$, there holds* RnC ⇒ HPnC.

Proof. Assume the hypothesis of HPnC, that is lhs(∗) of proposition 5.1 satisfied. Obviously, then $\text{roots}(\hat{m}_{nn}) \subseteq \text{roots}(\hat{m}_{n-1,n})$. Consequently by RnC and proposition 5.3, $c_1 = c_2 = \dots = c_n = 1$. Polynomial multiplication also tells us, that $\text{coeff}(\hat{m}_{nn}, x^0) \cdot \text{coeff}(\hat{m}_{n-1,n-1}, x^0) = \text{coeff}(\hat{m}_{n-1,n}, x^0)^2$. By the formulae in corollary 4.3, this says $c_{n-1} c_{n+1} = c_n^2$. So $c_{n+1} = 1$. □

6. Proofs for RnC and HPnC for $n \leq 8$

In this section we prove RnC and thus by proposition 5.4 also HPnC for all $n \leq 8$. We also show that proofs for RnC for larger n can in principle be tried by the same ideas as those we employ for $n = 7, 8$. We assume throughout $c_1 = c_2 = 1$ and will routinely use that the roots of by corollary 4.3d monic polynomials $\hat{m}_{ij}$, $i, j \in \{n-1, n\}$, determine them completely and that the sum of the multiplicities of the roots equals their degree $n-2$.

Lemma 6.1. *Let $c_1 = c_2 = 1$ and consider with indeterminates e_j and $\hat{e}_j$, the two systems of $(n-1) + (n-1)$ equations*

$$\sum\{c_{i_1} ... c_{i_{j+1}} : i_1 + \dots + i_{j+1} = n-1\} \;=\; \hat{e}_{n-2-j}, \quad j = 0, \dots, n-2;$$
$$\sum\{c_{i_1} ... c_{1+i_{j+1}} : i_1 + \dots + i_{j+1} = n-1\} \;=\; e_{n-2-j}, \quad j = 0, \dots, n-2.$$

Then these systems imply respectively

(a) $c_j \in \mathbb{Q}[\hat{e}_2, ..., \hat{e}_{j-1}]$, for $j = 3, ..., n-2, n-1$;
(b) $c_j \in \mathbb{Z}[e_1, ..., e_{j-2}]$, for $j = 3, ..., n$. *Assume additionally* $\mathrm{roots}(\hat{m}_{n,n}) = \{z'_1, ..., z'_{n-2}\}$, $\mathrm{roots}(\hat{m}_{n-1,n}) = \{z_1, ..., z_{n-2}\}$, *as multisets respecting multiplicities, and* $\hat{e}_j = e_j(z'_1, ..., z'_{n-2})$, $e_j = e_j(z_1, ..., z_{n-2})$, *where* $e_j(...)$ *denotes the* j*-th elementary symmetric function in* $n-2$ *variables.*

$$\begin{aligned} 0 &= \hat{e}_1 + 2 - n \\ 0 &= \hat{e}_2 + d_1(e_1) \\ 0 &= \hat{e}_3 + d_2(e_1, e_2) \\ &\vdots \\ 0 &= \hat{e}_{n-2} + d_{n-3}(e_1, e_2, ..., e_{n-3}), \end{aligned}$$

Then

(c) *The two systems of equations above express true equalities for complex numbers and;*
(d) *If* $\mathrm{roots}(\hat{m}_{n,n}) \subseteq \mathrm{roots}(\hat{m}_{n-1,n})$ *as sets, not necessarily respecting multiplicities, then these complex numbers satisfy* $n-2$ *relations of the form shown at the left with certain polynomials* $d_j \in \mathbb{Z}[x_1, \ldots, x_j]$, $j = 1, \ldots, n-3$.

Proof. For later note that by corollary 4.3, the left hand sides of the systems given describe the coefficient of $-^{n-2-j}x^j$ of $\hat{m}_{nn}(x)$ and $\hat{m}_{n-1,n}(x)$ respectively.
(a) We use ideas already found in the proof of proposition 5.3. For a fixed j, consider $(i_1, \ldots, i_{j+1})$ as ranging over the set $\mathcal{P} = \mathcal{P}(n-1, j+1)$ of all positive integer $j+1$-tuples of sum $n-1$. Then $\max\{\max(i_1, \ldots, i_j, i_{j+1}) : (i_1, \ldots, i_{j+1}) \in \mathcal{P}\} = n - j - 1$, and this value is achieved exactly when $(i_1, \ldots, i_j, i_{j+1})$ is a permutation of $(1, 1, ..., 1, n-j-1)$. There are $j+1$ such permutations. Consequently, and with the understanding $p_1 = p_{n-2} = 0$, $p_2()$:=number of partitions of n-1 into n-3 parts containing only entries 1 and 2, we can write, for certain integer polynomials p_j in $j-2$ variables, $\sum\{c_{i_1}...c_{i_{j+1}} : i_1 + \ldots + i_{j+1} = n-1\} = (j+1)c_{n-j-1} + p_{n-j-2}(c_3, ..., c_{n-j-2})$.
Thus we have $(j+1)c_{n-j-1} + p_{n-j-2}(c_3, ..., c_{n-j-2}) = \hat{e}_{n-j-2}$. Reading

this now for $j = n-3, ..., 0$ in succession, we find the system at the left below, called naturally $\hat{m}_{nn}$-system.

$$\begin{array}{rclcrcl}
\multicolumn{3}{c}{\hat{m}_{nn} - \text{system}} & \quad & \multicolumn{3}{c}{\hat{m}_{n-1,n} - \text{system}} \\
& & & & & & \\
(n-2) & = & \hat{e}_1 & & c_3 + q_2() & = & e_1 \\
(n-3)c_3 + p_2() & = & \hat{e}_2 & & c_4 + q_3(c_3) & = & e_2 \\
(n-4)c_4 + p_3(c_3) & = & \hat{e}_3 & & c_5 + q_4(c_3, c_4) & = & e_3 \\
(n-5)c_5 + p_4(c_3, c_4) & = & \hat{e}_4 & & c_5 + q_5(c_3, c_4, c_5) & = & e_3 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
2c_{n-2} + p_{n-3}(c_3, ..., c_{n-2}) & = & \hat{e}_{n-3} & & c_{n-1} + q_{n-2}(c_3, ..., c_{n-2}) & = & e_{n-3} \\
c_{n-1} & = & \hat{e}_{n-2} & & c_n & = & e_{n-2}
\end{array}$$

We will need the first of the equations of the $\hat{m}_{nn}$-system later. From the second of the equations one finds that c_3 is a polynomial in $\hat{e}_2$, then from the third, that c_4 a polynomial in $\hat{e}_2, \hat{e}_3$, and so forth. It is clear that the coefficients of these polynomials are all rationals, establishing part a.
(b) We apply similar reasoning, with the difference that one examines where the maximum entry of $(i_1, \ldots, i_{n-3}, 1+i_{j+1})$ is assumed as $(i_1, ..., i_{j+1})$ ranges over $\mathcal{P}$. One finds that there are polynomials q_j with coefficients in $\mathbb{Z}$ in $j-2$ variables, so that second system transforms into the $\hat{m}_{n-1,n}$-system as one chooses successively $j = n-3, n-1, ..., 0$. Inspection yields that here $q_2()$ =number of $n-2$-tuples of form $(i_1, \ldots, i_{n-3}, 1+i_{n-2})$ and of sum n containing only parts 1 and 2, while $(i_1, ..., i_{n-2})$ ranges over $\mathcal{P}(n-1, n-2)$. From the $\hat{m}_{n-1,n}$-system, similarly as before one finds that c_j can be written as a polynomial in $e_1, ..., e_{j-2}$, this time for $j = 3, ..., n$. Thanks to the fact that the c_j are introduced in the $\hat{m}_{n-1,n}$-system with coefficient 1, we can this time infer that the c_j are integer polynomials of the $e_1, ..., e_{j-2}$.
(c) Follows from the Vietá -formulae and the formulae for polynomials $\hat{m}_{n-1,n}$, $\hat{m}_{n,n}$ given in corollary 4.3.
(d) The first equation is evidently equivalent to the first equation of the $\hat{m}_{nn}$-system, the other equations follow from the remaining equations of that system and part b of the lemma. $\square$

We now proceed first to proving RnC for $n = 3, 4, 5$. The cases $n = 3, 4$ are very simple. The case 5 is also relatively easy and we need not establish the generic system of lemma 6.1d.

Case $n=3$. In this case $\hat{m}_{33}=-1+x$, $\hat{m}_{23}=-c_3+x$. So from the hypothesis of R3C, $\{1\}=\text{roots}(-1+x)\subseteq\text{roots}(-c_3+x)=\{c_3\}$. This yields $c_3=1$, proving R3C by proposition 5.3b, since ii there is true.

For $n\geq 4$, to prove RnC, we may assume that $\hat{m}_{nn}$ has a double root, for otherwise the hypothesis of RnC implies $\hat{m}_{n-1,n}=\hat{m}_{n,n}$, and so again by proposition 5.3b, we are done.

Case $n=4$. In this case assuming the degree 2 polynomial $\hat{m}_{44}$ has a double root, then in proposition 5.3b conclusion iv holds, so R4C is true.

(Alternatively, use $\hat{m}_{44}=c_3-2x+x^2$, $\hat{m}_{34}=c_4-(1+c_3)x+x^2$. If $\hat{m}_{44}$ has a double root, then its discriminant $\Delta=4-4c_3=0$, so $c_3=1$, and $\text{roots}(\hat{m}_{44})=\{1\}\subseteq\text{roots}(\hat{m}_{34})$ implies $c_4-2\cdot 1+1=0$ so $c_4=1$.)

Case $n=5$. Here $\hat{m}_{55}=-c_4+(1+2c_3)x-3x^2+x^3$, $\hat{m}_{45}=-c_5+(2c_3+c_4)x-(2+c_3)x^2+x^3$.

$$\begin{array}{rcl} 3 & \stackrel{1}{=} & 2a+b \\ 1+2c_3 & \stackrel{1'}{=} & a^2+2ab \\ c_4 & \stackrel{1''}{=} & a^2b \end{array} , \quad \begin{array}{rcl} 2+c_3 & \stackrel{2}{=} & a+b+g \\ 2c_3+c_4 & \stackrel{2'}{=} & ab+ag+bg \\ (c_5 & \stackrel{2''}{=} & abg) \end{array}$$

Let $\text{roots}(\hat{m}_{45})=\{a,b,g\}$. We need only consider the subcase 21, namely $\text{roots}(\hat{m}_{55})=\{a,a,b\}$. Then Vietá's formulae permit us to write the equations at the left.

Using '$\stackrel{1}{=}$' one has $b=3-2a$. Then '$\stackrel{1'}{=}$' yields $c_3=\frac{1}{2}(-3a^2+6a-1)$, and then by '$\stackrel{2}{=}$', $g=-\frac{3}{2}a^2+4a-\frac{3}{2}$, while '$\stackrel{1''}{=}$' gives $c_4=-2a^3+3a^2$. Substituting these expressions in a in '$\stackrel{2'}{=}$,' yields $0=\frac{7}{2}(a-1)^3$. Hence $a=1$. Thus $b=1$, and $g=1$, showing $\text{roots}(\hat{m}_{45})=\{1\}$.

For the remaining cases $n=6,7,8$ note that the hypothesis $\text{roots}(\hat{m}_{n,n})\subseteq\text{roots}(\hat{m}_{n-1,n})$ decomposes into various subcases that are naturally parameterized by the decreasing partitions of $n-2$. Namely, if we assume $\text{roots}(\hat{m}_{n-1,n})=\{z_1,...,z_{n-2}\}$, symmetry allows us to write

$$\text{roots}(\hat{m}_{n,n})=\{z_1,...,z_1,z_2,...,z_2,....,z_{n-2},...,z_{n-2}\}$$

where z_i occurs μ_i times with $\mu_1\geq\mu_2\geq...\geq\mu_{n-2}$ and $\sum\mu_i=n-2$. For $n=6,7,8$ we will consider the 'subcases $\mu_1...\mu_{n-2}$'. The subcases $\mu_1=n-1$ and $\mu_{n-2}=1$ correspond to the cases iv and iii of proposition 5.3 and need not be considered further, since by part b of that proposition, RnC is true under these additional hypothesis.

If we order the partitions lexicographically say, then two successive partitions differ in exactly two entries and such a transition changes corresponds to the (de)specialization of one variable. For example for $n = 8$, $n-2 = 6$, the transition from 3111 to 321 can be identified with passing from assuming roots$(\hat{m}_{nn}) = \{a, a, a, b, c, d\}$ to roots$(\hat{m}_{nn}) = \{a, a, a, b, b, c\}$, so d is specialized to b. In the generic system this corresponds to replacing $\hat{e}_j = e_j(a, a, a, b, c, d)$ by $\hat{e}_j = e_j(a, a, a, b, b, c)$. Since quite general, $e^n_j(.., u, ..) = e^{n-1}_j(..,,..) + ue^{n-1}_j(..,,..)$, where upper index denotes the number of variables, and ',,' means omission, such a replacement $e^n_j(.., u, ..) \to e^n_j(.., v, ..)$ corresponds to adding $(v-u)e^{n-1}_j(..,,..)$ to $e^n_j(.., u, ..)$; so one has not to change very much in each transition in the generic systems of lemma 6.1d. It also reinforces the belief that 'all roots equal to 1' is the only solution to the generic system, given that we know it is the only solution if we do no *a priori* specialization at all, and put $\hat{e}_j = e_j = e_j(z_1, ..., z_{n-2})$.

Case $n = 6$. Here polynomials $\hat{m}_{56}, \hat{m}_{66}$ are given by

$$\hat{m}_{56} = c_6 - (c_3^2 + 2c_4 + c_5)x + (1 + 4c_3 + c_4)x^2 - (3 + c_3)x^3 + x^4;$$

$$\hat{m}_{66} = c_5 - (2c_3 + 2c_4)x + (3 + 3c_3)x^2 - 4x^3 + x^4;$$

We can assume that roots$(\hat{m}_{5,6}) = \{a, b, g, h\}$ which we do not assume necessarily distinct. We have to show that each of the following subcases 31, 22, 211 implies $a = b = g = h = 1$.

$$\begin{array}{rcl} 4 &=& \hat{e}_1 \\ 3+3c_3 &=& \hat{e}_2 \\ 2c_3+2c_4 &=& \hat{e}_3 \\ c_5 &=& \hat{e}_4 \end{array}, \quad \begin{array}{rcl} 3+c_3 &=& e_1 \\ 1+4c_3+c_4 &=& e_2 \\ c_3^2+2c_4+c_5 &=& e_3 \\ (c_6 &=& e_4), \end{array} \quad \begin{array}{rcl} 0 &=& \hat{e}_1 - 4 \\ 0 &=& \hat{e}_2 - 3e_1 + 6 \\ 0 &=& \hat{e}_3 - 2e_2 + 6e_1 - 16 \\ 0 &=& \hat{e}_4 - e_3 + 2e_2 + e_1^2 \\ && -14e_1 + 31 \end{array}$$

The Vieta formulae yield the two systems of equations at the left, where in subcase 31 one has to read $\hat{e}_k = e_k(a, a, a, b)$, in subcase 22 $\hat{e}_k = e_k(a, a, b, b)$, etc., while in all cases, in the second system $e_k = e_k(a, b, g, h)$, $k = 1, 2, 3, 4$. By considerations as in lemma 6.1, one then arrives at the system at the right. In each of the subcases this is a system purely in a, b, g, h.

Subcase 211 (i.e. aabg): We could solve this subcase by similar systematic technique as the other two subcases below. But it is more illuminating to proceed as follows.

Note that the elementary symmetric functions of four variables can be written in terms of those of three variables as

$$e_j(x_1,x_2,x_3,x_4) = e_j(x_1,x_2,x_3) + x_4 e_{j-1}(x_1,x_2,x_3).$$

So, introducing $\check{e}_j = e_j(a,b,g)$, we find the relations

$$\hat{e}_j = \check{e}_j + a\check{e}_{j-1}, \quad e_j = \check{e}_j + h\check{e}_{j-1}, \quad j = 1,2,3,4,$$

with the conventions $\check{e}_4 = 0, \check{e}_0 = e_0 = 1$. Substituting these in the system above, we get 0 =xpri for i= $1,2,3,4$ below, while 0 =xpr5 is a consequence of a natural algebraic dependence of $a, \check{e}_1, \check{e}_2, \check{e}_3$.

$$\begin{array}{rcl}
0 &=& \text{xpr1} := -4 + a + \check{e}_1, \\
0 &=& \text{xpr2} := 6 - 3\check{e}_1 + a\check{e}_1 + \check{e}_2 - 3h, \\
0 &=& \text{xpr3} := -16 + 6\check{e}_1 - 2\check{e}_2 + a\check{e}_2 + \check{e}_3 + 6h - 2\check{e}_1 h, \\
0 &=& \text{xpr4} := 31 - 14\check{e}_1 + \check{e}_1^2 + 2\check{e}_2 - \check{e}_3 + a\check{e}_3 - 14h + 4\check{e}_1 h - \check{e}_2 h + h^2, \\
0 &=& \text{xpr5} := a^3 - a^2\check{e}_1 + a\check{e}_2 - \check{e}_3.
\end{array}$$

From xpr1, xpr2, xpr3, we successively obtain expressions for $\check{e}_1, \check{e}_2, \check{e}_3$, in terms of a, h; namely

$$\check{e}_1 = 4 - a, \ \ \check{e}_2 = 6 - 7a + a^2 + 3h, \ \ \check{e}_3 = 4 - 14a + 9a^2 - a^3 + 8h - 5ah.$$

Using these in the last two equations, they turn into

$$\begin{array}{rcl}
0 &=& \text{xpr4'} := -1 + 10a - 20a^2 + 10a^3 - a^4 - 6h + 16ah - 6a^2h - 2h^2, \\
0 &=& \text{xpr5'} := -4 + 20a - 20a^2 + 4a^3 - 8h + 8ah = 4(-1+a) \\
 & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 - 4a + a^2 + 2h)
\end{array}$$

Therefore $a = 1$ or $h = (-1 + 4a - a^2)/2$. In the latter case, substituting in xpr4', we find $0 = (3 * (-1 + a)^4)/2$. So $a = 1$ in any case. Then 0 =xpr4' yields $h = 1$. Consequently $\check{e}_1 = 3, \check{e}_2 = 3, \check{e}_3 = 1$. Since the values of the elementary symmetric functions determine the values of their variables up to permutation — this is a consequence of Vietá again — this yields $a = b = g = 1$.

We are somewhat dismayed, that we could not exhibit the following two subcases as specialization to the previous case, and so have to do everything all over again.

Subcase 22 (aabb): In this case the generic system reads

$$\begin{aligned}
0 &= \text{expr1} := -4 + 2a + 2b \\
0 &= \text{expr2} := 6 - 3a + a^2 - 3b + 4ab + b^2 - 3g - 3h \\
0 &= \text{expr3} := -16 + 6a + 6b - 2ab + 2a^2b + 2ab^2 + 6g - 2ag - 2bg + 6h \\
&\qquad -2ah - 2bh - 2gh \\
0 &= \text{expr4} := 31 - 14a + a^2 - 14b + 4ab + b^2 + a^2b^2 - 14g + 4ag + 4bg \\
&\qquad -abg + g^2 - 14h + 4ah + 4bh - abh + 4gh - agh - bgh + h^2
\end{aligned}$$

Again we do the obvious, substituting $b = 2 - a$ in expr2,expr3,expr4, obtaining after multiplication with suitable integers,

$$\begin{aligned}
0 &= \text{expr2n} := 4 + 4a - 2a^2 - 3g - 3h \\
0 &= \text{expr3n} := -4 + 4a - 2a^2 + 2g + 2h - 2gh \\
0 &= \text{expr4n} := 7 + 4a + 2a^2 - 4a^3 + a^4 - 6g - 2ag + a^2g + g^2 - 6h - 2ah \\
&\qquad +a^2h + 2gh + h^2
\end{aligned}$$

Next reducing expr3n and expr4n via expr2n, we get after multiplication with 3 and 9 respectively

$$\begin{aligned}
0 &= \text{expr3n1} = -4 + 20a - 10a^2 - 8g - 8ag + 4a^2g + 6g^2 \\
0 &= \text{expr4n1} = 7 - 28a + 42a^2 - 28a^3 + 7a^4 = 7(-1 + a)^4
\end{aligned}$$

Thus $a = 1$ is a root. From $0 =$expr1, $b = 1$; from $0 =$ expr3n1, $g = 1$, and from $0 =$ expr2n, $h = 1$.

Subcase 31 (aaab): Then the generic equations turn into

$$\begin{aligned}
0 &= \text{expr1} := -4 + 3a + b \\
0 &= \text{expr2} := 6 - 3a + 3a^2 - 3b + 3ab - 3g - 3h \\
0 &= \text{expr3} := -16 + 6a + a^3 + 6b - 2ab + 3a^2b + 6g - 2ag - 2bg + 6h \\
&\qquad -2ah - 2bh - 2gh \\
0 &= \text{expr4} := 31 - 14a + a^2 - 14b + 4ab + a^3b + b^2 - 14g + 4ag + 4bg \\
&\qquad -abg + g^2 - 14h + 4ah + 4bh - abh + 4gh - agh - bgh + h^2
\end{aligned}$$

We first do the obvious: using $0 =$expr1, we eliminate b. With this equations 2,3,4 become

$$\begin{aligned}
0 &= \text{expr2n} := -6 + 18a - 6a^2 - 3g - 3h \\
0 &= \text{expr3n} := 8 - 20a + 18a^2 - 8a^3 - 2g + 4ag - 2h + 4ah - 2gh \\
0 &= \text{expr4n} := -9 + 20a - 2a^2 + 4a^3 - 3a^4 + 2g - 12ag + 3a^2g + g^2 + 2h \\
&\qquad -12ah + 3a^2h + 2agh + h^2
\end{aligned}$$

We eliminate h from (new) expr3n, expr4n via reducing by expr2n. The results are new expressions expr3n1, expr4n1, shown here as the rhs of the following equations.

$$\begin{array}{rcl} 0 &=& \text{expr3n1} := 12-40a+46a^2-16a^3+4g-12ag+4a^2g+2g^2 \\ 0 &=& \text{expr4n1} := -9+32a-40a^2+22a^3-5a^4+4g-16ag+16a^2g \\ && \qquad -4a^3g+2g^2-2ag^2 \end{array}$$

Next, we reduce expr4n1 via expr3n1 obtaining

$$0=-21+84a-126a^2+84a^3-21a^4=-21(-1+a)^4.$$

Thus $a=1$ is a root. $0=$expr1 implies $b=1$. Then $0=$expr3n1 yields $g=1$ and this, then yields $h=1$ from $0=$expr4.

This concludes the proof of the case $n=6$.

If one does this case relying on automatic Groebner basis computations instead of interactivity it can be done within seconds.

The cases $n=7,8$ are currently viable only by computer.

Case $n=7$. In this case the generic system takes the form

$$\begin{array}{rcl} 0 &=& -5+\hat{e}_1, \\ 0 &=& 10-4e_1+\hat{e}_2, \\ 0 &=& -40+12e_1-3e_2+\hat{e}_3, \\ 0 &=& 150-54e_1+3e_1^2+6e_2-2e_3+\hat{e}_4, \\ 0 &=& -376+164e_1-16e_1^2-16e_2+2e_1e_2+2e_3-e_4+\hat{e}_5 \end{array}$$

Departing from here we did Groebner basis computations. We assume $\text{roots}(\hat{m}_{n-1,n}) = \{a,b,g,h,l\}$. We need to explore the several cases $\text{roots}(\hat{m}_{n,n}) \subseteq \{a,b,g,h,l\}$. The subcases are $5,41,32,311,221,2111,11111$, but the first and the last case need not be considered.

Let `ls` denote the list of polynomials on the rhs of above system. In any given case, read $\hat{e}_j$ as being obtained by substituting in $e_j(\underline{x})$, $\underline{x}=(x_1,x_2,x_3,x_4,x_5,x_6)$, by the corresponding sequence of roots; e.g. in case 411, in the list `ls` $\hat{e}_j=e_j(a,a,a,a,b,g)$; while in all cases, $e_j=e_j(a,b,g,h,l,m)$. In each case issue the Mathematica© command

```
gb=GroebnerBasis[ls,{l,h,g,b,a}];
```

The result is that a Groebner basis corresponding to inverse lex order is given for the ideal generated by ls. It would be too space consuming to give the full bases, so we limit ourselves to indicate the statistics for these cases. ‘Time’ indicates the time it took to compute `gb` , ‘NpolysGb’ is the number of polynomials in the Groebner basis found, ‘Lengths’ gives the list of the numbers of terms the polynomials in `gb` comprise, Factorization: gives the factorization of the first element in `gb` (this turned out to be always a polynomial in a only), finally max.coeff gives the modulus of the largest coefficient in any of the polynomials of `gb`.

Subcase 41 $\{a, a, a, a, b\}$: Time: 1s. NpolysGb: 5.
Lengths: $\{6, 3, 13, 13, 6\}$. Factorization: $(-1 + a)^5$ max.coeff: 6330.

Subcase 32 $\{a, a, a, b, b\}$: Time: 1s. NpolysGb: 5.
Lengths: $\{6, 3, 13, 13, 6\}$. Factorization: $(-1 + a)^5$ max.coeff: 18870.

Subcase 311 $\{a, a, a, b, g\}$: Time: 1.5s. NpolysGb: 6
Lengths: $\{8, 15, 15, 4, 16, 8\}$ Factorization: $(-1+a)^7$ max.coeff: 2715.

Subcase 221 $\{a, a, b, b, g\}$: Time: 2.04s. NpolysGb: 8
Lengths: $\{9, 15, 18, 18, 15, 4, 17, 8\}$ Factorization: $(-1 + a)^8$.
max.coeff: 4060850500.

Subcase 2111 $\{a, a, b, g, h\}$: Time: 3.46. NpolysGb: 8.
Lengths: $\{8, 22, 21, 28, 18, 31, 5, 11\}$. Factorization: $(-1 + a)^7$.
max.coeff: 9768.

In each of these cases one proceeds, given `gb`, by showing that the only solution to the system obtained by putting the polynomials of `gb` all equal to 0, is $a = b = g = h = l = 1$. This is done somewhat analogously as in the case $n = 6$ treated before. First, $(-1 + a)^{k_1} = 0$ allows us to say that every solution has $a = 1$. Using this a certain polynomial in `gb` specializes to $(-1 + b)^{k_2}$, so $b = 1$. Next using $a = b = 1$ one gets in `gb` a polynomial of the form $(-1 + g)^{k_3}$, so $g = 1$, etc.

Case $n = 8$. Here the generic system takes the form

$$\begin{aligned}
0 &= -6 + \hat{e}_1\\
0 &= 15 - 5e_1 + \hat{e}_2\\
0 &= -80 + 20e_1 - 4e_2 + \hat{e}_3\\
0 &= 441 - 132e_1 + 6e_1^2 + 12e_2 - 3e_3 + \hat{e}_4\\
0 &= -2076 + 750e_1 - 60e_1^2 - 60e_2 + 6e_1e_2 + 6e_3 - 2e_4 + \hat{e}_5\\
0 &= 6392 - 2740e_1 + 314e_1^2 - 6e_1^3 + 210e_2 - 36e_1e_2 + e_2^2 - 18e_3 + 2e_1e_3\\
&\quad +2e_4 - e_5 + \hat{e}_6.
\end{aligned}$$

In this case we assume the possible roots for $\hat{m}_{78}$ are named a, b, g, h, l, m. We have to consider the subcases

6, 51,42,411,33,321,3111, 222,2211,21111,111111, and, as always, discard the first and last again. The statistics for these cases, using the pattern familiar from the case $n = 7$ reads as follows. Note that most of the cases took less than 15 seconds to compute, only one took about 3 minutes.

Subcase 51. $\{a, a, a, a, a, b\}$: Time: 6s NpolysGb: 6.
Lengths: $\{7, 3, 19, 26, 19, 7\}$ Factorization: $(-1+a)^6$ max.coeff: 3942.

Subcase 42: $\{a, a, a, a, b, b\}$ Time: 7s. NpolysGb: 6.
Lengths: $\{7, 3, 18, 24, 18, 7\}$. Factorization $(-1+a)^6$ max.coeff: 1512.

Subcase 411: $\{a, a, a, a, b, g\}$. Time: 8s. NpolysGb: 8
Lengths: $\{10, 21, 20, 25, 4, 31, 24, 9\}$. Factorization $(-1 + a)^9$.
max.coeff: 42452.

Subcase 33: $\{a, a, a, b, b, b\}$ Time: 6s. NpolysGb: 6
Lengths: $\{7, 3, 17, 24, 18, 7\}$ Factorization $(-1 + a)^6$ max.coeff: 253.

Subcase 321: $\{a, a, a, b, b, g\}$ Time: 10s. NpolysGb: 10.
Lengths: $\{11, 19, 24, 26, 25, 21, 4, 32, 25, 9\}$. Factorization $(-1 + a)^{10}$.
max.coeff: 2897703183496025.

Subcase 3111: $\{a, a, a, b, g, h\}$ Time: 37s. NpolysGb: 14
Lengths: $\{11, 34, 44, 42, 43, 42, 55, 63, 58, 46, 35, 5, 29, 12\}$.
Factorization $(-1 + a)^{10}$. max.coeff: 126166071850.

Subcase 222: $\{a, a, b, b, g, g\}$ Time: 9s. NpolysGb: 8.
Lengths: $\{10, 21, 19, 25, 4, 30, 23, 9\}$. Factorization $(-1 + a)^9$.
max.coeff: 38120.

Subcase 2211: $\{a, a, b, b, g, h\}$. Time: 96s. NpolysGb: 16.
Lengths: $\{13, 23, 30, 37, 40, 44, 42, 36, 61, 64, 58, 46, 35, 5, 30, 12\}$.
Factorization $(-1 + a)^{12}$. max.coeff: 9727786053411235885.

Subcase 21111: $\{a, a, b, g, h, l\}$ Time: 188s. NpolysGb: 20.
Lengths: $\{10, 39, 42, 43, 42, 70, 98, 63, 96, 92, 98, 107, 82, 73, 85, 103, 32, 64, 6, 16\}$.
Factorization $(-1 + a)^9$. max.coeff: 1327205985.

One can finish each of these cases in a similar manner as in the case $n = 7$, showing this way that $a = b = g = h = l = m = 1$ is always the only solution. This way one establishes R8C. $\square$

7. Delimitations and other approaches tried

We report briefly on examples showing that certain reasonable generalizations of HPnC are false and also on approaches that may in the hands of others lead to some success, although we could not make them working.

Example 7.1. Two natural generalization of HPnC are false. Consider the symmetric matrix $S(x)$ and the Hankel matrix $H(x)$ below.

$$S(x) = \begin{bmatrix} & & x & c_1 & c_2 \\ & x & c_1 & c_2 & c_3 \\ x & c_1 & c_2 & -1 & 4 \\ c_1 & c_2 & -1 & 4 & 2 \\ c_2 & c_3 & 4 & 2 & 1 \end{bmatrix}, \; H(x) = \begin{bmatrix} 0 & x & -0.41 & 0.41 & -1 \\ x & -0.41 & 0.41 & -1 & 1 \\ -0.41 & 0.41 & -1 & 1 & -1 \\ 0.41 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix}$$

The system of equations that arises from requiring that the polynomial $\det S(x) \in \mathbb{C}[x]$ be 0 is solvable with

$$\begin{aligned} c_1 &\approx -0.00446226068514 3479, \\ c_2 &\approx -0.0873040997792691, \\ c_3 &\approx 0.831366078454159. \end{aligned}$$

With the concrete values given here the coefficients of $\det S(x)$ are in modulus all$\leq 10^{-14}$, but the last two columns evidently are not dependent. In the case of $H(x)$ read 0.41 as 0.4142135623730951. Then $H(x)$ is singular for all practical effects but the determinant of the right upper 3×3 matrix is 0.3431457505076199, so the last three columns are not dependent.

Remarks 7.2. Several approaches come to mind if one works on HPnC.
(a) Can one not try to prove HPnC inductively? We attempted to do so, but in fact, even backward induction $(*)$ HPnC$\Rightarrow$HP(n-1)C seems to be hard. Assuming HPnC there seems to be no easy way to see HP(n-1)C. Write as in section

$$\det H_n(x) = \sum_{j=0}^{n-2} cx_j \cdot x^j$$

and write

$$\det H_{n-1}(x) = \sum_{j=0}^{n-3} cx'_j \cdot x^j.$$

Let I =ideal($\{cx_0, ..., cx_{n-2}\}$). Experiments done for $n = 3, 4, 5$ indicate $(cx'_j)^{n-1} \in I$. If this is true in general then one would have $(*)$. We have no general proof, but even if we had, its value would be questionable.
(b) Another approach the authors tried includes to begin with a strong hypothesis and to weaken it gradually.

$$H_n(x_1, \dots, x_{n-2}) = \begin{bmatrix} & & & x_1 & c_1 & c_2 \\ & & x_2 & c_1 & c_2 & c_3 \\ & & & & \vdots & \vdots \\ x_{n-2} & c_1 & \dots & \dots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & \dots & \dots & c_{n-1} & c_n \\ c_2 & c_3 & \dots & \dots & c_n & c_{n+1} \end{bmatrix},$$

Namely, one may try considering the non-Hankel-matrix at the left and ask whether assuming $\det H_n(x_1, \dots, x_{n-2}) \equiv 0$ implies that the last two columns are dependent. This is actually easy; but the hypothesis is strong. One then could try adding gradually more and more equations of the form $x_i = x_j$, weakening thus the hypothesis, and see to which extent one still can deduce the desired conclusion. This appears to be a promising approach but as yet the authors have failed here as well.
(c) Finally, there is an approach that dispenses with considering the determinant altogether. What are the consequences of assuming that there exists a vector function

$$\mathbb{C} \ni x \mapsto v(x) \in S^{n-1} := \{(z_1, \dots, z_n)^T \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 = 1\}$$

such that $H_n(x)v(x) \equiv 0 \in \mathbb{C}^n$? Evidently this hypothesis is equivalent to $\det H_n(x) \equiv 0$. We originally thought to have a proof of HPnC based on this idea, but later found an error.

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