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JACOBI-NIJENHUIS ALGEBROIDS AND THEIR MODULAR CLASSES

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Abstract: Jacobi-Nijenhuis algebroids are defined as a natural generalization of Poisson-Nijenhuis algebroids, in the case where there exists a Nijenhuis operator on a Jacobi algebroid which is compatible with it. We study modular classes of Jacobi and Jacobi-Nijenhuis algebroids.

Keywords: Jacobi algebroid, Jacobi-Nijenhuis algebroid, modular class.
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1. Introduction

It is well known that the cotangent bundle T^*M of any Poisson manifold M admits a Lie algebroid structure and the pair (TM, T^*M) is a Lie bialgebroid over M. As a kind of reciprocal result, any Lie bialgebroid (A, A^*) induces a Poisson structure on its base manifold. A special kind of Lie bialgebroids are the triangular Lie bialgebroids. These are Lie algebroids (A, P) equipped with an A-bivector field P such that $[P, P] = 0$. The A-bivector field P induces a Lie algebroid structure on the dual vector bundle A^* so that the pair (A, A^*) is a Lie bialgebroid. Triangular bialgebroids are also called Lie algebroids with a Poisson structure.

If one moves from the Poisson to the Jacobi framework, these statements are not true. In fact, if M is a Jacobi manifold, its cotangent bundle T^*M is not, in general, a Lie algebroid. In order to associate a Lie algebroid to a Jacobi manifold, one has to consider the 1-jet bundle $T^*M \times \mathbb{R} \to M$. However, if we take the dual vector bundle $TM \times \mathbb{R} \to M$ endowed with its natural Lie algebroid structure, the pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ is not a Lie bialgebroid. Motivated by this, Iglesias and Marrero [7] and Grabowski and Marmo [3] introduced the concepts of Jacobi algebroid, i.e. a Lie algebroid A with a 1-cocycle ϕ, and of Jacobi bialgebroid, i.e. a pair $((A, \phi), (A^*, X))$ of Jacobi algebroids in duality satisfying a compatibility condition. Jacobi bialgebroids admit Lie bialgebroids as particular cases and are well adapted to

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the Jacobi context since every Jacobi manifold (M, Λ, E) has an associated Jacobi bialgebroid, $((TM \times \mathbb{R}, (0,1)), (T^*M \times \mathbb{R}, (-E, 0)))$. Imitating the Poisson case, Iglesias and Marrero introduced in [7] the notion of triangular Jacobi bialgebroid, as follows. If (A, ϕ) is a Jacobi algebroid and P is a Jacobi bivector field, i.e. an A-bivector field such that $[P, P]^{\phi_0} = 0$, then there exists a Lie algebroid structure on A^* with a 1-cocycle such that the pair of Jacobi algebroids in duality is a Jacobi bialgebroid. As it happens in the Poisson case, the base manifold of any Jacobi bialgebroid inherits a Jacobi structure.

On the other hand, Poisson-Nijenhuis structures on Lie algebroids, i.e. Poisson-Nijenhuis algebroids, were introduced by Grabowski and Urbanski in [6], as Lie algebroids equipped with a Poisson structure and a Nijenhuis operator fulfilling some compatibility conditions. In the first part of this paper, we extend this concept to the Jacobi framework and we study *Jacobi-Nijenhuis algebroids*.

The other main goal of this paper is to study modular classes, including modular classes of Jacobi-Nijenhuis algebroids. The modular class of a Poisson manifold was defined by Weinstein in [19], as an analogue in Poisson geometry of the modular automorphism group of a von Neumann algebra. In [2], Evens, Lu and Weinstein introduced the notion of modular class of a Lie algebroid A over M, using a representation of A on the line bundle $Q_A = \mathfrak{X}^{\text{top}}(A) \otimes \Omega^{\text{top}}(M)$. For the case of the cotangent Lie algebroid T^*M of a Poisson manifold M, they showed that its modular class is twice the modular class of M, in the sense of [19]. Modular classes of triangular Lie bialgebroids were studied in [11], from the point of view of generating operators for Batalin-Vilkovisky algebras.

Regarding the Jacobi context, the first work on modular classes is due to Vaisman, who introduced in [18] the concept of modular class of a Jacobi manifold. Then, in [8], modular classes of triangular Jacobi bialgebroids were studied.

In the second part of this paper, we consider a Jacobi algebroid (A, ϕ) and we define, using the 1-cocycle ϕ, a new representation of A on the line bundle $Q_A = \mathfrak{X}^{\text{top}}(A) \otimes \Omega^{\text{top}}(M)$, which leads to the definition of *modular class of a Jacobi algebroid*.

Modular classes of Poisson-Nijenhuis algebroids were defined in [1]. Inspired in [1], we define *modular class of a Jacobi-Nijenhuis algebroid*. We

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The vector field

$$\mathcal{M}_{(N,P)} = \mathcal{M}^{\nu}_{(A_N,\phi_1,P)} - N\mathcal{M}^{\nu}_{(A,\phi_0,P)}$$

does not depend on the top-section of A^* chosen and is related with $X_{(N,P)}$ by

$$\langle \alpha, \mathcal{M}_{(N,P)} \rangle = \langle \alpha, X_{(N,P)} \rangle + i_P \mathrm{d}_N \alpha - i_P \mathrm{d} N^* \alpha. \tag{58}$$

Example 23. *Consider a Jacobi-Nijenhuis manifold $(M, (\Lambda, E), \mathcal{N})$. The modular class of the Jacobi manifold $(M, (\Lambda, E))$ is defined by (see* [18, 8]*)*

$$2\left[V^{(\Lambda,E)}\right] = \mod (T^*M \times \mathbb{R}) - (n+1)\,[(E,0)]$$

so

$$\left[V^{\mathcal{N}(\Lambda,E)}\right] - \mathcal{N}\left[V^{(\Lambda,E)}\right] = \frac{1}{2}\left[\mathrm{d}_*(\mathrm{Tr}\,\mathcal{N})\right] = \frac{1}{2} \mod {}^{(\mathcal{N},(\Lambda,E))}(T^*M \times \mathbb{R})$$

and we have the analogous relation as in the Poisson case.

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obtain a hierarchy of vector fields on the Jacobi algebroid that covers a hierarchy of Jacobi structures on the base.

The paper is divided into five sections. Section 2 is devoted to Jacobi algebroids. We recall how to obtain a Lie algebroid structure on $A \times \mathbb{R}$ over $M \times \mathbb{R}$ from a Jacobi algebroid (A, ϕ) over M [7, 3]. The notion of compatibility of two Jacobi bivectors on a Jacobi algebroid is introduced, and we prove that these Jacobi bivectors cover two compatible Jacobi structures on the base manifold. In Section 3 we define Jacobi-Nijenhuis algebroid and we show that a Jacobi-Nijenhuis algebroid defines a hierarchy of compatible Jacobi bivectors on the Jacobi algebroid and a hierarchy of compatible Jacobi structures on the base manifold. Moreover, the dual vector bundle also inherits a hierarchy of Jacobi algebroid structures that provides the existence of a family of triangular Jacobi bialgebroids. As a particular case of this construction, we recover the notion of strong (or strict) Jacobi-Nijenhuis manifold [9, 17]. In Section 4, we introduce the notion of modular class of a Jacobi algebroid and we discuss the relation between modular class of a Jacobi algebroid (A, ϕ) over M and modular class of the Lie algebroid $A \times \mathbb{R}$ over $M \times \mathbb{R}$. Relations between modular forms of A^* and $A^* \times \mathbb{R}$, in the triangular case, as well as duality between modular classes of A and A^* are also discussed. At this point we relate our results with those obtained in [8]. In Section 5, we give the definition of modular class of a Jacobi-Nijenhuis algebroid and we prove a result which generalizes the corresponding one of [1]: there exists a hierarchy of A-vector fields that defines two hierarchies of vector fields, one on $M \times \mathbb{R}$ and another on M. These hierarchies determine a family of Jacobi structures on the manifold M.

Notation and conventions: Let $(A, \rho, [\,,\,])$ be a Lie algebroid over M. We denote by $\mathfrak{X}^k(A)$ (resp. $\Omega^k(A)$) the $C^\infty(M)$-module of A-k-vector fields (resp. A-k-forms), by $\mathfrak{X}(A) = \oplus_k \mathfrak{X}^k(A)$ (resp. $\Omega(A) = \oplus_k \Omega^k(A)$) the corresponding Gerstenhaber algebra of A-multivector fields (resp. A-forms) and by $\mathfrak{X}^{\mathrm{top}}(A)$ the top-degree sections of A. The DeRham differential is denoted by d while d stands for the Lie algebroid differential.

Regarding the conventions of sign for the Schouten bracket and for the interior product by a multivector field, we use the same conventions of [3, 11], which are different from those of [7, 8].

2. Jacobi algebroids

We begin by recalling some well known facts about Jacobi algebroids.

2.1. Jacobi algebroids. A *Jacobi algebroid* [3] or *generalized Lie algebroid* [7] is a pair (A, ϕ_0) where $A = (A, [\,,\,], \rho)$ is a Lie algebroid over a manifold M and $\phi_0 \in \Omega^1(A)$ is a 1-cocycle in the Lie algebroid cohomology with trivial coefficients, $\mathrm{d}\phi_0 = 0$. A Jacobi algebroid has an associated Schouten-Jacobi bracket on the graded algebra $\mathfrak{X}(A)$ of multivector fields on A given by

$$[P,Q]^{\phi_0} = [P,Q] + (p-1)P \wedge i_{\phi_0}Q - (-1)^{p-1}(q-1)i_{\phi_0}P \wedge Q, \tag{1}$$

for $P \in \mathfrak{X}^p(A), Q \in \mathfrak{X}^q(A)$.

This bracket $[\,,\,]^{\phi_0}$ satisfies the following properties (in fact it is totaly defined by them), with $X, Y \in \mathfrak{X}^1(A)$, $P \in \mathfrak{X}^p(A)$, $Q \in \mathfrak{X}^q(A)$ and $f \in C^\infty(M)$:

$$[X,f]^{\phi_0} = \rho^{\phi_0}(X)f, \tag{2}$$

$$[X,Y]^{\phi_0} = [X,Y]\,, \tag{3}$$

$$[P,Q]^{\phi_0} = -(-1)^{(p-1)(q-1)}\,[Q,P]^{\phi_0}\,, \tag{4}$$

$$[P,Q\wedge R]^{\phi_0} = [P,Q]^{\phi_0}\wedge R + (-1)^{(p-1)q}Q\wedge[P,R]^{\phi_0} - (-1)^{p-1}i_{\phi_0}P\wedge Q\wedge R, \tag{5}$$

$$\begin{aligned}(-1)^{(p-1)(r-1)}\left[P,[Q,R]^{\phi_0}\right]^{\phi_0} + (-1)^{(q-1)(p-1)}\left[Q,[R,P]^{\phi_0}\right]^{\phi_0} \\ + (-1)^{(r-1)(q-1)}\left[R,[P,Q]^{\phi_0}\right]^{\phi_0} = 0.\end{aligned} \tag{6}$$

In property (2), ρ^{ϕ_0} is the representation of the Lie algebra $\mathfrak{X}^1(A)$ on $C^\infty(M)$ given by

$$\rho^{\phi_0}(X)f = \rho(X)f + f\langle\phi_0, X\rangle.$$

The cohomology operator d^{ϕ_0} associated with this representation is called ϕ_0-*differential* of A and is given by

$$\mathrm{d}^{\phi_0}\omega = \mathrm{d}\omega + \phi_0 \wedge \omega, \quad \omega \in \Omega(A). \tag{7}$$

With the ϕ_0-differential we can define a ϕ_0-*Lie derivative*:

$$\mathcal{L}^{\phi_0}_X\omega = i_X\,\mathrm{d}^{\phi_0}\omega + (-1)^{p-1}\mathrm{d}^{\phi_0}\,i_X\omega, \quad X \in \mathfrak{X}^p(A), \omega \in \Omega(A). \tag{8}$$

In [7, 4] we can find a construction which allow us to obtain a Lie algebroid over $M \times \mathbb{R}$ from a Jacobi algebroid over M. This construction is very useful when we speak about Jacobi algebroids, in fact it contains the essence of philosophy adopted in the proofs in this paper, so we will explain it now.

This hierarchy covers a hierarchy of vector fields on M given by

$$X^{i+j}_M = -(N^iP)^\sharp_M(dh_j) = -(N^jP)^\sharp_M(dh_i), \tag{55}$$

and defines a hierarchy of vector fields on the Lie algebroid $TM \times \mathbb{R}$ given by

$$Y^{i+j} = X^{i+j}_M + \langle dh_j, E^i_M\rangle\frac{\partial}{\partial t}, \tag{56}$$

where $((N^iP)_M, E^i_M)$ are the Jacobi structures on M induced by the Jacobi bivectors N^iP on A.

Remark 22. *Some remarks should be made at this point. First, one should notice that even if N is degenerated the hierarchy exists but only for $i+j>1$, i.e.,*

$$X^{i+j}_{(N,P)} = \mathrm{d}_{N^iP}h_j = \mathrm{d}_{N^{j-1}P}h_{i+1}, \quad (0 \le i < j, 1 < j).$$

In case N is degenerated we can always consider a non-degenerated Nijenhuis operator of the form $N+\lambda I$, λ constant, and we obtain the same algebra of commuting integrals.

It is also important to observe that although the hierarchy of vector fields on A is defined by a Nijenhuis operator, we may not have a Nijenhuis operator on M nor on $M \times \mathbb{R}$ that generates neither one of the covered hierarchies.

We will finish with a relation between the sequence of modular vector fields of the Jacobi-Nijenhuis algebroid and the sequence of modular vector fields of the Jacobi bialgebroid (in the sense of [8]).

First recall the relation (44):

$$\mathcal{M}^\nu_{(A,\phi_0,P)}(\alpha) = \xi^{\nu\otimes\mu}_{A^*}(\alpha) + X_0(\alpha) + i_P\mathrm{d}\alpha - \mathrm{div}_\mu(\rho\circ P^\sharp(\alpha)). \tag{57}$$

Now we have

$$\begin{aligned}\mathcal{M}^\nu_{(A_N,\phi_1,P)}(\alpha) - N\mathcal{M}^\nu_{(A,\phi_0,P)}(\alpha) &= \xi^{\nu\otimes\mu}_{A^*_N}(\alpha) + X_1(\alpha) + i_P\mathrm{d}_N\alpha \\ &\quad - \mathrm{div}_\mu(\rho_N\circ P^\sharp(\alpha)) - N(\xi^{\nu\otimes\mu}_{A^*})(\alpha) - NX_0(\alpha) \\ &\quad - i_P\mathrm{d}N^*\alpha + \mathrm{div}_\mu(\rho\circ P^\sharp(N^*\alpha))) \\ &= \langle\alpha, \mathrm{d}_P(\mathrm{Tr}\,N)\rangle + i_P\mathrm{d}_N\alpha - i_P\mathrm{d}N^*\alpha\end{aligned}$$

or equivalently, since $i_{NP}\mathrm{d} = i_P\mathrm{d}_N$,

$$\begin{aligned}\mathcal{M}^\nu_{(A_N,\phi_1,P)}(\alpha) - N\mathcal{M}^\nu_{(A,\phi_0,P)}(\alpha) &= \\ &= \langle\alpha, \mathrm{d}_P(\mathrm{Tr}\,N)\rangle + i_{NP}\mathrm{d}\alpha - i_P\mathrm{d}N^*\alpha \\ &= \mathcal{M}^\nu_{(A,\phi_0,NP)}(\alpha) - N\mathcal{M}^\nu_{(A,\phi_0,P)}(\alpha).\end{aligned}$$

where

$$h_0 = \ln(\det N) \quad \text{and} \quad h_i = \frac{1}{i}\operatorname{Tr} N^i, \quad (i \neq 0, \quad i, j \in \mathbb{Z}). \tag{51}$$

These hierarchies cover two hierarchies, one on $M \times \mathbb{R}$ and another one on M:

The hierarchy on M is given by

$$X_M^{i+j} = \rho(X_{(N,P)}^{i+j}) = -P_M^{i\sharp}(dh_j) = -P_M^{j\sharp}(dh_i) \tag{52}$$

and the hierarchy on $M \times \mathbb{R}$ is given by

$$\begin{aligned}
\hat{X}_{i+j} &= \hat{\rho}(\hat{X}_{(N,\tilde{P})}^{i+j}) = \hat{\rho}(N^{i+j-1}\hat{X}_{(N,\tilde{P})}) \\
&= e^{-t}\hat{\rho}(X_{(N,P)}^{i+j}) \\
&= e^{-t}\left(\rho(X_{(N,P)}^{i+j}) + \langle \phi_0, X_{(N,P)}^{i+j}\rangle \frac{\partial}{\partial t}\right) \\
&= e^{-t}\left(X_M^{i+j} + \langle \mathrm{d}h_j, N^i P(\phi_0)\rangle \frac{\partial}{\partial t}\right) \\
&= e^{-t}\left(X_M^{i+j} + \langle dh_j, E_M^i\rangle \frac{\partial}{\partial t}\right) \\
&= e^{-t}\left(-(N^iP)_M^\sharp(dh_j) + \langle dh_j, E_M^i\rangle \frac{\partial}{\partial t}\right),
\end{aligned}$$

where $((N^iP)_M, E_M^i)$ is the Jacobi structure on M induced by the Jacobi algebroid (A, ϕ_0, N^iP) (see (23) and (24)).

This way we have proven the next theorem, which is a generalization to Jacobi-Nijenhuis algebroids of the analogous result for Poisson-Nijenhuis Lie algebroids [1] (see [15, 13] for the Poisson-Nijenhuis manifold case).

Theorem 21. *Let (A, ϕ_0, P, N) be a Jacobi-Nijenhuis algebroid with N a non-degenerated Nijenhuis operator compatible with P. Then the modular vector field $X_{(N,P)}$ is a d_{NP}-coboundary and determines a hierarchy of vector fields*

$$X_{(N,P)}^{i+j} = N^{i+j-1}X_{(N,P)} = \mathrm{d}_{N^iP}h_j = \mathrm{d}_{N^jP}h_i, \quad (i, j \in \mathbb{Z}) \tag{53}$$

where

$$h_0 = \ln(\det N) \quad \textit{and} \quad h_i = \frac{1}{i}\operatorname{Tr} N^i, \quad (i \neq 0). \tag{54}$$

Consider the natural vector bundle $\hat{A} = A \times \mathbb{R}$ over $M \times \mathbb{R}$. The sections of $\hat{A}$ may be seen as time-dependent sections of A and this space is generated as a $C^\infty(M \times \mathbb{R})$-module by the space of sections of A, which are simply the time-independent sections of $\hat{A}$.

The anchor

$$\hat{\rho}(X) = \rho(X) + \langle \phi_0, X\rangle \frac{\partial}{\partial t}, \quad X \in \mathfrak{X}^1(A), \tag{9}$$

and the bracket defined by $[\,,\,]$ for time independent multivectors

$$[X, Y]_{\hat{A}} = [X, Y]\,, \quad X, Y \in \mathfrak{X}(A), \tag{10}$$

define a Lie algebroid structure on $\hat{A}$ that we call the *induced Lie algebroid structure from A by ϕ_0*. If $\hat{\mathrm{d}}$ is the differential in $\hat{A}$, from (9) we get

$$\phi_0 = \hat{\mathrm{d}}t, \tag{11}$$

which means that the 1-cocycle ϕ_0 can be seen as an exact 1-form on $\hat{A}$.

Considering the gauging in $\mathfrak{X}(A)$ defined by

$$\tilde{X} = e^{-(p-1)t}X, \quad X \in \mathfrak{X}^p(A),$$

we have the following relation between the Lie bracket in $\mathfrak{X}(\hat{A})$ and the Jacobi bracket (1):

$$\left[\tilde{X}, \tilde{Y}\right]_{\hat{A}} = \widetilde{[X, Y]^{\phi_0}}. \tag{12}$$

Now consider a *Jacobi bivector* on A, i.e., a bivector $P \in \mathfrak{X}^2(A)$ such that

$$[P, P]^{\phi_0} = 0. \tag{13}$$

From relation (12) we deduce that $\tilde{P} = e^{-t}P$ is a Poisson bivector on $\hat{A}$ and, consequently, it defines a Lie algebroid structure over $M \times \mathbb{R}$ on $\hat{A}^*$ given by

$$[\alpha, \beta]_{\tilde{P}} = \widehat{\mathcal{L}}_{\tilde{P}\sharp_\alpha}\beta - \widehat{\mathcal{L}}_{\tilde{P}\sharp_\beta}\alpha - \hat{\mathrm{d}}\tilde{P}(\alpha, \beta), \tag{14}$$

$$\hat{\rho}_*(\alpha) = \hat{\rho} \circ \tilde{P}^\sharp(\alpha) \tag{15}$$

where $\alpha, \beta \in \mathfrak{X}^1(\hat{A}^*)$ and $\widehat{\mathcal{L}}$ is the Lie derivative in $\hat{A}$. In particular, for $\alpha,\, \beta \in \mathfrak{X}^1(A^*)$, we have

$$\left[e^t\alpha, e^t\beta\right]_{\tilde{P}} = e^t(\mathcal{L}_{P\sharp_\alpha}^{\phi_0}\beta - \mathcal{L}_{P\sharp_\beta}^{\phi_0}\alpha - \mathrm{d}^{\phi_0}P(\alpha, \beta)). \tag{16}$$

The Lie bracket

$$[\alpha, \beta]_P = \mathcal{L}_{P\sharp_\alpha}^{\phi_0}\beta - \mathcal{L}_{P\sharp_\beta}^{\phi_0}\alpha - \mathrm{d}^{\phi_0}P(\alpha, \beta), \tag{17}$$

together with the anchor

$$\rho_* = \rho \circ P^\sharp, \tag{18}$$

endows A^* with a Lie algebroid structure over M.

The section on A, $X_0 = -P^\sharp(\phi_0)$ is a 1-cocycle of A^*, and so (A^*, X_0) is a Jacobi algebroid. The pair $((A,\phi_0),(A^*,X_0))$ is a special kind of Jacobi bialgebroid called *triangular Jacobi bialgebroid* and we will denote it by (A,ϕ_0,P).

Recall that a *Jacobi bialgebroid* (see [7], [3]) is a pair of Jacobi algebroids in duality, $((A,\phi_0),(A^*,X_0))$, such that $\mathrm{d}_*^{X_0}$ is a derivation of $(\mathfrak{X}(A),[\,,\,]^{\phi_0})$ or, equivalently, d^{ϕ_0} is a derivation of $(\mathfrak{X}(A^*),[\,,\,]_*^{X_0})$.

The relation (16) can be generalized to multisections of A^* if we consider the gauging in $\Omega(A)$:

$$\hat{\omega} = e^{pt}\omega, \quad \omega \in \Omega^p(A). \tag{19}$$

Proposition 1. *Let α,β be multisections of A^*. Then*

$$\left[\hat{\alpha},\hat{\beta}\right]_{\tilde{P}} = \widehat{[\alpha,\beta]_P}. \tag{20}$$

One should also notice that the structure of Lie algebroid on $\hat{A}^*$ does not coincide with Lie algebroid structure induced from A^* by the 1-cocycle X_0 (at least not in the same way it was done with A and ϕ_0). In fact, the bracket of two time independent sections on $\hat{A}^*$, $\alpha,\beta \in \Omega^1(A)$, is given by

$$[\alpha,\beta]_{\tilde{P}} = e^{-t}([\alpha,\beta]_P - \langle\alpha,X_0\rangle\beta + \langle\beta,X_0\rangle\alpha) \tag{21}$$

and the anchor of $\hat{A}^*$ is defined by

$$\hat{\rho}_*(\alpha) = e^{-t}\left(\rho_*(\alpha) + \langle\alpha,X_0\rangle\frac{\partial}{\partial t}\right).$$

Any Jacobi bialgebroid $((A,\phi_0),(A^*,X_0))$ gives to M a structure of Jacobi manifold, i.e., it equips M with a bivector field P_M and a vector field E_M satisfying

$$[P_M,P_M] = -2E_M \wedge P_M, \quad [E_M,P_M] = 0, \tag{22}$$

or, equivalently, it defines a Jacobi bracket on $C^\infty(M)$ given by:

$$\{f,g\}_M = \langle \mathrm{d}^{\phi_0} f, \mathrm{d}_*^{X_0} g\rangle.$$

Since $(\hat{A},\tilde{P},N)$ is a Poisson-Nijenhuis Lie algebroid it has a modular vector field (see [1]) given by

$$\begin{aligned}\hat{X}_{(N,\tilde{P})} &= \xi_{\hat{A}^*_{N^*}} - N\xi_{\hat{A}^*}\\ &= \hat{\mathrm{d}}_{\tilde{P}}(\operatorname{Tr} N) = -\tilde{P}^\sharp(\hat{\mathrm{d}}\operatorname{Tr} N)\\ &= -e^{-t}P^\sharp(d\operatorname{Tr} N) = e^{-t}\mathrm{d}_P(\operatorname{Tr} N).\end{aligned}$$

This $\hat{A}$-vector field is independent of the $Q_{\hat{A}}$-section considered to compute the modular vector fields $\xi_{\hat{A}^*_{N^*}}$ and $\xi_{\hat{A}^*}$. So the equation (39) implies

$$\hat{X}_{(N,\tilde{P})} = e^{-t}(\xi^{\nu\otimes\mu}_{A^*_{N^*}} - N\xi^{\nu\otimes\mu}_{A^*})$$

and equation (40) implies

$$\hat{X}_{(N,\tilde{P})} = e^{-t}(\xi^{X_0,\nu\otimes\mu}_{A^*_{N^*}} - N\xi^{X_0,\nu\otimes\mu}_{A^*}),$$

therefore

$$\xi^{X_0,\nu\otimes\mu}_{A^*_{N^*}} - N\xi^{X_0,\nu\otimes\mu}_{A^*} = \mathrm{d}_P(\operatorname{Tr} N). \tag{50}$$

This relation motivates the next definition.

Definition 19. *The **modular vector field** of the Jacobi-Nijenhuis algebroid (A,ϕ_0,P,N) is defined by*

$$X_{(N,P)} = \xi_{A^*_{N^*}} - N\xi_{A^*} = \xi^{X_1}_{A^*_{N^*}} - N\xi^{X_0}_{A^*}$$

*and is independent of the section of Q_A chosen. Its cohomology class is called the **modular class** of (A,ϕ_0,P,N) and is denoted by $\operatorname{mod}^{(N,P)} A = \left[X_{(N,P)}\right]$.*

Remark 20. *In fact, the modular class defined above is $\operatorname{mod}(N^*)$, the relative modular class of the Lie algebroid morphism $N^* : A^*_{N^*} \to A^*$ [14]. As in the Poisson case, $\operatorname{mod}(N)$ and $\operatorname{mod}(N^*)$ are related by P:*

$$P^\sharp \operatorname{mod}(N) = -\operatorname{mod}(N^*).$$

Following [1], if N is non-degenerated, we have a hierarchy of $\hat{A}$-vector fields:

$$\hat{X}^{i+j}_{(N,\tilde{P})} = N^{i+j-1}\hat{X}_{(N,\tilde{P})} = \mathrm{d}_{N^i\tilde{P}}h_j = \mathrm{d}_{N^j\tilde{P}}h_i,$$

and a hierarchy of A-vector fields

$$X^{i+j}_{(N,P)} = N^{i+j-1}X_{(N,P)} = \mathrm{d}_{N^iP}h_j = \mathrm{d}_{N^jP}h_i,$$

and

$$\begin{aligned}\xi_A^{\eta\otimes\mu}(P^\sharp\alpha)\eta\otimes\mu &= \left[P^\sharp\alpha,\eta\right]\otimes\mu+\eta\otimes\mathcal{L}_{\rho(P^\sharp\alpha)}\mu\\ &= \langle\nu,\left[P^\sharp\alpha,\eta\right]\rangle\eta\otimes\mu+\eta\otimes\mathcal{L}_{\rho(P^\sharp\alpha)}\mu\\ &= -\langle\mathcal{L}_{P^\sharp\alpha}\nu,\eta\rangle\eta\otimes\mu+\eta\otimes\mathcal{L}_{\rho(P^\sharp\alpha)}\mu\\ &= \langle[\alpha,\nu]_P+2\,\alpha\wedge\mathrm{d}i_P\nu,\eta\rangle\eta\otimes\mu+\eta\otimes\mathcal{L}_{\rho(P^\sharp\alpha)}\mu\\ &= (\xi_{A^*}^{\nu\otimes\mu}(\alpha)+2\,\langle\alpha\wedge\mathrm{d}i_P\nu,\eta\rangle)\eta\otimes\mu.\end{aligned}$$

So, $P^\sharp(\xi_A^{\eta\otimes\mu})(\alpha)=-\xi_{A^*}^{\nu\otimes\eta}(\alpha)-2\langle\alpha\wedge\mathrm{d}i_P\nu,\eta\rangle$. ∎

Now let (A,ϕ_0) be a Jacobi algebroid of rank n and P a Jacobi bivector on A. The pair $(\hat{A},\tilde{P})$ is a triangular Lie bialgebroid and we can use the previous proposition to relate the modular classes of $\hat{A}$ and $\hat{A}^*$.

Consider $\eta\in\mathfrak{X}^n(A)$ and $\nu\in\Omega^n(A)$ such that $\langle\nu,\eta\rangle=1$, then we have

$$\tilde{P}^\sharp(\xi_{\hat{A}}^{\eta\otimes\tilde{\mu}})(\alpha)=-\xi_{\hat{A}^*}^{\nu\otimes\tilde{\mu}}(\alpha)-2\langle\alpha\wedge\hat{\mathrm{d}}i_{\tilde{P}}\nu,\eta\rangle.$$

Relations (38) and (40) imply that

$$\tilde{P}^\sharp(\xi_A^{\eta\otimes\mu})(\alpha)=-e^{-t}(\xi_{A^*}^{X_0,\,\nu\otimes\mu}(\alpha)-\langle\alpha,X_0\rangle)-2\langle\alpha\wedge\hat{\mathrm{d}}i_{\tilde{P}}\nu,\eta\rangle,$$

and, since $\alpha\wedge\hat{\mathrm{d}}i_{\tilde{P}}\nu=\alpha\wedge\hat{\mathrm{d}}(e^{-t}i_P\nu)=e^{-t}\,(\alpha\wedge\mathrm{d}i_P\nu-P(\phi_0,\alpha)\nu)$, we have

$$P^\sharp(\xi_A^{\eta\otimes\mu})(\alpha)=-\xi_{A^*}^{X_0,\,\nu\otimes\mu}(\alpha)-\langle\alpha,X_0\rangle-2\langle\alpha\wedge\mathrm{d}i_P\nu,\eta\rangle,\quad\alpha\in\Omega^1(A).$$

The previous equation is obviously equivalent to the duality equation written in [8]. It can also be rewritten as

$$P^\sharp(\xi_A^{\eta\otimes\mu})(\alpha)=-\xi_{A^*}^{\nu\otimes\mu}(\alpha)+(n-2)\langle\alpha,X_0\rangle-2\langle\alpha\wedge\mathrm{d}i_P\nu,\eta\rangle,\quad\alpha\in\Omega^1(A),$$

or as

$$P^\sharp(\xi_A^{\phi_0,\,\eta\otimes\mu})(\alpha)=-\xi_{A^*}^{X_0,\,\nu\otimes\mu}(\alpha)+(n-2)\langle\alpha,X_0\rangle-2\langle\alpha\wedge\mathrm{d}i_P\nu,\eta\rangle,\ \alpha\in\Omega^1(A).$$

5. Modular classes of Jacobi-Nijenhuis algebroids

Let (A,ϕ_0) be a Jacobi algebroid and N a Nijenhuis operator. Consider a Jacobi bivector P on A compatible with the Nijenhuis operator N. The sections $X_0=-P^\sharp(\phi_0)$ and $X_1=-NP^\sharp(\phi_0)=-P^\sharp N^*(\phi_0)$ are 1-cocycles of the Lie algebroid $A^*_{N^*}$.

In particular, if (A,ϕ_0,P) is a triangular Jacobi bialgebroid then (P_M,E_M) is defined by *

$$P_M(df,dg)=\rho^2P(df,dg)=P(\rho^*df,\rho^*dg)=P(\mathrm{d}f,\mathrm{d}g),\tag{23}$$

$$E_M=\rho\circ P^\sharp(\phi_0).\tag{24}$$

2.2. The triangular Jacobi bialgebroid of a Jacobi manifold. Let (M,Λ,E) be a Jacobi manifold, i.e, a manifold equipped with a bivector Λ and a vector field E such that

$$[\Lambda,\Lambda]=-2E\wedge\Lambda,\quad[E,\Lambda]=0.\tag{25}$$

The vector bundle $T^*M\times\mathbb{R}$ is endowed with a Lie algebroid structure over M [10]. The Lie bracket and the anchor are defined by

$$\begin{aligned}[(\alpha,f),(\beta,g)]_{(\Lambda,E)} = (&\mathcal{L}_{\Lambda^\sharp\alpha}\beta-\mathcal{L}_{\Lambda^\sharp\beta}\alpha-d(\Lambda(\alpha,\beta))+f\mathcal{L}_E\beta-g\mathcal{L}_E\alpha\\ &-i_E(\alpha\wedge\beta),\Lambda(\beta,\alpha)+\Lambda(\alpha,dg)-\Lambda(\beta,df)+fE(g)-gE(f))\end{aligned}\tag{26}$$

and

$$\widetilde{(\Lambda,E)}^\sharp(\alpha,f)=\Lambda^\sharp(\alpha)+fE.$$

In this Lie algebroid the differential is given by

$$\mathrm{d}_*(X,Y)=([\Lambda,X]+kE\wedge X+\Lambda\wedge Y,-[\Lambda,Y]-(k-1)E\wedge Y+[E,X]),$$

for $(X,Y)\in\mathfrak{X}^k(M)\oplus\mathfrak{X}^{k-1}(M)$. The section $X_0=(-E,0)$ is a 1-cocycle of $T^*M\times\mathbb{R}$ and the X_0-differential is

$$\begin{aligned}\mathrm{d}_*^{(-E,0)}(X,Y)=&([\Lambda,X]+(k-1)E\wedge X+\Lambda\wedge Y,\\ &-[\Lambda,Y]-(k-2)E\wedge Y+[E,X]),\end{aligned}$$

for $(X,Y)\in\mathfrak{X}^k(M)\otimes\mathfrak{X}^{k-1}(M)$.

Now consider the canonical vector bundle $TM\times R$ over M with its structure of Lie algebroid given by the Lie bracket

$$[(X,f),(Y,g)]=([X,Y],X(g)-Y(f))$$

and the anchor

$$\rho(X,f)=X.$$

*We denote by ρ^p the morphism $\rho^p:\mathfrak{X}^p(A)\to\mathfrak{X}^p(M)$, given by $\rho^pP(\alpha_1,\ldots,\alpha_p)=P(\rho^*\alpha_1,\ldots,\rho^*\alpha_p)$, with $\alpha_1,\ldots,\alpha_p\in\Omega^1(M)$. Since ρ is a Lie algebroid morphism, we have that $\rho^{p+q-1}[P,Q]=[\rho^pP,\rho^qQ]$, with $P\in\mathfrak{X}^p(A)$ and $Q\in\mathfrak{X}^q(A)$.

The differential d of this Lie algebroid is

$$\mathrm{d}(\alpha,\beta)=(d\alpha,-d\beta),\quad \alpha,\beta\in\Omega(M).$$

Obviously, $\phi_0=(0,1)$ is a 1-cocycle of $TM\times\mathbb{R}$. The ϕ_0-differential is given by

$$\mathrm{d}^{(0,1)}(\alpha,\beta)=(d\alpha,\alpha-d\beta),\quad \alpha,\beta\in\Omega(M).$$

A Jacobi bivector on the Jacobi algebroid $(TM\times\mathbb{R},(0,1))$ is a section (Λ,E) on $\mathfrak{X}^2(M)\oplus\mathfrak{X}^1(M)$ such that

$$[(\Lambda,E),(\Lambda,E)]^{(0,1)}=0. \tag{27}$$

Since (27) is equivalent to (25), (Λ,E) defines a Jacobi structure on the manifold M. Moreover, $(\Lambda,E)^\sharp(0,1)=(E,0)$, where $(\Lambda,E)^\sharp:T^*M\times\mathbb{R}\to TM\times\mathbb{R}$ is the vector bundle morphism defined by $(\Lambda,E)^\sharp(\alpha,f)=(\Lambda^\sharp\alpha+fE,-i_E\alpha)$.

The Lie algebroid structure $([\,,\,]_{(\Lambda,E)},\widetilde{(\Lambda,E)}^\sharp)$ in $T^*M\times\mathbb{R}$ coincides with the Lie algebroid structure defined by the Jacobi bivector (Λ,E). In fact one can check that

$$\begin{aligned}[(\alpha,f),(\beta,g)]_{(\Lambda,E)}=&\mathcal{L}^{(0,1)}_{(\Lambda,E)^\sharp(\alpha,f)}(\beta,g)-\mathcal{L}^{(0,1)}_{(\Lambda,E)^\sharp(\beta,g)}(\alpha,f)\\&-\mathrm{d}^{(0,1)}\left((\Lambda,E)((\alpha,f),(\beta,g))\right)\end{aligned}$$

and

$$\widetilde{(\Lambda,E)}^\sharp=\rho\circ(\Lambda,E)^\sharp.$$

So we may conclude that the pair $(\,(TM\times\mathbb{R},(0,1)),(T^*M\times\mathbb{R},(-E,0))\,)$ is a triangular Jacobi bialgebroid [7]. Moreover, the Jacobi structure induced on the base manifold coincides with the initial one.

2.3. Compatible Jacobi bivectors. With the construction presented in the section 2.1 the notion of compatible Jacobi bivectors appears naturally.

Definition 2. *Let (A,ϕ_0) be a Jacobi algebroid. Two Jacobi bivectors P and Q on A are said to be* ***compatible*** *if*

$$[P,Q]^{\phi_0}=0. \tag{28}$$

4.5. Duality between modular classes of A and A^*. Following the philosophy of this paper, we will find a relation between the modular classes of the Jacobi algebroids (A,ϕ_0) and (A^*,X_0) using relations on the associated Lie bialgebroid. So we begin by presenting some results about duality of modular classes on the Lie bialgebroids.

Proposition 17. *Let $(A,[\,,\,],\rho)$ be a Lie algebroid equipped with a Poisson bivector P, $([\,,\,]_P,\rho_*=\rho\circ P^\sharp)$ the Lie algebroid structure induced by P on A^* and ν a top-section on A^*. For all $\alpha\in\Omega^1(A)$, we have*

$$\begin{aligned}\mathcal{L}_{P^\sharp\alpha}\nu&=[\alpha,\nu]_P+2i_P(\mathrm{d}\alpha)\,\nu\\&=-[\alpha,\nu]_P-2\alpha\wedge\mathrm{d}i_P\nu.\end{aligned}$$

Proof: Since ν is a top-section of A^*, using Cartan's formula, we have

$$\mathcal{L}_{P^\sharp\alpha}\nu=\mathrm{d}i_{P^\sharp\alpha}\nu. \tag{47}$$

But $\alpha\wedge\nu=0$ and $i_P(\alpha\wedge\nu)=-i_{P^\sharp\alpha}\nu+\alpha\wedge i_P\nu$, so $i_{P^\sharp\alpha}\nu=\alpha\wedge i_P\nu$. Substituting in (47) we have

$$\mathcal{L}_{P^\sharp\alpha}\nu=\mathrm{d}\alpha\wedge i_P\nu-\alpha\wedge\mathrm{d}i_P\nu.$$

Again because ν is a top-section, we have that $i_P(\mathrm{d}\alpha\wedge\nu)=0$, so $i_P(\mathrm{d}\alpha)\nu=\mathrm{d}\alpha\wedge i_P\nu$ and

$$\mathcal{L}_{P^\sharp\alpha}\nu=i_P(\mathrm{d}\alpha)\nu-\alpha\wedge\mathrm{d}i_P\nu. \tag{48}$$

On the other hand, using the fact that $\partial_P=[\mathrm{d},i_P]$ is a generator of the Gerstenhaber algebra of A^*, we have

$$\begin{aligned}[\alpha,\nu]_P&=-i_P(\mathrm{d}\alpha)\,\nu-\alpha\wedge\mathrm{d}i_P\nu\\&=\mathcal{L}_{P^\sharp\alpha}\nu-2i_P(\mathrm{d}\alpha)\nu\end{aligned}$$

or, equivalently, $[\alpha,\nu]_P=-\mathcal{L}_{P^\sharp\alpha}\nu-2\alpha\wedge\mathrm{d}i_P\nu$. ∎

Proposition 18. *Let (A,A^*,P) be a triangular Lie bialgebroid. Then*

$$P^\sharp\xi_A^{\eta\otimes\mu}(\alpha)=-\xi_{A^*}^{\nu\otimes\mu}(\alpha)-2\langle\alpha\wedge\mathrm{d}i_P\nu,\eta\rangle,\quad \alpha\in\Omega^1(A), \tag{49}$$

where μ is a volume form of M, $\eta\in\mathfrak{X}^{\mathrm{top}}(A)$ and $\nu\in\Omega^{\mathrm{top}}(A)$ such that $\langle\nu,\eta\rangle=1$.

Proof: Since $\langle\nu,\eta\rangle=1$, we have

$$\langle\nu,[X,\eta]\rangle=-\langle\mathcal{L}_X\nu,\eta\rangle,\quad X\in\mathfrak{X}^1(A),$$

and relation (39), we obtain

$$\xi^{\nu\otimes\mu}_{A^*}(\alpha) = \mathcal{M}^{\nu}_{(A,\phi_0,P)}(\alpha) - i_P \mathrm{d}\alpha - \langle \alpha, X_0\rangle + \mathrm{div}_\mu(\rho(P^\sharp(\alpha))). \tag{44}$$

On the other hand, notice that relation (43) implies $\langle \hat{\mathrm{d}} f, \hat{X}^{\hat{\nu}}\rangle \hat{\nu} = \left[\hat{\mathrm{d}} f, \hat{\nu}\right]_{\tilde{P}}$, $f \in C^\infty(M\times\mathbb{R})$, so

$$\hat{\rho}(\xi^{\hat{\nu}\otimes\tilde{\mu}}_{\hat{A}^*}) = \hat{\rho}(\hat{X}^{\hat{\nu}}) + X^{T(M\times\mathbb{R})}, \tag{45}$$

where $X^{T(M\times\mathbb{R})}$ is the modular vector field of the Poisson manifold $M\times\mathbb{R}$ (endowed with the Poisson bivector induced from the triangular Lie bialgebroid $(\hat{A}, \tilde{P})$).

Since the 1-form ϕ_0 is closed, we have

$$\xi^{\nu\otimes\mu}_{A^*}(\phi_0) = \mathcal{M}^{\nu}_{(A,\phi_0,P)}(\phi_0) - \mathrm{div}_\mu(\rho(X_0)),$$

so

$$\begin{aligned}\hat{\rho}(\xi^{\hat{\nu}\otimes\tilde{\mu}}_{\hat{A}^*}) &= e^{-t}\hat{\rho}(\xi^{\nu\otimes\mu}_{A^*}) = e^{-t}\left(\rho(\xi^{\nu\otimes\mu}_{A^*}) + \langle \phi_0, \xi^{\nu\otimes\mu}_{A^*}\rangle \frac{\partial}{\partial t}\right)\\ &= e^{-t}\left(\rho(\xi^{\nu\otimes\mu}_{A^*}) + \left(\mathcal{M}^{\nu}_{(A,\phi_0,P)}(\phi_0) - \mathrm{div}_\mu\rho(X_0)\right)\frac{\partial}{\partial t}\right).\end{aligned}$$

On another hand, $\hat{\rho}(\hat{X}^{\hat{\nu}}) = e^{-t}\left(\rho(\mathcal{M}^{\nu}_{(A,\phi_0,P)}) + \mathcal{M}^{\nu}_{(A,\phi_0,P)}(\phi)\frac{\partial}{\partial t}\right)$ and equation (45) can be rewritten as

$$\rho(\xi^{\nu\otimes\mu}_{A^*}) = \rho(\mathcal{M}^{\nu}_{(A,\phi_0,P)}) + e^t X^{T(M\times\mathbb{R})} + \mathrm{div}_\mu\rho(X_0)\frac{\partial}{\partial t}. \tag{46}$$

Let (P_M, E_M) be the Jacobi structure on M induced by the triangular Jacobi algebroid (A, ϕ_0, P), i.e.,

$$P_M(df, dg) = P(\mathrm{d}f, \mathrm{d}g), \quad E_M = \rho\circ P^\sharp(\phi_0).$$

The modular field of the Jacobi manifold (M, P_M, E_M), $V^{(P_M,E_M)}$, was introduced in [18] and is defined as

$$V^{(P_M,E_M)} = e^t X^{T(M\times\mathbb{R})}.$$

So, equation (46) is equivalent to

$$\rho(\xi^{\nu\otimes\mu}_{A^*}) = \rho(\mathcal{M}_{(A,\phi_0,P)}) + V^{(P_M,E_M)} + \mathrm{div}_\mu\rho(X_0)\frac{\partial}{\partial t}.$$

Due to relation (12), compatible Jacobi bivectors P and Q on A are obviously associated with compatible Poisson bivectors on $\hat{A}$, $\tilde{P} = e^{-t}P$ and $\tilde{Q} = e^{-t}Q$:

$$\left[\tilde{P}, \tilde{Q}\right]_{\hat{A}} = 0.$$

Moreover, they cover compatible Jacobi structures on the base manifold M. Recall that two compatible Jacobi structures on a manifold M (see [16]) is a pair of Jacobi structures (Λ_1, E_1) and (Λ_2, E_2) such that $(\Lambda_1+\Lambda_2, E_1+E_2)$ is also a Jacobi structure, or, equivalently, they satisfy the following two conditions:

$$\begin{aligned}&[\Lambda_1, \Lambda_2] = -E_1\wedge\Lambda_2 - E_2\wedge\Lambda_1,\\ &[E_1, \Lambda_2] + [E_2, \Lambda_1] = 0.\end{aligned}$$

Theorem 3. *Let P and Q be compatible Jacobi bivectors on a Jacobi algebroid (A, ϕ_0). These bivectors cover two compatible Jacobi structures on the base manifold M.*

Proof: By definition of the Schouten-Jacobi bracket $[\,,\,]^{\phi_0}$, the compatibility condition $[P,Q]^{\phi_0} = 0$ is equivalent to

$$[P,Q] = -P^\sharp(\phi_0)\wedge Q - Q^\sharp(\phi_0)\wedge P = 0. \tag{29}$$

On another hand, as we mentioned, compatible Jacobi bivectors P and Q are associated with the compatible Poisson tensors $\tilde{P} = e^{-t}P$ and $\tilde{Q} = e^{-t}Q$ on $\hat{A}$. Since $\phi_0 = \hat{\mathrm{d}}t$, compatibility between these Poisson tensors implies that

$$\left[\tilde{P}^\sharp(\phi_0), \tilde{Q}\right]_{\hat{A}} + \left[\tilde{Q}^\sharp(\phi_0), \tilde{P}\right]_{\hat{A}} = 0,$$

or, using relation (12),

$$\left[P^\sharp(\phi_0), Q\right]^{\phi_0} + \left[Q^\sharp(\phi_0), P\right]^{\phi_0} = 0.$$

Now notice that

$$\left[P^\sharp(\phi_0), Q\right]^{\phi_0} = \left[P^\sharp(\phi_0), Q\right] - i_{\phi_0}P^\sharp(\phi_0)\wedge Q = \left[P^\sharp(\phi_0), Q\right],$$

so, compatibility between Jacobi bivectors also implies that

$$\left[P^\sharp(\phi_0), Q\right] + \left[Q^\sharp(\phi_0), P\right] = 0. \tag{30}$$

Now, let $(P_M = \rho^2 P, E^1_M = \rho(P^\sharp(\phi_0))$ and $(Q_M = \rho^2 Q, E^2_M = \rho(Q^\sharp(\phi_0))$ be the Jacobi structures on M induced by the triangular Jacobi algebroids (A, ϕ_0, P) and (A, ϕ_0, Q) (see (23) and (24)).

Since ρ is a Lie algebroid morphism, we have

$$\begin{aligned}[P_M, Q_M] &= \left[\rho^2 P, \rho^2 Q\right] = \rho^3 [P, Q] = \rho^3(-P^\sharp(\phi_0) \wedge Q - Q^\sharp(\phi_0) \wedge P) \\ &= -\rho(P^\sharp(\phi_0)) \wedge \rho^2 Q - \rho(Q^\sharp(\phi_0)) \wedge \rho^2 P \\ &= -E^1_M \wedge Q_M - E^2_M \wedge P_M\end{aligned}$$

and

$$\left[E^1_M, Q_M\right] + \left[E^2_M, P_M\right] = \rho^2(\left[P^\sharp(\phi_0), Q\right] + \left[Q^\sharp(\phi_0), P\right]) = 0.$$

So the given Jacobi structures on M are compatible. ∎

3. Jacobi-Nijenhuis algebroids

We begin this section exposing some well known results about Nijenhuis operators and compatible Poisson structures on Lie algebroids.

3.1. Poisson-Nijenhuis Lie algebroids. Let $(A, [\,,\,], \rho)$ be a Lie algebroid over a manifold M. Recall that a *Nijenhuis operator* is a bundle map $N : A \to A$ (over the identity) such that the induced map on the sections (denoted by the same symbol N) has vanishing torsion:

$$T_N(X, Y) := [NX, NY] - N[X, Y]_N = 0, \quad X, Y \in \mathfrak{X}^1(A), \tag{31}$$

where $[\,,\,]_N$ is defined by

$$[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y], \quad X, Y \in \mathfrak{X}^1(A).$$

Let us set $\rho_N := \rho \circ N$. For a Nijenhuis operator N, one easily checks that the triple $A_N = (A, [\,,\,]_N, \rho_N)$ is a new Lie algebroid, and then $N : A_N \to A$ is a Lie algebroid morphism.

Since N is a Lie algebroid morphism, its transpose gives a chain map of complexes of differential forms $N^* : (\Omega^k(A), \mathrm{d}_A) \to (\Omega^k(A_N), \mathrm{d}_{A_N})$. Hence we also have a map at the level of algebroid cohomology $N^* : H^\bullet(A) \to H^\bullet(A_N)$.

When the Lie algebroid A is equipped with a Poisson structure P and a Nijenhuis operator N which are *compatible*, it is called a *Poisson-Nijenhuis Lie algebroid*.

The compatibility condition between N and P means that NP is a bivector field and

$$[\,,\,]_{NP} = [\,,\,]^N_P,$$

These relations imply that

$$\begin{aligned}\xi^{\nu\otimes\tilde{\mu}}_{\hat{A}^*}(\alpha)\nu\otimes\tilde{\mu} &= [\alpha,\nu]_{\tilde{P}}\otimes\tilde{\mu} + \nu\otimes\mathcal{L}_{\hat{\rho}_*(\alpha)}(\mu\wedge \mathrm{d}t) \\ &= e^{-t}\,[\alpha,\nu]^{X_0}_P\otimes\tilde{\mu} + e^{-t}\nu\otimes\left(\mathcal{L}_{\rho(P^\sharp(\alpha))}\mu\wedge \mathrm{d}t - \langle\alpha, X_0\rangle\tilde{\mu}\right) \\ &= e^{-t}\left(\xi^{X_0,\,\nu\otimes\mu}_{A^*}(\alpha) - \langle\alpha, X_0\rangle\right)\nu\otimes\tilde{\mu}\end{aligned}$$

and relation (40) follows. ∎

4.4. Relation with the modular vector field of a triangular Jacobi bialgebroid. The definition of modular class of a triangular Jacobi bialgebroid was given in [8]. In this section we will present this definition using the approach we have chosen, relating it with the modular field of the triangular bialgebroid associated with the Jacobi bialgebroid.

Let $(\hat{A}, \tilde{P})$ be the triangular Lie bialgebroid associated with the triangular Jacobi algebroid (A, ϕ_0, P) of rank n and ν a section of $\wedge^n A^*$. The modular field of the triangular Lie bialgebroid $(\hat{A}, \tilde{P})$ with respect to $\hat{\nu} = e^{nt}\nu$ (see [11]) is the section $\hat{X}^{\hat{\nu}}$ of $\hat{A}$ given by

$$\begin{aligned}\hat{X}^{\hat{\nu}}(\alpha)\hat{\nu} &= -\alpha\wedge\hat{\mathrm{d}}\,i_{\tilde{P}}\hat{\nu} = -\alpha\wedge\hat{\mathrm{d}}i_{e^{-t}P}(e^{nt}\nu) \\ &= -\alpha\wedge\hat{\mathrm{d}}(e^{(n-1)t}i_P\nu) \\ &= -e^{(n-1)t}\alpha\wedge((n-1)\phi_0\wedge i_P\nu + \mathrm{d}i_P\nu), \quad \alpha\in\Omega^1(A).\end{aligned}$$

Comparing with the definition of $\mathcal{M}^{\nu}_{(A,\phi_0,P)}$, the modular vector field of the triangular Jacobi bialgebroid (A, ϕ_0, P) given in [8], we notice that

$$\hat{X}^{\hat{\nu}} = e^{-t}\mathcal{M}^{\nu}_{(A,\phi_0,P)}. \tag{42}$$

Since $[\,,\,]_{\tilde{P}}$ is generated by $\partial_{\tilde{P}} = \hat{\mathrm{d}}i_{\tilde{P}} - i_{\tilde{P}}\hat{\mathrm{d}}$, we have

$$\hat{X}^{\hat{\nu}}(\alpha)\hat{\nu} = [\alpha,\hat{\nu}]_{\tilde{P}} + e^{-t}(i_P\mathrm{d}\alpha)\hat{\nu}. \tag{43}$$

Moreover (see (41)),

$$\mathcal{L}_{\hat{\rho}(\tilde{P})^\sharp\alpha}\tilde{\mu} = e^{-t}(\mathrm{div}_\mu\,\rho(P^\sharp\alpha) - \langle\alpha, X_0\rangle)\tilde{\mu},$$

where $\tilde{\mu} = \mu\wedge dt$, $\mu\in\Omega^{\mathrm{top}}(M)$. Using the definition of modular form of a Lie algebroid (35):

$$\xi^{\hat{\nu}\otimes\tilde{\mu}}_{\hat{A}^*}(\alpha)\hat{\nu}\otimes\tilde{\mu} = [\alpha,\hat{\nu}]_{\tilde{P}}\otimes\tilde{\mu} + \hat{\nu}\otimes\mathcal{L}_{\hat{\rho}(\tilde{P}^\sharp(\alpha))}\tilde{\mu}$$

Proposition 16. *The modular form of the Lie algebroid A^* with respect to $\nu \otimes \mu$ is given by*

$$\xi_{A^*}^{\nu\otimes\mu}(\alpha) = e^t \xi_{\hat{A}^*}^{\hat{\nu}\otimes\tilde{\mu}}(\alpha), \quad \alpha \in \mathfrak{X}^1(A^*), \tag{39}$$

with $\hat{\nu} = e^{nt}\nu$ and $\tilde{\mu} = \mu \wedge \mathrm{d}t$.

The modular form of the Jacobi algebroid (A^*, X_0), where $X_0 = -P^\sharp(\phi_0)$, with respect to $\nu \otimes \mu$ is given by

$$\xi_{A^*}^{X_0,\,\nu\otimes\mu} = e^t \xi_{\hat{A}^*}^{\nu\otimes\tilde{\mu}} + X_0. \tag{40}$$

Proof: By definition of modular form and relation (20), for $\alpha \in \Omega^1(A)$, we have

$$\begin{aligned} e^t \xi_{\hat{A}^*}^{\hat{\nu}\otimes\tilde{\mu}}(\alpha)\hat{\nu}\otimes\tilde{\mu} &= \xi_{\hat{A}^*}^{\hat{\nu}\otimes\tilde{\mu}}(\hat{\alpha})\hat{\nu}\otimes\tilde{\mu} \\ &= \left[e^t\alpha, e^{nt}\nu\right]_{\tilde{P}} \otimes \tilde{\mu} + \hat{\nu}\otimes \mathcal{L}_{\hat{\rho}_*(e^t\alpha)}(\mu\wedge \mathrm{d}t) \\ &= e^{nt}[\alpha,\nu]_P \otimes \tilde{\mu} + \hat{\nu}\otimes \mathcal{L}_{\hat{\rho}(P^\sharp(\alpha))}(\mu\wedge \mathrm{d}t) \\ &= e^{nt}[\alpha,\nu]_P \otimes \tilde{\mu} + \hat{\nu}\otimes \mathcal{L}_{\rho(P^\sharp(\alpha))}(\mu)\wedge \mathrm{d}t \\ &= \xi_{A^*}^{\nu\otimes\mu}(\alpha)(\hat{\nu}\otimes\tilde{\mu}). \end{aligned}$$

So

$$\xi_{A^*}^{\nu\otimes\mu}(\alpha) = e^t \xi_{\hat{A}^*}^{\hat{\nu}\otimes\tilde{\mu}}(\alpha), \quad \alpha \in \Omega^1(A).$$

Since ν is a n-form of A, we have $\alpha \wedge i_{X_0}\nu = \langle \alpha, X_0\rangle \nu$, $\alpha \in \Omega^1(A)$, and using relation (21) we obtain

$$\begin{aligned} [\alpha,\nu]_{\tilde{P}} &= e^{-t}\left([\alpha,\nu]_P - n\langle\alpha, X_0\rangle\nu + \alpha\wedge i_{X_0}\nu\right) \\ &= e^{-t}\left([\alpha,\nu]_P - (n-1)\langle\alpha, X_0\rangle\nu\right) \\ &= e^{-t}\,[\alpha,\nu]_P^{X_0}. \end{aligned}$$

Also we have

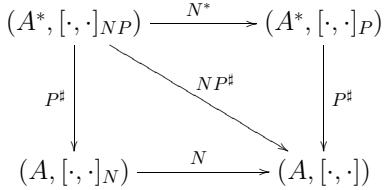
$$\begin{aligned} \mathcal{L}_{\hat{\rho}_*(\alpha)}\tilde{\mu} = \mathcal{L}_{e^{-t}\hat{\rho}(P^\sharp\alpha)}\tilde{\mu} &= e^{-t}\mathcal{L}_{\hat{\rho}(P^\sharp\alpha)}\tilde{\mu} + \langle \hat{\mathrm{d}}e^{-t}, \hat{\rho}(P^\sharp\alpha)\rangle\tilde{\mu} \\ &= e^{-t}\left(\mathcal{L}_{\rho(P^\sharp\alpha)+\langle\phi_0,P^\sharp\alpha\rangle\frac{\partial}{\partial t}}\tilde{\mu} - \langle\phi_0, P^\sharp\alpha\rangle\tilde{\mu}\right) \\ &= e^{-t}\left(\mathcal{L}_{\rho(P^\sharp\alpha)}\tilde{\mu} + \mathcal{L}_{\langle\phi_0,P^\sharp\alpha\rangle\frac{\partial}{\partial t}}\tilde{\mu} - \langle\phi_0, P^\sharp\alpha\rangle\tilde{\mu}\right) \\ &= e^{-t}(\mathcal{L}_{\rho(P^\sharp\alpha)}\mu\wedge \mathrm{d}t - \langle\alpha, X_0\rangle)\tilde{\mu} \\ &= e^{-t}\left(\mathcal{L}_{\rho(P^\sharp(\alpha))}\mu\wedge \mathrm{d}t - \langle\alpha, X_0\rangle\tilde{\mu}\right). \end{aligned} \tag{41}$$

where $[\,,\,]_{NP}$ is the bracket defined by the bivector field $NP \in \mathfrak{X}^2(A)$, and $[\,,\,]_P^N$ is the Lie bracket obtained from the Lie bracket determined by the bivector P, $[\,,\,]_P$, by deformation along the Nijenhuis tensor N^*.

As a consequence, NP defines a new Poisson structure on A, compatible with P:

$$[P, NP] = [NP, NP] = 0,$$

and one has a commutative diagram of morphisms of Lie algebroids:



In fact, we have a whole hierarchy N^kP ($k \in \mathbb{N}$) of pairwise compatible Poisson structures on A.

3.2. Jacobi-Nijenhuis algebroids. Let (A, ϕ_0) be a Jacobi algebroid and N a Nijenhuis operator on A. The definition of the Lie algebroid structure on $\hat{A} = A \times \mathbb{R}$ given by (9) and (10) allows us to say that N is also a Nijenhuis operator on $\hat{A}$. So we have an additional Lie algebroid structure on $\hat{A}$, $\hat{A}_N$.

Proposition 4. *The 1-form $\phi_1 = N^*\phi_0$ is a 1-cocycle of A_N. The Lie algebroid structure $\hat{A}_N$ coincides with the Lie algebroid structure on $\hat{A}$ induced from A_N by ϕ_1.*

Proof: First notice that, since $N : A_N \to A$ is a Lie algebroid morphism, $\mathrm{d}_N\phi_1 = \mathrm{d}_N N^*\phi_0 = N^*(\mathrm{d}\phi_0) = 0$, and then ϕ_1 is a 1-cocycle of A_N. Besides, for $X, Y \in \mathfrak{X}^1(A)$, we have NX, $NY \in \mathfrak{X}^1(A)$,

$$\begin{aligned} [X,Y]_{\hat{A}_N} &= [NX,Y]_{\hat{A}} + [X,NY]_{\hat{A}} - N\,[X,Y]_{\hat{A}} \\ &= [NX,Y] + [X,NY] - N\,[X,Y] = [X,Y]_N \end{aligned}$$

and

$$\begin{aligned} \hat{\rho}_N(X) = \hat{\rho}\circ N(X) &= \rho(NX) + \langle\phi_0, NX\rangle \\ &= \rho\circ N(X) + \langle N^*\phi_0, X\rangle \\ &= \widehat{\rho_N}(X). \end{aligned}$$

Since $\mathfrak{X}(\hat{A})$, as $C^\infty(M\times\mathbb{R})$-module, is generated by $\mathfrak{X}(A)$, we conclude that $\hat{A}_N$ and the Lie algebroid structure on $\hat{A}$ induced from A_N by ϕ_1 are the same. ∎

In fact we have a whole sequence of Lie algebroid structures on $\hat{A}$ given by N^k or, equivalently, by the 1-cocycle of A_{N^k}, $\phi_k = N^{*\,k}\phi_0$:

$$\hat{A}_{N^k} = (\hat{A},\, [\,,\,]_{N^k}\,, \hat{\rho}_{N^k} = \hat{\rho}\circ N^k), \quad k\in\mathbb{N}. \tag{32}$$

Now suppose $P\in\mathfrak{X}^2(A)$ is a Jacobi bivector, i.e., a bivector such that $[P,P]^{\phi_0}=0$.

Definition 5. *The Jacobi bivector P and the Nijenhuis operator N are **compatible** if the following two conditions are satisfied:*

(1) $NP = PN^*$;
(2) $[NP,P]^{\phi_0}=0$.

*In this case, the Jacobi algebroid (A,ϕ_0) is said to be a **Jacobi-Nijenhuis algebroid** and is denoted by (A,ϕ_0,P,N).*

Proposition 6. *On a Jacobi-Nijenhuis algebroid (A,ϕ_0,P,N), we have a hierarchy of compatible Jacobi bivectors.*

Proof: Consider the Poisson bivector $\tilde{P}=e^{-t}P$ on $\hat{A}$. The Jacobi-compatibility between N and P is equivalent to Poisson-compatibility between N and $\tilde{P}$ and we have a hierarchy of compatible Poisson bivectors $N^k\tilde{P}$, $k\in\mathbb{N}$, on $\hat{A}$. This hierarchy induces a hierarchy of compatible Jacobi bivectors on A, N^kP:

$$\left[N^iP, N^jP\right]^{\phi_0}=0, \quad (i,j\in\mathbb{N}). \tag{33}$$

∎

Corollary 7. *The Jacobi-Nijenhuis algebroid (A,ϕ_0,P,N) defines a hierarchy of compatible Jacobi structures on M.*

Proof: This is an immediate consequence of the above proposition and theorem 3. ∎

Also the compatibility conditions define a sequence of Lie algebroid structures on A^*.

4.3. Relation between the modular classes of A and $\hat{A}$ and of A^* and $\hat{A}^*$. Let (A,ϕ_0) be a Jacobi algebroid of rank n. In this section we compute modular forms of $\hat{A}$ and $\hat{A}^*$ (in the triangular case) and we establish relations between them and the modular forms of A and A^*. Let $\eta\in\mathfrak{X}^n(A)$ and $\mu\in\Omega^{\rm top}(M)$, then η is also a n-section of $\hat{A}$ and $\tilde{\mu}=\mu\wedge \mathrm{d}t$ is a volume form of $M\times\mathbb{R}$.

The Lie bracket on $\hat{A}$ coincides with the Lie bracket on A for time-independent multivectors, so

$$\begin{aligned}\xi_{\hat{A}}^{\eta\otimes\tilde{\mu}}(X)\eta\otimes\tilde{\mu} &= [X,\eta]_{\hat{A}}\otimes\tilde{\mu}+\eta\otimes\mathcal{L}_{\hat{\rho}(X)}\tilde{\mu}\\ &= [X,\eta]\otimes\tilde{\mu}+\eta\otimes\mathcal{L}_{\rho(X)+\langle\phi_0,X\rangle\frac{\partial}{\partial t}}\tilde{\mu}, \quad X\in\mathfrak{X}^1(A).\end{aligned}$$

Since

$$\mathcal{L}_{\rho(X)+\langle\phi_0,X\rangle\frac{\partial}{\partial t}}(\mu\wedge\mathrm{d}t)=\mathcal{L}_{\rho(X)}\mu\wedge\mathrm{d}t,$$

we have

$$\begin{aligned}\xi_{\hat{A}}^{\eta\otimes\tilde{\mu}}(X)\eta\otimes\tilde{\mu} &= [X,\eta]\otimes\tilde{\mu}+\eta\otimes\mathcal{L}_{\rho(X)}\mu\wedge\mathrm{d}t\\ &= \xi_A^{\eta\otimes\mu}(X)\eta\otimes\tilde{\mu}.\end{aligned} \tag{38}$$

Now consider the section of $\wedge^n\hat{A}$, $\tilde{\eta}=e^{-(n-1)t}\eta$. Using relation (12) and definition of the Schouten-Jacobi bracket $[\,,\,]^{\phi_0}$, we find that the modular form of $\hat{A}$ with respect to $\tilde{\eta}\otimes\tilde{\mu}$ is given by

$$\begin{aligned}\xi_{\hat{A}}^{\tilde{\eta}\otimes\tilde{\mu}}(X)\tilde{\eta}\otimes\tilde{\mu} &= \left(\xi_A^{\eta\otimes\mu}(X)-(n-1)\langle\phi_0,X\rangle\right)\tilde{\eta}\otimes\tilde{\mu}\\ &= \xi_A^{\phi_0,\,\eta\otimes\mu}(X)\tilde{\eta}\otimes\tilde{\mu}.\end{aligned}$$

Proposition 15. *Let (A,ϕ_0) be a Jacobi algebroid, then*

$$\left[\xi_{\hat{A}}\right]=\left[\xi_A^{\phi_0}\right]=\left[\xi_A\right].$$

It is clear that the cohomology considered in the previous proposition is the $\hat{A}$-cohomology. In $\hat{A}$ the 1-form ϕ_0 is exact, $\phi_0=\hat{\mathrm{d}}t$ and, generally, this is not the case in A.

Now suppose we also have a Jacobi bivector P on (A,ϕ_0). We saw that it induces a Poisson structure on $\hat{A}$, a Lie algebroid structure on $\hat{A}^*$ and another one on A^*. Consider $\nu\in\mathfrak{X}^n(A^*)$ a top-section on A^* and μ a volume form on M.

Proposition 12. *Let (A, ϕ_0) be a Jacobi algebroid. The bilinear map $D^{\phi_0} : \mathfrak{X}^1(A) \otimes Q_A \to Q_A$ defined by*

$$D^{\phi_0}_X(\eta \otimes \mu) = [X, \eta]^{\phi_0} \otimes \mu + \eta \otimes \mathcal{L}_{\rho(X)}\mu, \tag{37}$$

is a representation of the Lie algebroid A on Q_A.

Proof: By definition of $[\,,\,]^{\phi_0}$, we have

$$D^{\phi_0}_X(\eta \otimes \mu) = ([X, \eta] - (n-1)\langle \phi_0, X\rangle \eta) \otimes \mu + \eta \otimes \mathcal{L}_{\rho(X)}\mu,$$

so $D^{\phi_0} = D - (n-1)\phi_0$, where D is the representation of A on Q_A considered in [2] to define the modular class of the Lie algebroid A.

Obviously, for $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}^1(A)$ and $s \in \Gamma(Q_A)$, D^{ϕ_0} satisfies

$$D^{\phi_0}_{fX} s = f D^{\phi_0}_X s,$$

and

$$D^{\phi_0}_X(fs) = f D^{\phi_0}_X s + (\rho(X)f)s.$$

Moreover, since D is a representation and ϕ_0 a 1-cocycle of A,

$$\begin{aligned} D^{\phi_0}_X(D^{\phi_0}_Y s) - D^{\phi_0}_Y(D^{\phi_0}_X s) &= \\ &= D^{\phi_0}_X(D_Y s - (n-1)\langle \phi_0, Y\rangle s) - D^{\phi_0}_Y(D_X s - (n-1)\langle \phi_0, X\rangle s) \\ &= (D_X D_Y - D_Y D_X)s - (n-1)(\rho(X)\phi_0(Y) - \rho(Y)\phi_0(X))s \\ &= D_{[X,Y]}s - (n-1)\phi_0([X,Y])s = D^{\phi_0}_{[X,Y]}s. \end{aligned}$$

We conclude that D^{ϕ_0} is a representation of A on Q_A. ∎

Definition 13. *The **modular form of the Jacobi algebroid** (A, ϕ_0) with respect to $\eta \otimes \mu$ is the A-form $\xi_A^{\phi_0, \eta\otimes\mu}$ defined by*

$$\xi_A^{\phi_0, \eta\otimes\mu} = \xi_A^{\eta\otimes\mu} - (n-1)\phi_0.$$

Again, the cohomology class of a modular form is independent of the section of Q_A chosen.

Definition 14. *The **modular class of the Jacobi algebroid** (A, ϕ_0) is the cohomology class of a modular form. It will be denoted by* $\mathrm{mod}^{\phi_0} A = \left[\xi_A^{\phi_0, \eta\otimes\mu}\right]$.

Obviously $\mathrm{mod}^{\phi_0} A = \mathrm{mod}\, A$ if and only if ϕ_0 is exact.

Theorem 8. *Let (A, ϕ_0, P, N) be a Jacobi-Nijenhuis algebroid. Then A^* has a hierarchy of Jacobi algebroid structures (A^*, X_k), such that $((A, \phi_i), (A^*, X_k))$, $\phi_i = N^{*i}\phi_0$ and $X_k = N^k X_0$, $i \le k$, $k \in \mathbb{N}$, are triangular Jacobi bialgebroids.*

Proof: Last proposition guarantees that $N^i P$, $i \in \mathbb{N}$, is a hierarchy of compatible Jacobi bivectors.

Each one of the Poisson bivectors $N^k\tilde{P} = \widetilde{N^k P} = e^{-t}N^k P$ defines a Lie algebroid structure on $\hat{A}^*$,

$$\hat{A}^*_{N^k} = (\hat{A}^*, [\,,\,]_{N^k\tilde{P}}, \hat{\rho}_{k*} = \hat{\rho} \circ N^k\tilde{P}^\sharp),$$

and a Lie algebroid structure on A^*,

$$A^*_{N^k P} = (A^*, [\,,\,]_{N^k P}, \rho_{k*} = \rho \circ N^k P^\sharp),$$

where

$$[\alpha, \beta]_{N^k P} = e^{-t}\left[e^t\alpha, e^t\beta\right]_{N^k\tilde{P}}, \quad \alpha, \beta \in \mathfrak{X}^1(A^*).$$

Each Lie algebroid structure $A^*_{N^k P}$ coincides with the Lie algebroid structure obtained from the Jacobi algebroid (A, ϕ_{k-i}) by the Jacobi bivector $N^i P$, $i = 1, \ldots, k$. So, the pairs $((A, \phi_{k-i}), (A^*, X_k))$, $i = 1, \ldots, k$ are triangular Jacobi bialgebroids. ∎

On a Jacobi algebroid (A, ϕ_0) consider a Nijenhuis operator N and a Jacobi bivector P such that NP is a bivector.

Following [12], we define the *concomitant of P and N* as

$$C(P, N)(\alpha, \beta) = [\alpha, \beta]_{NP} - [\alpha, \beta]^N_P, \quad \alpha, \beta \in \Omega^1(A), \tag{34}$$

where $[\,,\,]_{NP}$ is the bracket on A^* obtained from (A, ϕ_0) by the bivector NP and $[\,,\,]^N_P$ is the bracket on A^* obtained from $(A_N, \phi_1 = N^*\phi_0)$ by the Jacobi bivector P, i.e.,

$$\begin{aligned} [\alpha, \beta]_{NP} &= \mathcal{L}^{\phi_0}_{NP^\sharp\alpha}\beta - \mathcal{L}^{\phi_0}_{NP^\sharp\beta}\alpha - \mathrm{d}^{\phi_0}NP(\alpha, \beta) \\ &= e^t[\alpha, \beta]_{N\tilde{P}} - \langle \alpha, NP^\sharp(\phi_0)\rangle\beta + \langle \beta, NP^\sharp(\phi_0)\rangle\alpha, \end{aligned}$$

and

$$\begin{aligned} [\alpha, \beta]^N_P &= \mathcal{L}^{N,\phi_1}_{P^\sharp\alpha}\beta - \mathcal{L}^{N,\phi_1}_{P^\sharp\beta}\alpha - \mathrm{d}^{\phi_1}_N P(\alpha, \beta) \\ &= e^t[\alpha, \beta]^N_{\tilde{P}} - \langle \alpha, P^\sharp(\phi_1)\rangle\beta + \langle \beta, P^\sharp(\phi_1)\rangle\alpha, \end{aligned}$$

where $[\,,\,]^N_{\tilde{P}}$ is the bracket on $\hat{A}^*$ obtained from $\hat{A}_N$ by $\tilde{P}$ and $\mathcal{L}^{N,\phi_1}$ is the ϕ_1-Lie derivative on A_N.

Recall that compatibility between the Poisson bivector $\tilde{P}$ and the Nijenhuis operator N means that $N\tilde{P}$ is a bivector and $C(\tilde{P}, N) = 0$. Observing that

$$C(P, N)(\alpha, \beta) = e^t C(\tilde{P}, N)(\alpha, \beta), \quad \alpha, \beta \in \Omega^1(A),$$

we conclude that N and P are compatible if and only if NP is a bivector and $C(P, N) = 0$.

This way we recover the definition of strong (or strict)Jacobi-Nijenhuis structure defined for Jacobi manifolds in [9] and [17].

Example 9. *Consider a Jacobi manifold $(M, (\Lambda, E))$ and the Lie algebroid $A = TM \times \mathbb{R}$ defined in section 2.2.* A strong (or strict) Jacobi-Nijenhuis structure *on M is given by a Nijenhuis operator on A, $\mathcal{N}$, compatible with (Λ, E) in the following sense:*

(i) $\mathcal{N} \circ (\Lambda, E)^\sharp = (\Lambda, E)^\sharp \circ \mathcal{N}^*$. This condition defines a new skew-symmetric bivector Λ_1 and a vector field E_1 such that $(\Lambda_1, E_1)^\sharp = \mathcal{N} \circ (\Lambda, E)^\sharp$.

(ii) The concomitant of (Λ, E) and $\mathcal{N}$, $\mathcal{C}((\Lambda, E), \mathcal{N})$, identically vanishes.

The concomitant $\mathcal{C}((\Lambda, E), \mathcal{N})$ is given in [9, 17] *by*

$$\begin{aligned}\mathcal{C}((\Lambda, E), \mathcal{N})((\alpha, f), (\beta, g)) = {} & [(\alpha, f), (\beta, g)]_{(\Lambda_1, E_1)} - [\mathcal{N}^*(\alpha, f), (\beta, g)]_{(\Lambda, E)} \\ & - [(\alpha, f), \mathcal{N}^*(\beta, g)]_{(\Lambda, E)} + \mathcal{N}^* [(\alpha, f), (\beta, g)]_{(\Lambda, E)},\end{aligned}$$

for $(\alpha, f), (\beta, g) \in \Omega^1(M) \oplus C^\infty(M)$, where the brackets $[\,,\,]_{(\Lambda, E)}$ and $[\,,\,]_{(\Lambda_1, E_1)}$ are defined in (26).

The concomitant can be rewritten as (see [12]*)*

$$\mathcal{C}((\Lambda, E), \mathcal{N})((\alpha, f), (\beta, g)) = [(\alpha, f), (\beta, g)]^{\mathcal{N}}_{(\Lambda, E)} - [(\alpha, f), (\beta, g)]_{(\Lambda_1, E_1)}$$

and we obtain the symmetric of (34).

We conclude that a strong Jacobi-Nijenhuis structure is a pair of compatible Nijenhuis and Jacobi structures in the sense of definition 5.

As in the Poisson case, N^* is a Nijenhuis operator of A^* and we have a commutative relation between duality by P and deformation along N^*.

Proposition 10. *Let (A, ϕ_0, P, N) be a Jacobi-Nijenhuis algebroid and consider the Lie algebroid structure on A^* given by (17) and (18). The operator N^* is a Nijenhuis operator on A^*.*

Proof: Since relation (20) holds and N^* is a Nijenhuis operator on $\hat{A}^*$, we have

$$\begin{aligned}T^{A^*}_{N^*}(\alpha,\beta) &= [N^*\alpha, N^*\beta]_P - N^*([N^*\alpha, \beta]_P + [\alpha, N^*\beta]_P - N^*[\alpha, \beta]_P) \\ &= e^{-t}\left([N^*\hat{\alpha}, N^*\hat{\beta}]_{\tilde{P}} - N^*([N^*\hat{\alpha}, \hat{\beta}]_{\tilde{P}} + [\hat{\alpha}, N^*\hat{\beta}]_{\tilde{P}} - N^*[\hat{\alpha}, \hat{\beta}]_{\tilde{P}})\right) \\ &= e^{-t} T^{\hat{A}^*}_{N^*}(\hat{\alpha}, \hat{\beta}) = 0,\end{aligned}$$

where $\hat{\alpha} = e^t\alpha$, $\hat{\beta} = e^t\beta$ and $\alpha, \beta \in \Omega^1(A)$. ∎

This way A^* can be deformed by N^* into $A^*_{N^*}$ and one can easily check that this is exactly the Lie algebroid A^*_{NP}.

Proposition 11. *The Lie algebroid $A^*_{N^k P}$ coincides with the Lie algebroid $A^*_{N^{*k}}$, obtained from A^* by deformation along N^{*k}.*

4. Modular classes of Jacobi algebroids

4.1. Modular class of a Lie algebroid. Let $(A, [\,,\,], \rho)$ be a Lie algebroid over the manifold M. For simplicity we will assume that both M and A are orientable, so that there exist non-vanishing sections $\eta \in \mathfrak{X}^{\mathrm{top}}(A)$ and $\mu \in \Omega^{\mathrm{top}}(M)$.

The *modular form* of the Lie algebroid A with respect to $\eta \otimes \mu$ (see [2]) is the 1-form $\xi_A^{\eta\otimes\mu} \in \Omega^1(A)$, defined by

$$\langle \xi_A^{\eta\otimes\mu}, X\rangle \eta \otimes \mu = \mathcal{L}_X \eta \otimes \mu + \eta \otimes \mathcal{L}_{\rho(X)}\mu, \quad X \in \mathfrak{X}^1(A). \tag{35}$$

This is a 1-cocycle of the Lie algebroid cohomology of A. If one makes a different choice of sections η' and μ', then $\eta' \otimes \mu' = f\eta \otimes \mu$, for some non-vanishing smooth function $f \in C^\infty(M)$. One checks easily that the modular form $\xi_A^{\eta'\otimes\mu'}$ associated with this new choice is given by:

$$\xi_A^{\eta\otimes\mu} = \xi_A^{\eta'\otimes\mu'} - \mathrm{d}\ln|f|, \tag{36}$$

so that the cohomology class $[\xi_A^{\eta\otimes\mu}] \in H^1(A)$ is independent of the choice of η and μ. This cohomology class is called the *modular class of the Lie algebroid* A and we will denoted it by $\mathrm{mod}\, A := [\xi_A^{\eta\otimes\mu}]$.

4.2. Modular classes of a Jacobi algebroid. Let (A, ϕ_0) be a Jacobi algebroid of rank n. The Schouten-Jacobi bracket $[\,,\,]^{\phi_0}$, given by (1), allows us to define a representation of A on $Q_A = \mathfrak{X}^n(A) \otimes \Omega^{\mathrm{top}}(M)$.