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INFINITE-VERTEX FREE PROFINITE SEMIGROUPOIDS AND SYMBOLIC DYNAMICS

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ABSTRACT: Some fundamental questions about infinite-vertex (free) profinite semigroupoids are clarified, putting in evidence differences with the finite-vertex case. This is done with examples of free profinite semigroupoids generated by the graph of a subshift. It is also proved that for minimal subshifts, the infinite edges of such free profinite semigroupoids form a connected compact groupoid.

KEYWORDS: Subshift; profinite semigroupoid, category, groupoid, semigroup.
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1. Introduction

Since the 1960's that the theory of finite semigroups and their pseudovarieties has seen substantial developments motivated by its applications in computer science through the theories of finite automata and regular languages [10, 20, 21, 26]. Since the mid 1980's, profinite semigroups, and particularly relatively free profinite semigroups, have been shown to play an important role in the study of pseudovarieties: free profinite semigroups over a pseudovariety V capture the common properties of semigroups in V; formal equalities between their elements serve to define subpseudovarieties of V; V-recognizable languages are the traces over finite words of their clopen subsets [6]. Yet, one of the main difficulties in the profinite approach is that, in general, very little is known about the structure of relatively free profinite semigroups.

Symbolic dynamics first came into this picture as a toolkit to exhibit elements of relatively free profinite semigroups with suitable properties [2, 8] and to explore structure features of such semigroups [8, 5]. Conversely, profinite conjugacy invariants have been found in relatively free profinite semigroups

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and some finite computable conjugacy invariants for sofic subshifts were deduced [17, 18, 15].

Through the work of Tilson [38], see also [36], finite categories and semigroupoids (categories without the requirement of local identities) have been shown to play a crucial role in the study of certain operations on pseudovarieties, such as various forms of semidirect products. The merge of this idea with the profinite approach was first attempted in [9]. At first sight, there is for categories and semigroupoids a similar theory of pseudovarieties and their relatively free profinite structures over given profinite graphs [25, 9]. But, as this paper shows, there are some significant differences in case the set of vertices is infinite. In many applications, the finite-vertex case is sufficient [3, 39]. Nevertheless, the general case is also of interest [7, 34].

This paper brings together symbolic dynamics and relatively free profinite semigroupoids. The latter are used to establish some profinite conjugacy invariants, a theme which will be further explored in forthcoming papers. The former serves as a tool to construct examples which clarify some difficulties in the theory of profinite semigroupoids, which is the main subject of this work.

Given a profinite graph Γ, let Γ^+ denote the semigroupoid freely generated by Γ and $\overline{\Omega}_\Gamma\mathsf{Sd}$ the profinite semigroupoid freely generated by Γ. A natural example of profinite graph is the graph $\Sigma(\mathcal{X})$ of the shift function on a subshift $\mathcal{X}$, whose discrete connected components are the orbits of $\mathcal{X}$. We use such examples to exhibit profinite graphs Γ such that Γ^+ is not dense in $\overline{\Omega}_\Gamma\mathsf{Sd}$, whose existence is apparently noted here for the first time.

Consider the graph $\lceil\Gamma\rceil_\beta$, where β is an ordinal, recursively defined as follows: $\lceil\Gamma\rceil_0 = \Gamma$; $\lceil\Gamma\rceil_{\beta^+}$ is the closure in $\overline{\Omega}_\Gamma\mathsf{Sd}$ of the subsemigroupoid generated by $\lceil\Gamma\rceil_\beta$; if β is a limit ordinal then $\lceil\Gamma\rceil_\beta = \bigcup_{\gamma\in\beta}\lceil\Gamma\rceil_\gamma$. There exists an ordinal β such that $\lceil\Gamma\rceil_\beta = \overline{\Omega}_\Gamma\mathsf{Sd}$. Let $\mathfrak{o}(\Gamma)$ be the least of those ordinals. As we said at the end of the previous paragraph, there are subshifts $\mathcal{X}$ such that $\mathfrak{o}(\Sigma(\mathcal{X})) > 1$. In fact, we prove that for every countable ordinal β, there is a countable subshift $\mathcal{X}$ such that $\mathfrak{o}(\Sigma(\mathcal{X})) > \beta$.

On the other hand, we study some forms of obtaining upper bounds for $\mathfrak{o}(\Sigma(\mathcal{X}))$, and in fact we obtain its exact value for several cases. For doing this we first observe that the graph $\Sigma(\mathcal{X})$ is the projective limit of the Rauzy graphs $\Sigma_{2n}(\mathcal{X})$ and that $\overline{\Omega}_{\Sigma(\mathcal{X})}\mathsf{Sd}$ embeds into $\varprojlim \overline{\Omega}_{\Sigma_{2n}(\mathcal{X})}\mathsf{Sd}$. In particular, it follows that if $\mathcal{X}$ is of finite type then $\mathfrak{o}(\Sigma(\mathcal{X})) = 1$. Other classes of subshifts demand more tools. We take the labelling map $\mu : \Sigma(\mathcal{X}) \to A$, where A is the alphabet of $\mathcal{X}$, and consider the natural continuous extension $\hat{\mu}$ of μ from $\varprojlim \overline{\Omega}_{\Sigma_{2n}(\mathcal{X})}\mathsf{Sd}$ to $\overline{\Omega}_A\mathsf{S}$, where $\overline{\Omega}_A\mathsf{S}$ is the free profinite semigroup generated by A. Let $L(\mathcal{X})$ be the language of the finite words appearing in elements of $\mathcal{X}$. Then

$$\overline{L(\mathcal{X})} = \hat{\mu}\Big(\overline{\Sigma(\mathcal{X})^+}\Big). \tag{1.1}$$

On the other hand, if $\mathscr{M}(\mathcal{X})$ is the set of elements of $\overline{\Omega}_A\mathsf{S}$ whose finite factors belong to $L(\mathcal{X})$ then

$$\mathscr{M}(\mathcal{X}) = \hat{\mu}\Big(\varprojlim \overline{\Omega}_{\Sigma_{2n}(\mathcal{X})}\mathsf{Sd}\Big). \tag{1.2}$$

Equality (1.2) is the framework for proving the following key result (Theorem 5.9): if $\hat{\mu}(\lceil\Sigma(\mathcal{X})\rceil_\beta) = \mathscr{M}(\mathcal{X})$ then $\lceil\Sigma(\mathcal{X})\rceil_\beta = \varprojlim \overline{\Omega}_{\Sigma_{2n}(\mathcal{X})}\mathsf{Sd}$, thus $\overline{\Omega}_{\Sigma(\mathcal{X})}\mathsf{Sd} = \varprojlim \overline{\Omega}_{\Sigma_{2n}(\mathcal{X})}\mathsf{Sd}$ and $\mathfrak{o}(\Sigma(\mathcal{X})) \leq \beta$ (it should be noticed that we do not know if there is $\mathcal{X}$ such that $\overline{\Omega}_{\Sigma(\mathcal{X})}\mathsf{Sd} \neq \varprojlim \overline{\Omega}_{\Sigma_{2n}(\mathcal{X})}\mathsf{Sd}$). We deduce this key result using two ingredients: first we prove that $\hat{\mu}$ is faithful (that is, co-terminal edges are mapped injectively), and second we prove that if q is an edge of $\lceil\Sigma(\mathcal{X})\rceil_\beta$ then a factorization of $\hat{\mu}(q)$ is naturally reflected in a *good* factorization of q in $\lceil\Sigma(\mathcal{X})\rceil_\beta$ (Theorem 4.9). We further deduce some general technical conditions under which one has $\hat{\mu}(\lceil\Sigma(\mathcal{X})\rceil_2) = \mathscr{M}(\mathcal{X})$, thus $\mathfrak{o}(\Sigma(\mathcal{X})) \leq 2$. For example, if $\mathcal{X}$ is the even subshift then $\mathcal{X}$ satisfies such conditions and from that it follows that $\mathfrak{o}(\Sigma(\mathcal{X})) = 2$. Those conditions are also satisfied when $\mathcal{X}$ is minimal; and from the fact that $\mathfrak{o}(\Sigma(\mathcal{X})) \leq 2$, we deduce that $\overline{L(\mathcal{X})} = \mathscr{M}(\mathcal{X})$ (an equality proved used a very different approach in [5]), and thus $\mathfrak{o}(\Sigma(\mathcal{X})) = 1$ by (1.1); these two properties are shared by the finite type subshifts, but to obtain them for minimal subshifts it was necessary a significantly larger effort.

Many of the above results are valid not only for free profinite semigroupoids, but also for their counterparts relatively to proper subpseudovarieties with suitable properties.

This paper is divided in six sections. Section 2 presents some preliminaries on semigroups, subshifts and graphs. Section 3 is dedicated to the construction of a good definition of relatively free profinite semigroupoids generated by profinite graphs. Section 4 specializes to relatively free profinite semigroupoids generated by the graph of a subshift. There we study fundamental properties of the labeling map $\hat{\mu}$ with which in Section 5 we investigate upper and lower bounds for the ordinal $\mathfrak{o}(\Sigma(\mathcal{X}))$, computing its exact value in some cases. Finally, in Section 6 we focus on the case where $\mathcal{X}$ is minimal: after proving that $\overline{L(\mathcal{X})} = \mathscr{M}(\mathcal{X})$ and $\mathfrak{o}(\Sigma(\mathcal{X})) = 1$, we use this fact to show that $\overline{\Omega}_{\Sigma(\mathcal{X})}\mathsf{Sd} \setminus \Sigma(\mathcal{X})^+$ is a connected compact groupoid.

Our basic reference for symbolic dynamics is the book of Lind and Marcus [27]. For background on profinite semigroups and semigroupoids see the introductory text [6].

2. Preliminaries

2.1. Some remarks about topology. Throughout this article all topologies are considered to be Hausdorff. In absence of confusion, finite sets are endowed with the discrete topology. Familiarity with nets is assumed. Let I be a directed set (that is, I is endowed with a partial order $\leq$ such that for every $i, j \in I$ there is $k \in I$ such that $i \leq k$ and $j \leq k$). A *directed system* of topological spaces $(X_i)_{i\in I}$ is a family $(\varphi_{j,i} : X_j \to X_i)_{i,j\in I,\, i\leq j}$ of continuous maps such that $\varphi_{i,i}$ is the identity map and $\varphi_{j,i} \circ \varphi_{k,j} = \varphi_{k,i}$ whenever $i, j, k \in I, i \leq j \leq k$. The corresponding *projective limit* is the topological space

$$\varprojlim_{i\in I} X_i = \{(s_i)_i \in \textstyle\prod_{i\in I} X_i \mid i \leq j \Rightarrow \varphi_{j,i}(s_j) = s_i\}.$$

Note that if φ_i is the canonical projection of $\varprojlim_{i\in I} X_i$ into X_i, then $\varphi_i = \varphi_{j,i} \circ \varphi_j$. If the maps $\varphi_{j,i}$ are onto then we speak about an *onto directed system* and an *onto projective limit.* It is well known that $\varprojlim_{i\in I} X_i$ is a closed subset of $\prod_{i\in I} X_i$, which is nonempty if the spaces X_i are compact, and that the canonical projections of an onto projective limit are onto: see [22, Section 3.2], for instance. The following proposition is easy to prove.

Proposition 2.1. *Let Y be a subset of $\varprojlim_{i\in I} X_i$. If for every $i \in I$ there is $k \geq i$ such that the canonical projection of Y into X_k is onto, then Y is dense in $\varprojlim \mathscr{F}$.*

2.2. Pseudovarieties of semigroups. We require some very basic knowledge about the definitions of semigroups, topological semigroups, alphabets, letters, words, languages, rational languages. This that can be found in [26, 31, 14]. Anyway, we shall recall some of the terminology and notation. For instance, given a semigroup S which is not a monoid, S^1 denotes the monoid obtained from S by adding an extra neutral element 1; if S is a monoid then $S^1 = S$. The length of a word u is denoted by $|u|$. The cardinal of a set X is also denoted by $|X|$. As usually, the free semigroup generated by an alphabet A is denoted by A^+, the empty word is denoted by 1, and

A^* is the monoid $A^+ \cup \{1\}$. Recall that a language L of A^+ is *recognized* by a semigroup S if there is some semigroup homomorphism $\varphi : A^+ \to S$ such that $L = \varphi^{-1}\varphi(L)$. If $\mathscr{C}$ is a class of semigroups, then we say L is *$\mathscr{C}$-recognizable* if L is recognized by some element of $\mathscr{C}$.

A *pseudovariety of semigroups* is a class of finite semigroups closed under taking homomorphic images, subsemigroups and finite direct products. Denote by $\mathcal{V}A^+$ the set of V-recognizable languages, and by $\mathcal{V}$ the family $(\mathcal{V}B^+)_B$ where B runs in the class of finite alphabets. Eilenberg proved that the correspondence $\mathsf{V} \to \mathcal{V}$ is a latticeisomorphim between the set of pseudovarieties of semigroups and the set of the so called *varieties of rational languages*, thus opening a vast research program linking the algebraic theory of finite semigroups with the combinatorial theory of languages.

In contrast with Birkhoff's varietaltheory of free algebras [13], a theory of free objects in a pseudovariety V leads to the consideration of topological semigroups. A map $\psi : X \to F$ *separates* two elements x and y of the set X if $\psi(x) \neq \psi(y)$. A *pro-V semigroup* is a compact semigroup such that every pair of distinct elements is separated by a continuous homomorphism into a semigroup of V. This is equivalent to being the projective limit of an onto directed system of semigroups of V [30]. If V is the class S of all finite semigroups then one usually uses the designation *profinite* instead of pro-S. We shall use the fact that for every element s of a profinite semigroup the sequence $(s^{n!})_n$ converges to an idempotent denoted by s^ω [6, pg. 20].

A map κ from A into a topological semigroup T is a *generating map of T* if the subsemigroup of T generated by $\kappa(A)$ is dense in T. A pro-V semigroup T is a *free pro-V semigroup generated by A, with generating map $\kappa : A \to T$,* if for every map φ from A into a pro-V semigroup S there is a unique continuous semigroup homomorphism $\hat{\varphi} : T \to S$ satisfying $\hat{\varphi} \circ \kappa = \varphi$ (which means Diagram (2.1) commutes).

$$\begin{tikzcd} A \arrow[r, "\kappa"] \arrow[dr, "\varphi"'] & T \arrow[d, dashed, "\hat{\varphi}"] \\ & S \end{tikzcd} \tag{2.1}$$

By the usual abstract nonsense, up to isomorphism of topological semigroups, there is no more than one free pro-V semigroup generated by A. In fact there is always such a semigroup: roughly speaking, it is the projective limit of all A-generated semigroups of V. It is denoted by $\overline{\Omega}_A\mathsf{V}$. By *relatively* *free profinite semigroup* we mean a semigroup of the form $\overline{\Omega}_A\mathsf{V}$, for some pseudovariety V. If V has nontrivial semigroups then A embeds into $\overline{\Omega}_A\mathsf{V}$, and if V contains the pseudovariety N of finite *nilpotent semigroups* (semigroups whose idempotents are all equal to a zero element) then A^+ embeds as a dense subset of $\overline{\Omega}_A\mathsf{V}$, and the elements of A^+ are isolated points in $\overline{\Omega}_A\mathsf{V}$; for these reasons the elements of $\overline{\Omega}_A\mathsf{V}$ are also called *pseudowords* (or *profinite words*), and the elements of $\overline{\Omega}_A\mathsf{V} \setminus A^+$ are the *infinite pseudowords*. The following proposition [1, Theorem 3.6.1] establishes an important connection between the topology of $\overline{\Omega}_A\mathsf{V}$ and V-recognizable languages, when V contains N.

Proposition 2.2. *Let V be a pseudovariety of semigroups containing N. Let A be a finite alphabet. A language L of A^+ is V-recognizable if and only if its topological closure in $\overline{\Omega}_A\mathsf{V}$ is open. The topology of $\overline{\Omega}_A\mathsf{V}$ is generated by the topological closures of V-languages of A^+.*

2.3. Some operations on pseudovarieties. Several algebraic operations on pseudovarieties of semigroups reflect important combinatorial operations on varieties of rational languages. One of such algebraic operations is the *Mal'cev product* of two pseudovarieties V and W, denoted by $\mathsf{V} \circledcirc \mathsf{W}$ (see the survey [32]). The pseudovariety $\mathsf{V} \circledcirc \mathsf{W}$ contains V and W. A semigroup whose subgroups are trivial is called *aperiodic*. Let A be the pseudovariety of finite aperiodic semigroups. Note that $\mathsf{N} \subseteq \mathsf{A}$. A variety of languages $\mathcal{V}$ is *closed under concatenation product* if $\mathcal{V}A^+$ contains the concatenation of its elements, for every finite alphabet A. The pseudovariety $\mathsf{A} \circledcirc \mathsf{V}$ is the least pseudovariety containing V whose corresponding variety of languages is closed under concatenation product. Since $\mathsf{A} \circledcirc \mathsf{V} = \mathsf{A} \circledcirc (\mathsf{A} \circledcirc \mathsf{V})$ [31, Exercise 5.10], the variety of V-recognizable languages is closed under concatenation product if and only if $\mathsf{V} = \mathsf{A} \circledcirc \mathsf{V}$. This result is a particular instance of a more general result from [16], which in turn generalizes a similar result from [37] proved for pseudovarieties of monoids.

Lemma 2.3. *Let V be a pseudovariety of semigroups containing N. Let A be a finite alphabet. The multiplication in $\overline{\Omega}_A\mathsf{V}$ is an open map if and only if $\mathsf{V} = \mathsf{A} \circledcirc \mathsf{V}$.*

Proof: Let $\mathcal{V}$ be the variety of V-recognizable languages. Then $\{\overline{L} \mid L \in \mathcal{V}A^+\}$ is a basis for the topology of $\overline{\Omega}_A\mathsf{V}$, by Proposition 2.2. Therefore $\{\overline{L} \times \overline{K} \mid L, K \in \mathcal{V}A^+\}$ is a basis for the topology of $\overline{\Omega}_A\mathsf{V} \times \overline{\Omega}_A\mathsf{V}$. For all subsets P and Q of A^+ we have $\overline{P} \cdot \overline{Q} = \overline{PQ}$. Hence the multiplication in $\overline{\Omega}_A \mathsf{V}$ is an open map if and only if $\overline{LK}$ is open for every $L, K \in \mathcal{V}A^+$. The set $\overline{LK}$ is open if and only if $LK \in \mathcal{V}A^+$, by Proposition 2.2. Hence the multiplication in $\overline{\Omega}_A \mathsf{V}$ is an open map if and only if $\mathcal{V}A^+$ is closed under concatenation. ∎

Another key operation which is considered in detail in [1, Chapter 10], the *semidirect product* (also designated *wreath product*) of two pseudovarieties V and W is a pseudovariety denoted by $\mathsf{V} * \mathsf{W}$. It contains V and W. Both the Mal'cev and semidirect products respect inclusion in both factors (that is, for example, if $\mathsf{V} \subseteq \mathsf{U}$ then $\mathsf{V} * \mathsf{W} \subseteq \mathsf{U} * \mathsf{W}$). While both the Mal'cev and semidirect products are non-commutative operations having the pseudovariety I of trivial semigroups as neutral element, the latter is associative, unlike the first. A set of semidirect products that deserved special attention is the set of pseudovarieties of the form $\mathsf{V} * \mathsf{D}$, where D is the pseudovariety of semigroups whose idempotents are right zeros. At the variety of languages level, this operation is closely related with the operation of taking inverse images under maps of the form Φ_k which we next define. Given an alphabet A and $k \geq 1$, consider the alphabet A^k of words on A of length k; to avoid ambiguities, we represent an element $w_1 \cdots w_n$ of $(A^k)^+$ (with $w_i \in A^k$) by $\langle w_1, \cdots, w_n \rangle$; for $k \geq 0$ the map Φ_k from A^+ to $(A^{k+1})^*$ is given by

$$\Phi_k(a_1 \cdots a_n) = \begin{cases} 1 & \text{if } n \leq k, \\ \langle a_{[1,k+1]}, a_{[2,k+2]}, \cdots a_{[n-k-1,n-1]}, a_{[n-k,n]} \rangle & \text{if } n > k, \end{cases}$$

where $a_i \in A$ and $a_{[i,j]} = a_i a_{i+1} \cdots a_{j-1} a_j$.

A *semilattice* is a commutative semigroup whose elements are all idempotents. The pseudovariety of finite semilattices is denoted by Sl.

Theorem 2.4. *Let* V *be a pseudovariety containing* Sl *and* N. *For every finite alphabet* A *and nonnegative integer* k, *the map* $\Phi_k : A^+ \to (A^{k+1})^*$ *has a unique continuous extension from* $\overline{\Omega}_A(\mathsf{V} * \mathsf{D})$ *to* $(\overline{\Omega}_{A^{k+1}} \mathsf{V})^1$, *which we denote by* Φ_k^{V}.

Theorem 2.4 was proved by the first author for the pseudovariety S of all finite semigroups [1, Lemma 10.6.11]. The general case then follows easily from results of [1, Section 10], as argued in [15]. We shall deal with a lot of pseudovarieties of the form $\mathsf{V} = \mathsf{V} * \mathsf{D}$. Since $\mathsf{D} * \mathsf{D} = \mathsf{D}$ and the semidirect product of pseudovarieties is associative, the solutions of the equation $\mathsf{V} =$

$\mathsf{V}*\mathsf{D}$ on the variable V are precisely the pseudovarieties of the form $\mathsf{V}*\mathsf{D}$. For every pseudovariety of semigroups V, the class $\mathscr{L}\mathsf{V}$ of all finite semigroups whose subsemigroups that are monoids belong to V is a pseudovariety of semigroups. One has always $\mathsf{V}*\mathsf{D} \subseteq \mathscr{L}\mathsf{V}$ (sometimes $\mathsf{V}*\mathsf{D} \neq \mathscr{L}\mathsf{V}$) and $\mathscr{L}\mathsf{V} = \mathscr{L}\mathsf{V}*\mathsf{D}$. One of the most celebrated results of finite semigroup theory is the equality $\mathsf{Sl}*\mathsf{D} = \mathscr{L}\mathsf{Sl}$ [12, 29, 42, 41].

2.4. Subshifts. Suppose the alphabet A is finite. Let $A^{\mathbb{Z}}$ be the set of sequences of letters of A indexed by $\mathbb{Z}$. The *shift* in $A^{\mathbb{Z}}$ is the bijective map σ_A (or just σ) from $A^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$ defined by $\sigma_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. The *orbit* of $x \in A^{\mathbb{Z}}$ is the set $\mathcal{O}(x) = \{\sigma^k(x) \mid k \in \mathbb{Z}\}$. We endow $A^{\mathbb{Z}}$ with the product topology with respect to the discrete topology of A. Note that $A^{\mathbb{Z}}$ is compact, since A is finite. A *symbolic dynamical system* of $A^{\mathbb{Z}}$ is a nonempty closed subset $\mathcal{X}$ that contains the orbits of its elements. Symbolic dynamical systems are also called *shift spaces* or *subshifts*.

Two subshifts $\mathcal{X} \subseteq A^{\mathbb{Z}}$ and $\mathcal{Y} \subseteq B^{\mathbb{Z}}$ are *topologically conjugate* if there is a homeomorphism $\varphi : \mathcal{X} \to \mathcal{Y}$ commuting with shift: $\varphi \circ \sigma_A = \sigma_B \circ \varphi$. Such a homeomorphism is also called a *topological conjugacy*. Since we will consider no other form of conjugacy, we drop the reference to its topological nature.

Let $x \in A^{\mathbb{Z}}$. By a *factor of* $(x_i)_{i\in\mathbb{Z}}$ we mean a word $x_i x_{i+1} \cdots x_{i+n-1} x_{i+n}$ (briefly denoted by $x_{[i,i+n]}$), where $i \in \mathbb{Z}$ and $n \geq 1$. If $\mathcal{X}$ is a subset of $A^{\mathbb{Z}}$ then we denote by $L(\mathcal{X})$ the set of factors of elements of $\mathcal{X}$, and by $L_n(\mathcal{X})$ the set of elements of $L(\mathcal{X})$ with length n. A subset K of a semigroup S is *factorial* if it is closed under taking factors, and it is *prolongable* if for every element u of K there are $a, b \in S$ such that $aub \in K$. It is easy to prove that the correspondence $\mathcal{X} \mapsto L(\mathcal{X})$ is a bijection between the subshifts of $A^{\mathbb{Z}}$ and the nonempty factorial prolongable languages of A^+ [27, Proposition 1.3.4].

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$ and V a pseudovariety of semigroups containing N. Since $\overline{K} \cap A^+ = K$ for every language K of A^+ (where $\overline{K}$ is the closure of K in $\overline{\Omega}_A\mathsf{V}$), the correspondence $\mathcal{X} \mapsto \overline{L(\mathcal{X})}$ is one-to-one. This suggests the exploration of the algebraic-topological properties of $\overline{\Omega}_A\mathsf{V}$ (in general much richer than those of A^+) to obtain information about $\mathcal{X}$. This program has been implemented by both authors in previous papers [5, 4, 17, 18, 15]. The following result had not appeared before, and its interest is obvious in this context.

Proposition 2.5. *Let* V *be a pseudovariety of semigroups such that* $\mathsf{V} = \mathsf{A} ⓜ \mathsf{V}$. *If* L *is a factorial language of* A^+ *then* $\overline{L}$ *is a factorial subset of* $\overline{\Omega}_A\mathsf{V}$.

For proving Proposition 2.5 we first prove an useful lemma.

Lemma 2.6. *Let* S *be a topological semigroup whose topology is generated by a metric. Suppose the multiplication is an open map. Let* $u, v \in S$. *Let* $(w_n)_n$ *be a sequence of elements of* S *converging to* uv. *Then there is a subsequence* $(w_{n_k})_k$ *and sequences* $(u_k)_k$, $(v_k)_k$ *such that* $w_{n_k} = u_k v_k$ *for all* k, *and* $\lim u_k = u$ *and* $\lim v_k = v$.

Proof: We denote by $B(t, \varepsilon)$ the open ball in S with center t and radius ε. Let $k \in \mathbb{Z}^+$. Since the multiplication is an open map, the set $B\big(u, \frac{1}{k}\big)B\big(v, \frac{1}{k}\big)$ is an open neighborhood of uv. Hence there is p_k such that $w_n \in B\big(u, \frac{1}{k}\big)B\big(v, \frac{1}{k}\big)$ if $n \geq p_k$. Let n_k be the strictly increasing sequence recursively defined by $n_1 = p_1$ and $n_k = \max\{n_{k-1} + 1, p_k\}$ if $k > 1$. For each $k \in \mathbb{Z}^+$ there are $u_k \in B\big(u, \frac{1}{k}\big)$ and $v_k \in B\big(v, \frac{1}{k}\big)$ such that $w_{n_k} = u_k v_k$. We have $\lim u_k = u$ and $\lim v_k = v$. ■

Proof of Proposition 2.5: Suppose $uv \in \overline{L}$. Let $(w_n)_n$ be a sequence of elements of L converging to uv. By Lemmas 2.3 and 2.6 there are a subsequence $(w_{n_k})_k$ and sequences $(u_k)_k$, $(v_k)_k$ such that $w_{n_k} = u_k v_k$ for all k, $\lim u_k = u$ and $\lim v_k = v$. Since $w_{n_k} \in A^+$, necessarily $u_k, v_k \in A^+$. And since $w_{n_k} \in L$ and L is factorial in A^+, we have $u_k, v_k \in L$. Hence $u, v \in \overline{L}$. ■

2.5. Prefixes and suffixes of pseudowords. Take [1, Sections 3.7 and 5.2] as reference for this subsection. By a *prefix* of an element t of a semigroup T we mean a left factor of t, that is, an element p of T such that $s = px$ for some $x \in T^1$. Dually, a *suffix* is a right factor.

Let w be a word of A^+ and n a positive integer. If $|w| \geq n$ then we denote by $\mathrm{t}_n(w)$ (respectively $\mathrm{i}_n(w)$) the unique suffix (respectively prefix) of w with length n; if $|w| < n$ then we let $\mathrm{t}_n(w) = \mathrm{i}_n(w) = w$. If V is a pseudovariety of semigroups containing D, then the map $\mathrm{t}_n : A^+ \to A^+$ has a unique extension to a continuous homomorphism from $\overline{\Omega}_A\mathsf{V}$ to A^+ relatively to the discrete topology of A^+. We also denote this extension by t_n. Replacing D by its dual pseudovariety, usual denoted by K, similar considerations hold for i_n. The least pseudovariety containing D and K is $\mathscr{L}\mathsf{I}$.

Endow $A^{\mathbb{Z}_0^+} \cup A^+$ (respectively $A^{\mathbb{Z}^-} \cup A^+$) with the topology defined as follows: $A^{\mathbb{Z}_0^+}$ (respectively $A^{\mathbb{Z}^-}$) is closed and endowed with the product topology, the elements of A^+ are isolated points, and a sequence $(u_n)_n$ of elements of A^+ converges to an element x of $A^{\mathbb{Z}_0^+}$ (respectively $A^{\mathbb{Z}^-}$) if and only if for all k the words $\mathrm{t}_k(u_n)$ and $x_{[0,k-1]}$ (respectively $\mathrm{i}_k(u_n)$ and $x_{[-k,-1]}$) are equal for all sufficiently large n. The topological space $A^{\mathbb{Z}_0^+} \cup A^+$ becomes a compact semigroup if we declare the elements of $A^{\mathbb{Z}_0^+}$ as left zeros and the remaining possible products as given by concatenation. In this way, $A^{\mathbb{Z}_0^+} \cup A^+$ is isomorphic with $\overline{\Omega}_A\mathsf{K}$. The dual characterization holds for $\overline{\Omega}_A\mathsf{D}$.

Finally, endow $A^{\mathbb{Z}} \cup A^+$ with the topology where $A^{\mathbb{Z}}$ is closed and endowed with the product topology, the elements of A^+ are isolated points, and a sequence $(u_n)_n$ of elements of A^+ converges to an element x of $A^{\mathbb{Z}}$ if and only if $(\mathrm{t}_k(u_n), \mathrm{i}_k(u_n))_n$ converges to $(x_{]-\infty,-1]}, x_{[0,+\infty[})$ for all k. Take the natural identification between $A^{\mathbb{Z}}$ and $A^{\mathbb{Z}^-} \times A^{\mathbb{Z}_0^+}$ Consider in $A^{\mathbb{Z}} \cup A^+$ the following multiplication: for $w \in A^+$, $x, x' \in A^{\mathbb{Z}^-}$ and $y, y' \in A^{\mathbb{Z}_0^+}$, we have

$$(x, y) \cdot w = (xw, y), \quad w \cdot (x, y) = (x, wy), \quad (x, y) \cdot (x', y') = (x', y).$$

With this multiplication, $A^{\mathbb{Z}} \cup A^+$ becomes a compact semigroup isomorphic with $\overline{\Omega}_A\mathscr{L}\mathsf{I}$.

If V contains $\mathscr{L}\mathsf{I}$ then $\overline{\Omega}_A\mathsf{K}$ is pro-V. Let $w \mapsto \overrightarrow{w}$ denote the canonical projection of $\overline{\Omega}_A\mathsf{V}$ in $\overline{\Omega}_A\mathsf{K}$, that is, the unique continuous homomorphism from $\overline{\Omega}_A\mathsf{V}$ to $\overline{\Omega}_A\mathsf{K}$ extending the identity in A. Dually, denote by $w \mapsto \overleftarrow{w}$ the canonical projection of $\overline{\Omega}_A\mathsf{V}$ in $\overline{\Omega}_A\mathsf{D}$. Note that $\mathrm{i}_n(w) = \mathrm{i}_n(\overrightarrow{w})$ and $\mathrm{t}_n(w) = \mathrm{t}_n(\overleftarrow{w})$ for all n. An element (x, y) of $A^{\mathbb{Z}^-} \times A^{\mathbb{Z}_0^+}$ will also be denoted by $x.y$.

For a word u, the left infinite sequence $\ldots uuuu$ is denoted by $u^{-\infty}$; dually, $u^{+\infty} = uuuu\ldots$; and $u^{-\infty}.v^{+\infty}$ denotes the bi-infinite sequence $\ldots uuuu.vvvv\ldots$. Finally, u^{∞} denotes $u^{-\infty}.u^{+\infty}$.

2.6. Graphs. By a *graph* we mean a directed multigraph, that is a disjoint union $G = V_G \cup E_G$ of a set V_G of *vertices* with a set E_G of *edges* together with two *incidence maps* α, ω from E_G to V_G. The pictorial meaning of the incidence maps is best described by writing $\alpha(e) \xrightarrow{e} \omega(e)$, (or alternatively $e : \alpha(e) \to \omega(e)$), and by saying that e goes from $\alpha(e)$ to $\omega(e)$, or that the edge starts at $\alpha(e)$ and ends at $\omega(e)$, and so on. Two edges e and f on a graph are *co-terminal* if $\alpha(e) = \alpha(f)$ and $\omega(e) = \omega(f)$. The set of edges from a vertex x to a vertex y is denoted by $E_G(x, y)$. Two edges e and f are said to be *consecutive* (by this order) if $\omega(e) = \alpha(f)$. A *path* on a graph is a finite sequence of consecutive edges.

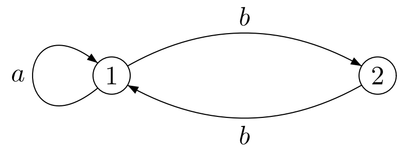

Figure 1. Presentation of the even subshift.

A function between graphs is a *graph homomorphism* if it maps vertices to vertices, edges to edges, and respects incidence maps. A graph homomorphism is *faithful* if it maps co-terminal edges injectively, and it is *quotient* if it is bijective in the set of vertices and onto in the set of edges.

A *labeled graph on* A is a pair (G, λ) where G is a graph and λ is a mapping assigning to each edge of G a letter of A. One can regard a labeled graph as an automaton whose vertices are all both initial and final states. A subshift $\mathcal{X}$ is called *sofic* if the language $L(\mathcal{X})$ is recognized by a finite labeled graph. In fact, $\mathcal{X}$ is sofic if and only if $L(\mathcal{X})$ is a rational language. Such a graph is said to be a *presentation* of the symbolic system. The graph of Figure 1 labeled with the letters a and b presents a familiar sofic system called the *even subshift*.

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. The *Rauzy graph of order* n *of* $\mathcal{X}$ [33] is the graph $\Sigma_n(\mathcal{X})$ where the vertices are the elements of $L_n(\mathcal{X})$, the edges are the elements of $L_{n+1}(\mathcal{X})$, and the incidence maps are given by $\alpha(a_1a_2\cdots a_na_{n+1}) = a_1a_2\cdots a_n$ and $\omega(a_1a_2\cdots a_na_{n+1}) = a_2\cdots a_na_{n+1}$.

By a *(compact) topological graph* we mean a graph G endowed with a (compact) Hausdorff topology such that α_G and ω_G are continuous maps, and V_G and E_G are closed sets. Note that V_G and E_G are also open sets, since G is the disjoint union of V_G and E_G. The product of topological graphs is a topological graph with respect to the product topology.

For a subshift $\mathcal{X}$, let $\Sigma(\mathcal{X})$ denote the graph whose set of vertices is $\mathcal{X}$, whose set of edges is $\{(x, \sigma(x)) \in \mathcal{X} \times \mathcal{X} \mid x \in \mathcal{X}\}$, and such that the edge $(x, \sigma(x))$ starts in x and ends in $\sigma(x)$. Considering in $E_{\Sigma(\mathcal{X})}$ the topology induced from the product topology of $\mathcal{X} \times \mathcal{X}$, the maps α and ω are continuous, whence $\Sigma(\mathcal{X})$ has a structure of topological graph determined by the topology of $\mathcal{X}$. We call $\Sigma(\mathcal{X})$ the *graph of* $\mathcal{X}$. If two subshifts are conjugate then $\Sigma(\mathcal{X})$ and $\Sigma(\mathcal{Y})$ are isomorphic topological graphs.

A graph is *profinite* if every pair of distinct elements is separated by a continuous graph homomorphism into a finite graph. This is equivalent to being the projective limit of an onto directed system of finite graphs.

Let n and m be positive integers such that $m \geq n$. The following map, denoted by $\pi_{m,n}$, is an onto graph homomorphism:

$$\begin{array}{rcll} \Sigma_{2m}(\mathcal{X}) & \to & \Sigma_{2n}(\mathcal{X}) & \\ x_{[-m,m-1]} \in L_{2m}(\mathcal{X}) & \mapsto & x_{[-n,n-1]} \in L_{2n}(\mathcal{X}), & x \in \mathcal{X}. \\ x_{[-m,m]} \in L_{2m+1}(\mathcal{X}) & \mapsto & x_{[-n,n]} \in L_{2n+1}(\mathcal{X}), & x \in \mathcal{X}, \end{array}$$

The family of graph homomorphisms $\{\pi_{m,n} \mid n \leq m\}$ defines an onto directed system. Its projective limit and $\Sigma(\mathcal{X})$ are isomorphic and will be identified, according to the fact that the map

$$\begin{array}{rcll} \Sigma(\mathcal{X}) & \to & \varprojlim \Sigma_{2n}(\mathcal{X}) & \\ x & \mapsto & (x_{[-n,n-1]})_n & \\ (x, \sigma(x)) & \mapsto & (x_{[-n,n]})_n, & x \in \mathcal{X}. \end{array}$$

is a continuous graph isomorphism. The graph $\Sigma(\mathcal{X})$ is therefore profinite.

3. Relatively free profinite semigroupoids

3.1. Semigroupoids. Let S be a graph with a nonempty set of edges. Denote by D_S the set of pairs of consecutive edges of S. We say S is a *semigroupoid* if the set of edges of S is endowed with a partial binary operation "$\cdot$" usually called *composition*, such that:

(1) given edges s and t of S, the product $s \cdot t$ is an edge which is defined if and only if $(s, t) \in D_S$;
(2) if $(s, t) \in D_S$ then $\alpha(s \cdot t) = \alpha(s)$ and $\omega(s \cdot t) = \omega(t)$;
(3) if $(s, t) \in D_S$ and $(t, r) \in D_S$ then $(s \cdot t) \cdot r = s \cdot (t \cdot r)$.

The product $s \cdot t$ of two consecutive edges will be denoted by st whenever it is clear that we are not speaking about the path made of s and t.

A subgraph T of a semigroupoid S is a *subsemigrupoid* of S if T is a semigroupoid whose composition is the restriction of the operation of S. Given a nonempty subgraph X of the semigroupoid S, the intersection of all subsemigroupoids of S containing X is a semigroupoid, called the *subsemigroupoid of S generated by X*, and denoted by $\langle X \rangle$. Note that $V_{\langle X \rangle} = V_X$ and that

$$E_{\langle X \rangle} = \bigcup_{n \geq 1} \{s_1 s_2 \cdots s_n \mid s_1, s_2, \ldots, s_n \text{ are consecutive edges of } X\}. \qquad (3.1)$$

Given two semigroupoids S and T, a *homomorphism of semigroupoids* from S to T is a homomorphism of graphs $\varphi : S \to T$ such that $\varphi(s \cdot t) = \varphi(s) \cdot \varphi(t)$ for every $(s, t) \in D_S$. If the restriction of φ to the set S of vertices is injective

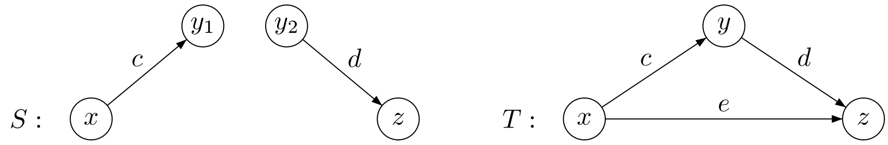

FIGURE 2. The homomorphic image of S in T is not a subsemigroupoid.

then for every subsemigroupoid R of S the set $\varphi(R)$ is a subsemigroupoid of T. However, it may happen that $\varphi(S)$ is not a subsemigroupoid of T.

Example 3.1. *Consider the graphs S and T represented in Figure 2. The set D_S is empty, hence S is a semigroupoid for the empty binary operation. On the other hand, $D_T = \{(c, d)\}$ and T is a semigroupoid for the operation $(c, d) \mapsto e$. Since $D_S = \emptyset$, any graph homomorphism from S to T is a semigroupoid homomorphism. That is the case of the map $\varphi : S \to T$ such that $\varphi(y_1) = \varphi(y_2) = y$ and $\varphi(s) = s$ for all $s \in S \setminus \{y_1, y_2\}$. The graph $\varphi(S)$ is not a subsemigroupoid of T, because $\varphi(c) \cdot \varphi(d) = c \cdot d = e \notin \varphi(S)$.*

Given a set C, it is convenient to identify C with the graph $G(C)$ with a single vertex x not belonging to C and such that $E_{G(C)}(x, x) = C$. Accordingly, if H is a graph, a graph homomorphism from H to C will be understood as a map from E_H to C. Likewise, a semigroup S will be identified with the semigroupoid having $G(S)$ as underlying graph and whose composition is the semigroup operation of S. Conversely, if T is a semigroupoid and $E_T(x, x) \neq \emptyset$, then $E_T(x, x)$ is a semigroup for the composition operation, called the *local semigroup of T in x*.

Let Γ be a graph. The graph Γ^+ is the graph whose vertices are those of Γ and whose edges from a vertex x to a vertex y are the paths of Γ from x to y. Note that Γ is a subgraph of Γ^+. Under the operation of concatenation of paths, Γ^+ is the *free semigroupoid generated by* Γ, in the same sense that justified the use of the terminology of free semigroup. In fact, if Γ is a set then Γ^+ is actually the free semigroup generated by Γ. Given a homomorphism φ of graphs from Γ to a semigroupoid S, we shall denote by φ^+ the unique semigroupoid homomorphism from Γ^+ to S extending φ.

A *congruence* on a semigroupoid S is an equivalence relation θ on S such that:

(1) if x is a vertex of S then $x/\theta = \{x\}$.
(2) for all edges s and t of S, if $s \mathrel{\theta} t$ then s are t co-terminal edges;
(3) for all edges s, t and r of S, if $s \mathrel{\theta} t$ and $\omega(r) = \alpha(s)$ then $rs \mathrel{\theta} rt$;

FIGURE 3. A sofic subshift $\mathscr{Z}$ such that $\overline{\Sigma(\mathscr{Z})^+}$ is not a subsemigroupoid of any compact semigroupoid in which $\Sigma(\mathscr{Z})^+$ embeds.

(4) for all edges s, t and r of S, if $s\ \theta\ t$ and $\alpha(r) = \omega(s)$ then $sr\ \theta\ tr$.

The relation identifying co-terminal edges is a congruence, called *co-terminality congruence*. If θ is a congruence on a semigroupoid S then the quotient graph S/θ is naturally endowed with a structure of semigrupoid. The usual isomorphism theorems hold. It is important to note that if θ is an equivalence relation on S identifying distinct vertices albeit satisfying the remaining three conditions we gave for defining a congruence, then it may be impossible to endow the graph S/θ with a semigroupoid structure. For instance, in Example 3.1 the quotient graph $S/\operatorname{Ker}\varphi$ is not a semigroupoid because $c_1/\operatorname{Ker}\varphi$ and $c_2/\operatorname{Ker}\varphi$ are consecutive edges, but there is no edge in $S/\operatorname{Ker}\varphi$ from $\alpha(c_1/\operatorname{Ker}\varphi)$ to $\omega(c_2/\operatorname{Ker}\varphi)$.

Let G be a topological graph. Then, for any $x, y \in V_G$, the set $E_G(x, y)$ is closed; the set D_G is also closed. If the topology of V_G is the discrete one then $E_G(x, y)$ and D_G are open. A *(compact) topological semigroupoid* is a semigroupoid S whose underlying graph is a (compact) topological graph and whose composition is continuous, which means that if $(s_i, t_i)_{i\in I}$ is a net of elements of D_S converging to (s, t), then $(s_i t_i)_{i\in I}$ converges to st (note that D_S is closed, hence (s, t) belongs to D_S). The product of topological semigroupoids is a semigroupoid with respect to the product topology and to the composition defined componentwise.

3.2. The closed subsemigroupoid generated by a graph. Let R be a topological semigroupoid and X a nonempty subgraph of R. Let $\mathscr{Q}$ be the set of closed subsemigroupoids of R containing X. Note that $R \in \mathscr{Q}$. Let $\lceil X \rceil$ be the intersection of all elements of $\mathscr{Q}$. Then $\lceil X \rceil \in \mathscr{Q}$. We say that $\lceil X \rceil$ is the *closed subsemigroupoid of* R *generated by* X. It is routine to check that if D_R is open then $\lceil X \rceil = \overline{\langle X \rangle}$.

Proposition 3.2. *For a two-letter alphabet $\{a, b\}$, let $\mathscr{Z}$ be the sofic subshift of $\{a, b\}^{\mathbb{Z}}$ presented in Figure 3. Suppose $\Sigma(\mathscr{Z})^+$ is a subsemigroupoid of a compact semigroupoid S such that $\mathscr{Z}$ is a topological subspace of V_S. Then $\overline{\Sigma(\mathscr{Z})^+}$ is not a subsemigroupoid of S.*

Proof: For each positive integer n, let s_n be the unique edge of $\Sigma(\mathcal{Z})^+$ from $a^{-\infty}.b^{+\infty}$ to $\sigma^n(a^{-\infty}.b^{+\infty})$, and let t_n be the unique edge of $\Sigma(\mathcal{Z})^+$ from $\sigma^{-n}(b^{-\infty}.a^{+\infty})$ to $b^{-\infty}.a^{+\infty}$. Since S is compact, the sequences $(s_n)_n$ and $(t_n)_n$ have accumulation points s and t in S, respectively. Due to the continuity of α and ω, we have

$$\alpha(s) = a^{-\infty}.b^{+\infty}, \quad \omega(s) = b^{\infty} = \alpha(t), \quad \omega(t) = b^{-\infty}.a^{+\infty}.$$

Since s and t are consecutive edges, the product $s \cdot t$ exists in S.

Suppose $\overline{\Sigma(\mathcal{Z})^+}$ is a subsemigroupoid of S. Then, since $s, t \in \overline{\Sigma(\mathcal{Z})^+}$, we have $s \cdot t \in \overline{\Sigma(\mathcal{Z})^+}$. Hence, there is a net $(e_i)_{i \in I}$ of edges of $\Sigma(\mathcal{Z})^+$ converging to $s \cdot t$. Due to the continuity of α and ω, the nets $(\alpha(e_i))_{i \in I}$ and $(\omega(e_i))_{i \in I}$ converge to $a^{-\infty}.b^{+\infty}$ and $b^{-\infty}.a^{+\infty}$, respectively. Note that $a^{-\infty}.b^{+\infty}$ and $b^{-\infty}.a^{+\infty}$ are isolated points of $\mathcal{Z}$, hence there is $i \in I$ such that $\alpha(e_i) = a^{-\infty}.b^{+\infty}$ and $\omega(e_i) = b^{-\infty}.a^{+\infty}$. But in $\Sigma(\mathcal{Z})^+$ there is no edge from $a^{-\infty}.b^{+\infty}$ to $b^{-\infty}.a^{+\infty}$. We thus reach a contradiction, which shows that $s \cdot t \notin \overline{\Sigma(\mathcal{Z})^+}$. ∎

Later on we shall verify that the semigroupoid $\Sigma(\mathcal{Z})^+$ indeed embeds into a compact semigroupoid (cf. Corollary 3.25). Once this is done, Proposition 3.2 gives an example of a nonempty subgraph X of a compact semigroupoid R such that $\overline{\langle X \rangle} \subsetneqq \lceil X \rceil$: just take $X = \Sigma(\mathcal{Z})^+$ and note that $\langle \Sigma(\mathcal{Z})^+ \rangle = \Sigma(\mathcal{Z})^+$.

Returning to an abstract setting, let X be a nonempty subgraph of a topological semigroupoid R. Consider the following definition, by transfinite recursion, of sets denoted by $\lceil X \rceil_\beta$, where β is an ordinal:

- $\lceil X \rceil_0 = X$;
- $\lceil X \rceil_{\beta^+}$ is the closure in R of the subsemigroupoid generated by $\lceil X \rceil_\beta$;
- if β is a limit ordinal then $\lceil X \rceil_\beta = \bigcup_{\gamma \in \beta} \lceil X \rceil_\gamma$.

Note that $X \subseteq \lceil X \rceil_\beta \subseteq \lceil X \rceil$ for every ordinal β, which is easily proved by transfinite induction.

For the sake of conciseness, in the following lines the set $\lceil X \rceil_\beta$ is denoted by y_β.

Lemma 3.3. *Let β_0 be an ordinal such that $y_{\beta_0^+} = y_{\beta_0}$. Then $\lceil X \rceil = y_{\beta_0}$.*

Proof: We have $\langle y_{\beta_0} \rangle \subseteq \overline{\langle y_{\beta_0} \rangle} = y_{\beta_0}$, thus $y_{\beta_0} \in \mathcal{Q}$. Moreover, $y_{\beta_0} \subseteq \lceil X \rceil$. ∎

Lemma 3.4. *If $\mathfrak{d}$ is a cardinal greater than the cardinal of $\lceil X \rceil$ then there is an ordinal β_0 belonging to $\mathfrak{d}$ such that $y_{\beta_0^+} = y_{\beta_0}$.*

Proof: Let β and γ be distinct ordinals. Then $\beta \in \gamma$ or $\gamma \in \beta$. Suppose $\beta \in \gamma$. Then $\beta^+ \subseteq \gamma$. One can easily prove by transfinite induction that the operator y preserves order, thus $y_{\beta^+} \subseteq y_\gamma$. Similarly, if $\gamma \in \beta$ then $y_{\gamma^+} \subseteq y_\beta$. Anyway, we have $(y_{\beta^+} \setminus y_\beta) \cap (y_{\gamma^+} \setminus y_\gamma) = \emptyset$. Therefore the following correspondence is well defined:

$$\begin{array}{rcl} f: \lceil X \rceil & \rightarrow & \mathfrak{d} \\ x & \mapsto & \begin{cases} \beta & \text{if } \beta \in \mathfrak{d} \text{ and } x \in y_{\beta^+} \setminus y_\beta, \\ 0 & \text{in the remaining cases.} \end{cases} \end{array}$$

Suppose the lemma is false. Then, by Lemma 3.3, for every ordinal β belonging to $\mathfrak{d}$, there is an element x_β of $y_{\beta^+} \setminus y_\beta$. Note that $x_\beta \in \lceil X \rceil$, since $y_\gamma \subseteq \lceil X \rceil$ for every ordinal γ. Therefore $\beta = f(x_\beta)$, for every ordinal β belonging to $\mathfrak{d}$. Hence f is onto, and therefore $\mathfrak{d} \leq |\lceil X \rceil|$. This contradicts the hypothesis $|\lceil X \rceil| < \mathfrak{d}$. ∎

Lemma 3.5. *Let R and S be topological semigroupoids. Consider a nonempty subgraph X of R such that $R = \lceil X \rceil$. Let ψ and η be continuous homomorphisms of semigroupoids from R to S. If $\psi|_X = \eta|_X$ then $\psi = \eta$.*

Proof: By Lemmas 3.4 and 3.3, it is sufficient to prove by transfinite induction that $\psi|_{y_\beta} = \eta|_{y_\beta}$ for every ordinal β, which is a pure routine task. ∎

3.3. Pseudovarieties of semigroupoids. A semigroupoid S is a *divisor* of a semigroupoid T if there are a faithful homomorphism $\varphi : R \to T$ and a quotient homomorphism $\varphi : R \to S$ for some semigroupoid R. A *pseudovariety of semigroupoids* is a class of finite semigroupoids containing the trivial semigroup and the divisors and finite direct products of its elements*. The intersection of semigroupoid pseudovarieties is also a semigroupoid pseudovariety. The *pseudovariety generated* by a class $\mathscr{C}$ of finite semigroupoids is the intersection of those pseudovarieties containing $\mathscr{C}$, and its elements

* Tilson's original definition [38] includes the need of a pseudovariety of semigroupoids to contain the finite disjoint unions of its elements. This results from Tilson's preference for an equational theory with graph-identities on finite connected graphs. In [9] it is not imposed any restriction about connectedness. However, in the same article the definition of semigroupoid pseudovariety is Tilson's one. Tilson's hypothesis about unions can be dropped in order to have a coherent equational theory with graph-identities over non-connected graphs. This implies some minor changes in certain arguments made in [9]. For example, in the proof of Theorem 2.7 of [9] where one finds "let A be a subgraph of C generating C" one puts "let $A_1, \ldots, A_n$ be subgraphs of C generating the connected components $C_1, \ldots, C_n$ of C, respectively", since it is not assumed C is connected. Anyway, choosing or not Tilson's definition is irrelevant for our purposes.

are the divisors of finite direct products of members of $\mathscr{C}$ (cf. [9, Section 2]). The pseudovariety of semigroupoids generated by a pseudovariety V of semigroups, called the *global* of V, is denoted by gV.

Let V be a pseudovariety of semigroupoids. A semigroupoid S is *pro*-V if every pair of distinct elements of S is separated by a continuous semigroupoid homomorphism into a semigroupoid of V. If V is the class of all finite semigroupoids then S is said to be *profinite*.

Note that the projective limit of a directed system of compact semigroupoids is a compact semigroupoid. We call a directed system of quotient homomorphisms of semigroupoids a *directed quotient system*.

Theorem 3.6 (cf. [25, Theorem 4.1])**.** *Let V be a pseudovariety of semigroupoids. Let S be a finite-vertex topological semigroupoid. Then S is pro-V if and only if S is isomorphic to a projective limit of a directed quotient system of semigroupoids of V, if and only if S is isomorphic to a projective limit of a directed system of semigroupoids of V.*

Problem 3.7. *Let V be a pseudovariety of finite semigroupoids. A pro-V semigroupoid is a projective limit of semigroupoids of V?*

Without going into all the details, the difficulty in generalizing the proof of Theorem 3.6 relies on the fact that in general one cannot consider congruences identifying vertices (or in another perspective of the same phenomena, that the image of a semigroupoid homomorphism is not necessarily a semigroupoid).

The *consolidate* of a semigroupoid S is the semigroup S_{cd} whose elements are the edges of S and, if S has pairs of nonconsecutive edges, an extra element 0, the product in S_{cd} of two consecutive edges of S being their composition, and the remaining products being equal to 0. If S is a topological semigroupoid then we endow S_{cd} with the topology of E_S together with 0 as an isolated point.

Remark 3.8. *If S is a finite-vertex topological semigroupoid then S_{cd} is a topological semigroup.*

Proof: Let $(s_i, t_i)_{i\in I}$ be a net of pairs of elements of S_{cd} converging to (s, t).

If $st = 0$ then $(s, t) \notin D_S$. Since D_S is closed and 0 is an isolated point, the set

$$U = (E_S \times E_S) \setminus D_S \cup E_S \times \{0\} \cup \{0\} \times E_S \cup \{(0, 0)\}$$

is an open neighborhood of (s,t) in $S_{cd} \times S_{cd}$. Hence there is $i_0 \in I$ such that if $i \leq i_0$ then $(s_i, t_i) \in U$, thus $s_i t_i = 0$. Therefore $(s_i t_i)_{i\in I}$ converges to st.

If $st \neq 0$ then $(s,t) \in D_S$. Since D_S is open, there is $i_0 \in I$ such that if $i \leq i_0$ then $(s_i, t_i) \in D_S$, thus $s_i t_i \in E_S$. By the definition of topological semigroupoid, the net $(s_i t_i)_{i\in I}$ converges to st. ■

The semigroup B_2 is the *syntactic semigroup* (see [26] for the definition) of the language $(ab)^+$ on the two-letter alphabet $\{a, b\}$.

Proposition 3.9. *Let* V *be a pseudovariety of semigroups containing* B_2*. Let* S *be a finite semigroupoid. Then* $S \in \mathsf{gV}$ *if and only if* $S_{cd} \in \mathsf{V}$*.*

See [9, Corollary 7.7] for a proof of Proposition 3.9. The converse implication is trivial, and it follows from it that gS is the pseudovariety Sd of all finite semigroupoids.

Suppose $\varphi : S \to T$ is a continuous quotient homomorphism of topological semigroupoids. Clearly $0 \in S_{cd}$ if and only if $0 \in T_{cd}$. Consider the map $\varphi_{cd} : S_{cd} \to T_{cd}$ such that $\varphi_{cd}(s) = \varphi(s)$ for every $s \in E_S$, and $\varphi_{cd}(0) = 0$ if $0 \in S_{cd}$. Then φ_{cd} is a continuous homomorphism. If $\varphi : S \to T$ separates s and t then so does φ_{cd}. Conversely, if $\psi : S_{cd} \to T_{cd}$ separates s and t then so does $\psi \circ \gamma$, where $\gamma : S \to S_{cd}$ is the identity map on the edges. These simple facts justify the following corollary of Proposition 3.9.

Corollary 3.10. *Let* V *be a pseudovariety of semigroups containing* B_2*. Let* S *be a finite-vertex topological semigroupoid. Then* S *is pro-*gV *if and only if* S_{cd} *is pro-*V*.*

3.4. Relatively free profinite finite-vertex semigroupoids. Consider a graph Γ and a pseudovariety V of semigroupoids. A map κ from Γ into a topological semigroupoid T is a *generating map of* T if $\lceil \kappa(\Gamma) \rceil = T$. A pro-$\mathsf{V}$ semigroupoid T is a *free pro-*V *semigroupoid generated by* Γ*, with generating map* $\kappa : \Gamma \to T$*,* if for every graph homomorphism φ from Γ into a pro-V semigroupoid S there is a unique continuous semigroupoid homomorphism $\hat{\varphi} : T \to S$ satisfying $\hat{\varphi} \circ \kappa = \varphi$. Note that it suffices to consider semigroupoids S such that $S = \lceil \varphi(\Gamma) \rceil$, and in particular if Γ is finite-vertex then one may always suppose S is finite-vertex.

By the usual abstract nonsense, up to isomorphism of topological semigroupoids, there is no more than one free pro-V semigroupoid generated by Γ. For the case where Γ is finite-vertex, we describe in the following lines a semigroupoid that turns out to be the free pro-V semigroupoid generated by Γ. Note that when Γ is a one-vertex graph and $\mathsf{V} = \mathsf{gW}$ for some pseudovariety W of semigroups, such a semigroupoid is the free pro-W semigroup generated by Γ. Let $Con_\Gamma\mathsf{V}$ be the set of congruences θ on Γ^+ such that Γ^+/θ belongs to V. If ϑ is the co-terminality congruence then Γ^+/ϑ divides the trivial semigroup, hence $Con_\Gamma\mathsf{V}$ is nonempty if and only if Γ is finite-vertex. The intersection of congruences is also a congruence, hence $Con_\Gamma\mathsf{V}$ endowed with the partial order $\supseteq$ is a directed set. The family

$$\{q_{\theta,\rho} : \Gamma^+/\theta \to \Gamma^+/\rho \mid \rho, \theta \in Con_\Gamma\mathsf{V}, \rho \supseteq \theta\}$$

is a directed system of quotient homomorphisms. Its projective limite is a pro-V semigroupoid, denoted by $\overline{\Omega}_\Gamma\mathsf{V}$. If Γ is finite then $Con_\Gamma\mathsf{V}$ is countable, and therefore the topological space $\overline{\Omega}_\Gamma\mathsf{V}$ is generated by a metric [40, Theorem 22.3].

Let $\iota : \Gamma \to \overline{\Omega}_\Gamma\mathsf{V}$ be the map defined by $\iota(a) = ([a]_\theta)_{\theta \in Con_\Gamma\mathsf{V}}$. The subsemigroupoid of $\overline{\Omega}_\Gamma\mathsf{V}$ generated by $\iota(\Gamma)$ is the set $\iota^+(\Gamma^+)$, denoted by $\Omega_\Gamma\mathsf{V}$.

Theorem 3.11 (cf. [25, Theorem 6.3])**.** *Let* V *be a pseudovariety of semigroupoids and let* Γ *be a finite-vertex graph. The semigroupoid* $\overline{\Omega}_\Gamma\mathsf{V}$ *is a free pro-*V *semigroupoid generated by* Γ*, with generating map* ι*.*

Lemma 3.12. *Let* Γ *be a graph and* u *a path on* Γ*. Then there is a semigroup* S *in* N *and a semigroupoid homomorphism* $\varphi : \Gamma^+ \to S$ *such that* $\varphi^{-1}\varphi(u) = \{u\}$*.*

Proof: Let Λ be the set of edges of Γ which are factors of u. Let F be the set of paths of Λ with length less or equal to that of u. Then $I = E_\Gamma^+ \setminus F$ is an ideal of E_Γ^+ (for the definition of semigroup ideal and Rees quotient see [26]). The Rees quotient E_Γ^+/I belongs to N. The natural semigroupoid homomorphism $\varphi : \Gamma^+ \to E_\Gamma^+/I$ satisfies $\varphi^{-1}\varphi(u) = \{u\}$. ■

Proposition 3.13. *Let* V *be a pseudovariety of semigroupoids and let* Γ *be a finite-vertex graph. If* V *contains nontrivial semigroups then* $\iota : \Gamma \to \Omega_\Gamma\mathsf{V}$ *is an embedding. If* V *contains* N*, then* ι^+ *is a semigroupoid isomorphism from* Γ^+ *to* $\Omega_\Gamma\mathsf{V}$*, and the elements of* $\Omega_\Gamma\mathsf{V}$ *are isolated points of* $\overline{\Omega}_\Gamma\mathsf{V}$*.*

Proof: Let u and v be distinct edges of Γ. Suppose V contains a nontrivial semigroup S. Then there is a graph homomorphism $\psi : \Gamma \to S$ such that $\psi(u) \neq \psi(v)$. There is a unique continuous semigroupoid homomorphism $\hat{\psi} : \overline{\Omega}_\Gamma\mathsf{V} \to S$ such that $\hat{\psi} \circ \iota = \psi$, thus $\iota(u) \neq \iota(v)$. Hence ι is an embedding.

Suppose V contains N. The map $\iota^+ : \Gamma^+ \to \Omega_\Gamma \mathsf{V}$ is a quotient semigroupoid homomorphism. We want to prove it is injective. Let u and v be distinct edges of Γ^+. By Lemma 3.12 there are a semigroup S in N and a semigroupoid homomorphism $\varphi : \Gamma^+ \to S$ such that $\varphi(u) \neq \varphi(v)$. Since $\mathsf{N} \subseteq \mathsf{V}$, there is a unique continuous semigroupoid homomorphism $\hat{\varphi}$ from $\overline{\Omega}_\Gamma \mathsf{V}$ to S such that $\hat{\varphi} \circ \iota = \varphi|_\Gamma$. Then $\hat{\varphi} \circ \iota^+ = \varphi$, thus $\iota^+(u) \neq \iota^+(v)$. Therefore ι^+ is an isomorphism.

We identify Γ^+ with $\Omega_\Gamma \mathsf{V}$ through ι^+. Take an arbitrary edge u of Γ^+. Let $(u_\tau)_{\tau \in \mathcal{T}}$ be a net of edges of Γ^+ converging to u. Let φ be as in Lemma 3.12. Since $\hat{\varphi}$ is continuous and $\hat{\varphi}|_{\Gamma^+} = \varphi$, there is $\tau_0 \in \mathcal{T}$ such that if $\tau_0 \leq \tau$ then $\varphi(u_\tau) = \varphi(u)$. Since $\varphi^{-1}\varphi(u) = \{u\}$, if $\tau_0 \leq \tau$ then $u_\tau = u$. Since Γ^+ is dense in $\overline{\Omega}_\Gamma \mathsf{V}$, this proves the last assertion. ■

3.5. Relatively free profinite semigroupoids generated by profinite graphs. Let Γ be a profinite graph. A pro-V semigroupoid T is a *free pro-*V *semigroupoid generated by* Γ, if there is a continuous generating map $\kappa : \Gamma \to S$ such that for every continuous graph homomorphism φ from Γ into a pro-V semigroupoid S there is a unique continuous semigroupoid homomorphism $\hat{\varphi} : T \to S$ satisfying $\hat{\varphi} \circ \kappa = \varphi$. Note that, up to isomorphism of topological semigroupoids, there is no more than one free pro-V semigroupoid generated by Γ. We shall prove in this section that such semigroupoid always exist when Γ is profinite. If Γ is a finite graph, then we already know this is true by Theorem 3.11.

From hereon, Γ is a projective limit of finite graphs defined by a directed system $\{\delta_{j,i} : \Gamma_j \to \Gamma_i \mid i, j \in I, i \leq j\}$ of onto graph homomorphisms. The canonical projection $\Gamma \to \Gamma_i$ is denoted by δ_i.

Lemma 3.14. *If φ is a continuous graph homomorphism from Γ into a finite graph S then the set $I_\varphi = \{i \in I \mid \forall x, y \in \Gamma,\ \delta_i(x) = \delta_i(y) \Rightarrow \varphi(x) = \varphi(y)\}$ is nonempty.*

Proof: Suppose $I_\varphi = \emptyset$. Then for every $i \in I$ there are $x_i, y_i \in \Gamma$ such that $\delta_i(x_i) = \delta_i(y_i)$ and $\varphi(x_i) \neq \varphi(y_i)$. Since Γ is compact, the nets $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ have subnets $(x_{\lambda(j)})_{j \in J}$ and $(y_{\lambda(j)})_{j \in J}$ converging to some elements x and y of Γ, respectively. Since φ is continuous and S is finite, $\varphi(x) \neq \varphi(y)$. Hence $x \neq y$. Therefore there is $k \in I$ such that $\delta_k(x) \neq \delta_k(y)$. The set $\{(u, v) \in \Gamma_k \times \Gamma_k \mid u = v\}$ is closed in $\Gamma_k \times \Gamma_k$. Hence, since

$$\lim_{j \in J} (\delta_k(x_{\lambda(j)}), \delta_k(y_{\lambda(j)})) = (\delta_k(x), \delta_k(y)),$$

there is $j_0 \in J$ such that if $j_0 \leq j$ then $\delta_k(x_{\lambda(j)}) \neq \delta_k(y_{\lambda(j)})$. There is $j_1 \in J$ such that $j_0 \leq j_1$ and $k \leq \lambda(j_1)$. Let $l = \lambda(j_1)$. Then

$$\delta_{l,k}(\delta_l(x_l)) = \delta_k(x_l) \neq \delta_k(y_l) = \delta_{l,k}(\delta_l(y_l)).$$

But this contradicts the equality $\delta_l(x_l) = \delta_l(y_l)$. ■

Corollary 3.15. *Let φ be a continuous graph homomorphism from Γ into a finite graph S. There is $i \in I$ for which it exists a unique continuous graph homomorphism $\varphi_i : \Gamma_i \to S$ such that $\varphi_i \circ \delta_i = \varphi$.*

Proof: Take $i \in I_\varphi$. ■

If i and j are distinct elements of I such that $i \leq j$ then, by Theorem 3.11, there is a unique continuous semigroupoid homomorphism $\hat{\delta}_{j,i}$ such the following diagram is commutative, where ι_k denotes the generating map of $\overline{\Omega}_{\Gamma_k}\mathsf{V}$:

$$\begin{tikzcd}
\Gamma_j \arrow[r, "\iota_j"] \arrow[d, "\delta_{j,i}"'] & \overline{\Omega}_{\Gamma_j}\mathsf{V} \arrow[d, "\hat{\delta}_{j,i}"] \\
\Gamma_i \arrow[r, "\iota_i"] & \overline{\Omega}_{\Gamma_i}\mathsf{V}
\end{tikzcd}$$

The family $\{\hat{\delta}_{j,i} : \overline{\Omega}_{\Gamma_j}\mathsf{V} \to \overline{\Omega}_{\Gamma_i}\mathsf{V} \mid i, j \in I, i \leq j\}$ is therefore a directed system of continuous homomorphisms of profinite semigroupoids. Denote by $\hat{\delta}_i$ the canonical projection of $\varprojlim_{j\in I} \overline{\Omega}_{\Gamma_j}\mathsf{V}$ on $\overline{\Omega}_{\Gamma_i}\mathsf{V}$, and by ι the map from Γ into $\varprojlim_{j\in I} \overline{\Omega}_{\Gamma_j}\mathsf{V}$ defined by $\iota(x) = (\iota_i \circ \delta_i(x))_{i\in I}$. Note that $\hat{\delta}_i \circ \iota = \iota_i \circ \delta_i$.

Lemma 3.16. *Let φ be a continuous graph homomorphism from Γ into a finite semigroupoid S. Then there is a continuous semigroupoid homomorphism $\bar{\varphi}$ from $\varprojlim_{j\in I} \overline{\Omega}_{\Gamma_j}\mathsf{V}$ into S such that $\bar{\varphi} \circ \iota = \varphi$.*

Proof: Let $\varphi_i : \Gamma_i \to S$ be as in Corollary 3.15. By Theorem 3.11 there is a unique continuous semigroupoid homomorphism $\hat{\varphi}_i$ from $\overline{\Omega}_{\Gamma_i}\mathsf{V}$ into S such that $\hat{\varphi}_i \circ \iota_i = \varphi_i$. The following diagram is commutative:

$$\begin{tikzcd}
\Gamma \arrow[rr, "\iota"] \arrow[dr, "\delta_i"] \arrow[ddrr, bend right, "\varphi"'] & & \varprojlim_{j\in I} \overline{\Omega}_{\Gamma_j}\mathsf{V} \arrow[d, "\hat{\delta}_i"] \\
& \Gamma_i \arrow[r, "\iota_i"] \arrow[dr, "\varphi_i"] & \overline{\Omega}_{\Gamma_i}\mathsf{V} \arrow[d, "\hat{\varphi}_i"] \\
& & S
\end{tikzcd}$$

It suffices to take $\bar{\varphi} = \hat{\varphi}_i \circ \hat{\delta}_i$. ■

Theorem 3.17. *Let φ be a continuous graph homomorphism from Γ into a semigroupoid S of V. Then there is a unique continuous semigroupoid homomorphism $\hat{\varphi} : \lceil \iota(\Gamma) \rceil \to S$ such that $\hat{\varphi} \circ \iota = \varphi$.*

Proof: It is an immediate consequence of Lemmas 3.16 and 3.5. ■

By Theorems 3.11 and 3.17, if Γ has a finite number of vertices then $\overline{\Omega}_\Gamma \mathsf{V}$ and $\lceil \iota(\Gamma) \rceil$ are isomorphic compact semigroupoids. For that reason, there is no ambiguity in denoting $\lceil \iota(\Gamma) \rceil$ by $\overline{\Omega}_\Gamma \mathsf{V}$ when Γ is a finite-vertex graph. We shall also denote by $\Omega_\Gamma \mathsf{V}$ the subsemigroupoid of $\overline{\Omega}_\Gamma \mathsf{V}$ generated by $\iota(\Gamma)$.

Theorem 3.18. *Let V be a pseudovariety of semigroupoids and let Γ be a profinite graph. The semigroupoid $\overline{\Omega}_\Gamma \mathsf{V}$ is a free pro-V semigroupoid generated by Γ, with generating map ι.*

For proving Theorem 3.18 we need some auxiliary results.

Lemma 3.19. *If S is a pro-V semigroupoid then there are a family $\mathscr{F}$ of semigroupoids of V and a continuous embedding $\Psi : S \to \prod_{F \in \mathscr{F}} F$.*

Proof: Let $\mathcal{P}_2(S)$ be the set of the subsets of S with two elements. Since S is pro-V, for each element $\{u, v\}$ of $\mathcal{P}_2(S)$ there is a continuous semigroupoid homomorphism $\psi_{\{u,v\}}$ from S to a semigroupoid $F_{\{u,v\}}$ of V such that $\psi_{\{u,v\}}(u) \neq \psi_{\{u,v\}}(v)$. The map

$$\begin{aligned} \Psi : \; S \; &\to \; \textstyle\prod_{\{s,t\} \in \mathcal{P}_2(S)} F_{\{u,v\}} \\ s \; &\mapsto \; \big(\psi_{\{u,v\}}(s)\big)_{\{u,v\} \in \mathcal{P}_2(S)}. \end{aligned}$$

is a continuous embedding of semigroupoids.

■

Lemma 3.20. *Let $\psi : S \to T$ be a continuous homomorphism of topological semigroupoids. Let X be a nonempty subgraph of S. Then, for every ordinal β,*

$$\psi\big(\lceil X \rceil_\beta\big) \subseteq \lceil \psi(X) \rceil_\beta \tag{3.2}$$

and

$$\psi\big(\langle \lceil X \rceil_\beta \rangle\big) \subseteq \langle \lceil \psi(X) \rceil_\beta \rangle. \tag{3.3}$$

If $\psi|_{V_S}$ is injective then $\psi\big(\lceil X \rceil_\beta\big) = \lceil \psi(X) \rceil_\beta$ and $\psi\big(\langle \lceil X \rceil_\beta \rangle\big) = \langle \lceil \psi(X) \rceil_\beta \rangle$.

Proof: Let us prove (3.2) by transfinite induction on β. The case $\beta = 0$ is trivial. Suppose (3.2) is verified. Since ψ is a continuous map of compact spaces, we have

$$\psi\big(\lceil X\rceil_{\beta^+}\big) = \psi\Big(\overline{\langle\lceil X\rceil_\beta\rangle}\Big) = \overline{\psi\big(\langle\lceil X\rceil_\beta\rangle\big)}. \tag{3.4}$$

And since ψ is a homomorphism of semigroupoids, according to equality (3.1) (page 12) we have

$$\psi\big(\langle\lceil X\rceil_\beta\rangle\big) \subseteq \big\langle\psi\big(\lceil X\rceil_\beta\big)\big\rangle. \tag{3.5}$$

Hence, from (3.4) and (3.2) we deduce

$$\psi\big(\lceil X\rceil_{\beta^+}\big) \subseteq \overline{\big\langle\psi\big(\lceil X\rceil_\beta\big)\big\rangle} \subseteq \overline{\big\langle\lceil\psi(X)\rceil_\beta\big\rangle} = \lceil\psi(X)\rceil_{\beta^+},$$

concluding the successor case of the inductive step of (3.2). The limit case is immediate.

By (3.2) and (3.5), we have $\psi\big(\langle\lceil X\rceil_\beta\rangle\big) \subseteq \big\langle\psi\big(\lceil X\rceil_\beta\big)\big\rangle \subseteq \langle\lceil\psi(X)\rceil_\beta\rangle$ for every ordinal β, which proves (3.3).

If $\psi|_{V_S}$ is injective then the proof of the equalities in the statement is similarly done, the difference being that in (3.5) we now have an equality. ■

Corollary 3.21. *Let $\psi : S \to T$ be a continuous homomorphism of compact semigroupoids. Let X be a nonempty subgraph of S. Then $\psi\big(\lceil X\rceil\big) \subseteq \lceil\psi(X)\rceil$. If $\psi|_{V_S}$ is injective then $\psi\big(\lceil X\rceil\big) = \lceil\psi(X)\rceil$.* ■

Proof of Theorem 3.18: Let S be a pro-V semigroupoid. Let Ψ and $\mathscr{F}$ be as in Lemma 3.19. For each $T \in \mathscr{F}$, let ρ_T be the canonical projection $\prod_{F\in\mathscr{F}} F \to T$. Take an arbitrary continuous graph homomorphism $\varphi : \Gamma \to S$. By Theorem 3.17, for each $T \in \mathscr{F}$ there is a unique continuous semigroupoid homomorphism ζ_T from $\overline{\Omega}_\Gamma\mathsf{V}$ to T such that $\zeta_T \circ \iota = \rho_T \circ \Psi \circ \varphi$. Consider the map $\zeta : \overline{\Omega}_\Gamma\mathsf{V} \to \prod_{F\in\mathscr{F}} F$ such that $\zeta(u) = (\zeta_F(u))_{F\in\mathscr{F}}$.

$$\begin{tikzcd}
\Gamma \arrow[r, "\iota"] \arrow[d, "\varphi"'] & \overline{\Omega}_\Gamma\mathsf{V} \arrow[d, "\zeta"] \arrow[dr, "\zeta_T"] & \\
S \arrow[r, "\Psi"'] & \prod_{F\in\mathscr{F}} F \arrow[r, "\rho_T"'] & T
\end{tikzcd} \tag{3.6}$$

Since for all $T \in \mathscr{F}$ we have $\rho_T \circ \zeta \circ \iota = \zeta_T \circ \iota = \rho_T \circ \Psi \circ \varphi$, we conclude that $\zeta \circ \iota = \Psi \circ \varphi$, thus Diagram (3.6) commutes. Then, by Corollary 3.21,

$$\zeta(\overline{\Omega}_\Gamma\mathsf{V}) = \zeta\big(\lceil\iota(\Gamma)\rceil\big) \subseteq \lceil\zeta(\iota(\Gamma))\rceil = \lceil\Psi(\varphi(\Gamma))\rceil \subseteq \lceil\Psi(S)\rceil = \Psi(S).$$

Hence we can consider the map $\hat{\varphi} = \Psi^{-1} \circ \zeta$, a continuous semigroupoid homomorphism from $\overline{\Omega}_\Gamma \mathsf{V}$ to S. Then $\hat{\varphi} \circ \iota = \varphi$. The uniqueness of $\hat{\varphi}$ follows from Lemma 3.5. ■

Problem 3.22. *Is there some projective limit $\Gamma = \varprojlim_{i\in I} \Gamma_i$ of finite graphs such that $\overline{\Omega}_\Gamma \mathsf{V} \neq \varprojlim_{i\in I} \overline{\Omega}_{\Gamma_i} \mathsf{V}$?*

3.6. Pseudovarieties containing the finite nilpotent semigroups. If $i \leq j$, let $\delta_{j,i}{}^+$ be the unique semigroupoid homomorphism for which the following diagram commutes:

$$\begin{tikzcd} \Gamma_j \arrow[r, hook] \arrow[d, "\delta_{j,i}"'] & \Gamma_j^+ \arrow[d, "\delta_{j,i}{}^+"] \\ \Gamma_i \arrow[r, hook] & \Gamma_i^+ \end{tikzcd}$$

The family $\{\delta_{j,i}{}^+ : \Gamma_j^+ \to \Gamma_i^+ \mid i, j \in I, i \leq j\}$ is a directed system of semigroupoid homomorphisms. Denote by $\delta_i{}^+$ the canonical projection from $\varprojlim_{j\in I} \Gamma_j^+$ to Γ_i^+. The graph Γ is a subgraph of $\varprojlim_{j\in I} \Gamma_j^+$.

Lemma 3.23. *The semigroupoids Γ^+ and $\varprojlim_{i\in I} \Gamma_i^+$ can be identified, in the sense that the unique semigroupoid homomorphism $\jmath$ from Γ^+ to $\varprojlim_{i\in I} \Gamma_i^+$ extending the inclusion is an isomorphism.*

Proof: Clearly $\jmath$ is a bijection between the sets of vertices. Let $w = w_1 \cdots w_k$ be a path on Γ, where $w_1, \ldots, w_k$ are edges of Γ. Given $i \in I$, we have

$$\delta_i{}^+ \circ \jmath(w) = \delta_i(w_1) \cdots \delta_i(w_k). \tag{3.7}$$

Suppose $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_m$ are paths on Γ, where $u_1, \ldots, u_n, v_1, \ldots, v_m$ are edges of Γ. If $\jmath(u) = \jmath(v)$ then $\delta_i(u_1) \cdots \delta_i(u_n) = \delta_i(v_1) \cdots \delta_i(v_m)$ by (3.7). Hence $n = m$ and $\delta_i{}^+(u_l) = \delta_i{}^+(v_l)$, for any $l \in \{1, \ldots, n\}$. Since i is arbitrary, we conclude that $u_l = v_l$, for any $l \in \{1, \ldots, n\}$. That is, $u = v$.

On the other hand, let q be an element of $\varprojlim_{i\in I} \Gamma_i^+$. Since the directed system defining Γ is surjective, for every $i \in I$ there are $q_{i,1}, \ldots, q_{i,n_i} \in \Gamma$ such that $\delta_i{}^+(q) = \delta_i(q_{i,1}) \cdots \delta_i(q_{i,n_i})$. If $i \leq j$ then, since $\delta_i{}^+ = \delta_{j,i}{}^+ \circ \delta_j{}^+$, we have

$$\delta_i(q_{i,1}) \cdots \delta_i(q_{i,n_i}) = \delta_i(q_{j,1}) \cdots \delta_j(q_{j,n_j}).$$

Therefore

$$j \geq i \Rightarrow \big(n_j = n_i \quad \text{and} \quad \delta_i(q_{i,l}) = \delta_i(q_{j,l}) \quad \forall l \in \{1, \ldots, n_i\}\big). \tag{3.8}$$

In particular, if i_1 and i_2 are arbitrary elements of I, then $n_{i_1} = n_{i_2} = n_{i_0}$, for every i_0 such that $i_1 \le i_0$ and $i_2 \le i_0$. Since I is directed, such i_0 always exist, thus the net $(n_i)_{i\in I}$ has constant value n. Let F be a finite subset of I. Then there is $k \in I$ such that $i \le k$ for any $i \in F$. By (3.8), for all $i \in F$ we have $q_{k,l} \in \bigcap_{i\in F} \delta_i^{-1}\delta_i(q_{i,l})$. The set $\delta_i^{-1}\delta_i(q_{i,l})$ is closed for every $i \in I$. Then, since Γ is compact and $\bigcap_{i\in F} \delta_i^{-1}\delta_i(q_{i,l}) \neq \emptyset$ for every finite subset F of I, the set $\bigcap_{i\in I} \delta_i^{-1}\delta_i(q_{i,l})$ is nonempty. Let q_l be one of its elements. For $l < n$,

$$\omega(q_l) = (\omega(\delta_i(q_l)))_{i\in I} = (\omega(\delta_i(q_{i,l})))_{i\in I} = (\alpha(\delta_i(q_{i,l+1})))_{i\in I} = \cdots = \alpha(q_{l+1}).$$

Since $q_1, \ldots, q_n$ are consecutive edges, we can consider the element $\jmath(q_1 \cdots q_n)$ of the image of $\jmath$. Then

$$\delta_i^+(\jmath(q_1 \cdots q_n)) = \delta_i(q_1) \cdots \delta_i(q_n) = \delta_i(q_{i,1}) \cdots \delta_i(q_{i,n}) = {\delta_i}^+(q).$$

Since i is arbitrary, we conclude that $q = \jmath(q_1 \cdots q_n)$. Hence $\jmath$ is surjective. ■

Proposition 3.24. *Let* V *be a pseudovariety of semigroupoids and let* Γ *be a profinite graph. If* V *contains nontrivial semigroups then* $\iota : \Gamma \to \Omega_\Gamma \mathsf{V}$ *is an embedding. If* V *contains* N *then* ι^+ *is a semigroupoid isomorphism from* Γ^+ *onto* $\Omega_\Gamma\mathsf{V}$.

Proof: Suppose V contains nontrivial semigroups. Let u and v be distinct elements of Γ. Then there is $i \in I$ such that $\delta_i(u) \neq \delta_i(v)$. The graph homomorphism ι_i is an embedding, by Proposition 3.13. Hence $\iota_i(\delta_i(u)) \neq \iota_i(\delta_i(v))$. Since $\iota(w) = (\iota_i \circ \delta_i(w))_{i\in I}$, this proves ι is an embedding.

Suppose V contains N. The map $\iota^+ : \Gamma^+ \to \Omega_\Gamma\mathsf{V}$ is a quotient homomorphism of semigroupoids. We want to prove that it is injective. Let $w = w_1 \ldots w_n$ be a path on Γ, where $w_1, \ldots, w_n$ are consecutive edges of Γ. Then, for every $i \in I$,

$$\hat{\delta}_i(\iota^+(w)) = \hat{\delta}_i(\iota(w_1)) \cdots \hat{\delta}_i(\iota(w_n)) = \iota_i(\delta_i(w_1)) \cdots \iota_i(\delta_i(w_n)) = {\iota_i}^+({\delta_i}^+(w)).$$

Hence if u and v are edges of Γ^+ and $\iota^+(u) = \iota^+(v)$ then ${\iota_i}^+({\delta_i}^+(u)) = {\iota_i}^+({\delta_i}^+(v))$ for all $i \in I$. From Proposition 3.13 we deduce ${\delta_i}^+(u) = {\delta_i}^+(v)$ for all $i \in I$. Then $u = v$ by Lemma 3.23. ■

We could not prove Proposition 3.24 directly using the arguments in the proof of Proposition 3.13 because in general one can not expect the homomorphism in Lemma 3.12 to be continuous. According to Proposition 3.24, one may consider Γ^+ as a subsemigroupoid of $\overline{\Omega}_\Gamma\mathsf{V}$.

Corollary 3.25. *For every pseudovariety of semigroupoids* V *containing* N, *there are profinite graphs* Γ *such that* Γ^+ *is not dense in* $\overline{\Omega}_\Gamma\mathsf{V}$.

Proof: Take the graph $\Sigma(\mathcal{Z})$ in Proposition 3.2 and apply Propositions 3.2 and 3.24. ■

4. Relatively free profinite semigroupoids defined by subshifts

From here on $\mathcal{X}$ designates a generic subshift of $A^{\mathbb{Z}}$ and V a pseudovariety of semigroups containing $\mathscr{L}\mathsf{I}$. This allow us to define the maps i_n and t_n with domain $\overline{\Omega}_A\mathsf{V}$. The canonical projection $\Sigma(\mathcal{X}) \to \Sigma_{2n}(\mathcal{X})$ will be denoted by π_n. We shall denote by $\widehat{\Sigma}(\mathcal{X})$ the semigroupoid $\overline{\Omega}_{\Sigma(\mathcal{X})}\mathsf{gV}$. Since gV contains N, we can consider $\Sigma(\mathcal{X})^+$ as a subgraph of $\widehat{\Gamma}$, by Proposition 3.24. Note that since $\Sigma(\mathcal{X})$ is a complete conjugacy invariant then so is $\widehat{\Sigma}(\mathcal{X})$.

4.1. Labelling. Assign to each edge $q = a_1 \cdots a_{2n}a_{2n+1}$ (where $a_i \in A$) of $\Sigma_{2n}(\mathcal{X})$ the letter a_n, denoted by $\mu_n(q)$. We say that $\mathcal{X}$ is a $2n$*-step subshift of finite type* if $L(\mathcal{X})$ is recognized by the labeled graph $(\Sigma_{2n}(\mathcal{X}), \mu_n)$. This means that $\mathcal{X} = \{x \in A^{\mathbb{Z}} : L_{2n+1}(x) \subseteq L(\mathcal{X})\}$. A system is of *finite type* if it is $2n$-step finite type for some n.

According to Proposition 3.2, there is a subshift $\mathcal{Z}$ such that $\overline{\Sigma(\mathcal{Z})^+} \neq \widehat{\Sigma}(\mathcal{Z})$. This situation is in contrast with the following proposition:

Proposition 4.1. *If* $\mathcal{X}$ *is a finite type subshift then* $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X}) = \widehat{\Sigma}(\mathcal{X}) = \overline{\Sigma(\mathcal{X})^+}$.

Proof: There is an integer N such that $\mathcal{X}$ is $2n$-step for every $n \geq N$. Consider a path $q = q_1 \cdots q_k$ in $\Sigma_{2n}(\mathcal{X})$. There is $x \in \mathcal{X}$ such that $q_i = x_{[-n+i-1,n+i-1]}$. Let p be the unique path in $\Sigma(\mathcal{X})$ from x to $\sigma^k(x)$. We have $\hat{\pi}_{2n}(p) = q$. Hence $\hat{\pi}_{2n}(\Sigma(\mathcal{X})^+) = \Sigma_{2n}(\mathcal{X})^+$, thus $\hat{\pi}_{2n}\Big(\overline{\Sigma(\mathcal{X})^+}\Big) = \overline{\Sigma_{2n}(\mathcal{X})^+}$. Moreover, $\overline{\Sigma_{2n}(\mathcal{X})^+} = \widehat{\Sigma}_{2n}(\mathcal{X})$ by Theorem 3.11, because $\Sigma_{2n}(\mathcal{X})$ is finite-vertex. The result follows from Proposition 2.1. ■

We shall denote by μ the continuous graph homomorphism from $\Sigma(\mathcal{X})$ to A mapping each edge $(x, \sigma(x))$ of $\Sigma(\mathcal{X})$ to the letter x_0. We have $\mu_n \circ \pi_n = \mu$, and if $n \leq m$ then $\mu_n \circ \pi_{m,n} = \mu_m$. Since $\overline{\Omega}_A\mathsf{V}$ is a pro-V semigroup, by Theorem 3.11 there is a unique continuous semigroupoid homomorphism $\hat{\mu}_n$ from $\widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}})$ to $\overline{\Omega}_A\mathsf{V}$ such that $\hat{\mu}_n|_{\Sigma_{2n}(\mathcal{A}^{\mathbb{Z}})} = \mu_n$. If $n \leq m$ then $\hat{\mu}_n \circ \hat{\pi}_{m,n}$

is a continuous semigroupoid homomorphism whose restriction to $\Sigma_{2m}(\mathcal{A}^{\mathbb{Z}})$ coincides with μ_m, thus $\hat{\mu}_n \circ \hat{\pi}_{m,n} = \hat{\mu}_m$. Then

$$\hat{\mu}_m \circ \hat{\pi}_m = (\hat{\mu}_1 \circ \hat{\pi}_{m,1}) \circ \hat{\pi}_m = \hat{\mu}_1 \circ (\hat{\pi}_{m,1} \circ \hat{\pi}_m) = \hat{\mu}_1 \circ \hat{\pi}_1.$$

Therefore if q is an edge of $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}})$ then the sequence $(\hat{\mu}_n(\hat{\pi}_n(q)))_n$ has a constant value which we call the *label of* q and denote by $\hat{\mu}(q)$. The mapping $\hat{\mu}$ thus defined is a continuous semigroupoid homomorphism from $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}})$ to $\overline{\Omega}_A \mathsf{V}$.

Lemma 4.2. *Let* $q : x_{[-n,n-1]} \to y_{[-n,n-1]}$ *be an edge of* $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$*, where* $x, y \in \mathcal{X}$*. Let* $u = \hat{\mu}(q)$*. If* $k = \min\{|u|, n\}$ *then* $x_{[0,k-1]} = \mathrm{i}_k(u)$ *and* $y_{[-k,-1]} = \mathrm{t}_k(u)$*.*

Proof: The result is clear if $q \in \Sigma_{2n}(\mathcal{X})^+$. The general case is straightforwardly proved once we realize that $\Sigma_{2n}(\mathcal{X})^+$ is dense in $\widehat{\Sigma}_{2n}(\mathcal{X})$, which is true by Theorem 3.11 because $\Sigma_{2n}(\mathcal{X})$ is finite-vertex. ■

Lemma 4.3. *Let* $q : x \to y$ *be an edge of* $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$*. Let* $u = \hat{\mu}(q)$*. If* $u \in \overline{\Omega}_A \mathsf{V} \setminus A^+$ *then* $\overrightarrow{u} = x_{[0,+\infty[}$ *and* $\overleftarrow{u} = y_{]-\infty,-1]}$*. If* $u \in A^+$ *then* q *is the unique edge of* $\Sigma(\mathcal{X})^+$ *from* x *to* $\sigma^{|u|}(x)$*.*

Proof: Let n be a positive integer. We have $\alpha(\hat{\pi}_n(q)) = \hat{\pi}_n(\alpha(q)) = x_{[-n,n-1]}$. Likewise, $\omega(\hat{\pi}_n(q)) = y_{[-n,n-1]}$. Let $k = \min\{|u|, n\}$. Since $\hat{\mu}_n(\hat{\pi}_n(q)) = u$, by Lemma 4.2 we have $x_{[0,k-1]} = \mathrm{i}_k(u)$ and $y_{[-k,-1]} = \mathrm{t}_k(u)$.

If $u \notin A^+$ then $k = n$. Since n is arbitrary, we deduce that $\overrightarrow{u} = x_{[0,+\infty[}$ and $\overleftarrow{u} = y_{]-\infty,-1]}$.

Suppose $u \in A^+$. Let $(q_l)_l$ be a sequence of elements of $\Sigma_{2n}(\mathcal{X})^+$ converging to $\hat{\pi}_n(q)$. Then $\hat{\mu}_n(q_l) = u$ for l sufficiently large. Hence, taking subsequences if necessary, we may suppose that $|q_l|_l$ is constant equal to $|u|$. Since there is only a finite number of elements of $\Sigma_{2n}(\mathcal{X})^+$ with length $|u|$, we deduce that $\hat{\pi}_n(q) \in \Sigma_{2n}(\mathcal{X})^+$. Hence $q \in \Sigma(\mathcal{X})^+$, because n is arbitrary (cf. Lemma 3.23). Clearly q is the unique edge of $\Sigma(\mathcal{X})^+$ from x to $\sigma^{|q|}(x)$. Finally, $|q| = |\hat{\mu}(q)| = |u|$. ■

Denote by $\mathscr{M}_n(\mathcal{X})$ the set of pseudowords of $\overline{\Omega}_A \mathsf{V}$ whose finite factors of length n belong to $L(\mathcal{X})$. Note that $\mathscr{M}_{2n+1}(\mathcal{X}) \cap A^+$ is the language recognized by $(\Sigma_{2n}(\mathcal{X}), \mu_n)$. As observed in [17, Section 3.2], if V contains $\mathscr{L}\mathsf{SI}$ then $\mathscr{M}_n(\mathcal{X})$ is both closed and open. We denote by $\mathscr{M}(\mathcal{X})$ the intersection $\bigcap_{n \geq 1} \mathscr{M}_n(\mathcal{X})$, which in [17, 18] was called the *mirage of* $\mathcal{X}$. One always has

$\overline{L(\mathcal{X})} \subseteq \mathscr{M}(\mathcal{X})$, and the equality holds if $\mathcal{X}$ is of finite type; however if $\mathcal{Z}$ is the symbolic system presented in Figure 1 then $\overline{L(\mathcal{Z})} \neq \mathscr{M}(\mathcal{Z})$ if $L(\mathcal{Z})$ is V-recognizable [17].

Clearly, $\mathscr{M}(\mathcal{X})$ is factorial. It is also easy to see that if $u \in \mathscr{M}(\mathcal{X})$ then there are $a, b \in A$ such that $aub \in \mathscr{M}(\mathcal{X})$: if $u \notin A^+$ and $x, y \in \mathcal{X}$ are such that $\overrightarrow{u} = x_{[0,+\infty[}$ and $\overleftarrow{u} = y_{]-\infty,-1]}$, take $a = x_{-1}$ and $b = y_0$. And since $\mathscr{M}(\mathcal{X})$ is closed, one deduces the following:

Lemma 4.4. *If $u \in \mathscr{M}(\mathcal{X})$ then there are $v, w \in \overline{\Omega}_A\mathsf{V} \setminus A^+$ such that $vuw \in \mathscr{M}(\mathcal{X})$.*

Since $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$ is a projective limit of a countable family of metric spaces, its topology is generated by a metric [40, Theorem 22.3]. Hence one can use sequences instead of nets, as we do in the proof of the following proposition.

Proposition 4.5. *Consider a pseudovariety of semigroups V containing $\mathscr{L}\mathsf{Sl}$. Then $\overline{L(\mathcal{X})} = \hat{\mu}\Big(\overline{\Sigma(\mathcal{X})^+}\Big)$ and $\mathscr{M}(\mathcal{X}) = \hat{\mu}\Big(\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})\Big)$.*

Proof: Clearly $\hat{\mu}(\Sigma(\mathcal{X})^+) = L(\mathcal{X})$, thus $\overline{L(\mathcal{X})} = \hat{\mu}\Big(\overline{\Sigma(\mathcal{X})^+}\Big)$ by continuity of $\hat{\mu}$.

Let q be an edge of $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$. Let $u = \hat{\mu}(q)$. Consider an arbitrary positive integer n. Then $u = \hat{\mu}_n(\hat{\pi}_n(q))$. Since $\hat{\mu}_n(\Sigma_{2n}(\mathcal{X})^+) \subseteq \mathscr{M}_{2n+1}(\mathcal{X})$, $\overline{\Sigma_{2n}(\mathcal{X})^+} = \widehat{\Sigma}_{2n}(\mathcal{X})$ and $\mathscr{M}_{2n+1}(\mathcal{X})$ is closed, it follows from the continuity of $\hat{\mu}_n$ that $u \in \mathscr{M}_{2n+1}(\mathcal{X})$. Therefore $u \in \bigcap_{n\geq 1} \mathscr{M}_{2n+1}(\mathcal{X}) = \mathscr{M}(\mathcal{X})$.

Conversely, suppose u belongs to $\mathscr{M}(\mathcal{X})$. By Lemma 4.4 there are $v, w \in \overline{\Omega}_A\mathsf{V} \setminus A^+$ such that $vuw \in \mathscr{M}(\mathcal{X})$. Let $(v_k)_k$, $(u_k)_k$ and $(w_k)_k$ be sequences of elements of A^+ converging to v, u and w, respectively. For each k, the graph $\Sigma(A^{\mathbb{Z}})$ has consecutive paths p_k, q_k and r_k such that $\hat{\mu}(p_k) = v_k$, $\hat{\mu}(q_k) = u_k$ and $\hat{\mu}(r_k) = w_k$. Let n be an arbitrary positive integer. Since $vuw \in \mathscr{M}_{2n+1}(\mathcal{X})$ and $\mathscr{M}_{2n+1}(\mathcal{X})$ is open, and since v and w have infinite length, there is N such that if $k \geq N$ then $v_k u_k w_k \in \mathscr{M}_{2n+1}(\mathcal{X})$ and v_k, w_k have length greater than n. Then the edges forming the path $\hat{\pi}_n(q_k)$ belong to $L_{2n+1}(\mathcal{X})$. Hence $\hat{\pi}_n(q_k) \in \Sigma_{2n}(\mathcal{X})^+$. Let q be an accumulation point of $(q_k)_k$. Then $\hat{\pi}_n(q) \in \widehat{\Sigma}_{2n}(\mathcal{X})$, for every n. That is, $q \in \varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$. Finally, note that $\hat{\mu}(q) = u$. ■

4.2. Fidelity. Two co-terminal edges of $\Sigma(\mathcal{X})^+$ with the same length are equal, by Lemma 4.3. Next we generalize this property proving that two co-terminal edges of $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$ with the same label are equal.

Proposition 4.6. *Let* V *be a pseudovariety of semigroups containing* B_2 *and such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$. *Then the homomorphism* $\hat{\mu}_n : \widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}}) \to \overline{\Omega}_A\mathsf{V}$ *is faithful.*

Proof: Since $\widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}})$ has a finite number of vertices, we can consider the topological semigroup $T = (\widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}}))_{cd}$ (cf. Remark 3.8). By Corollary 3.10, we know that T is pro-V. Hence there is a unique continuous homomorphism $\Theta : \overline{\Omega}_{A^{2n+1}}\mathsf{V} \to T$ such that $\Theta(u) = u$ for every $u \in A^{2n+1} = E_{\widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}})}$. Since $\mathsf{V} = \mathsf{V}*\mathsf{D}$, by Theorem 2.4 the graph homomorphism $\Psi : \widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}}) \to \overline{\Omega}_{A^{2n+1}}\mathsf{V}$ assigning to each edge q of $\widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}})$ the pseudoword $\Phi^{\mathsf{V}}_{2n}[\mathrm{i}_n(\alpha(q)) \cdot \hat{\mu}_n(q) \cdot \mathrm{t}_n(\omega(q))]$ is well defined and continuous. One easily verifies by induction on the length of q that $\Theta(\Psi(q)) = q$, for any $q \in E_{\Sigma_{2n}(\mathcal{A}^{\mathbb{Z}})^+}$. Since Ψ is a continuous map and $\overline{\Sigma_{2n}(\mathcal{A}^{\mathbb{Z}})^+} = \widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}})$, we conclude that $\Theta(\Psi(q)) = q$, for every $q \in E_{\widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}})}$. Clearly, if q_1 and q_2 are co-terminal edges with the same label then $\Psi(q_1) = \Psi(q_2)$, thus $q_1 = q_2$. ■

Corollary 4.7. *Let* V *be a pseudovariety of semigroups containing* B_2 *and such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$. *Then the homomorphism* $\hat{\mu} : \varprojlim \widehat{\Sigma}_{2n}(\mathcal{A}^{\mathbb{Z}}) \to \overline{\Omega}_A\mathsf{V}$ *is faithful.* ■

If $\mathsf{V} = \mathsf{V}*\mathsf{D}$ then $\mathsf{Sl} \subseteq \mathsf{V}$ if and only if $\mathscr{L}\mathsf{Sl} \subseteq \mathsf{V}$. Therefore, if $\mathsf{V} = \mathsf{V}*\mathsf{D}$ then $B_2 \in \mathsf{V}$ if and only if $\mathscr{L}\mathsf{Sl} \subseteq \mathsf{V}$, because Sl is generated by a subsemigroup of B_2 and $B_2 \in \mathscr{L}\mathsf{Sl}$.

4.3. Good factorizations. Let q be an edge of $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$. Suppose $q_1, \ldots, q_n$ are consecutive edges of $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$ such that $q = q_1 \cdots q_n$. Let G be a subgraph of $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$. If the set $\left\{\prod_{i=k}^{l} q_i \mid 1 \leq k \leq l \leq n\right\}$ of factors of q is contained in E_G then we say that $q_1 \cdots q_n$ is a *good factorization of* q *in* G. Note that $q \in G$ if q has a good factorization in G.

Lemma 4.8. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{A} \,ⓜ\, \mathsf{V}$. *Let* $u, v, w, t \in \overline{\Omega}_A\mathsf{V}$ *be such that* $uv = wt$. *Then there is* $z \in (\overline{\Omega}_A\mathsf{V})^1$ *for which at least one of the following situations occurs:* $u = wz$ *and* $zv = t$, *or* $uz = w$ *and* $v = zt$.

Proof: Let $(u_n)_n$ and $(v_n)_n$ be sequences of elements of A^+ converging to u and v, respectively. The sequence $(u_nv_n)_n$ converges to wt. Then, by Lemmas 2.3 and 2.6, there is a subsequence $(u_{n_k}v_{n_k})_k$ and sequences $(w_n)_n$ and $(t_n)_n$ of elements of A^+ such that $u_{n_k}v_{n_k} = w_kt_k$, $\lim w_k = w$ and $\lim t_k = t$. It is clear that for every k there is $z_k \in A^*$ such that one of the following situations holds: $u_{n_k} = w_kz_k$ and $z_kv_{n_k} = t_k$, or $u_{n_k}z_k = w_k$ and $v_{n_k} = z_kt_k$. Therefore at least one of the sets

$$P = \{k : u_{n_k} = w_kz_k \text{ and } z_kv_{n_k} = t_k\}, \quad Q = \{k : u_{n_k}z_k = w_k \text{ and } v_{n_k} = z_kt_k\},$$

is infinite. Suppose P is infinite. Let z be a limit point of the subsequence $(z_k)_{k\in P}$. Then $u = wz$ and $zv = t$. Similarly, if Q is infinite then $uz = w$ and $v = zt$ for some $z \in (\overline{\Omega}_A\mathsf{V})^1$. ■

Theorem 4.9. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{A} ⓜ \mathsf{V}$. *Let* $q \in \varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$. *Suppose* $\hat{\mu}(q) = u_1 \cdots u_n$, *where* $u_i \in \overline{\Omega}_A\mathsf{V}$. *For an ordinal* β, *let* G *be one of the graphs* $\lceil\Sigma(\mathcal{X})\rceil_\beta$ *or* $\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle$. *If* $q \in G$ *then there is a good factorization* $q = q_1 \cdots q_n$ *in* G *such that* $\hat{\mu}(q_i) = u_i$, *for every* $i \in \{1, \ldots, n\}$.

Proof: Consider the following propositions:

$P(G, q, n)$: *"Suppose* $\hat{\mu}(q) = u_1 \cdots u_n$, *where* $u_i \in \overline{\Omega}_A\mathsf{V}$. *Then there is a good factorization* $q = q_1 \cdots q_n$ *in* G *such that* $\hat{\mu}(q_i) = u_i$, *for every* $i \in \{1, \ldots, n\}$."
$R(\beta)$: $\forall q \in \lceil\Sigma(\mathcal{X})\rceil_\beta, \forall n,\ P(\lceil\Sigma(\mathcal{X})\rceil_\beta, q, n)$.
$S(\beta)]$ $\forall q \in \langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle, \forall n,\ P(\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle, q, n)$.

We want to prove $R(\beta)\wedge S(\beta)$ for every ordinal β. We shall do it by transfinite induction on β. The case $\beta = 0$ is trivial, and the limit case of the inductive step offers no difficulties.

Let us see the successor case. Take an ordinal β such that $R(\beta) \wedge S(\beta)$ is true. Let $q \in \lceil\Sigma(\mathcal{X})\rceil_{\beta^+}$ and let $\hat{\mu}(q) = u_1 \cdots u_n$, where $u_i \in \overline{\Omega}_A\mathsf{V}$. Then there is a sequence $(q_k)_k$ of elements of $\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle$ converging to q. By Lemma 2.6, there is a subsequence $(q_{k_l})_l$ and sequences $(u_{i,l})_l$ of elements of $\overline{\Omega}_A\mathsf{V}$ converging to u_i such that $\hat{\mu}(q_{k_l}) = u_{1,l}u_{2,l} \cdots u_{n-1,l}u_{n,l}$. Since $S(\beta)$ is true, there is a good factorization $q = q_{1,l} \cdots q_{n,l}$ in $\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle$ such that $\hat{\mu}(q_{i,l}) = u_{i,l}$, for every $i \in \{1, \ldots, n\}$. Since $\overline{\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle}$ is compact, the sequence $(q_{1,k}, \ldots, q_{n,k})_k$ has some subsequence converging to a n-tuple $(q_1, \ldots, q_n)$ of consecutive edges of $\overline{\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle}$. Clearly $q_1 \cdots q_n$ is a good factorization of q in $\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}$ and $\hat{\mu}(q_i) = u_i$ for every $i \in \{1, \ldots, \}$. Hence $R(\beta^+)$ is true.

Let $q \in \langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle$. There are consecutive edges $q_1, \ldots, q_l$ of $\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}$ such that $q = q_1 \cdots q_l$. Let $\lambda(q)$ be the least possible value for l. Next we prove $P(\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle, q, n)$ by transfinite induction on $\lambda(q) + n$. If $\lambda(q) = 1$ then $q \in \lceil\Sigma(\mathcal{X})\rceil_{\beta^+}$, hence $P(\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle, q, n)$ is true for every n, because $R(\beta^+)$ is true. On the other hand, $P(\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle, q, 1)$ is obviously true, for every q. Therefore $P(\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle, q, n)$ is true when $\min\{\lambda(q), n\} = 1$. For a positive integer k, suppose $P(\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle, q, n)$ is true when $\lambda(q) + n < k$. Let q and n be such that $\lambda(q) + n = k$ and $\min\{\lambda(q), n\} > 1$. Suppose $\hat{\mu}(q) = u_1 \cdots u_n$, where $u_i \in \overline{\Omega}_A\mathsf{V}$. Let $q_1, \ldots, q_{\lambda(q)}$ be consecutive edges of $\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}$ such that $q = q_1 \cdots q_{\lambda(q)}$. Consider the edge $q' = q_1 \cdots q_{\lambda(q)-1}$. Since $\hat{\mu}(q')\hat{\mu}(q_{\lambda(q)}) = (u_1 \cdots u_{n-1})u_n$, by Lemma 4.8 there is $z \in (\overline{\Omega}_A\mathsf{V})^1$ for which at least one of the following conditions holds:

(1) $\hat{\mu}(q') = u_1 \cdots u_{n-1}z$ and $z\hat{\mu}(q_{\lambda(q)}) = u_n$,
(2) $\hat{\mu}(q')z = u_1 \cdots u_{n-1}$ and $\hat{\mu}(q_{\lambda(q)}) = zu_n$.

Suppose the first condition holds. Since $\lambda(q') + n < \lambda(q) + n$, by the induction hypothesis q' has a good factorization $s_1 \cdots s_{n-1}t$ in $\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle$ such that $\hat{\mu}(s_i) = u_i$ (for $i \in \{1, \ldots, n-1\}$) and $\hat{\mu}(t) = z$ (if $z = 1$ then consider t as an empty path). Let $s_n = tq_{\lambda(q)}$. Then $s_1 \cdots s_{n-1}s_n$ is a good factorization of $q'q_{\lambda(q)} = q$ in $\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle$. Since $\hat{\mu}(s_i) = u_i$ for every $i \in \{1, \ldots, n\}$, this proves $P(\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle, q, n)$.

Suppose the second condition holds. Since $R(\beta^+)$ is true, there are edges $r, t \in \lceil\Sigma(\mathcal{X})\rceil_{\beta^+}$ such that $q_{\lambda(q)} = rt$, $\hat{\mu}(r) = z$ and $\hat{\mu}(t) = u_n$. We have $\lambda(q'r) \leq \lambda(q') + 1 \leq \lambda(q)$, thus $\lambda(q'r) + (n-1) < \lambda(q) + n$. Since $\hat{\mu}(q'r) = u_1 \cdots u_{n-1}$, by inductive hypothesis $q'r$ has a good factorization $s_1 \cdots s_{n-1}$ in $\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle$ such that $\hat{\mu}(s_i) = u_i$, for every $i \in \{1, \ldots, n-1\}$. Hence $s_1 \cdots s_{n-1}t$ is a good factorization of q in $\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle$ whose i-th factor has label u_i. Hence $P(\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle, q, n)$ holds, concluding the inductive step on $\lambda(q) + n$. Therefore $S(\beta^+)$ is true.

Recapitulating, we proved that $R(\beta^+) \wedge S(\beta^+)$ is true, concluding the proof verification of the successor case of the inductive step on β. ∎

Corollary 4.10. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{A} \textcircled{m} \mathsf{V} = \mathsf{V} * \mathsf{D}$. *For an ordinal* β, *let* G *be one of the graphs* $\lceil\Sigma(\mathcal{X})\rceil_\beta$ *or* $\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle$. *Let* $p, q, r \in \varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$ *be such that* $p = qr$. *If* $p \in G$ *then* $q, r \in G$.

Proof: If $p \in G$ then there is a good factorization $p = q'r'$ in G such that $\hat{\mu}(q) = \hat{\mu}(q')$ and $\hat{\mu}(r) = \hat{\mu}(r')$. By Lemma 4.3, q and q' are co-terminal, and r and r' are also co-terminal. Hence $q = q'$ and $r = r'$, since $\hat{\mu}$ is faithful by Corollary 4.7. ∎

There are several examples of pseudovarieties of semigroups that are solutions of the chain of equalities $\mathsf{V} = \mathsf{A} ⓜ \mathsf{V} = \mathsf{V} * \mathsf{D}$ on the variable V. If H is a pseudovariety of groups then the pseudovariety $\overline{\mathsf{H}}$ of semigroups whose subgroups lie in H is a solution; in fact, not only $\overline{\mathsf{H}} = \overline{\mathsf{H}} * \mathsf{D}$ but also $\overline{\mathsf{H}} = \mathscr{L}\overline{\mathsf{H}}$. The *complexity pseudovarieties* C_n, recursively defined by $\mathsf{C}_0 = \mathsf{A}$ and $\mathsf{C}_n = \mathsf{A}*\mathsf{G}*\mathsf{C}_{n-1}$ if $n \geq 1$ are also solutions; in contrast with the previous example, one has $\mathsf{C}_n \neq \mathscr{L}\mathsf{C}_n$ if $n > 0$ (see [35] for a recent account on the complexity pseudovarieties).

A subshift $\mathcal{X}$ is *irreducible* if for every $u, v \in L(\mathcal{X})$ there is a word w such that $uwv \in L(\mathcal{X})$.

Corollary 4.11. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{A} ⓜ \mathsf{V}$. *If* $\mathcal{X}$ *is irreducible then* $\overline{\Sigma(\mathcal{X})^+} \setminus \Sigma(\mathcal{X})^+$ *is a strongly connected graph.*

Proof: Let x and y be arbitrary elements of $\mathcal{X}$. Since $\mathcal{X}$ is irreducible, for each $n \geq 1$ there is $z_n \in A^+$ such that the word $w_n = x_{[-n,n]} z_n y_{[-n,n]}$ belongs to $L(\mathcal{X})$. Let w be an accumulation point of $(w_n)_n$. Then $w = u_1 u_2 u_3$ for some accumulations points of the sequences $(x_{[-n,-1]})_n$, $(x_{[0,n]} z_n y_{[-n,-1]})_n$ and $(y_{[0,n]})_n$, respectively. Since $w \in \overline{L(\mathcal{X})} \setminus A^+$, there is $q \in \overline{\Sigma(\mathcal{X})^+} \setminus \Sigma(\mathcal{X})^+$ such that $\hat{\mu}(q) = w$, by Proposition 4.5. Then by Theorem 4.9 there is a good factorization $q = q_1 q_2 q_3$ in $\overline{\Sigma(\mathcal{X})^+}$ such that $\hat{\mu}(q_i) = u_i$, for all $i \in \{1, 2, 3\}$. By Lemma 4.3, we have $\alpha(q_2) = \overleftarrow{u_1}.\overrightarrow{u_2} = x$. Similarly, $\omega(q_2) = y$. Since $\hat{\mu}(q_2) \notin A^+$, q_2 is an edge of $\overline{\Sigma(\mathcal{X})^+} \setminus \Sigma(\mathcal{X})^+$ from x to y. ∎

The converse of Corollary 4.11 is false. For an example see the subshift of Proposition 3.2 and the corresponding proof.

5. The ordinal $\mathfrak{o}(\Sigma(\mathcal{X}))$

Let Γ be a nonempty subgraph of a compact semigroupoid. By Lemmas 3.3 and 3.4 the set of those ordinals β such that $|\beta| \leq |\lceil\Gamma\rceil|$ and $\lceil\Gamma\rceil_\beta = \lceil\Gamma\rceil$ is nonempty. Its infimum is denoted by $\mathfrak{o}(\Gamma)$.

Since $\Sigma(\mathcal{X})$ is a conjugacy invariant, the ordinal $\mathfrak{o}(\Sigma(\mathcal{X}))$ is also a conjugacy invariant. According to Proposition 4.1, if $\mathcal{X}$ is a finite type subshift then $\mathfrak{o}(\Sigma(\mathcal{X})) = 1$. In Proposition 3.2, we saw a sofic subshift $\mathcal{Z}$ such that

$\mathfrak{o}(\Sigma(\mathcal{X})) > 1$. We proceed to try to determine $\mathfrak{o}(\Sigma(\mathcal{X}))$ for some cases, or at least to find lower and upper bounds for $\mathfrak{o}(\Sigma(\mathcal{X}))$.

5.1. The ordinal $\mathfrak{o}(\Sigma(\mathcal{X}))$ can be very large. We first need some lemmas on word combinatorics.

Lemma 5.1. *Let $u, v, z \in A^+$ be such that $z^2u = vz^2$ and $|u| < |z|$. If the length of z is a prime number then $z \in a^+$ for some $a \in A$.*

Proof: Since $z^2u = vz^2$, there is $v' \in A^*$ such that $zu = v'z$. Since $|v'| = |u|$ and $|u| < |z|$, the prefix of z with length $|u|$ is v'. Since $z^2u = vz^2$, it is also true that the prefix of z with length $|u|$ is v. Therefore $v' = v$ and $vz^2 = z^2u = zvz$. Hence $vz = zv$, which by [26, Corollary 5.3] implies there is $w \in A^+$ and $k, l > 0$ such that $z = w^k$ and $v = w^l$. Since $|z| = k|w|$ and $|z|$ is prime, we have $k = 1$ or $|w| = 1$. If $k = 1$ then $z = w$ and $|v| = l|w| \geq |z|$, a contradiction. Hence $w \in A$. ∎

Lemma 5.2. *Let z be a word of A^+ whose length is a prime number, and suppose that z is not a power of an element of A. Let $k \geq 4$ and $u, v \in A^+$. If u and v are respectively a suffix and a prefix of some elements of Az^k then $uv \notin Az^k$.*

Proof: Suppose the lemma is false. That means that there are $a, b, c \in A$ such that u is a suffix of az^k, v is a prefix of bz^k, and $uv = cz^k$. Since $v \neq 1$, there are $i \geq 0$ and a strict prefix v' of z such that $v = bz^iv'$; and there are $j \geq 1$ and a strict suffix u' of z such that $u = u'z^j$. Hence

$$cz^k = u'z^jbz^iv'.$$

If $u' = 1$ then z is a prefix of cz, thus z is a power of the letter c, which is impossible. Hence $u' \neq 1$. We have $k|z| = (i + j)|z| + |u'| + |v'|$, thus $|u'| + |v'|$ is a multiple of $|z|$. Since $0 < |u'| + |v'| < |z| + |z| = |2z|$, we have $|u'| + |v'| = |z|$. Therefore $i + j = k - 1$. If $i \geq 2$ then z^2v' is a suffix of z^k, which is impossible by Lemma 5.1. Therefore $j \geq 2$, since $k \geq 4$. Since $u' \neq 1$, there is $u'' \in A^*$ such that $u' = cu''$. Then $z^k = u''z^jbz^iv'$, and $u''z^2$ is a prefix of z^k. Hence $u'' = 1$ by Lemma 5.1. Therefore $z^{k-j} = bz^iv'$. If $i \neq 0$ then $bz \in zA$, thus z is a power of b, which can not happen. Hence $i = 0$, $j = k - 1$ and $bv' = z$. But v' is a prefix of z, thus $bv' \in v'A$. This implies $v' \in b^+$, and therefore $z \in b^+$, which is impossible. ∎

It follows from Lemma 5.2 that the set Az^k in its statement is a *circular code* [11].

Given $v \in A^*$, denote by ψ_v the following mapping from $A^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$:

$$\dots x_{-2}x_{-1}.x_0x_1x_2x_3\dots \mapsto \dots vx_{-2}vx_{-1}v.x_0vx_1vx_2vx_3v\dots.$$

Note that ψ_1 is the identity on $A^{\mathbb{Z}}$. Observe also that $\psi_v \circ \sigma = \sigma^{|v|+1} \circ \psi_v$. It is easy to prove that $\mathcal{X}_v = \bigcup_{x\in\mathcal{X}} \mathcal{O}(\psi_v(x))$ is the least subshift of $A^{\mathbb{Z}}$ containing $\psi_v(\mathcal{X})$.

Lemma 5.3. *Let z be a word of A^+ whose length is a prime number, and suppose z is not a power of a letter. Let $k \geq 4$. If $x, y \in \mathcal{X}$ and $n \in \mathbb{Z}$ are such that $\psi_{z^k}(y) = \sigma^n(\psi_{z^k}(x))$ then n is a multiple of $k|z|+1$.*

Proof: There are $q, r \in \mathbb{Z}$ such that $n = q(k|z|+1)+r$ and $0 \leq r < k|z|+1$. Note that

$$\psi_{z^k}(y) = \sigma^n \circ \psi_{z^k}(x) = \sigma^r \circ \sigma^{q(k|z|+1)} \circ \psi_{z^k}(x) = \sigma^r \circ \psi_{z^k} \circ \sigma^q(x).$$

If $y = (a_i)_{i\in\mathbb{Z}}$ and $\sigma^q(x) = (b_i)_{i\in\mathbb{Z}}$ then

$$\psi_{z^k}((a_i)_{i\in\mathbb{Z}}) = \dots a_{-3}z^ka_{-2}z^ka_{-1}z^k.a_0z^ka_1z^ka_2z^ka_3z^k\dots =$$
$$\sigma^r \circ \psi_{z^k}((b_i)_{i\in\mathbb{Z}}) = \dots b_{-3}z^kb_{-2}z^kb_{-1}z^ku.vb_1z^kb_2z^kb_3z^k\dots.$$

where u, v are elements of A^+ such that $b_0z^k = uv$ and $|u| = r$. Since u is a suffix of $a_{-1}z^k$ and v is a prefix of a_0z^k, from Lemma 5.2 we deduce that $r = 0$. ■

Lemma 5.4. *Let z be a word of A^+ whose length is a prime number, and suppose z is not a power of a letter. Let $k \geq 4$. Let $x \in \mathcal{X}$. If $(y^{(n)})_n$ is a sequence of elements of $\mathcal{X}_{z^k}$ converging to $\psi_{z^k}(x)$ then there is a sequence $(x^{(m)})_m$ of elements of $\mathcal{X}$ converging to x and a subsequence $(y^{(n_m)})_m$ such that $y^{(n_m)} = \psi_{z^k}(x^{(m)})$, for any m.*

Proof: Since $y^{(n)} \in \mathcal{X}_{z^k}$, there are $x^{(n)} \in \mathcal{X}$ and an integer r_n such that $y^{(n)} = \sigma^{r_n}\psi_{z^k}(x^{(n)})$ and $0 \leq r_n < k|z|+1$. The sequence $(x^{(n)})_n$ has some subsequence $(x^{(n_i)})_i$ converging to an element x' of $\mathcal{X}$. Since $(r_{n_i})_i$ is a bounded sequence, it has some subsequence $(r_{n_{i_j}})_j$ with constant value C. Then

$$\sigma^C\psi_{z^k}(x') = \lim_{j\to+\infty} \sigma^C\psi_{z^k}(x^{(n_{i_j})}) = \lim_{j\to+\infty} y^{(n_{i_j})} = \psi_{z^k}(x).$$

Hence $C = 0$, by Lemma 5.3. Since ψ_{z^k} is injective, we deduce that $x' = x$. Therefore $(x^{(n_{i_j})})_j$ converges to x and $\psi_{z^k}(x^{(n_{i_j})}) = y^{(n_{i_j})}$ for all j. ■

Let $v \in A^+$ and $x \in \mathcal{X}$. According to Lemma 4.3, there is a unique path of $\Sigma(\mathcal{X}_v)^+$ with length $|v|+1$ from $\psi_v(x)$ to $\sigma^{|v|+1}(\psi_v(x)) = \psi_v(\sigma(x))$. Denote it by $\big(\psi_v(x), \psi_v(\sigma(x))\big)$. Clearly, the mapping

$$\begin{array}{rrcl} \Psi_v : & \Sigma(\mathcal{X}) & \to & \Sigma(\mathcal{X}_v)^+ \\ & x & \mapsto & \psi_v(x) \\ & (x, \sigma(x)) & \mapsto & \big(\psi_v(x), \psi_v(\sigma(x))\big), \qquad x \in \mathcal{X}, \end{array}$$

is a graph homomorphism. Let $\hat{\Psi}_v$ be the unique continuous semigroupoid homomorphism from $\widehat{\Sigma}(\mathcal{X})$ to $\widehat{\Sigma}(\mathcal{X}_v)$ extending Ψ_v.

Proposition 5.5. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{A} \circledm \mathsf{V}$. *Let z be a word of A^+ whose length is a prime number, and suppose that z is not a power of an element of A. Let $k \geq 4$. For every ordinal β we have*

$$\hat{\Psi}_{z^k}(E_{\lceil \Sigma(\mathcal{X}) \rceil_\beta}(x,y)) = E_{\lceil \Sigma(\mathcal{X}_{z^k}) \rceil_\beta}(\psi_{z^k}(x), \psi_{z^k}(y)),$$
$$\hat{\Psi}_{z^k}(E_{\langle \lceil \Sigma(\mathcal{X}) \rceil_\beta \rangle}(x,y)) = E_{\langle \lceil \Sigma(\mathcal{X}_{z^k}) \rceil_\beta \rangle}(\psi_{z^k}(x), \psi_{z^k}(y)),$$

for all $x, y \in \mathcal{X}$.

Proof: For every ordinal β and for every word v, by Lemma 3.20 we know that $\hat{\Psi}_v(\lceil \Sigma(\mathcal{X}) \rceil_\beta) \subseteq \lceil \Psi_v(\Sigma(\mathcal{X})) \rceil_\beta$ and $\hat{\Psi}_v(\langle \lceil \Sigma(\mathcal{X}) \rceil_\beta \rangle) \subseteq \langle \lceil \Psi_v(\Sigma(\mathcal{X})) \rceil_\beta \rangle$. Hence it remains to prove the conjunction of the following properties:

$$P(\beta) : \forall x, y \in \mathcal{X},\ E_{\lceil \Sigma(\mathcal{X}_{z^k}) \rceil_\beta}(\psi_{z^k}(x), \psi_{z^k}(y)) \subseteq \hat{\Psi}_{z^k}(E_{\lceil \Sigma(\mathcal{X}) \rceil_\beta}(x,y)),$$
$$Q(\beta) : \forall x, y \in \mathcal{X},\ E_{\langle \lceil \Sigma(\mathcal{X}_{z^k}) \rceil_\beta \rangle}(\psi_{z^k}(x), \psi_{z^k}(y)) \subseteq \hat{\Psi}_{z^k}(E_{\langle \lceil \Sigma(\mathcal{X}) \rceil_\beta \rangle}(x,y)).$$

We shall prove $P(\beta) \wedge Q(\beta)$ by transfinite induction on β.

By Lemma 5.3, we have $\psi_{z^k}(y) \neq \sigma(\psi_{z^k}(x))$, thus $E_{\Sigma(\mathcal{X})}(\psi_{z^k}(x), \psi_{z^k}(y)) = \emptyset$, which proves $P(0)$. Suppose $s \in E_{\Sigma(\mathcal{X}_{z^k})^+}(\psi_{z^k}(x), \psi_{z^k}(y))$. Then $\psi_{z^k}(y) = \sigma^{|s|}(\psi_{z^k}(x))$. By Lemma 5.3, there is a positive integer n such that $|s| = n(k|z|+1)$. Then $\psi_{z^k}(y) = \psi_{z^k}(\sigma^n(x))$. Since ψ_{z^k} is injective, it follows that $y = \sigma^n(x)$. Hence $E_{\Sigma(\mathcal{X})^+}(x,y)$ has an element s' with length n. The length of $\hat{\Psi}_{z^k}(s')$ is equal to $|s'|(k|z|+1)$, by the definition of Ψ_{z^k}. Hence s and $\hat{\Psi}_{z^k}(s')$ are elements of $E_{\Sigma(\mathcal{X}_{z^k})^+}(\psi_{z^k}(x), \psi_{z^k}(y))$, with the same length, thus $s = \hat{\Psi}_{z^k}(s')$ (cf. Lemma 4.3). This proves $P(0) \wedge Q(0)$.

Suppose $P(\beta) \wedge Q(\beta)$ is true. Let s be an element of $E_{\lceil \Sigma(\mathcal{X}_{z^k}) \rceil_{\beta+}}(\psi_{z^k}(x), \psi_{z^k}(y))$. Then there is a sequence $(s_n)_n$ of elements of $\langle \lceil \Sigma(\mathcal{X}) \rceil_\beta \rangle$ converging to s.

The sequences $(\alpha(s_n))_n$ and $(\omega(s_n))_n$ converge respectively to $\psi_{z^k}(x)$ and $\psi_{z^k}(y)$. By Lemma 5.4, taking subsequences if necessary, we may assume that $\alpha(s_n) = \psi_{z^k}(x^{(n)})$ and $\omega(s_n) = \psi_{z^k}(y^{(n)})$ for every n, for some sequences $(x^{(n)})_n$ and $(y^{(n)})_n$ of elements of $\mathcal{X}$ converging to x and y, respectively. Since $Q(\beta)$ is true, for each n there is $s'_n \in E_{\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle}(x^{(n)}, y^{(n)})$ such that $s_n = \hat{\Psi}_{z^k}(s'_n)$. If s' is a limit point of $(s'_n)_n$ then $s' \in E_{\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}}(x, y)$ and $\hat{\Psi}_{z^k}(s') = \lim s_n = s$, which proves $P(\beta^+)$.

For each positive integer l let $\langle\lceil\Sigma(\mathcal{X}_{z^k})\rceil_{\beta^+}\rangle_l$ be the set of all edges of $\widehat{\Sigma}(\mathcal{X})$ of the form $q_1 \cdots q_l$, where $q_1, \ldots, q_l$ are consecutive edges of $\lceil\Sigma(\mathcal{X}_{z^k})\rceil_{\beta^+}$. Note that

$$\langle\lceil\Sigma(\mathcal{X}_{z^k})\rceil_{\beta^+}\rangle = \bigcup_{l\geq 1}\langle\lceil\Sigma(\mathcal{X}_{z^k})\rceil_{\beta^+}\rangle_l.$$

Hence $Q(\beta)$ shall be proved once we prove by induction on l the following sentence:

$$Q(\beta, l):\ \forall x, y \in \mathcal{X},\ E_{\langle\lceil\Sigma(\mathcal{X}_{z^k})\rceil_{\beta^+}\rangle_l}(\psi_{z^k}(x), \psi_{z^k}(y)) \subseteq \hat{\Psi}_{z^k}(E_{\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle_l}(x, y)).$$

The initial step $l = 1$ corresponds to proposition $P(\beta^+)$, which we know is true. Suppose $l > 1$ and that $Q(\beta, l')$ is true when $l' < l$. Let r be an element of $E_{\langle\lceil\Sigma(\mathcal{X}_{z^k})\rceil_{\beta^+}\rangle_l}(\psi_{z^k}(x), \psi_{z^k}(y))$. Then there are consecutive edges $r_1, \ldots, r_l$ of $\lceil\Sigma(\mathcal{X}_{z^k})\rceil_{\beta^+}$ such that $r = r_1 \cdots r_l$. Since $Q(0)$ is true, we may assume that $r \notin \Sigma(\mathcal{X})^+$. Then there is $i \in \{1, \ldots, l\}$ such that $r_i \notin \Sigma(\mathcal{X})^+$. Since $l > 1$, we have $i < l$ or $i > 1$. Let us suppose that $i < l$ (the case $i > 1$ is similar). There is a positive integer m such that $\omega(r_i) = \sigma^m(\psi_{z^k}(x'))$ for some $x' \in \mathcal{X}$. Let $u = \mathrm{t}_m(\hat{\mu}(r_i))$. Since $r_i \notin \Sigma(\mathcal{X})^+$, the word u has length m. Let $(p_n)_n$ and $(q_n)_n$ be sequences of elements of $\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle$ converging to r_i and r_{i+1}, respectively. Since $(\overline{\Omega}_A\mathsf{V})u$ is open, we may assume that for every n there is $w_n \in \overline{\Omega}_A\mathsf{V}$ such that $\hat{\mu}(p_n) = w_n u$. By Theorem 4.9, there are edges p'_n e p''_n belonging to $\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle$ such that $p_n = p'_n p''_n$, $\hat{\mu}(p'_n) = w_n$ and $\hat{\mu}(p''_n) = u$. For each n, let q'_n be the unique edge of $\Sigma(\mathcal{X})^+$ from $\sigma^{-m}(\alpha(q_n))$ to $\alpha(q_n)$. Let (p', p'', q') be a limit point of the sequence $(p'_n, p''_n, q'_n)_n$. Since $(|q'_n|)_n$ is the sequence with constant value m, and since there is a only a finite number of paths on $\Sigma(\mathcal{X})$ with length m, we deduce that q' is a path of $\Sigma(\mathcal{X})$ from $\sigma^{-m}(\omega(q'))$ to $\omega(q')$. On the other hand, since $\hat{\mu}(p'') = u \in A^+$, by Lemma 4.3 we know that p'' is the unique path of $\Sigma(\mathcal{X})$ from $\sigma^{-m}(\omega(p''))$ to $\omega(p'')$. Since

$$\omega(p'') = \omega(r_i) = \alpha(r_{i+1}) = \lim_{n\to\infty} \omega(q'_n) = \omega(q'),$$

one concludes that $p'' = q'$. Therefore

$$r = (r_1 \cdots r_{i-1}p')((q'r_{i+1})r_{i+2} \cdots r_l).$$

Note that $p' \in \lceil \Sigma(\mathcal{X}) \rceil_{\beta^+}$ and that

$$\omega(p') = \alpha(p'') = \sigma^{-m}(\omega(p'')) = \sigma^{-m}(\omega(r_i)) = \psi_{z^k}(x'),$$

whence

$$r_1 \cdots r_{i-1}p' \in E_{\langle \lceil \Sigma(\mathcal{X}_{z^k}) \rceil_{\beta^+} \rangle_i}(\psi_{z^k}(x), \psi_{z^k}(x')).$$

On the other hand, since $q'_n q''_n \in \langle \lceil \Sigma(\mathcal{X}) \rceil_\beta \rangle$ and $q'r_{i+1}$ is a limit point of the sequence $(q'_n q''_n)_n$, we have $q'r_{i+1} \in \lceil \Sigma(\mathcal{X}) \rceil_{\beta^+}$. Therefore

$$(q'r_{i+1})r_{i+2} \cdots r_l \in E_{\langle \lceil \Sigma(\mathcal{X}_{z^k}) \rceil_{\beta^+} \rangle_{l-i}}(\psi_{z^k}(x'), \psi_{z^k}(y)).$$

Since properties $Q(\beta, i)$ and $Q(\beta, l-i)$ hold by the induction hypothesis, we conclude that

$$r_1 \cdots r_{i-1}p' \in \hat{\Psi}_{z^k}(E_{\langle \lceil \Sigma(\mathcal{X}) \rceil_{\beta^+} \rangle_i}(x, x')), \quad (q'r_{i+1})r_{i+2} \cdots r_l \in \hat{\Psi}_{z^k}(E_{\langle \lceil \Sigma(\mathcal{X}) \rceil_{\beta^+} \rangle_{l-i}}(x', y)),$$

thus $r_1 \cdots r_l \in \hat{\Psi}_{z^k}(E_{\langle \lceil \Sigma(\mathcal{X}) \rceil_{\beta^+} \rangle_l}(x, y))$, proving $Q(\beta, l)$. Hence $Q(\beta^+)$ is true.

The limit case of the inductive step of the proof of $P(\beta) \wedge Q(\beta)$ is trivial. ■

Lemma 5.6. *Let z be a word of A^+ which is not the power of a letter. Let k and l be integers such that $0 < k < l$, and $k|z| + 1$ and $l|z| + 1$ are coprime. Then there is $n_0 > 0$ such that if $n > n_0$ then $L_n((A^{\mathbb{Z}})_{z^k}) \cap L_n((A^{\mathbb{Z}})_{z^l}) = \emptyset$.*

Proof: What we want to prove can be reformulated as $(A^{\mathbb{Z}})_{z^k} \cap (A^{\mathbb{Z}})_{z^l} = \emptyset$ (the statement's formulation will be convenient later). Suppose $(A^{\mathbb{Z}})_{z^k} \cap (A^{\mathbb{Z}})_{z^l} \neq \emptyset$. Then there are sequences $(a_i)_{i \geq 1}$ and $(b_i)_{i \geq 1}$ of elements of A such that $z^k a_1 z^k a_2 z^k a_3 \ldots = v z^l b_1 z^l b_2 z^l b_3 \ldots$. for some $v \in A^+$. Since $k|z| + 1$ and $l|z| + 1$ are coprime, there are integers $r, s > 1$ such that $r(k|z| + 1) - s(l|z| + 1) = |v|$. Hence

$$|z^k a_1 z^k a_2 z^k \cdots a_{r-1} z^k| = r(k|z|+1) - 1 = |v| + s(l|z|+1) - 1 = |v z^l b_1 z^l b_2 z^l \cdots b_{s-1} z^l|,$$

thus $z^k a_1 z^k a_2 z^k \cdots a_{r-1} z^k = v z^l b_1 z^l b_2 z^l \cdots b_{s-1} z^l$. Since $0 < k < l$, there is $c \in A$ such that $z a_{r-1} = cz$, thus $z = c^{|z|}$, contradicting the hypothesis. ■

The following lemma can be proved quite similarly.

Lemma 5.7. *Let z be a word of A^+ which is not the power of a letter. For every $k > 0$, there is $n_0 > 0$ such that if $n > n_0$ then $L_n((A^{\mathbb{Z}})_{z^k}) \cap L_n(z^\infty) = \emptyset$.*

Theorem 5.8. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{A} ⓜ \mathsf{V}$. *Let* A *be a two-letter alphabet. If* β *is a countable ordinal then there is a countable subshift* $\mathcal{X}$ *of* $A^{\mathbb{Z}}$ *such that* $\mathfrak{o}(\Sigma(\mathcal{X})) > \beta$.

Proof: Take $A = \{a, b\}$. Let $\mathcal{Y}$ be the subshift $\{a^\infty\}$. Consider the following property:

$Q(\beta, \mathcal{X}, \mathcal{Z}, c)$: *$\beta$ is a countable ordinal, $\mathcal{X}$ and $\mathcal{Z}$ are subshifts of $A^{\mathbb{Z}}$, and $c \in A^+$, such that*

(1) $\mathcal{Y} \cup \mathcal{Z} \subseteq \mathcal{X}$, $\mathcal{Y} \cap \mathcal{Z} = \emptyset$ *and* $\mathcal{X}$ *is countable;*
(2) $b^\infty \in \mathcal{X}$ *and* $c^\infty \in \mathcal{Z}$;
(3) the graphs $\lceil \Sigma(\mathcal{Y}) \rceil_1$ *and* $\lceil \Sigma(\mathcal{Z}) \rceil_1$ *are strongly connected;*
(4) $\{s \in E_{\langle \lceil \Sigma(\mathcal{X}) \rceil_{\beta+} \rangle} : \alpha(s) \in \mathcal{Y}$ *and* $\omega(s) \in \mathcal{Z}\} \neq \emptyset$;
(5) $\{s \in E_{\langle \lceil \Sigma(\mathcal{X}) \rceil_{\beta+} \rangle} : \alpha(s) \in \mathcal{Y}$ *and* $\omega(s) \in \mathcal{Z}\} \cap \langle \lceil \Sigma(\mathcal{X}) \rceil_\beta \rangle = \emptyset$.

We denote the set $\{s \in E_{\langle \lceil \Sigma(\mathcal{X}) \rceil_{\beta+} \rangle} : \alpha(s) \in \mathcal{Y}$ and $\omega(s) \in \mathcal{Z}\}$ by $E_\beta(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.

Let $P(\beta)$ be the proposition "$\exists \mathcal{X} \exists \mathcal{Y} \exists \mathcal{Z} \exists c\, Q(\beta, \mathcal{X}, \mathcal{Z}, c)$". If $Q(\beta, \mathcal{X}, \mathcal{Z}, c)$ is true then $\mathcal{X}$ is a countable subshift of $A^{\mathbb{Z}}$ such that $\mathfrak{o}(\Sigma(\mathcal{X})) > \beta$. Therefore the theorem will be proved once we prove $P(\beta)$ by transfinite induction.

Let us verify the initial step $\beta = 0$. Consider the subshifts $\mathcal{Z} = \{b^\infty\}$ and $\mathcal{X} = \overline{\mathcal{O}(a^{-\infty}.b^{+\infty})}$. The set of edges of $\widehat{\Sigma}(\mathcal{X})$ from a^∞ to b^∞ does not contain any element of $\Sigma(\mathcal{X})^+ = \langle \lceil \Sigma(\mathcal{X}) \rceil_0 \rangle$, thus $E_0(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \cap \langle \lceil \Sigma(\mathcal{X}) \rceil_0 \rangle = \emptyset$. On the other hand, denoting by q_n the unique path of $\Sigma(\mathcal{X})^+$ from $\sigma^{-n}(a^{-\infty}.b^{+\infty})$ to $\sigma^n(a^{-\infty}.b^{+\infty})$, if q is an accumulation point of $(q_n)_n$ then q belongs to $E_0(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. Hence $P(0)$ is true.

Suppose $P(\beta)$ holds. Take subshifts $\mathcal{X}$ and $\mathcal{Z}$ of $A^{\mathbb{Z}}$ and a word c of A^+ such that $Q(\beta, \mathcal{X}, \mathcal{Z}, c)$ is true. Since $|\mathcal{X}| < |A^{\mathbb{Z}}|$, there is $z \in A^+ \setminus L(\mathcal{X})$. If necessary prolonging z, we can suppose $|z|$ is a prime number. By Dirichlet's Theorem [24, Section 16.1], the sequence $(n|z| + 1)_n$ has infinite prime numbers. For each positive integer k, let e_k be the k-th positive integer greater than 3 such that $e_k|z| + 1$ is prime. We let $e_0 = 0$.

Let $h > 0$ and $c_1, \ldots, c_h \in A$ be such that $c = c_1 \cdots c_h$. For each nonnegative integer k, take

$$\begin{aligned} t_k &= \psi_{z^{e_k}}(c)_{]-\infty,-1]}.\psi_{z^{e_{k+1}}}(a^\infty)_{[0,+\infty[} \\ &= \ldots c_1 z^{e_k} c_2 z^{e_k} \ldots c_{h-1} z^{e_k} c_h z^{e_k} c_1 z^{e_k} c_2 z^{e_k} \ldots c_{h-1} z^{e_k} c_h z^{e_k} . a z^{e_{k+1}} a z^{e_{k+1}} a z^{e_{k+1}} \ldots . \end{aligned}$$

Denote by $\mathcal{Z}'$ the subshift $\Big[\bigcup_{d\in A\,:\,d\text{ is a factor of }c}\mathcal{O}(z^{-\infty}.dz^{+\infty})\Big]\cup\mathcal{O}(z^{\infty})$. The least subshift $\mathcal{X}'$ containing $\bigcup_{k\geq 0}(\mathcal{X}_{z^{e_k}}\cup\{t_k\})$ is the set

$$\mathcal{X}'=\Big[\bigcup_{k\geq 0}(\mathcal{X}_{z^{e_k}}\cup\mathcal{O}(t_k))\Big]\cup\mathcal{Z}'.$$

Note that $\mathcal{Y}\cup\mathcal{Z}'\subseteq\mathcal{X}'$, $\mathcal{Y}\cap\mathcal{Z}'=\emptyset$ and that $\lceil\Sigma(\mathcal{Z}')\rceil_1$ is strongly connected. Moreover $\mathcal{X}'$ is countable. These observations are the first steps for proving $Q(\beta^+,\mathcal{X}',\mathcal{Z}',z)$.

For each $k\geq 0$ and $n>0$, let $q_{k,n}$ be the unique path on $\Sigma(\mathcal{X}')^+$ from $\sigma^{-n}(t_k)$ to $\sigma^n(t_n)$. Let q_k be an accumulation point of the sequence $(q_{k,n})_n$. Then the origin of q_k is an element of the orbit of $\psi_{z^{e_k}}(c^\infty)$, and its terminus is an element of the orbit of $\psi_{z^{e_{k+1}}}(a^\infty)$. Note that $q_k\in\lceil\Sigma(\mathcal{X}')\rceil_1$.

Acording to items (3) and (4) describing $Q(\beta,\mathcal{X},\mathcal{Z},c)$, there is an edge s_0 of $\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle$ from an element of $\mathcal{Y}$ to an element of $\alpha(q_0)$. By the same items, and by Proposition 5.5, for each $k\geq 1$ there is an edge s_k of $\langle\lceil\Sigma(\mathcal{X}_k)\rceil_{\beta^+}\rangle$ from $\omega(q_{k-1})$ to $\alpha(q_k)$ (see Figure 4). For each k, the sequence $s_0q_0s_1q_1s_2q_2\cdots s_kq_k$ is an element of $\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle$. Let q be a limit point of $(s_0q_0s_1q_1s_2q_2\cdots s_kq_k)_k$. Then $\omega(q)\in\mathcal{Z}'$ and $q\in\lceil\Sigma(\mathcal{X}')\rceil_{(\beta^+)^+}$, thus $E_{\beta^+}(\mathcal{X}',\mathcal{Y},\mathcal{Z}')$ is nonempty.

Suppose there is an element of $E_{\beta^+}(\mathcal{X}',\mathcal{Y},\mathcal{Z}')$ belonging to $\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}\rangle$. Such an element has some factor p belonging to $\lceil\Sigma(\mathcal{X})\rceil_{\beta^+}$ starting at some element of $\mathcal{X}'\setminus\mathcal{Z}'$ and ending at some element of $\mathcal{Z}'$. There is $k\geq 0$ such that $\alpha(p)\in U_k=\mathcal{O}(t_{k-1})\cup\mathcal{X}_{z^{e_k}}\cup\mathcal{O}(t_k)$, where $\mathcal{O}(t_{-1})$ designates the empty set. By Lemmas 5.6 and 5.7, if $k\neq l$ then $\mathcal{X}_{z^{e_k}}\cap\mathcal{X}_{z^{e_l}}=\emptyset$, and $\mathcal{X}_{z^{e_k}}\cap\mathcal{Z}'=\emptyset$, for all $k,l\geq 0$. Therefore, relatively to the topology of $\mathcal{X}'$, the sets U_k and

$$V_k=\Big[\bigcup_{r\geq k+4}(\mathcal{X}_{z^{e_r}}\cup\mathcal{O}(t_r))\Big]\cup\mathcal{Z}',$$

are open neighborhoods of $\alpha(p)$ and $\omega(p)$, respectively. Let $(p_n)_n$ be a sequence of edges of $\langle\lceil\Sigma(\mathcal{X})\rceil_{\beta}\rangle$ converging to p. Since α and ω are continuous maps, there is N such that if $n\geq N$ then $\alpha(p_N)\in U_k$ and $\omega(p_N)\in V_k$. If necessary changing the value of k by adding one, we can suppose that

$$\alpha(p_N)\in\mathcal{O}(t_{k-1})\cup\mathcal{X}_{z^{e_k}}\quad\text{and}\quad\omega(p_N)\in\mathcal{X}_{z^{e_r}}\cup\mathcal{O}(t_r)\cup\mathcal{Z}',$$

for some $r\geq k+3$.

Let us start by the case $k>0$. Let m be a positive integer. Since $\alpha(p_N)\in\mathcal{O}(t_{k-1})\cup\mathcal{X}_{z^{e_k}}$, every finite prefix of $\hat{\mu}(p_N)$ with sufficiently large length has

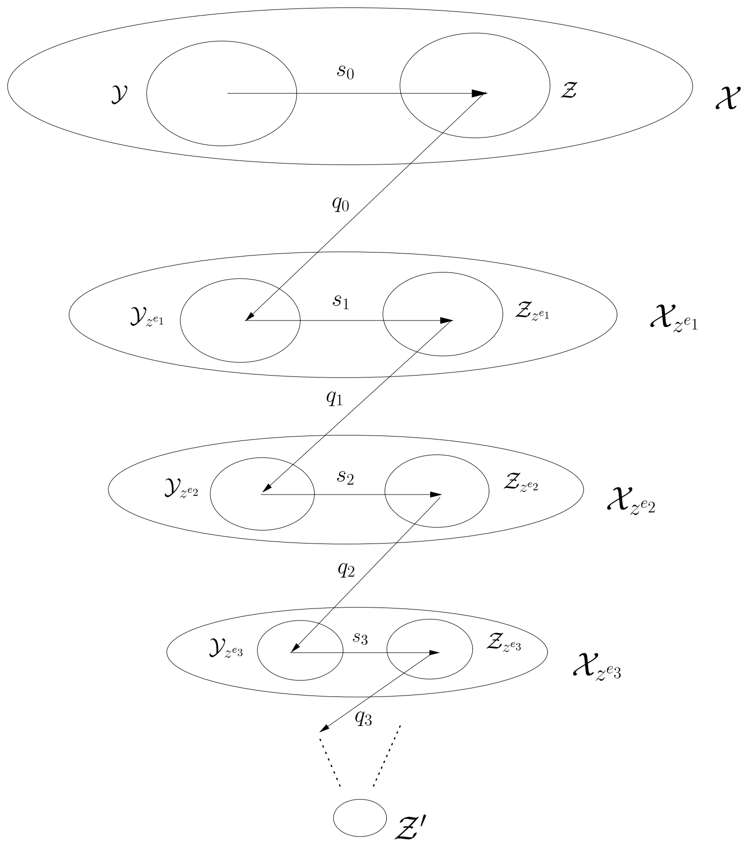

FIGURE 4. One step in the proof of Theorem 5.8.

some factor belonging to $(Az^{e_k})^m$ (cf. Lemma 4.3). And since

$$A^*(Az^{e_k})^m A^* = (A^*(Az^{e_k})^m)(A^* \setminus Az^{e_k}A^*),$$

there are $\rho_m \in (\overline{\Omega}_A\mathsf{S})(Az^{e_k})^m$ and $\nu_m \in (\overline{\Omega}_A\mathsf{S})^1 \setminus Az^{e_k}(\overline{\Omega}_A\mathsf{S})^1$ such that $\hat{\mu}(p_N) = \rho_m\nu_m$. Note that if $m \geq n$ then $\rho_m \in (\overline{\Omega}_A\mathsf{S})(Az^{e_k})^n$. Let ρ and ν be limit points of the sequences $(\rho_m)_m$ and $(\nu_m)_m$, respectively. Then

$$\rho \in \bigcap_{n\geq 1}(\overline{\Omega}_A\mathsf{S})(Az^{e_k})^n \quad \text{and} \quad \nu \in (\overline{\Omega}_A\mathsf{S})^1 \setminus Az^{e_k}(\overline{\Omega}_A\mathsf{S})^1.$$

The pseudoword has factors of length n for all $n \geq 1$, thus it is infinite. By Lemma 4.3 we have $\overleftarrow{\rho\nu} = \omega(p_N)_{]-\infty,-1]} \in \mathcal{X}_{z^{e_r}} \cup \mathcal{O}(t_r) \cup \mathcal{Z}'$. If ν is finite then $(Az^{e_k})^n \subseteq L((A^{\mathbb{Z}})_{z^{e_r}})$ for all $n \geq 1$, or $(Az^{e_k})^n \subseteq L(\mathcal{Z}')$ for all $n \geq 1$. But the first case contradicts Lemma 5.6, and the second contradicts Lemma 5.7. Hence ν is an infinite pseudoword.

Let $x = \overleftarrow{\rho}\,.\overrightarrow{\nu}$. Since $\hat{\mu}(p_N) \in \mathscr{M}(\mathcal{X}')$ by Proposition 4.5, we know that $x \in \mathcal{X}'$. We have

$$x_{]-\infty,-1]} = \dots a_{-3}z^{e_k}a_{-2}z^{e_k}a_{-1}z^{e_k}, \quad \text{for some } a_{-1}, a_{-2}, a_{-3}, \dots \in A, \tag{5.1}$$

and

$$x_{[0,e_k|z|]} \notin Az^{e_k}. \tag{5.2}$$

From (5.1) and Lemma 5.7 we deduce that $x \notin \mathcal{Z}'$.

Suppose there is $l \geq 0$ such that $x \in \mathcal{X}_{z^{e_l}}$. Then, by (5.1),

$$(Az^{e_k})^n \cap L(\mathcal{X}_{z^{e_l}}) \neq \emptyset, \quad \forall n \geq 1. \tag{5.3}$$

Hence $k = l$, by Lemma 5.6. Therefore there is a sequence $(b_i)_{i\in\mathbb{Z}}$ of elements of A and words $u, v \in A^*$ such that $uv = b_0 z^{e_k}$ and

$$x = \dots b_{-3}z^{e_k}b_{-2}z^{e_k}b_{-1}z^{e_k}u.vb_1z^{e_k}b_2z^{e_k}b_3z^{e_k}\dots \tag{5.4}$$

By (5.1), there is a suffix w of $b_{-1}z^{e_k}$ such that $wu = a_{-1}z^{e_k}$. By (5.2) and (5.4), we have $u, w \neq 1$. But since $e_k \geq 4$, this is impossible by Lemma 5.2. The absurd resulted from supposing that $x \in \mathcal{X}_{z^{e_l}}$ for some $l \geq 0$. Therefore $x \in \mathcal{O}(t_l)$, for some $l \geq 0$. Then by (5.1) we have (5.3), thus $k = l$ by Lemma 5.6.

Until now we supposed that $k > 0$. Next take $k = 0$. Then z is not a factor of $\alpha(p_N)$. Since z is a factor of $\omega(p_N)_{]-\infty,-1]}$, and $A^*zA^* = (A^+ \setminus A^*zA^*)zA^*$, there are pseudowords $\rho \in \overline{\Omega}_A\mathsf{S} \setminus (\overline{\Omega}_A\mathsf{S})^1 z(\overline{\Omega}_A\mathsf{S})^1$ and $\nu \in z(\overline{\Omega}_A\mathsf{S})^1$ such that $\hat{\mu}(p_N) = \rho\nu$. Since $\alpha(p_N) \in \mathcal{X}$, the word z is not a factor of any prefix of $\hat{\mu}(p_N)$, by Lemma 4.3. Hence ρ is infinite. If ν were finite then z would be a factor of $\omega(p_N)_{]-\infty,-1]}$ only a finite number of times (by Lemma 4.3), which is impossible. Hence ν is infinite. Since z is a factor of $\overleftarrow{\rho}\,.\overrightarrow{\nu}$ but not of $\overleftarrow{\rho}$, necessarily $\overleftarrow{\rho}\,.\overrightarrow{\nu} \in \mathcal{O}(t_0)$.

In any case, $k = 0$ or $k > 0$, there are infinite pseudowords ρ, ν such that $\hat{\mu}(p_N) = \rho\nu$ and $\overleftarrow{\rho}\,.\overrightarrow{\nu} \in \mathcal{O}(t_k)$. Hence the idempotent $f = (az^{e_{k+1}})^\omega$ is a factor of ν, whence $\hat{\mu}(p_N) = \rho' f \nu'$ for some pseudowords ρ' and ν'. By Theorem 4.9, there is a good factorization $p_N = s_1s_2$ in $\langle E_{\lceil \Sigma(\mathcal{X})\rceil_\beta}\rangle$ such that $\hat{\mu}(s_1) = \rho' f$ and $\hat{\mu}(s_2) = f\nu'$. Then $\alpha(s_2) = \overleftarrow{f}\,.\overrightarrow{f} = \psi_{z^{e_{k+1}}}(a^\infty) \in \mathcal{X}_{z^{e_{k+1}}}$.

Applying to s_2 the same arguments that where applied to p_N, we conclude that $\hat{\mu}(s_2) = \rho''\nu''$ for some pseudowords ρ'' and ν'' such that $\overleftarrow{\rho''}.\overrightarrow{\nu''} \in \mathcal{O}(t_{k+1})$. The idempotent

$$g = (c_1z^{e_{k+1}}c_2z^{e_{k+1}} \cdots c_{h-1}z^{e_{k+1}}c_hz^{e_{k+1}})^\omega$$

is a factor of ρ''. Hence, applying again Theorem 4.9, one concludes that there is a good factorization $s_2 = s_1's_2'$ in $\langle E_{\lceil\Sigma(\mathcal{X})\rceil_\beta}\rangle$ such that $\omega(s_1') = \overleftarrow{g}\,.\,\overrightarrow{g} = \psi_{z^{e_{k+1}}}(c^\infty)$. Therefore s_1' belongs to $E_{\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle}(\psi_{z^{e_{k+1}}}(a^\infty), \psi_{z^{e_{k+1}}}(c^\infty))$. Then, by Proposition 5.5, the set $E_{\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle}(a^\infty, c^\infty)$ is nonempty. This contradicts item (5) describing $Q(\beta, \mathcal{X}, \mathcal{Z}, c)$. The absurd resulted from the assumption that $E_\beta(\mathcal{X}', \mathcal{Y}, \mathcal{Z}') \cap \langle\lceil\Sigma(\mathcal{X}')\rceil_\beta\rangle \neq \emptyset$. Hence property $Q(\beta^+, \mathcal{X}', \mathcal{Z}', z)$ holds. Therefore $P(\beta^+)$ is true.

Suppose now that β is a countable limit ordinal and that $P(\gamma)$ is true for every ordinal $\gamma \in \beta$. For each $\gamma \in \beta$, let $\mathcal{X}_\gamma$, $\mathcal{Z}_\gamma$ be subshifts of $A^{\mathbb{Z}}$ and let $c_\gamma \in A^+$ be such that $Q(\beta, \mathcal{X}_\gamma, \mathcal{Z}_\gamma, c_\gamma)$ is true. Since β is countable, the set $X = \bigcup_{\gamma\in\beta} \mathcal{X}_\gamma$ is countable. Hence there is $z \in A^+$ such that $z \notin L(X)$ and $|z|$ is prime. Likewise in the proof of the successor case of the inductive step, we define the sequence $(e_k)_k$ as follows: $e_0 = 0$, and if $k > 0$ then e_k is the k-th positive integer greater than 3 such that $e_k|z| + 1$ is prime. Take an enumeration $\gamma_1, \gamma_2, \gamma_3, \dots$ of the elements of β. For each nonnegative integer k, let $t_k = \psi_{z^{e_k}}(c_{\gamma_k})_{]-\infty,-1]}.\psi_{z^{e_{k+1}}}(a^\infty)_{[0,+\infty[}$. Let D the set of letters d of A such that $\{\gamma \in \beta \mid c_\gamma \in A^*dA^*\}$ is infinite. Let $\mathcal{Z}_\beta$ be the subshift $\Big[\bigcup_{d\in D} \mathcal{O}(z^{-\infty}.dz^{+\infty})\Big] \cup \mathcal{O}(z^\infty)$. Consider the countable subshift $\mathcal{X}_\beta = \Big[\bigcup_{k\geq 0} (\mathcal{X}_{\gamma_k})_{z^{e_k}} \cup \mathcal{O}(t_k))\Big] \cup \mathcal{Z}_\beta$. Then the proposition $Q(\beta, \mathcal{X}_\beta, \mathcal{Z}_\beta, z)$ is true, which one proves similarly as we did for the successor case of the inductive step. Therefore $P(\beta)$ holds for every ordinal β. ∎

5.2. Upper bounds for $\mathfrak{o}(\Sigma(\mathcal{X}))$. We seek properties on $\mathcal{X}$ that imply bounds for $\mathfrak{o}(\Sigma(\mathcal{X}))$. We attack this problem using the trivial observation that if $\lceil\Sigma(\mathcal{X})\rceil_\beta = \varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$ then $\lceil\Sigma(\mathcal{X})\rceil = \varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$ and $\mathfrak{o}(\Sigma(\mathcal{X})) \leq \beta$.

Theorem 5.9. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{A} \textcircled{m} \mathsf{V} = \mathsf{V} * \mathsf{D}$. *Let* G *be a subgraph of* $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$ *equal to* $\lceil\Sigma(\mathcal{X})\rceil_\beta$ *or to* $\langle\lceil\Sigma(\mathcal{X})\rceil_\beta\rangle$, *for some ordinal* β. *If* $\hat{\mu}(G) = \mathscr{M}(\mathcal{X})$ *then* $G = \varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$.

Proof: Suppose $\hat{\mu}(G) = \mathscr{M}(\mathcal{X})$. Consider an edge $q : x \to y$ of $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$. Let $u = \hat{\mu}(q)$. Then $u \in \mathscr{M}(\mathcal{X})$, by Proposition 4.5. We want to prove that $q \in G$. We have $\Sigma(\mathcal{X})^+ \subseteq G$, since $\hat{\mu}(\Sigma(\mathcal{X})) = L_1(\mathcal{X}) \neq \mathscr{M}(\mathcal{X})$. Hence we can suppose that $q \notin \Sigma(\mathcal{X})^+$. Therefore $u \notin A^+$, by Lemma 4.3. Let v and w be accumulation points of $(x_{[-n,-1]})_n$ and $(y_{[0,n]})_n$ in $\overline{\Omega}_A\mathsf{V}$, respectively. Then $vuw \in \mathscr{M}(\mathcal{X})$. By hypothesis, there is an edge p of G such that

$\hat{\mu}(p) = vuw$. By Theorem 4.9, there is a good factorization $p = p_1p_2p_3$ in G such that $\hat{\mu}(p_1) = v$, $\hat{\mu}(p_2) = u$ and $\hat{\mu}(p_3) = w$. By Lemma 4.3, we have $\alpha(p_2) = \overleftarrow{v}.\overrightarrow{u} = x$ and $\omega(p_2) = \overleftarrow{u}.\overrightarrow{w} = y$. Therefore $p_2 = q$, since $\hat{\mu}$ is faithful, by Corollary 4.7. Hence $q \in G$. ■

It would be interesting to know if there is some subshift $\mathcal{X}$ such that $\lceil\Sigma(\mathcal{X})\rceil \neq \varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$. Its existence would solve Problem 3.22. If $\mathcal{X}$ is such a system and $\mathsf{V} = \mathsf{A} ⓜ \mathsf{V} = \mathsf{V} * \mathsf{D}$ then, since $\hat{\mu}(\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})) = \mathscr{M}(\mathcal{X})$, by Theorem 5.9 there would exist pseudowords in $\mathscr{M}(\mathcal{X})$ quite "far away" from $\overline{L(\mathcal{X})}$, in the sense that they would not belong to $\hat{\mu}(\lceil\Sigma(\mathcal{X})\rceil_\beta)$ for every ordinal β.

Lemma 5.10. *Let $(f(k))_k$ be a bounded sequence of integers greater than 1. Take a sequence $(u_{k,1}, u_{k,2}, \cdots, u_{k,f(k)-1}, u_{k,f(k)})_k$ of tuples of words of A^+ such that*

(1) $\lim_{k\to+\infty} \min\{|u_{k,i}|_i : 1 \leq i \leq f(k)\} = +\infty$,
(2) $u_{k,i}u_{k,i+1} \in L(\mathcal{X})$, *for every* $i \in \{1, \ldots, f(k) - 1\}$.

Then the accumulation points of the sequence $(u_{k,1}u_{k,2} \cdots u_{k,f(k)-1}u_{k,f(k)})_k$ belong to $\hat{\mu}\Big(\Big\langle\overline{\Sigma(\mathcal{X})^+}\Big\rangle\Big)$.

Proof: Let $w_k = \prod_{i=1}^{f(k)} u_{k,i}$. Let w be an accumulation point of the sequence $(w_k)_k$. Taking subsequences if necessary, on may assume that $\lim_{k\to+\infty} w_k = w$ and that $(f(k))_k$ is a constant sequence of value n.

For every $i \in \{1, \ldots, n\}$, let $p_{k,i}, s_{k,i} \in A^*$ be such that $u_{k,i} = p_{k,i}s_{k,i}$ e $||p_{k,i}| - |s_{k,i}|| \leq 1$. Let $(v_{k,j})_{j=1,\ldots,2n}$ be the sequence of words given by:

$$v_{k,2i-1} = p_{k,i}, \quad v_{k,2i} = s_{k,i}, \quad i \in \{1, \ldots, n\}.$$

Then $w_k = \prod_{j=1}^{2n} v_{k,j}$. Let $v_{k,0} = v_{k,2n+1} = 1$. For each $j \in \{1, \ldots, 2n\}$ the word $v_{k,j-1}v_{k,j}v_{k,j+1}$ belongs to $L(\mathcal{X})$, by Condition (2). Hence there are $z_{k,j} \in A^{\mathbb{Z}^-}$ and $t_{k,j} \in A^{\mathbb{Z}_0^+}$ such that $z_{k,j}v_{k,j-1}.v_{k,j}v_{k,j+1}t_{k,j}$ is an element of $\mathcal{X}$, briefly denoted by $x_{k,j}$. Let $q_{k,j}$ be the unique edge of $\Sigma(\mathcal{X})^+$ from $x_{k,j}$ to $\sigma^{|v_{k,j}|}(x_{k,j})$. Note that $\hat{\mu}(q_{k,j}) = v_{k,j}$. Taking subsequences if necessary, we may assume that the following limit exists:

$$\lim_{k\to+\infty} (q_{k,1}, q_{k,2}, \ldots, q_{k,2n-1}, q_{k,2n}) = (q_1, q_2, \ldots, q_{2n-1}, q_{2n})$$

Moreover, for every $j \in \{1, \dots, 2n-1\}$ we have $\lim_{k\to+\infty} |v_{k,j}| = \lim_{k\to+\infty} |v_{k,j+1}| = +\infty$, by Condition (1). Hence

$$\omega(q_j) = \lim_{k\to+\infty} \omega(q_{k,j}) = \lim_{k\to+\infty} x_{k,j+1} = \lim_{k\to+\infty} \alpha(q_{k,j+1}) = \alpha(q_{j+1}).$$

Therefore $q = q_1 q_2 \cdots q_{2n-1} q_{2n}$ is an edge of $\left\langle \overline{\Sigma(\mathcal{X})^+} \right\rangle$. Finally,

$$\hat{\mu}(q) = \hat{\mu}(q_1)\hat{\mu}(q_2)\cdots\hat{\mu}(q_{2n-1})\hat{\mu}(q_{2n}) = \lim_{k\to+\infty} v_{k,1}v_{k,2}\cdots v_{k,2n-1}v_{k,2n} = w. \quad \blacksquare$$

Lemma 5.11. *If S is a finite semigroup then for every finite collection $s_1, \dots, s_n$ of elements of S there is a subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$ with at most $|S|$ elements such that $s_1 \cdots s_n = s_{i_1} \cdots s_{i_k}$.*

Proof: Apply the pigeon-hole principle. $\blacksquare$

Proposition 5.12. *Let V be a pseudovariety of semigroups containing $\mathscr{L}\mathsf{SI}$. Suppose $(f(n))_{n\geq 1}$ is an unbounded sequence of positive integers. Let $L_f(\mathcal{X})$ be the set $\bigcup_{n\geq 1}\{u \in L(\mathcal{X}) : |u| = f(n)\}$. Suppose there are families of words $(p_u)_{u\in L_f(\mathcal{X})}$, $(z_u)_{u\in L_f(\mathcal{X})}$ and $(s_u)_{u\in L_f(\mathcal{X})}$ such that:*

(1) $u = p_u z_u s_u$ for every $u \in L_f(\mathcal{X})$;
(2) for every $u, v \in L_f(\mathcal{X})$, if $|u| = |v|$ then $z_u s_v \in L(\mathcal{X})$;
(3) $\lim_{n\to+\infty}\Big(\min_{u\in L_{f(n)}(\mathcal{X})} |p_u|\Big) = \lim_{n\to+\infty}\Big(\min_{u\in L_{f(n)}(\mathcal{X})} |z_u|\Big) = \lim_{n\to+\infty}\Big(\min_{u\in L_{f(n)}(\mathcal{X})} |s_u|\Big) = +\infty.$

Then $\mathscr{M}(\mathcal{X}) = \hat{\mu}(\lceil\Sigma(\mathcal{X})\rceil_2)$.

Proof: Let $v \in \mathscr{M}(\mathcal{X})$. If $v \in A^+$, then $v \in L(\mathcal{X})$ and therefore $v \in \hat{\mu}(\Sigma(\mathcal{X})^+)$.

Suppose $v \notin A^+$. Let $(v_n)_n$ be a sequence of elements of A^+ converging to v. Since $\mathscr{M}_{3f(k)}(\mathcal{X})$ is an open neighborhood of v, there is an integer N_k such that

$$n \geq N_k \Rightarrow (v_n \in \mathscr{M}_{3f(k)}(\mathcal{X}) \text{ and } |v_n| \geq 3f(k)).$$

Let n_k be the sequence of integers recursively defined by $n_1 = N_1$ and $n_k = \max\{n_{k-1}+1, N_k\}$ if $k > 1$. Then $(v_{n_k})_k$ is a subsequence of $(v_n)_n$ such that $v_{n_k} \in \mathscr{M}_{3f(k)}(\mathcal{X})$ and $|v_{n_k}| \geq 3f(k)$, for every k. The word v_{n_k} has a factorization of the following type:

$$v_{n_k} = v_{k,1}v_{k,2}\cdots v_{k,r_k-1}v_{k,r_k},\ |v_{k,1}| = |v_{k,2}| = \cdots = |v_{k,r_{k-1}}| = f(k),$$
$$f(k) \leq |v_{k,r_k}| < 2f(k),\ r_k \geq 3.$$

Then

$$v_{n_k} = p_{v_{k,1}} z_{v_{k,1}} \cdot \left(\prod_{i=1}^{r_k-2} s_{v_{k,i}} p_{v_{k,i+1}} z_{v_{k,i+1}} \right) \cdot s_{v_{k,r_k-1}} v_{k,r_k}.$$

Let K be a V-recognizable language of A^+. Then there is a homomorphism $\varphi : A^+ \to S$ from A^+ into a semigroup S of V such that $K = \varphi^{-1}\varphi(K)$. By Lemma 5.11 there exists $t_k \leq |S|$ and a subset $\{i_1, \dots, i_{t_k}\}$ of $\{1, \dots, r_k - 2\}$ such that

$$\varphi(v_{n_k}) = \varphi\left(p_{v_{k,1}} z_{v_{k,1}} \cdot \left(\prod_{j=1}^{t_k} s_{v_{k,i_j}} p_{v_{k,i_j+1}} z_{v_{k,i_j+1}} \right) \cdot s_{v_{k,r_k-1}} v_{k,r_k} \right). \tag{5.5}$$

The equality (5.5) suggests that we consider the following tuple:

$$\begin{aligned} \lambda_k = (&p_{v_{k,1}}, z_{v_{k,1}}, s_{v_{k,i_1}}, p_{v_{k,i_1+1}}, z_{v_{k,i_1+1}}, s_{v_{k,i_2}}, p_{v_{k,i_2+1}}, z_{v_{k,i_2+1}}, s_{v_{k,i_3}}, \dots \\ &\dots, s_{v_{k,i_{t_k}}}, p_{v_{k,i_{t_k}+1}}, z_{v_{k,i_{t_k}+1}}, s_{v_{k,r_k-1}}, v_{k,r_k}). \end{aligned}$$

The number of components of λ_k is $3t_k+4 \leq 3|S|+4$. The product of any two consecutive components of λ_k is either a factor of a word of the form $v_{k,i}v_{k,i+1}$ — which belongs to $L(\mathcal{X})$ because $|v_{k,i}v_{k,i+1}| < 3f(k)$ and $v_{n_k} \in \mathscr{M}_{3f(k)}(\mathcal{X})$ — or of the form $z_{u_1}s_{u_2}$ with $u_1, u_2 \in L_{f(k)}(\mathcal{X})$. Applying Condition (2), we conclude that the product of any two consecutive components of λ_k belongs to $L(\mathcal{X})$. On the other hand, since

$$\lim_{k\to+\infty} \min\{|v_{k,i}| : 1 \leq i \leq r_k\} = \lim_{k\to+\infty} f(k) = +\infty,$$

by Condition (3), we deduce

$$\lim_{k\to+\infty} \min\{|(\lambda_k)_i| \,:\, 1 \leq i \leq 3t_k + 4\} = +\infty.$$

Let $w_k = \prod_{i=1}^{3t_k+4} (\lambda_k)_i$. Then by Lemma 5.10 there is an element w of $\hat{\mu}\Big(\Big\langle\overline{\Sigma(\mathcal{X})^+}\Big\rangle\Big)$ which is the limit of a subsequence $(w_{k_l})_l$ of $(w_k)_k$. Let $\hat{\varphi}$ be the unique continuous homomorphism from $\overline{\Omega}_A\mathsf{V}$ to S extending φ. From (5.5) we deduce that

$$\hat{\varphi}(v) = \lim_{l\to+\infty} \varphi(v_{n_{k_l}}) = \lim_{l\to+\infty} \varphi(w_{k_l}) = \hat{\varphi}(w).$$

Hence

$$\hat{\varphi}^{-1}\hat{\varphi}(v) \cap \hat{\mu}\Big(\Big\langle\overline{\Sigma(\mathcal{X})^+}\Big\rangle\Big) \neq \emptyset. \tag{5.6}$$

Since $\hat{\varphi}^{-1}\varphi(K)$ is closed and open in $\overline{\Omega}_A\mathsf{V}$, and A^+ is dense in $\overline{\Omega}_A\mathsf{V}$, we have

$$\hat{\varphi}^{-1}\varphi(K) = \overline{\hat{\varphi}^{-1}\varphi(K) \cap A^+} = \overline{\varphi^{-1}\varphi(K)} = \overline{K}. \tag{5.7}$$

Therefore, if $\overline{K}$ contains v then $\overline{K} \cap \hat{\mu}\Big(\Big\langle\overline{\Sigma(\mathcal{X})^+}\Big\rangle\Big) \neq \emptyset$, by (5.6) and (5.7). According to Proposition 2.2 the topology of $\overline{\Omega}_A\mathsf{V}$ is generated by the closure of the V-recognizable languages, whence $v \in \overline{\hat{\mu}\Big(\Big\langle\overline{\Sigma(\mathcal{X})^+}\Big\rangle\Big)} = \hat{\mu}(\lceil\Sigma(\mathcal{X})\rceil_2)$. ■

Corollary 5.13. *Let* V *be a pseudovariety of semigroups containing* $\mathscr{L}\mathsf{Sl}$. *Let* $\mathcal{X}$ *be a sofic subshift presented by a labeled graph* G *for which there are a vertex* i *and an integer* N *such that every path on* G *with length* N *contains* i. *Then* $\mathscr{M}(\mathcal{X}) = \hat{\mu}(\lceil\Sigma(\mathcal{X})\rceil_2)$.

Proof: Let u be an element of $L(\mathcal{X})$ with length greater than $4N$. Take a path q on G labeled u. Then there are paths q_1, q_2, q_3 and r such that $q = q_1q_2rq_3$, $|q_1| = |q_2| = |r| = N$ e $|q_3| > N$. By hypothesis, there are paths r_1 and r_2 such that $\omega(r_1) = \alpha(r_2) = i$ and $r = r_1r_2$. Let p_u, z_u and s_u be the labels of q_1, q_2r_1 and r_2q_3, respectively. Consider the map $f(n) = n + 4N$, $n \geq 1$. The families $(p_u)_{u\in L_f(\mathcal{X})}$, $(z_u)_{u\in L_f(\mathcal{X})}$ and $(s_u)_{u\in L_f(\mathcal{X})}$ satisfy the conditions of Proposition (5.12). ■

Corollary 5.14. *Let* V *be a pseudovariety of semigroups containing* $\mathscr{L}\mathsf{Sl}$. *Let* $\mathcal{X}$ *be a subshift such that for each positive integer* n *there is a word of length* n *uniformly recurrent in* $L(\mathcal{X})$. *Then* $\mathscr{M}(\mathcal{X}) = \hat{\mu}(\lceil\Sigma(\mathcal{X})\rceil_2)$.

Proof: For each positive integer n let w_n be a word of length n uniformly recurrent in $L(\mathcal{X})$. Let $g(n)$ be a positive integer such that every word of $L(\mathcal{X})$ with length $g(n)$ has w_n as factor. Let $(f(n))_n$ be the strictly increasing sequence recursively defined by $f(1) = 2 + g(1)$ and $f(n) = \max\{f(n-1) + 1, 2n + g(n)\}$ if $n > 1$. For each $u \in L_{f(n)}(\mathcal{X})$ there are words u_1, u_2, u_3 such that $u = u_1u_2u_3$, $|u_1| = |u_3| = n$ and $|u_2| \geq g(n)$. Then w_n is a factor of u_2, thus $u = p_uw_ns_u$ for some words p_u and s_u with length greater or equal to n. Letting $z_u = w_n$, the families $(p_u)_{u\in L_f(\mathcal{X})}$, $(z_u)_{u\in L_f(\mathcal{X})}$ and $(s_u)_{u\in L_f(\mathcal{X})}$ satisfy the conditions of Proposition 5.12. ■

Corollary 5.15. *Whenever* $\mathsf{V} = \mathsf{A} ⓜ \mathsf{V} = \mathsf{V}*\mathsf{D}$ *and* $\mathcal{X}$ *satisfies the conditions described in Corollary 5.14 or in Corollary 5.13 then* $\mathfrak{o}(\Sigma(\mathcal{X})) \leq 2$.

Proof: Apply Theorem 5.9 together with Corollary 5.14 or Corollary 5.13 ■

The following result gives an example of a subshift $\mathcal{Z}$ such that $\mathfrak{o}(\Sigma(\mathcal{Z})) = 2$. Note that the language $a^+ \cup a^*ba^*$, being factorial and prolongable, is the language of the finite factors of a unique subshift of $A^{\mathbb{Z}}$.

Proposition 5.16. *Consider a pseudovariety of semigroups* V *containing* A *and such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$*. Let* A *be the two-letter alphabet* $\{a, b\}$*. Let* $\mathcal{Z}$ *be the subshift of* $A^{\mathbb{Z}}$ *such that* $L(\mathcal{Z}) = a^+ \cup a^*ba^*$*. Then*

$$\overline{\Sigma(\mathcal{Z})^+} \subsetneqq \left\langle \overline{\Sigma(\mathcal{Z})^+} \right\rangle \subsetneqq \lceil \Sigma(\mathcal{Z}) \rceil_2 = \widehat{\Sigma}(\mathcal{Z}) = \varprojlim \widehat{\Sigma}_{2n}(\mathcal{Z}). \tag{5.8}$$

Proof: Suppose $ba^\omega b \in \overline{L(\mathcal{Z})}$. The languages $L(\mathcal{Z})$ and ba^*b are A-recognizable, thus $\overline{L(\mathcal{Z})} \cap \overline{ba^*b}$ is an open neighborhood of $ba^\omega b$ by Proposition 2.2. Hence $\overline{L(\mathcal{Z})} \cap \overline{ba^*b} \cap A^+ \neq \emptyset$, because A^+ is dense in $\overline{\Omega}_A\mathsf{V}$. But $\overline{L(\mathcal{Z})} \cap \overline{ba^*b} \cap A^+ = L(\mathcal{Z}) \cap ba^*b = \emptyset$. Therefore $ba^\omega b \notin \overline{L(\mathcal{Z})}$.

Since $ba^{n!+n}$ belongs to $L(\mathcal{Z})$, there are consecutive paths q_n, p_n on $\Sigma(\mathcal{Z})$ such that $\hat{\mu}(q_n) = ba^{n!}$ and $\hat{\mu}(p_n) = a^n$. Let q and p be accumulation points of $(q_n)_n$ and $(p_n)_n$ respectively. Then q and p are edges of $\overline{\Sigma(\mathcal{Z})^+}$ such that $\omega(q) = \alpha(p) = a^\infty$, and $\hat{\mu}(q) = ba^\omega$. Similarly, there is an edge r of $\overline{\Sigma(\mathcal{Z})^+}$ such that $\alpha(r) = a^\infty$ and $\hat{\mu}(r) = a^\omega b$. Then q and r are consecutive edges of $\overline{\Sigma(\mathcal{Z})^+}$ such that $\hat{\mu}(qr) = ba^\omega b$. Therefore $ba^\omega b$ is an element of $\hat{\mu}\left(\left\langle \overline{\Sigma(\mathcal{Z})^+} \right\rangle\right)$ not in $\overline{L(\mathcal{Z})}$.

Next, let $u = b(a^\omega b)^\omega = \lim b(a^{n!}b)^{n!}$. Let K_n be the language $b(A^+b)^n$. Then $u \in \overline{K_n}$. Suppose $u \in \overline{L(\mathcal{Z})^n}$. The languages K_n and $L(\mathcal{Z})^n$ are A-recognizable, since they are the concatenation of the A-recognizable languages $L(\mathcal{Z})$, A^+ and $\{b\}$. Hence $\overline{K_n} \cap \overline{L(\mathcal{Z})^n}$ is open, and since A^+ is dense in $\overline{\Omega}_A\mathsf{V}$, we conclude that $\overline{K_n} \cap \overline{L(\mathcal{Z})^n} \cap A^+ \neq \emptyset$. But $\overline{K_n} \cap \overline{L(\mathcal{Z})^n} \cap A^+ = K_n \cap L(\mathcal{Z})^n = K_n \cap L(\mathcal{Z})^n = \emptyset$. Hence $u \notin \overline{L(\mathcal{Z})^n}$, for all n. Having in mind Proposition 4.5 and that $\overline{L(\mathcal{Z})^n} = \left(\overline{L(\mathcal{Z})}\right)^n$, we conclude that $u \notin \hat{\mu}\left(\left\langle \overline{\Sigma(\mathcal{Z})^+} \right\rangle\right)$. On the other hand, $u \in \mathscr{M}(\mathcal{Z})$.

Recapitulating,

$$\overline{L(\mathcal{Z})} \subsetneqq \hat{\mu}\left(\left\langle \overline{\Sigma(\mathcal{Z})^+} \right\rangle\right) \subsetneqq \mathscr{M}(\mathcal{Z}).$$

The word a^n is uniformly recurrent in $L(\mathcal{Z})$. We have $\overline{L(\mathcal{Z})} = \hat{\mu}\left(\overline{\Sigma(\mathcal{Z})^+}\right)$ and $\mathscr{M}(\mathcal{Z}) = \hat{\mu}(\lceil \Sigma(\mathcal{Z}) \rceil_2)$ by Proposition 4.5 and Corollary 5.14. Then we deduce (5.8) using Theorem 5.9. ∎

For certain pseudovarieties (like the pseudovariety of all finite semigroups), the property described in Proposition 5.16 also holds for the even subshift. This is proved with Corollary 5.13 and similar arguments as detailed in [19].

6. Minimal subshifts

A subshift $\mathcal{X}$ is *minimal* if $\mathcal{X}$ does not contain subshifts different from $\mathcal{X}$. The subshift $\mathcal{X}$ is minimal if and only if all words in $L(\mathcal{X})$ are uniformly recurrent in $L(\mathcal{X})$ [23]. Using Corollary 5.14, we shall prove that $\mathfrak{o}(\Sigma(\mathcal{X})) = 1$, whenever $\mathcal{X}$ is minimal and $\mathsf{V} = \mathsf{A} ⓜ \mathsf{V} = \mathsf{V} * \mathsf{D}$.

Two elements of a semigroup are $\mathcal{J}$*-equivalent* if they are a factor of each other. A $\mathcal{J}$-class is *regular* if it contains an idempotent. If moreover it contains the idempotent factors of its elements then it is called *maximal regular*. Since every infinite pseudoword has idempotent factors [1, Corollary 5.6.2], the maximal regular $\mathcal{J}$-classes of $\overline{\Omega}_A\mathsf{V}$ are the $\mathcal{J}$-classes of infinite pseudowords whose factors not $\mathcal{J}$-equivalent with them are finite words.

Using the uniform recurrence property, it is not difficult to prove that if $\mathcal{X}$ is minimal then $\overline{L(\mathcal{X})} \setminus A^+$ is contained in a regular $\mathcal{J}$-class, which we denote by $\mathfrak{J}(\mathcal{X})$, whenever $\mathsf{V} \subseteq \mathscr{L}\mathsf{SI}$. More precisely, the correspondence $\mathcal{X} \mapsto \mathfrak{J}(\mathcal{X})$ is a bijection between the set of minimal subshifts and the set of maximal regular $\mathcal{J}$-classes of $\overline{\Omega}_A\mathsf{V}$. This was proved in [5] under the hypothesis $\mathsf{V} = \mathsf{S}$, but the proof also holds for $\mathsf{V} \subseteq \mathscr{L}\mathsf{SI}$. A rather different proof appears in [19].

The algebraic structure of a semigroup is normally described in terms of *Green's relations*, one of which is the relation $\mathcal{J}$. We describe the others. Two elements of a semigroup are $\mathcal{R}$*-equivalent* (respectively, $\mathcal{L}$*-equivalent*) if they are a prefix (respectively, suffix) of each other. The intersection of the $\mathcal{R}$- and $\mathcal{L}$-equivalences is called the $\mathcal{H}$*-equivalence* and their join, which by associativity is also their composite in any order, is called the $\mathcal{D}$*-equivalence*. A $\mathcal{D}$-class contains an idempotent if and only if each of its $\mathcal{R}$-classes and $\mathcal{L}$-classes contains an idempotent. The $\mathcal{H}$-classes of a semigroup S which contain idempotents are precisely the maximal subgroups of S. Green's Lemma states that if s and st are $\mathcal{R}$-equivalent then the correspondence $x \mapsto xt$ defines a bijection between the $\mathcal{L}$-classes of s and st. The following propositions are applications of Green's Lemma:

Proposition 6.1. *For two $\mathcal{D}$-equivalent elements s and t, $s \mathrel{\mathcal{R}} st \mathrel{\mathcal{L}} t$ if and only if there is an idempotent e such that $s \mathrel{\mathcal{L}} e \mathrel{\mathcal{R}} t$.*

Proposition 6.2. *If e and f are idempotents of a semigroup, then for all $x \in e/\mathcal{R} \cap f/\mathcal{L}$ there is a unique $y \in f/\mathcal{R} \cap e/\mathcal{L}$ such that $xy = e$ and $yx = f$.*

Another application of Green's Lemma is that all maximal subgroups within a $\mathcal{D}$-class are isomorphic.

It is well known that, in a compact semigroup, if s is a prefix of t and t is a factor of s then t is also a prefix of s. This property, which is known as *right stability*, together with its dual imply that the $\mathcal{D}$- and $\mathcal{J}$-equivalences coincide. For further information and the significance of Green's relations in semigroup theory see, for instance, [26].

The following theorem was proved in [5, Theorem 2.6] by the first author in a substantially different manner. The new proof exemplifies how the semigroupoid $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X})$ may be useful for studying relatively free profinite semigroups.

Theorem 6.3. *Consider a pseudovariety of semigroups V containing $\mathscr{L}\mathsf{Sl}$. Suppose $\mathcal{X}$ is a minimal subshift. Then $\mathfrak{J}(\mathcal{X}) = \mathscr{M}(\mathcal{X}) \setminus A^+$.*

Proof: Since $\overline{L(\mathcal{X})} \setminus A^+ \subseteq \mathfrak{J}(\mathcal{X})$, we have $\mathfrak{J}(\mathcal{X}) \subseteq \mathscr{M}(\mathcal{X}) \setminus A^+$.

Let u and v be elements of $\mathfrak{J}(\mathcal{X})$ such that $uv \in \mathscr{M}(\mathcal{X})$. Let s and p be accumulation points of the sequences $(\mathrm{t}_n(u))_n$ and $(\mathrm{i}_n(v))_n$, respectively. Then $u = u's$ and $v = pv'$, for some pseudowords u' and v'. Note also that $sp \in \overline{L(\mathcal{X})}$. Since s and p are infinite pseudowords, there are factorizations $s = s_1 e s_2$ and $p = p_1 f p_2$ such that e and f are idempotents [1, Corollary 5.6.2]. Consider the pseudowords $x = u's_1e$, $y = es_2p_1f$ and $z = fp_2v''$. The elements of the set $W = \{e, f, x, y, z\}$ are infinite factors of elements of $\mathfrak{J}(\mathcal{X})$, thus $W \subseteq \mathfrak{J}(\mathcal{X})$. Since $x = xe$, $y = ey$ and $\overline{\Omega}_A\mathsf{V}$ is stable, we have $x \mathrel{\mathcal{L}} e$ and $y \mathrel{\mathcal{R}} e$. Hence $xy \in \mathfrak{J}(\mathcal{X})$, by Proposition 6.1. Similarly, since $xy = xyf$ and $z = fz$, we have $xyz \in \mathfrak{J}(\mathcal{X})$. Note that $xyz = uv$. Therefore,

$$(u, v \in \mathfrak{J}(\mathcal{X}) \text{ and } uv \in \mathscr{M}(\mathcal{X})) \Rightarrow uv \in \mathfrak{J}(\mathcal{X}). \tag{6.1}$$

Suppose next that $u \in L(\mathcal{X})$, $v \in \mathfrak{J}(\mathcal{X})$ and $uv \in \mathscr{M}(\mathcal{X})$ (the case $vu \in \mathscr{M}(\mathcal{X})$ is similar). Since $\mathfrak{J}(\mathcal{X})$ is regular, there is an idempotent e such that $v \mathrel{\mathcal{R}} e$. There is $t \in \overline{\Omega}_A\mathsf{V}$ such that $v = et$. It follows that $ev = et = v$. Let w be an accumulation point of the sequence $(u\,\mathrm{i}_n(e))_n$. Then $w \in \overline{L(\mathcal{X})} \setminus A^+$, and hence $w \in \mathfrak{J}(\mathcal{X})$; on the other hand, $uv = uev = wsv$ for some suffix s of e. The pseudoword sv is an infinite factor of v, thus belongs to $\mathfrak{J}(\mathcal{X})$. Hence $wsv = uv \in \mathfrak{J}(\mathcal{X})$, by (6.1). This concludes the proof of the following implication:

$$(u, v \in L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X}) \text{ and } uv \in \mathscr{M}(\mathcal{X})) \Rightarrow uv \in L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X}). \tag{6.2}$$

Let $q_1, \ldots, q_n$ be consecutive edges of $\overline{\Sigma(\mathcal{X})^+}$. We shall prove by induction on n that $\hat{\mu}(q_1 \cdots q_n) \in L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X})$. By Proposition 4.5 we have $\hat{\mu}\left(\overline{\Sigma(\mathcal{X})^+}\right) = \overline{L(\mathcal{X})}$. Since $\overline{L(\mathcal{X})} \subseteq L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X})$, the initial step is proved. Suppose $n > 1$ and that $\hat{\mu}(q_1 \cdots q_{n-1}) \in L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X})$. Since $\hat{\mu}(q_n) \in L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X})$ and, by Proposition 4.5, $\hat{\mu}(q_1 \cdots q_{n-1}q_n) \in \mathscr{M}(\mathcal{X})$, from (6.2) we deduce $\hat{\mu}(q_1 \cdots q_{n-1}q_n) \in L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X})$. That is,

$$\hat{\mu}\left(\left\langle \overline{\Sigma(\mathcal{X})^+} \right\rangle\right) \subseteq L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X}).$$

Since $\hat{\mu}$ is continuous, $\mathfrak{J}(\mathcal{X})$ is closed and $\overline{L(\mathcal{X})} \subseteq L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X})$, it follows that

$$\hat{\mu}\left(\overline{\left\langle \overline{\Sigma(\mathcal{X})^+} \right\rangle}\right) \subseteq L(\mathcal{X}) \cup \mathfrak{J}(\mathcal{X}).$$

Hence $\mathscr{M}(\mathcal{X}) \setminus A^+ = \mathfrak{J}(\mathcal{X})$, by Corollary 5.14. ■

Corollary 6.4. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{A} \,ⓜ\, \mathsf{V}$. *If* $\mathcal{X}$ *is a minimal subshift then* $\mathscr{M}(\mathcal{X}) = \overline{L(\mathcal{X})}$.

Proof: We already know that $\overline{L(\mathcal{X})} \subseteq \mathscr{M}(\mathcal{X})$ and $\overline{L(\mathcal{X})} \cap \mathfrak{J}(\mathcal{X}) \neq \emptyset$. The set $\overline{L(\mathcal{X})}$ is factorial, by Proposition 2.5, thus $\mathfrak{J}(\mathcal{X}) \subseteq \overline{L(\mathcal{X})}$. Since $\mathscr{M}(\mathcal{X}) \cap A^+ = L(\mathcal{X})$, the result follows from Theorem 6.3. ■

Corollary 6.5. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{A} \,ⓜ\, \mathsf{V} = \mathsf{V} * \mathsf{D}$. *If* $\mathcal{X}$ *is a minimal subshift then* $\varprojlim \widehat{\Sigma}_{2n}(\mathcal{X}) = \widehat{\Sigma}(\mathcal{X}) = \overline{\Sigma(\mathcal{X})^+}$.

Proof: Apply Corollary 6.4, Proposition 4.5 and Theorem 5.9. ■

The two previous corollaries exhibit properties of minimal subshifts shared by finite type subshifts (cf. Proposition 4.1). However, differently with the finite type case, it is not reasonable to expect a proof of Corollary 6.5 using Proposition 2.1. Let us see why. Suppose there is a positive integer n such that $\hat{\pi}_n(\widehat{\Sigma}(\mathcal{X})) = \widehat{\Sigma}_{2n}(\mathcal{X})$. Then

$$\overline{L(\mathcal{X})} = \hat{\mu}\big(\hat{\pi}_n(\widehat{\Sigma}(\mathcal{X}))\big) = \hat{\mu}\big(\widehat{\Sigma}_{2n}(\mathcal{X})\big) = \mathscr{M}_{2n+1}(\mathcal{X}).$$

That is, $L(\mathcal{X}) = \mathscr{M}_{2n+1}(\mathcal{X}) \cap A^+$, thus $\mathcal{X}$ is of finite type. But if $|A| > 1$ then there are $\aleph_0$ finite type subshifts of $A^{\mathbb{Z}}$, while there are $2^{\aleph_0}$ minimal subshifts of $A^{\mathbb{Z}}$ [28, Chapter 2].

Lemma 6.6. *Suppose $\mathcal{X}$ is a minimal subshift. Let $u, v \in \mathfrak{J}(\mathcal{X})$. Then $u \mathrel{\mathcal{R}} v$ if and only if $\overrightarrow{u} = \overrightarrow{v}$. Dually, $u \mathrel{\mathcal{L}} v$ if and only if $\overleftarrow{u} = \overleftarrow{v}$.*

Proof: Suppose $\overrightarrow{u} = \overrightarrow{v}$. Let e be an accumulation point of the sequence $(\mathrm{i}_n(u))_n$. By hypothesis $\mathrm{i}_n(u) = \mathrm{i}_n(v)$, for every n. Hence e is a common prefix of u and v. By the $\mathcal{J}$-maximality of $\mathfrak{J}(\mathcal{X})$ and the stability of $\overline{\Omega}_A\mathsf{V}$, we conclude that e, u, v are $\mathcal{R}$-equivalent. The converse is immediate. ∎

A semigroupoid C is a *category* if for every vertex x of C there is an edge 1_x such that $1_x s = s$ and $t 1_x = t$, for all edges s and t of C such that $\alpha(s) = x$ and $\omega(t) = x$. A *groupoid* is a category G such that for every edge $s : x \to y$ there is an edge $s' : y \to x$ for which $ss' = 1_x$ and $s's = 1_y$. Note that the local semigroups of groupoids are groups.

The graph $\widehat{\Sigma}(\mathcal{X}) \setminus \Sigma(\mathcal{X})^+$ will be briefly denoted by $\hat{\Sigma}_\infty(\mathcal{X})$. Note that $\hat{\Sigma}_\infty(\mathcal{X})$ is a closed subsemigroupoid of $\widehat{\Sigma}(\mathcal{X})$.

Theorem 6.7. *Consider a pseudovariety of semigroups V such that $\mathsf{V} = \mathsf{A} \textcircled{m} \mathsf{V} = \mathsf{V} * \mathsf{D}$. If $\mathcal{X}$ is a minimal subshift then $\hat{\Sigma}_\infty(\mathcal{X})$ is a connected groupoid.*

Proof: Every minimal subshift is irreducible, hence $\hat{\Sigma}_\infty(\mathcal{X})$ is strongly connected by Corollary 4.11. It remains to prove that $\hat{\Sigma}_\infty(\mathcal{X})$ is a groupoid.

Let z be an arbitrary element of $\mathcal{X}$. Since $\hat{\Sigma}_\infty(\mathcal{X})$ is strongly connected, there are edges from z to z, hence one can consider the local semigroup S_z of $\hat{\Sigma}_\infty(\mathcal{X})$ at z. Since S_z is compact, it contains at least one idempotent ε_z [14, Theorem 3.5].

Let $q : x \to y$ be an arbitrary edge of $\hat{\Sigma}_\infty(\mathcal{X})$. Then $\overrightarrow{\hat{\mu}(\varepsilon_x q)} = x_{[0,+\infty[} = \overrightarrow{\hat{\mu}(q)}$, and so $\hat{\mu}(\varepsilon_x q)$ is $\mathcal{R}$-equivalent to $\hat{\mu}(q)$ by Lemma 6.6. Therefore $\hat{\mu}(q) = \hat{\mu}(\varepsilon_x q)w$ for some $w \in (\overline{\Omega}_A\mathsf{V})^1$. Hence

$$\hat{\mu}(\varepsilon_x q) = \hat{\mu}(\varepsilon_x)\hat{\mu}(q) = \hat{\mu}(\varepsilon_x)\hat{\mu}(\varepsilon_x q)w = \hat{\mu}(\varepsilon_x^2 q)w = \hat{\mu}(\varepsilon_x q)w = \hat{\mu}(q).$$

Then $\varepsilon_x q = q$, since $\hat{\mu}$ is faithful. Dually $q\varepsilon_y = q$. This proves $\hat{\Sigma}_\infty(\mathcal{X})$ is a category.

By Proposition 6.2, there is $v \in \hat{\mu}(\varepsilon_x)/\mathcal{L} \cap \hat{\mu}(\varepsilon_y)/\mathcal{R}$ such that $v\hat{\mu}(q) = \hat{\mu}(\varepsilon_y)$ and $\hat{\mu}(q)v = \hat{\mu}(\varepsilon_x)$. Since $\hat{\mu}(\varepsilon_x)$ and $\hat{\mu}(\varepsilon_x)$ are idempotents, $v \in \hat{\mu}(\varepsilon_x)/\mathcal{L} \cap \hat{\mu}(\varepsilon_y)/\mathcal{R}$ implies that $v = \hat{\mu}(\varepsilon_y)v\hat{\mu}(\varepsilon_x)$. By Proposition 4.5 there is an edge p of $\overline{\Sigma(\mathcal{X})^+}$ such that $\hat{\mu}(p) = v$. Then by Theorem 4.9 there is a good factorization $p = p_1p_2p_3$ in $\overline{\Sigma(\mathcal{X})^+}$ such that $\hat{\mu}(p_1) = \hat{\mu}(\varepsilon_y)$, $\hat{\mu}(p_2) = v$ and

$\hat{\mu}(p_3) = \hat{\mu}(\varepsilon_x)$. We have $\alpha(p_2) = \overleftarrow{\hat{\mu}(\varepsilon_y)}.\overrightarrow{v} = \overleftarrow{\hat{\mu}(\varepsilon_y)}.\overrightarrow{\hat{\mu}(\varepsilon_y)} = y$, by Lemma 4.3. Hence q and p_2 are consecutive. And $\hat{\mu}(qp_2) = \hat{\mu}(q)v = \hat{\mu}(\varepsilon_x)$. Similarly, $\omega(p_2) = x$ and $\hat{\mu}(p_2 q) = v\hat{\mu}(q) = \hat{\mu}(\varepsilon_y)$. Since qp_2 and ε_x are co-terminal and equally labeled, one has $qp_2 = \varepsilon_x$, because $\hat{\mu}$ is faithful. Similarly, $p_2 q = \varepsilon_y$. ■

In a forthcoming paper we will show that the local groups of $\hat{\Sigma}_\infty(\mathcal{X})$ are isomorphic to the maximal subgroup of $\mathfrak{J}(\mathcal{X})$. Note that this implies that the maximal subgroup of $\mathfrak{J}(\mathcal{X})$ is a conjugacy invariant, a fact that is a particular case of a more general result proved by the second author using rather different methods [17]. The maximal subgroup of $\mathfrak{J}(\mathcal{X})$ has been computed for several classes of minimal subshifts by the first author [5]. Hopefully, the groupoid $\hat{\Sigma}_\infty(\mathcal{X})$ may add a new geometric perspective on $\mathfrak{J}(\mathcal{X})$, and $\mathcal{X}$ itself.

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