Continuous modelling by PDEs

Critical domain size

Computational Biology

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Critical domain size

- Reaction-diffusion equations are used to estimate the size of a habitat that can support a population. In general, it is not possible to establish a stable surviving population on an island that is too small.
- For pests, like the spruce budworm, information about the critical patch size can be used to determine how to split a woodland into small enough patches so as to prevent the budworms from settling in.

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Fisher's reaction-diffusion equation

To illustrate the use of reaction-diffusion equations in this context, we use Fisher's equation, which shows all necessary features:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \mu u (1 - u),$$

where u(x, t) is the density of the gene in the population at time t and location x.

- The term µu(1 u) is (the well known) Verhulsts law of growth with saturation.
- A partial differential equation on a bounded interval needs boundary conditions.

Boundary conditions

 Appropriate island boundary conditions are the homogeneous Dirichlet boundary conditions given by (for x ∈ [0, L])

$$u(0,t)=u(L,t)=0.$$

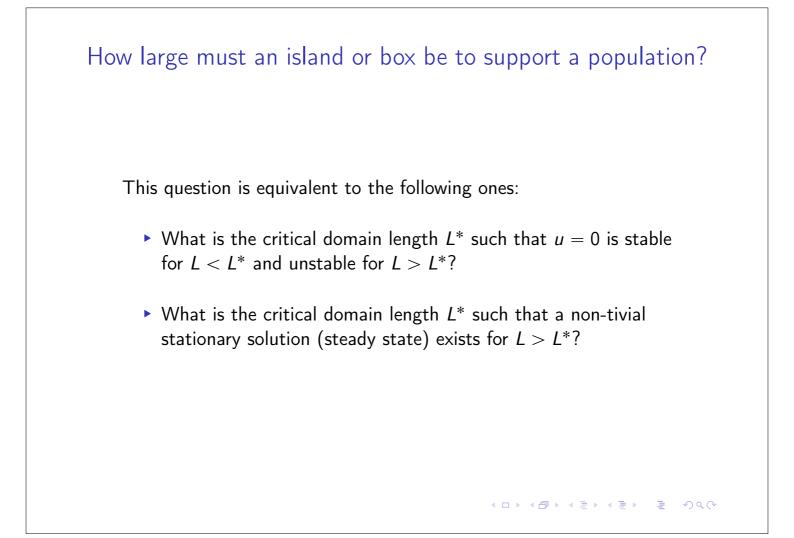
 We can also study a valley or a box, or a patch with sealing walls. Then no individual can leave the patch. Appropriate box boundary conditions are the homogeneous Neumann boundary conditions given by

$$u_{x}(0,t) = u_{x}(L,t) = 0.$$

Obviously, combinations of island and box boundary conditions can occur if, for example, the patch is bounded by a wall on the one side and by water on the other. We could also include some semi-permeable walls such that only a fraction of the population can leave the domain, etc.

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Steady-states

A steady state satisfies

$$\frac{\partial u}{\partial t} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\mu}{D}u(1-u)$$

We are looking for solutions u(x) ≠ 0 which satisfy the correct boundary conditions, and we will use phase plane analysis the equation. With a new variable v = ∂u/∂x and considering the appropriate boundary conditions, we obtain the system of ODEs

$$\frac{du}{dx} = v, \qquad \frac{dv}{dx} = -\frac{\mu}{D}u(1-u).$$

We consider Dirichlet boundary conditions (island)

$$u(0)=0, \qquad u(L)=0$$

or with Neumann boundary conditions (box)

$$v(0) = 0, v(L) = 0.$$

Homework #10: Critical size domain

Exercise 3.2: Using the same analysis for ODEs, prove that (see Textbook [1]):

- 1. a box of any size supports a population up to the carrying capacity (which is 1 in this case). The corresponding solution is u(x, t) = 1;
- 2. an island can support a population if its length L satisfies

$$L > L^* = \pi \sqrt{D/\mu}.$$

If $L < L^*$ each initial population will die out.

Chemotaxis

- Motile cells can move in response to gradients in chemical concentrations, a process known as chemotaxis. This leads to slightly more complicated transport equations.
- The diffusive flux for the population density of the cells, u, is as previously: $J_D = -D\nabla u$.
- The flux due to chemotaxis s taken to be of the form:

$$J_C = u\chi(a)\nabla a$$

where *a* is the chemical concentration and $\chi(a) > 0$ is the chemotactic sensitivity (assuming it is an attractant rather than a repellent).

The cells move by diffusion and in response to a gradient of the chemical. Thus the total flux is

$$J_D + J_C = -D\nabla u + u\chi(a)\nabla a.$$

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Keller-Segel model

Combining the transport of the motile cells, together with a term describing their reproduction and/or death, plus an equation for the chemical which also diffuses and, typically, is secreted and degrades leads to the following equations

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div} (D\nabla u - u\chi(a)\nabla a) + f(u, a) \\\\ \frac{\partial a}{\partial t} = \operatorname{div} (D_a \nabla a) + g(u, a) \end{cases}$$

known as Keller-Segel model for chemotaxis, where f(u, a) is often taken to be a logistic growth,

$$g(u,a) = uh(a) - \frac{V_m u}{K_m u}$$

reflecting the production of chemoattractant by the cells and its degradation according to the Michaelis-Menten kinetics, and $\chi(a)$ is the term describing the chemotaxis.

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