Continuous modelling by PDEs

Critical domain size

Computational Biology

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Critical domain size

- Reaction-diffusion equations are used to estimate the size of a habitat that can support a population. In general, it is not possible to establish a stable surviving population on an island that is too small.
- For pests, like the spruce budworm, information about the critical patch size can be used to determine how to split a woodland into small enough patches so as to prevent the budworms from settling in.

Fisher's reaction-diffusion equation

- To illustrate the use of reaction-diffusion equations in this context, we use Fisher's equation, which shows all necessary features:
$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \mu u(1-u),$$
where $u(x,t)$ is the density of the gene in the population at time t and location x.
- The term $\mu u(1-u)$ is (the well known) Verhulsts law of growth with saturation.
- A partial differential equation on a bounded interval needs boundary conditions.

Boundary conditions

- Appropriate island boundary conditions are the homogeneous Dirichlet boundary conditions given by (for $x \in [0, L]$)
$$u(0,t) = u(L,t) = 0.$$
- We can also study a valley or a box, or a patch with sealing walls. Then no individual can leave the patch. Appropriate box boundary conditions are the homogeneous Neumann boundary conditions given by
$$u_x(0,t) = u_x(L,t) = 0.$$
- Obviously, combinations of island and box boundary conditions can occur if, for example, the patch is bounded by a wall on the one side and by water on the other. We could also include some semi-permeable walls such that only a fraction of the population can leave the domain, etc.

How large must an island or box be to support a population?

This question is equivalent to the following ones:

- What is the critical domain length L^* such that $u = 0$ is stable for $L < L^*$ and unstable for $L > L^*$?
- What is the critical domain length L^* such that a non-tivial stationary solution (steady state) exists for $L > L^*$?

Steady-states

- A steady state satisfies

$$\frac{\partial u}{\partial t} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\mu}{D} u(1-u)$$

- We are looking for solutions $u(x) \neq 0$ which satisfy the correct boundary conditions, and we will use phase plane analysis the equation. With a new variable $v = \frac{\partial u}{\partial x}$ and considering the appropriate boundary conditions, we obtain the system of ODEs

$$\frac{du}{dx} = v, \qquad \frac{dv}{dx} = -\frac{\mu}{D} u(1-u).$$

- We consider Dirichlet boundary conditions (island)

$$u(0) = 0, \qquad u(L) = 0$$

or with Neumann boundary conditions (box)

$$v(0) = 0, \qquad v(L) = 0.$$

Homework #10: Critical size domain

Exercise 3.2: Using the same analysis for ODEs, prove that (see Textbook [1]):

1. a box of any size supports a population up to the carrying capacity (which is 1 in this case). The corresponding solution is $u(x,t) = 1$;
2. an island can support a population if its length L satisfies

$$L > L^* = \pi\sqrt{D/\mu}.$$

If $L < L^*$ each initial population will die out.

Chemotaxis

- Motile cells can move in response to gradients in chemical concentrations, a process known as chemotaxis. This leads to slightly more complicated transport equations.
- The diffusive flux for the population density of the cells, u, is as previously: $J_D = -D\nabla u$.
- The flux due to chemotaxis s taken to be of the form:

$$J_C = u\chi(a)\nabla a$$

where a is the chemical concentration and $\chi(a) > 0$ is the chemotactic sensitivity (assuming it is an attractant rather than a repellent).
- The cells move by diffusion and in response to a gradient of the chemical. Thus the total flux is

$$J_D + J_C = -D\nabla u + u\chi(a)\nabla a.$$

Keller-Segel model

- Combining the transport of the motile cells, together with a term describing their reproduction and/or death, plus an equation for the chemical which also diffuses and, typically, is secreted and degrades leads to the following equations

$$\begin{cases} \dfrac{\partial u}{\partial t} & = & \mathrm{div}\,(D\nabla u - u\chi(a)\nabla a) + f(u,a) \\ \\ \dfrac{\partial a}{\partial t} & = & \mathrm{div}\,(D_a\nabla a) + g(u,a) \end{cases}$$

known as Keller-Segel model for chemotaxis, where $f(u,a)$ is often taken to be a logistic growth,

$$g(u,a) = uh(a) - \frac{V_m u}{K_m u}$$

reflecting the production of chemoattractant by the cells and its degradation according to the Michaelis-Menten kinetics, and $\chi(a)$ is the term describing the chemotaxis.