Continuous modelling by PDEs

Traveling waves

Computational Biology

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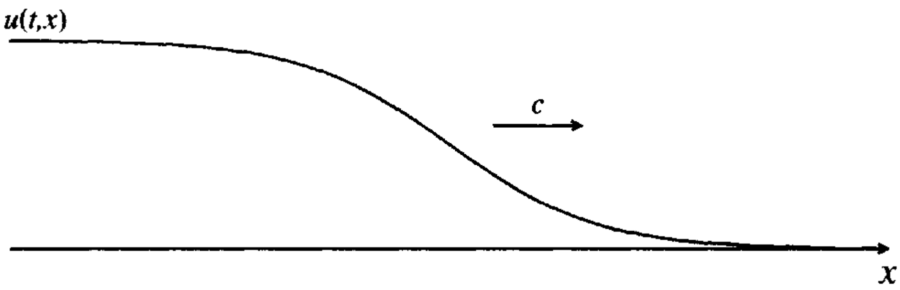
How species can invade new habitats

- To illustrate the use of reaction-diffusion equations in this context, we use Fisher's equation:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \mu u(1-u),$$

in all $\mathbb{R}$, i.e., for $x \in \mathbb{R}$.

- We seek solutions $u(x,t)$ that have the form shown in the next figure, and then move with constant speed c.

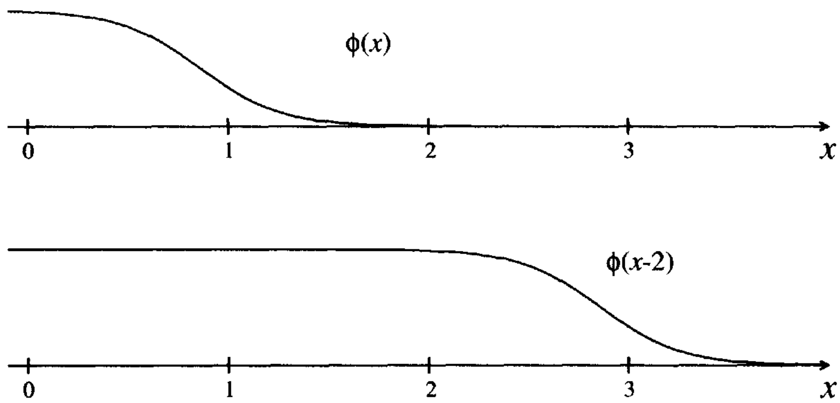


Traveling waves

- A solution of this type can be expressed as

$$u(x,t) = \phi(x - ct).$$

- For $c > 0$, the function $\phi(x - ct)$ is the function $\phi(x)$ shifted to the right by ct (see next figure).



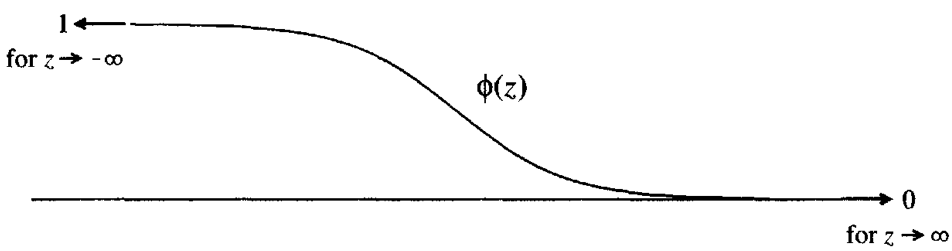
- The parameter c is the wave speed, the new variable $z := x - ct$ is called the wave variable, and the function $\phi(z)$ is called the wave profile.

Traveling wave ansatz

- A solution of this type can be expressed as

$$u(x,t) = \phi(x - ct), \quad \phi(-\infty) = 1, \quad \phi(+\infty) = 0.$$

Instead of boundary conditions, we have conditions at $\pm\infty$.



- For $x \to -\infty$, the population already has grown to its carrying capacity (1 in this case), and for $x \to +\infty$, the population has not arrived yet.

Steady-states

- From the previous equation we wave

$$\frac{\partial u}{\partial t}(x,t) = -c\phi'(x-ct), \quad \frac{\partial^2 u}{\partial x^2}(x,t) = \phi''(x-ct),$$

  where $\phi' = \frac{d\phi}{dz}$, with $z = x - ct$.
- The Fisher equation reduces to the following ODE for $\phi(z)$

$$-c\phi' = D\phi'' + \mu\phi(1-\phi).$$

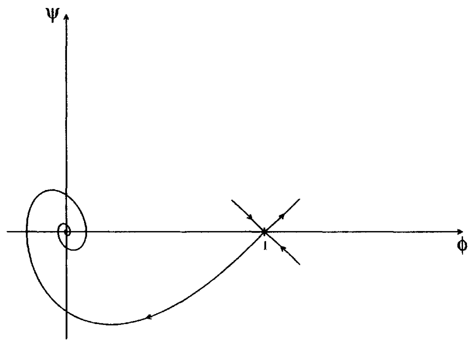
- Introducing a new variable $\psi = \phi$ we have the system of ODEs

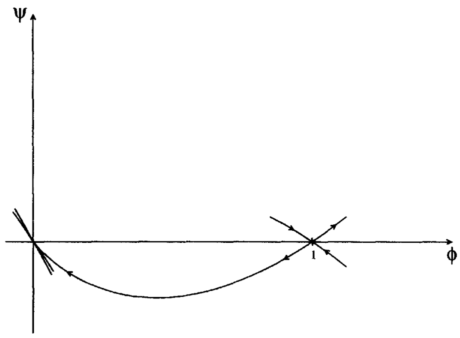
$$\phi' = \psi, \qquad \psi' = -\frac{c}{D}\psi - \frac{\mu}{D}\phi(1-\phi).$$

- Exercise 3.3: The equilibria of the system are $P_1 = (0,0)$ and $P_2 = (1,0)$. Using the linearization, prove that the point $P_1 = (0,0)$ is stable for $c > 0$. It is a stable spiral for $c < 2\sqrt{D\mu}$ and a stable node for $c > 2\sqrt{D\mu}$. The point $P_2 = (1,0)$ is always a saddle.

Minimal speed

- In the phase portrait of system of ODEs , we have to find a connection from the saddle $(1,0)$ to the stable point $(0,0)$. We show these connections for $c < 2\sqrt{D\mu}$ in the next figure (left), and for $c > 2\sqrt{D\mu}$ in next figure (right).





- The function ϕ is the profile of the population density; hence it has to be nonnegative. Thus solutions for $c < 2\sqrt{D\mu}$ are not biologically relevant. They correspond to an oscillating front. We obtain that the minimal speed c^* for which a wave front solution exists is given by $c^* = 2\sqrt{D\mu}$.

Minimal speed and linear conjecture

- As we saw, the minimal wave speed c^* is exactly that value where $(0,0)$ changes from spiral into node.
- Exercise 3.4: By linearization, prove that the solution near $(0,0)$ behaves like $e^{-c^*/(2D)}$, where $c^* = 2\sqrt{D\mu}$.
- Linear conjecture: In many cases, it is enough to measure the decay rate of the profile for large x to get a good approximation for the minimal wave speed c^*.
- Exercise 3.5: Prove that the minimal wave speed for the general Fisher equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u),$$

is $c^* = 2\sqrt{Df'(0)}$.

Homework #11: Dingoes in Australia (see Textbook [1])

Exercise 3.6: A dingo population which lives in the eastern parts of Australia is prevented from invasion to the west by a fence which runs north-south. Consider the case in which the fence breaks somewhere (at time $t = 0$). Two farms, A and B are located on the west side of the fence. The distance from farm A to the fence is 100 miles, and the distance from farm A to B is another 100 miles. The farmers would like to know how long it would take for the dingoes to reach their farms. We model the spread of the dingo population with Fisher's equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u(1-u).$$

1. The region between farm A and the fence is flat and the diffusion constant is $D_1 = 100$ (miles^2/month). When does the dingo population reach farm A?
2. The region between farm A and B has rocks and slope; hence there the diffusion constant is $D_2 = 50$ (miles^2/month). When does the dingo population reach farm B?