Discrete-Time Models

Systems of discrete-time equations

Computational Biology

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Love affairs

- R_n: Romeo's love/hate for Juliet on day n; J_n Juliet's love/hate for Romeu on day n.
- $R_n > 0$: Romeo loves Juliet; $R_n < 0$: Romeo hates Juliet; $R_n = 0$: Romeo is neutral towards Juliet.
- $J_n > 0$: Juliet loves Romeo; $J_n < 0$: Juliet hates Romeu; $J_n = 0$: Juliet is neutral towards Romeo.

Love affairs: mathematical model

- Add linear terms that represent the response of Romeo and Juliet to the feelings of the other:

$$\begin{cases} R_{n+1} = a_R R_n + p_R J_n, \\ J_{n+1} = a_J J_n + p_J R_n, \end{cases} \qquad a_R, a_J > 0, \;\; p_R, p_J \in \mathbb{R}.$$

- The sign of the p parameter determines a particular romantic style:
  - if $p_R > 0$, then Romeo gets excited by Juliet's love for him, while he gets discouraged by Juliet's hate for him;
  - if $p_R < 0$, then Juliet's hate for him contributes to his love for her, while Juliet's love for him contributes to his hate for her.

Love affairs: dynamics

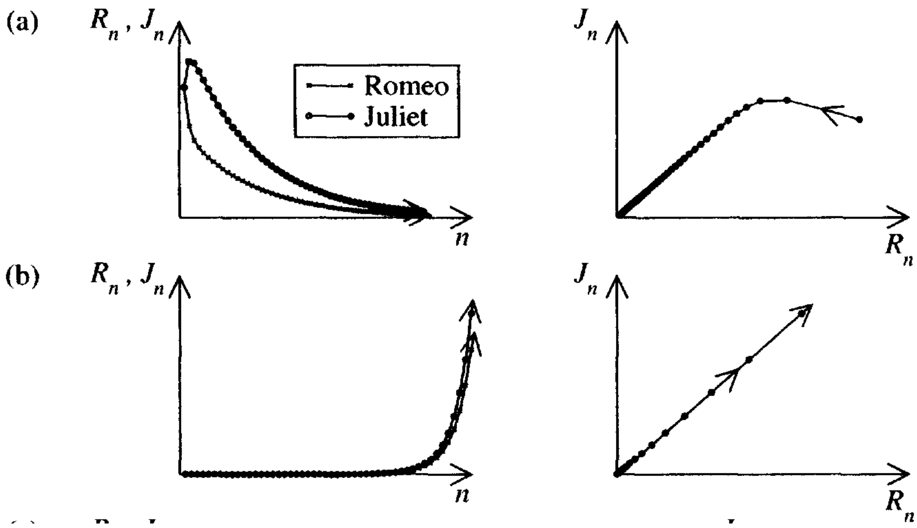
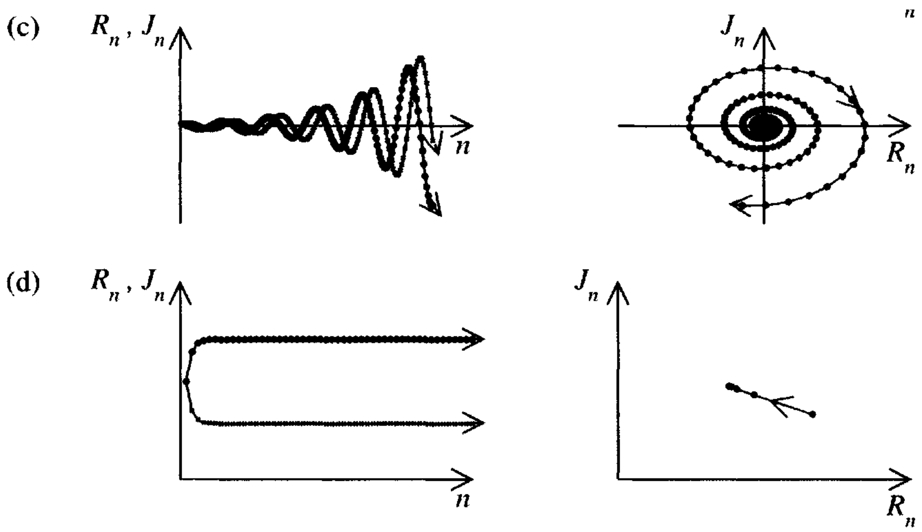


Figure: Graphs in the left column show R_n and J_n as functions of n. Graphs in the right column show the orbits in the (R_n, J_n) phase plan. $R_0, J_0 = 1$ a) $a_R = 0.5, a_J = 0.7,\ p_R = 0.2,\ p_J = 0.5$. (b) $a_R = 0.5,$ $a_J = 0.7,\ p_R = 0.7,\ p_J = 0.9$.

Love affairs: dynamics



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Fixed points and linear stability

Consider the two-dimensional discrete-time system:

$$\begin{cases} x_{n+1} = f(x_n, y_n), \\ y_{n+1} = g(x_n, y_n). \end{cases}$$

Fixed points: all (x^*, y^*) such that $x^* = f(x^*, y^*)$ and $y^* = g(x^*, y^*)$.

Consider Jacobian matrix at the fixed point (x^*, y^*):

$$J = \begin{bmatrix} \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\ \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*) \end{bmatrix}$$

The dynamics are determined by the size of the eigenvalues of J:

- (x^*, y^*) is stable if all eigenvalues of J have magnitude < 1;
- (x^*, y^*) is unstable if at least one of the eigenvalues has magnitude > 1.

Eigenvalues

To calculate the eigenvalues, λ we solve the characteristic equation

$$\det(J - \lambda I) = 0.$$

For two-dimensional systems, that is

$$\lambda^2 - \mathrm{tr}J\lambda + \det J = 0.$$

Jury conditions: For two-dimensional systems,

$$|\mathrm{tr}J| < 1 + \det J < 2,$$

are necessary and sufficient conditions for all eigenvalues of J to have magnitude less than 1, that is, for the fixed point in question to be stable.

Host-parasitoid models

Host-parasitoid models: models that address the life cycles of two interacting species of insects, one a host and the other a parasitoid.

Example of a Host-parasitoid model:

- Parasitoids are insects whose females lay their eggs in or on the bodies of the host insects. Parasitoid eggs develop into parasitoid larvae at the expense of their host.
- Hosts that have been parasitized thus give rise to the next generation of parasitoids, while only hosts that are not parasitized will give rise to the next generation of hosts.

Example of a Host-parasitoid model

Let H_n (resp. P_n) be the number of the hosts (resp. parasitoids) at generation n and $f(H_n, P_n)$ be the fraction of hosts that are not parasitized.

- $f(H_n, P_n)H_n$ = number of hosts not parasitized,
- $(1 - f(H_n, P_n))H_n$ = number of hosts parasitized.

Two assumptions:

- The host population grows geometrically in the absence of the parasitoids, with reproductive rate $k > 1$.
- The average number of eggs laid in a single host that give rise to adult parasitoids is c.

Host-parasitoid model:

$$\begin{cases} H_{n+1} = kf(H_n, P_n)H_n \\ P_{n+1} = c(1 - f(H_n, P_n))H_n. \end{cases}$$

Nicholson and Bailey's model

We assume that:

- encounters between hosts and parasitoids occur at random and are independent;
- the number of encounters is proportional to the product H_nP_n, that is, aH_nP_n.

The average number of encounters per host is thus

$$\nu = \frac{aH_nP_n}{H_n} = aP_n.$$

Since the encounters are random and independent, they follow a Poisson process

$$p(i) = \frac{\nu^i e^{-\nu}}{i!} = \frac{(aP_n)^i e^{-aP_n}}{i!},$$

where $p(i)$ is the probability that a host experiences i encounters.

Nicholson and Bailey's model

The fraction of hosts not parasitized is

$$f(H_n, P_n) = p(0) = e^{-\nu} = e^{-aP_n}.$$

Nicholson and Bailey's classic model:

$$\begin{cases} H_{n+1} = ke^{-aP_n} H_n, \\ P_{n+1} = c(1 - e^{-aP_n}) H_n. \end{cases}$$

Nicholson and Bailey's model:dynamics

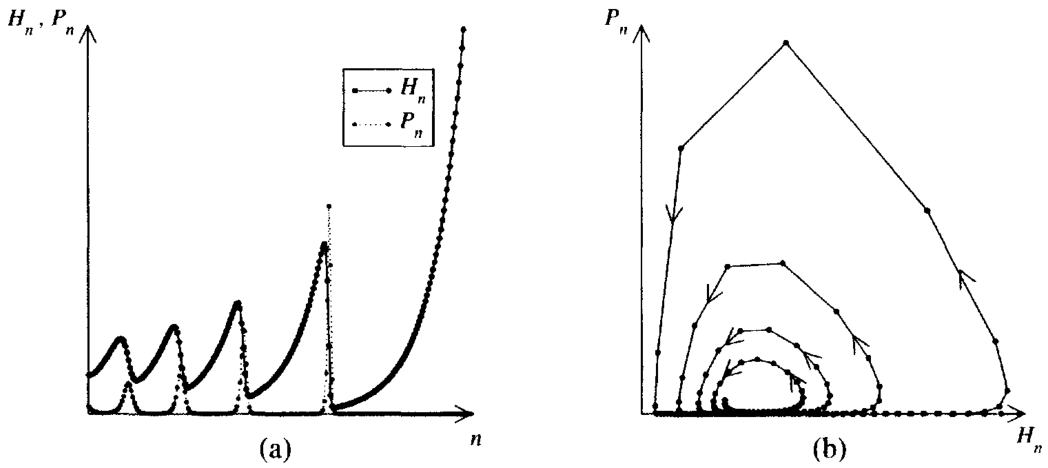

Figure: The Nicholson-Bailey modelexhibits growing oscilations. The left panel shows a typical solution for H_n and P_n as functions of n. The right panel shows the orbit in the (P_n, H_n) phase plane. Model parameters used are $k = 1.05$, $a = 0.005$, and $c = 3$, and initial conditions are $H_0 = 50$ and $P_0 = 10$.

Fixed points and stability

The Nicholson-Bailey model has two fixed points:

- the trivial fixed point, $(H_1^*, P_1^*) = (0, 0)$;
- the nontrivial fixed point

$$(H_2^*, P_2^*) = \left(\frac{k \ln k}{ac(k-1)}, \frac{\ln k}{a} \right),$$

provided that $k > 1$.

Important question: Do we have a situation where there is coexistence of the two insect species?

To answer we have to study the stability of the nontrivial fixed point.

Fixed points and stability

The Jacobian matrix, evaluated at the nontrivial fixed point, is

$$J(H_2^*, P_2^*) = \begin{bmatrix} 1 & -\dfrac{k \ln k}{c(k-1)} \\ \dfrac{c(k-1)}{k} & \dfrac{\ln k}{k-1} \end{bmatrix},$$

so that

$$\mathrm{tr} J = 1 + \frac{\ln k}{k-1}, \qquad \det J = \ln k + \frac{\ln k}{k-1}.$$

Since $k > 1$:

- the first of the Jury conditions always is satisfied;
- the second Jury condition can never be satisfied ($\det J > 1$).

Fixed points and stability

- The nontrivial fixed point, (H_2^*, P_2^*), is always unstable.
- Instability of the nontrivial steady state in itself does not preclude coexistence of the two insect species. For example, coexistence could come in the form of a stable cycle.
- However, for the Nicholson-Bailey model, no choice of parameter values leads to coexistence.

Beddington model

Let us modify the equation for the host population to

$$H_{n+1} = e^{r(1-H_n/K)} H_n.$$

Their full host-parasitoid model thus reads

$$\begin{cases} H_{n+1} = e^{r(1-H_n/K)} e^{-aP_n} H_n, \\ P_{n+1} = c(1 - e^{-aP_n}) H_n. \end{cases}$$

Beddington model:dynamics

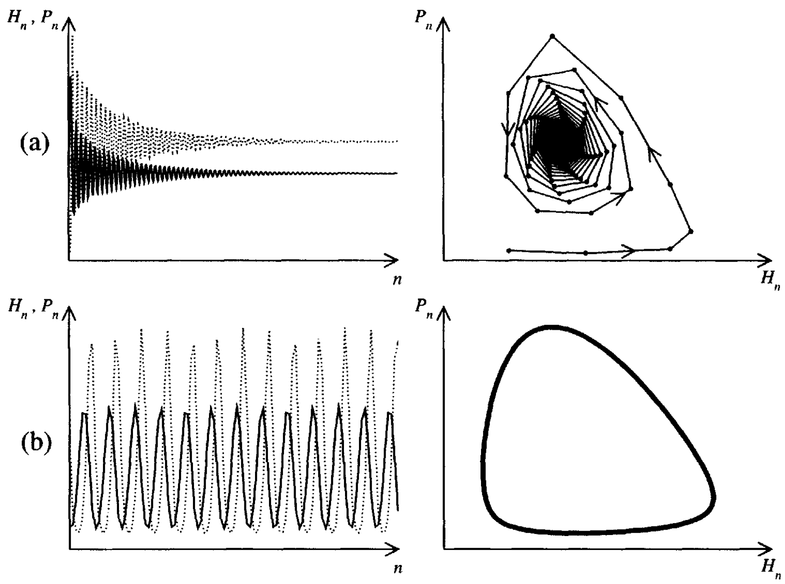

Figure: Two types of behavior exhibited by the Beddington model. Graphs in the left column show H_n and P_n as functions of n; graphs in the right column show corresponding orbits in the (H_n, P_n) phase plane, (a) The host and parasitoid coexist at a stable fixed point ($K = 200$). (b) The host and parasitoid coexist in a stable cycle ($K = 250$). Other model parameters are $r = 1.1$, $a = 0.005$, and $c = 3$.

Homework #3

Exercise 1.7: Study the stability of the nontrivial fixed points of the Beddington model, reproducing the figures in the previous slide. Explain the situations where the host and parasitoid coexist.

Exercise 1.8: Consider the discrete-time model developed for the relationship between Romeo and Juliet.

1. Study the stability of the fixed point $(R^*, J^*) = (0, 0)$.
2. Draw the trajectories and the phase plane for different sets of parameters in order to illustrate the different types of behaviour. Hint: use the parameters corresponding to the plots in the figures.